ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

W. ALLEGRETTO

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 25, nº 1 (1971), p. 41-46
<http://www.numdam.org/item?id=ASNSP_1971_3_25_1_41_0>

A COMPARISON THEOREM FOR NONLINEAR OPERATORS

W. ALLEGRETTO

Sturmian comparison theorems for the solutions of non-selfadjoint linear uniformly elliptic equations and inequalities were first obtained by M. H. Protter [5], in 1959, and more recently, by C. A. Swanson [6], [7]. These results stated that if a certain differential equation or inequality was satisfied in a domain G by a function which vanished on the boundary of G, then every solution v of a related non-selfadjoint differential equation or inequality, would vanish somewhere in $\bar{G}$. K. Kreith [4], by means of the Hopf maximum principle, obtained a « strong » version of the results of C. A. Swanson for domains which have a boundary with bounded curvature. That is, he showed that under the above assumptions v would vanish in G rather than in $\bar{G}$. He also obtained, [3], by spectral considerations, strong comparison theorems for special pairs of differential inequalities, under more general boundary conditions, but also under the assumption that conditions were so regular that a Green's function could be constructed for the operators in question.

The main purpose of this paper is to obtain strong comparison theorems for the generalized solutions of nonlinear elliptic-parabolic equations, without assuming any regularity properties of the boundaries of the domains involved, and without imposing conditions that ensure the validity of the maximum principle, or the existence of a Green's function. The conditions on the regularity of the boundary are replaced by the weaker assumption that the functions involved belong to suitable Sobolev spaces. Specialization of the results obtained to the case of a smooth domain and a linear uniformly elliptic inequality still yields sharper results than those previously available, since an extension of the basic Lemma in [6] is also obtained.

Pervenuto alla Redazione l'11 Marzo 1970.

Let G denote a domain, not necessarily bounded, of the n dimensional Euclidean space E^n. The points of E^n are denoted by $x = (x_1, \dots, x_n)$ and differentiation with respect to x_i, in the L^2 sense, by D_i for $i = 1, \dots, n$. We consider the nonlinear elliptic-parabolic operator L with real coefficients and formally defined by:

$$Lv = -\sum_{i,j=1}^{n} D_i [A_{ij}(x, v) D_j v] + 2 \sum_{j=1}^{n} B_j(x, v) D_j v + C(x, v) v .$$

Let D denote the subset of $H^1(G)$ such that if $v \in D$ then:

(i) $A_{ij}(x, v(x))$, $B_j(x, v(x))$, $C(x, v(x))$ are measurable and essentially bounded in G for $i, j = 1, \dots, n$, and

(ii) there exists an essentially bounded measurable function H such that the matrix $M = \begin{pmatrix} (A_{ij}(x, v)) & (B_j(x, v))^T \\ (B_j(x, v)) & H \end{pmatrix}$ is symmetric non-negative definite a. e. G.

We note that a sufficient condition for H to exist is that the matrix (A_{ij}) be uniformly positive definite, but weaker conditions can be stated by using limiting procedures. It is, in fact, easy to construct examples of parabolic non selfadjoint operators for which such a function exists.

We now associate with L the functionals B, F where for $u, w \in H^1(G)$ and $v \in D$,

$$B(v, u, w) = \int_G \Sigma A_{ij}(x, v) D_i u D_j w + 2w \Sigma B_j(x, v) D_j u + C(x, v) uw$$

$$F(v, u, w) = B(v, u, w) + \int_G Huw.$$

The function $v \in D$ is then said to satisfy the inequality $Lv \geq 0$ (resp. $Lv \leq 0$) in G iff $B(v, v, \Phi) \geq 0$ (resp. $B(v, v, \Phi) \leq 0$) for every $\Phi \in C_0^\infty(G)$, $\Phi \geq 0$ in G. If simultaneously $Lv \geq 0$ and $Lv \leq 0$, then v is said to satisfy the identity $Lv = 0$ in G.

Since we are concerned, unlike the previous authors, with the behaviour of a generalized solution v of a differential equation, the classical conclusion of a Sturmian theorem, i. e. that there exists $x_0 \in G$ (or $\bar{G}$) such that $v(x_0) = 0$, is somewhat vacuous. We therefore replace it by its natural extension, which is to show that the subset of G where v is non-negative and the subset where v is non-positive have positive measure. If conditions are sufficiently regular so that the Hopf maximum principle can be applied, then this is known to be equivalent to v vanishing somewhere in G, [4].

THEOREM 1. Let $u \in H_0^1(G)$, $v \in D$ and $F(v, u, u) < 0$. If $Lv \geq 0$ (resp. $Lv \leq 0$) then $\mu\{x : v(x) \leq 0\} > 0$ (resp. $\mu\{x : v(x) \geq 0\} > 0$) where μ denotes the Lebesque measure.

PROOF. Assume $Lv \geq 0$, $v > 0$ a. e. G. Since $u \in H_0^1(G)$, there exists a function $\Phi \in C_0^\infty(G)$ such that $F(v, \Phi, \Phi) < 0$. We note that for any positive constant ε, the functions $\Phi^2(v+\varepsilon)^{-1}$, $\Phi(v+\varepsilon)^{-1}$ belong to $H_0^1(G)$ and their derivatives are given by the classical formulas. Now by an identity of C. A. Swanson, [6], we obtain:

$$(1) \int_G (v+\varepsilon)^2 \Sigma A_{ij} D_i(\Phi/v+\varepsilon) D_j(\Phi/v+\varepsilon) + 2\Phi(v+\varepsilon)\Sigma B_j D_j(\Phi/v+\varepsilon) + H\Phi^2 =$$
$$= F(v, \Phi, \Phi) - B(v, v+\varepsilon, \Phi^2/v+\varepsilon)$$

and by the choice of H we have:

$$0 \leq F(v, \Phi, \Phi) - B(v, v, \Phi^2/v+\varepsilon) - \int_G C\Phi^2 \frac{\varepsilon}{v+\varepsilon}$$

and therefore for every ε,

$$0 \leq F(v, \Phi, \Phi) - \int_G C\Phi^2 \frac{\varepsilon}{v+\varepsilon}.$$

But $0 \leq \varepsilon/v+\varepsilon \leq 1$ and $\lim_{\varepsilon \to 0^+} (\varepsilon/v+\varepsilon) = 0$ a. e. G. Passing to the limit as $\varepsilon \to 0$, obtain $0 \leq F(v, \Phi, \Phi)$. The contradiction establishes the theorem. If $Lv \leq 0$, then $B(v, -v, \Phi) \geq 0$ and we proceed as before.

COROLLARY 2. Let u satisfy the conditions of Theorem 1 and assume $Lv = 0$. Then $\mu\{x : v(x) \leq 0\} > 0$ and $\mu\{x : v(x) \geq 0\} > 0$.

Simple examples can be constructed to show that the conclusion of Theorem 1 cannot be strengthened to $\mu\{x : v(x) < 0\} > 0$ (resp. $\mu\{x : v(x) > 0\} > 0$) even if v is assumed non-trivial, since the maximum principle need not hold.

If the operator is essentially elliptic, then we may choose $H = \sum_i B_i h_i$ where $h_i = \sum_j B_j A^{ij}$ and $(A^{ij}) = (A_{ij})^{-1}$. In this case the following result, where the condition on F is weaker, is valid. (A one dimensional version of which has been obtained by C. A. Swanson [8]).

THEOREM 3 (*Elliptic Case*). Let $u \in H_0^1(G) \cap C^1(G)$, $v \in D$, $u > 0$ in G, $F(v, u, u) \leq 0$ and $Lv \geq 0$. Furthermore assume that (A_{ij}) is essentially positive definite and that $h_i \in C^1(G)$ for $i = 1, \dots, n$. Then $v > 0$ a. e. G iff $v = e^w u$ a. e. G for some C^1 function w such that grad $(w) = (h_1, \dots, h_n)$.

PROOF: Assume $v > 0$ a. e. G. Then, except for a set of measure zero, $G = \bigcup_1^\infty G_\beta$ where $G_\beta = \{x : v(x) > \beta^{-1}, \ |D_i v| < \beta$ for $i = 1, \dots, n\}$.

Let $\{\Phi_m\}$ be a sequence of $C_0^\infty(G)$ functions such that $\Phi_m \to u$ in $H^1(G)$, then we obtain from the left hand side of (1), for any integer $\beta > 0$,

$$\lim_{m \to \infty} \lim_{\varepsilon \to 0} \left[\int_{G_\beta} (v + \varepsilon)^2 \, \Sigma A_{ij} D_i (\Phi_m / v + \varepsilon) D_j (\Phi_m / v + \varepsilon) + \right.$$

$$\left. + 2 \Phi_m (v + \varepsilon) \Sigma B_j D_j (\Phi_m / v + \varepsilon) + H \Phi_m^2 \right] = 0.$$

Since the matrix M is symmetric, non-negative definite, we conclude that in G_β the $n + 1$-vector $\{vD_1 u - uD_1 v, \dots, vD_n u - uD_n v, uv\}$ must essentially lie in the kernel of M. A simple computation then shows that $vD_i u - uD_i v = - h_i uv$ a. e. G_β for every integer β and therefore a. e. G. Now let S denote an open sphere such that $\bar{S} \subset G$. Then $v/u \in H^1(S)$ and therefore $D_i(v/u) = h_i v/u$ a. e. S. As a consequence we obtain $D_i(h_j) = D_j(h_i)$ in S for $i, j = 1, \dots, n$ and therefore, by Poincare's Lemma, there exists a function w_s such that grad $(w_s) = (h_1, \dots, h_n)$. Clearly we must have $D_i(e^{-w_s} v/u) = 0$ a. e. S for $i = 1, \dots, n$ and therefore by Sobolev's integral identity, it follows that $v = e^{w_s} u$ a. e. S with the integration constant absorbed in w_s. Now since G can be written as the countable union of spheres, $G = \bigcup_1^\infty S_i$, we can define a function w as follows: if $x \in G$ then $x \in S_i$ for some i, and we set $w(x) = w_{s_i}(x)$. It follows that $v = e^w u$ a. e. G.

We remark that the condition to be satisfied by F in the theorem is more general than the one previously required for the basic Lemma in [6], and [7], where a strict inequality is wanted. From the proof of Theorem 3 the following corollary is immediate:

COROLLARY 4. Let the conditions of Theorem 3 hold and furthermore assume there exists $x_0 \in G$ and also integers i, j such that $D_i(h_j)(x_0) \neq D_j(h_i)(x_0)$. Then $\mu\{x : v(x) \leq 0\} > 0$.

If the operator is «symmetric», i. e. $(B_j) \equiv 0$, then $H \equiv 0$ and Theorem 3 reduces to:

COROLLARY 5. (*Symmetric Case*) Let the conditions of Theorem 3 hold. Then $v > 0$ a. e. G iff $v = \lambda u$ a. e. G for some constant λ.

If we assume that L is linear, G is bounded, and conditions are sufficiently regular then the theory of Krasnoselskii [2], is valid, and we can conclude that, for vanishing boundary conditions, L has a real eigenvalue λ_0 in G with a positive eigenvector v_0. It is known [1], that $\lambda_0 \leq \mu_0$ where μ_0 is the smallest eigenvalue of the symmetric operator $\dfrac{L+L^*}{2}+H$, L^* denoting the formal adjoint of L. By means of Theorem 3 we can now state :

COROLLARY 6. $\lambda_0 = \mu_0$ iff there exists a function w such that grad $(w) = (h_1, \ldots, h_n)$. Furthermore, in such a case, v_0 is given by $v_0 = \tau e^w u_0$ where τ is an arbitrary non-zero constant and u_0 is the eigenvector corresponding to μ_0.

The above results yield immediate comparison theorems. Let L' denote the operator formally defined by :

$$L' u = -\Sigma D_i [a_{ij}(x, u) D_j u] + 2 \Sigma b_j(x, u) D_j u + c(x, u) u.$$

By D' we mean the subset of $H^1(G)$ such that $u \in D'$ implies $a_{ij}(x, u(x))$, $b_j(x, u(x))$, $c(x, u(x))$ are measurable and essentially bounded for $i, j = 1, \ldots, n$.

THEOREM 7. Let u be a non-trivial function in $H_0^1(G) \cap D'$ such that $L' u = 0$, and assume $v \in D$ satisfies $Lv = 0$. If the matrix

$$V(x, v, u) = \begin{pmatrix} (a_{ij}(x, u)) & (b_j(x, u))^T \\ (b_j(x, u)) & c(x, u) \end{pmatrix} - \begin{pmatrix} (A_{ij}(x, v)) & (B_j(x, v))^T \\ (B_j(x, v)) & C(x, v) + H \end{pmatrix}$$

is positive definite a. e. G then $\mu\{x : v(x) \leq 0\} > 0$ and $\mu\{x : v(x) \geq 0\} > 0$.

If the operator L is also assumed to be elliptic, then clearly theorems analogous to Theorem 7 can be stated with weaker conditions imposed on V. It is also evident that the pointwise conditions on V can be replaced by more general integral conditions.

In conclusion, we remark that, in a sense, the above results are really linear, since once v and u are given, the operators involved become linear. However, the comparison theorems are also meaningful for those pairs of nonlinear operators L, L' for which the matrix V can be shown to be positive definite regardless of u and v. Of particular interest are the cases where one of the operators is linear, for then the known linear theory can be drawn upon to obtain oscillation or nonoscillation criteria for the other operator. Simple physical examples of such cases are furnished by Mathieu's equation and by Duffing's equation.

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ADDED IN PROOF: Since the completion of this paper, new related results have been obtained by several authors:

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