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E. B. FABES
N. M. RIVIERE

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L^p-ESTIMATES NEAR THE BOUNDARY FOR SOLUTIONS OF THE DIRICHLET PROBLEM

by E. B. FABES and N. M. RIVIERE

Introduction.

In this paper we consider solutions of the initial-Dirichlet boundary value problem in the cylinder, $\Omega \times (0, T)$, for parabolic equations of order $2b$, and solutions of the Dirichlet problem in the domain Ω for strongly elliptic equations of order $2b$. (b is any integer ≥ 1). The coefficients of each operator in question are assumed to be bounded and measurable and in addition those of highest order are taken to be Hölder continuous in the closure of the domain of definition.

For the parabolic case we estimate the L^p-norm $(1 < p \leq \infty)$ of the solution, $u(\bar{x}, t)$, over all lateral surfaces near and parallel to $\partial\Omega \times (0, T)$ ($\partial\Omega =$ boundary of Ω) by a constant times the sum of the L^p-norm over Ω of the initial value, $u(\bar{x}, 0)$, the L^p-norm over $\partial\Omega \times (0, T)$ of u, and certain « negative » norms over $\partial\Omega \times (0, T)$ of the remaining Dirichlet data. In the elliptic case we estimate the L^p-norm of the solution, $u(\bar{x})$, over any surface near and parallel to $\partial\Omega$ by a constant times the sum of the

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L^p-norm of u over $\partial\Omega$, « negative » norms over $\partial\Omega$ of the remaining Dirichlet data, and the L^1-norm over Ω of u. (See statement of theorems in section 2). For elliptic operators one of the inequalities we state (theorem 5.3) was announced by Agmon in [1] but there the coefficients of highest order were assumed to have derivatives up to that order continuous in $\bar{\Omega}$.

The estimates in the parabolic case allow us to prove existence and uniqueness results for the initial-Dirichlet problem when the restriction of the solution bolongs only to L^p (theorems 4.1.6, 4.1.7, 4.2.2, 4.2.3).

Section 1 of this paper is concerned with defining the type of operators which will be used in the statements and proofs of the results in this work. Section 2 states the main results of the paper. These results are proved in sections 4 and 5. In section 3 we examine the problem in the half-space and obtain the necessary estimates for sections 4 and 5.

1. Basic definitions.

(1.1) Definition of parabolic singular integral operator, symbol, and the operators Λ^{-k}, Λ^k on $R^n \times (0, T)$.

In this section and elsewhere we will denote points in R^n by x, z, or w and positive real numbers by t or s. If f and g are measurable functions defined in some Euclidean space we denote by $f * g$ the convolution of f with g and by $\mathcal{F}(f)$ the Fourier transform of f. For $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i non-negative integer, we set $|\alpha| = \sum_{i=1}^{n} \alpha_i$, $D_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. We set $S_T = R^n \times (0, T)$ and $R_+^{n+1} = R^n \times (0, \infty)$.

DEFINITION. A parabolic singular integral operator (p. s. i. o.) has the form

$$(1.1.1) \qquad Sf(x, t) = a(x, t) f(x, t) + \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_{R^n} k(x, t; x - z, t - s) f(z, s)\, dz\, ds,$$

where $a(x, t)$ is bounded, uniformly continuous on S_T, and $k(x, t; z, s)$ satisfies

a) $k(x, t : z, s) = 0$ for $s < 0$

b) for every $\lambda > 0$, $k(x, t; \lambda z, \lambda^{2b} s) = \lambda^{-n-2b} k(x, t; z, s)$

c) $\int k(x, t; z, 1)\, dz = 0$

d) $\mathcal{F}_\xi (k(x, t; \xi, 1))(z) = \widehat{\Omega}(x, t; z)$ satisfies

($\mathcal{F}_\xi$ means Fourier transform only in ξ)

$$|D_z^\alpha \Omega(x,t;z)| \leq B_\alpha \exp(-A_\alpha |z|^{2b}),$$

with A_α and B_α depending only on α, and

$$|D_z^\alpha \widehat{\Omega}(x_1,t_1;z) - \widehat{\Omega}(x_2,t_2;z)| \leq \omega_\alpha(|x_1-x_2| + |t_1-t_2|^{1/2b}) \exp(-A_\alpha|z|^{2b})$$

where $\lim\limits_{\delta \to 0+} \omega_\alpha(\delta) = 0$.

Under the above assumptions on the kernel the limit in 1.1.1 is known to exist in $L^p(S_T)$ $1 < p < \infty$. In this paper the functions $a(x,t)$ which arise will satisfy a uniform Hölder-continuity condition in $\overline{S}_T$ and $\omega_\alpha(\delta) \leq C_\alpha \delta^\gamma$ where γ is a fixed number satisfying $0 < \gamma < 1$.

(1.1.2) Definition. We will denote by $\mathcal{J}_p(S_T)$ $(1 \leq p \leq \infty)$ the class of operators J mapping $L^p(S_T) \to L^p(S_T)$ and satisfying for any $a \geq 0$; if $X_{(a,b)}$ denotes the characteristic function of (a,b), then

i) $JX_{(a,\infty)} = X_{(a,\infty)} JX_{(a,\infty)}$ (i. e. if $f(x,t) = 0$ for $t < a$ then $Jf(x,t) = 0$ for $t < a$).

ii) $\|X_{(a,a+\varepsilon)} J(X_{(a,a+\varepsilon)} f)\|_{L^p(S_T)} \leq \omega(\varepsilon) \|X_{(a,a+\varepsilon)} f\|_{L^p(S_T)}$ where $\omega(\varepsilon) \to 0$ with ε, uniformly in $a \geq 0$. We set $\mathcal{J}(S_T) = \bigcap\limits_{1 \leq p \leq \infty} \mathcal{J}_p(S_T)$.

If an operator S is of the form 1.1.1 we define the «symbol of S», denoted by $\sigma(S)(x,t;z,s)$ to be the function.

$$(1.1.3) \quad \ldots a(x,t) + \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_\varepsilon^R \int_{R_n} k(x,t;w,r)\, e^{i(z,w)} e^{isr}\, dw\, dr, =$$

$$= a(x,t) + \int_0^\infty \frac{\widehat{\Omega}(x,t;zr^{1/2b})}{r} e^{isr}\, dr,$$

the above limit existing for $(x,t) \in \overline{S}_T$ and $(z,s) \neq (0,0)$.

Throughout this paper we will rely on the results in ([4], [8]) relating the notions of a parabolic singular integral operator, its symbol, and the operators $\mathcal{J}_p(S_T)$.

In this section we also want to define an operator Λ^{-k} $(k > 0)$ which will act as a «fractional» integration operator for functions defined on S_T. We will also define its inverse, denoted by Λ^k.

We begin by using the fundamental solution of $(-1)^b \Delta^b + D_t$ $(\Delta = \Sigma D_{x_i}^2)$ defined by $\Gamma(x,t) = \mathcal{F}(e^{-|z|^{2b}t})(x)$ for $t > 0$ and 0 for $t \leq 0$.

(1.1.4) For $k > 0$ set $\Lambda^{-k}(x, t) = \frac{1}{\gamma(k/2b)} t^{\frac{k}{2b}-1} \Gamma(x, t)$ where $\gamma(\cdot)$ is the gamma function.

It is not difficult to see that $\Lambda^{-k}(x, t) \in \mathcal{S}'(R^{n+1})$ = space of tempered distributions on R^{n+1} and that for k a positive integer $\leq 2b$, the Fourier transform of $\Lambda^{-k}(x, t)$ in the distribution sense is $(|x|^{2b} - it)^{-k/2b}$ (see [5]).

For a given $g \in L^p(S_T)$, $1 \leq p \leq \infty$, we set

$$\Lambda^{-k}(g)(x, t) = (\Lambda^{-k} * g)(x, t) = \int_0^t \int_{R^n} \Lambda^{-k}(x - z, t - s)\, g(z, s)\, dz\, ds. \tag{1.1.5}$$

We define Λ^0 to be the identity. The function $u(x, t) = \Lambda^{-k}(g)(x, t)$ has the property that for $g \in L^p(S_T)$, $1 < p < \infty$, $D_x^\alpha u(x, t) \in L^p S_T)$ for $|\alpha| \leq k$ and $D_t \Lambda^{-2b+k}(u) \in L^p(S_T)$. In fact this last statement characterizes those functions $u(x, t) \in L^p(S_T)$ for which $u(x, t) = \Lambda^{-k}(g)$ for some $g \in L^p(S_T)$ $(1 < p < \infty)$. (See [5]).

Now for k a positive integer $\leq 2b$ we define the inverse to the operator Λ^{-k}, denoted by Λ^k, by

$$\ldots \Lambda^k(f)(x, t) = \sum_{|\beta| = k} K_\beta D_x^\beta f + D_t \Lambda^{-2b+k} f \tag{1.1.6}$$

where K_β is the p. s. i. o. with $\sigma(K_\beta)(x, t) = i^{2b-k} \frac{P_\beta(x)}{(|x|^{2b} - it)^{1-k/2b}}$ and $P_\beta(x)$ is the homogeneous polynomial of degree $2b - k$ defined by $|x|^{2b} = \sum_{|\beta|=k} P_\beta(x) x^\beta$. The operator Λ^k is well defined on functions $f \in \mathcal{S}(R^{n+1})$ (= space of rapidly decreasing functions on R^{n+1}) which vanish for $t \leq 0$. For such functions f it is easy to see that $\mathcal{F}(\Lambda^k f)(x, t) = (|x|^{2b} - it)^{k/2b} \mathcal{F}(f)$.

(1.2) Definitions of Λ^{-k} and Λ^k on $\partial\Omega \times (0. T)$.

Throughout this paper Ω will denote a bounded, smooth domain in R^{n+1}. More precisely we assume there is a number $\delta_0 > 0$ such that if $\Omega_\delta = \{\bar{x} \in \Omega : \operatorname{dist}(\bar{x}, \partial\Omega) > \delta\}$ then the compact set $\bar{\Omega} \setminus \Omega_{\delta_0}$ can be covered by a finite number of open sets U_i with the property that $\bar{U}_i \cap \bar{\Omega}$ can be mapped in a $1 - 1$ manner onto the closure of the hemisphere $B_{r_i}^+ = \{(x, y) : |x|^2 + y^2 < r_i^2, y > 0\}$, $0 < r_i \leq 1$, in $(n + 1)$ space, with $U_i \cap \partial\Omega$ mapping onto the flat part of the hemisphere. We assume that the mapping together with its inverse, which we will denote by F_i, are assumed to have continuous and bounded derivatives up to order $2b + 1$. Moreover, we can choose the mapping so that $D_y F_i(x, y) = N_Q$ where $Q = F_i(x, 0) \in \partial\Omega$ and N_Q denotes the unit inner normal to $\partial\Omega$ at Q. We also assume that $U_i \cap \cap \bar{\Omega} \subset \bar{\Omega} \setminus \Omega_{4\delta_0}$ and that each $\bar{x} \in \bar{\Omega} \setminus \Omega_{4\delta_0}$ can be uniquely written as $\bar{x} = Q + rN_Q$, with $Q \in \partial\Omega$, and $0 \leq r \leq 4\delta_0$.

We set $\Omega_T = \Omega \times (0, T)$, $\partial\Omega_T = \partial\Omega \times (0, T)$. We let $\{\varphi_i\}$ denote a (fixed) partition of unity subordinate to the cover $\{U_i\}$ of $\bar{\Omega} \setminus \Omega_{\delta_0}$, and we denote by $\{\psi_i\}$ a sequence of functions for which $\psi_i \in C_0^\infty(U_i)$ and $\psi_i = 1$ in a neighborhood of the support of φ_i.

If $u(Q, t) \in L^p(\partial\Omega_T)$, $1 \leq p \leq \infty$, and k is a non-negative integer $\leq 2b$, we define

$$(1.2.1) \qquad \ldots \Lambda^{-k}(u)(Q, t) = \sum_i \psi_i(Q) \Lambda^{-k}[(u\varphi_i) \circ F_i](F_i^{-1}(Q), t).$$

$$\left(\Lambda^{-k}((u\varphi_i) \circ F_i)(x, y, t) = \int_0^t \int_{R^n} \Lambda^{-k}(x - z, t - s)(u\varphi_i)(F_i(z, y)s)\, dz\, ds\right).$$

Also for $u(Q, t) \in C^\infty \partial\Omega \times (0, \infty)$ which is zero for t near zero we define

$$(1.2.2) \qquad \ldots \Lambda^k(u)(Q, t) = \sum_i \psi_i(Q) \Lambda^k((u\varphi_i) \circ F_i)(F_i^{-1}(Q), t)$$

$$(\mathcal{F}_{z,s}[\Lambda^k(u\varphi_i \circ F_i(z, y, s))](x, t) = (|x|^{2b} - it)^{k/2b} \mathcal{F}_{z,s}[u\varphi_i \circ F_i(z, y, s)](x, t).$$

As we have described in the parentheses above, the operators Λ^{-k}, Λ^k which appear in the summations of 1.2.1 and 1.2.2 are those operators on S_T described in the previous section. We have used the same notation for the corresponding operators on $\partial\Omega_T$ since in context there should be no confusion.

$\Lambda^{-k}\Lambda^k$ is not the identity on functions $f(Q, t) \in C^\infty(\partial\Omega \times (0, T])$ which vanish near $t = 0$, but it is invertible in this class. One way of seeing this is to observe that $\Lambda^{-k}\Lambda^k$ is an invertible p.s.i.o on $\partial\Omega_T$ as defined in [6] which maps the above class into itself. We have chosen not to go through the somewhat lengthy details of this result as we feel it is not an integral part of the techniques of this paper.

As in the case of $\partial\Omega_T$ if $u(Q) \in L^p(\partial\Omega)$ we define $G_k(u)(Q) = \sum_i \psi_i(Q) \cdot \cdot [G_k * (u\varphi_i \circ F_i)](F_i^{-1}(Q))$ where $G_k(x)$ is defined on R^n and $\mathcal{F}(G_k) = (1 + |x|^2)^{-k/2}$. $G_k * f$ is the familiar Bessel potential of f.

2. Summary of Main Results.

We set $L = \sum_{|\alpha| \leq 2b} a_\alpha(\bar{x}, t) D^\alpha_{\bar{x}} - D_t$, $\bar{x} \in \Omega$, $t \in (0, T)$, and we assume L is parabolic in the Petrovsky sense, i. e. $\operatorname{Re}\left(\sum_{|\alpha| = 2b} a_\alpha(\bar{x}, t)(i\xi)^\alpha\right) < - C |\xi|^{2b}$

where $\xi \in R^{n+1} \setminus \{0\}$ and $C > 0$ and independent of $(\bar{x}, t) \in \Omega_T$. About the coefficients we will assume that $a_\alpha(\bar{x}, t)$ is bounded and measurable for all α and that for $|\alpha| = 2b$, a_α is Hölder continuous in $\bar{\Omega} \times [0, T]$.

For a given p, $1 < p \leq \infty$, and given γ, $0 < \gamma < 1$, set $d_p(\bar{x}, t) = \min[\operatorname{dist}(\bar{x}, \partial\Omega), t^{\gamma_p/2b}]$ where $\gamma_\infty = 1$ and for $1 < p < \infty$, γ_p is any number ≤ 1 for which $1 - \frac{1}{p} < \gamma_p < \frac{2b}{2b - \gamma}\left(1 - \frac{1}{p}\right)$. For $\bar{x} = Q + rN_Q \in \Omega \setminus \Omega_{4\delta_0}$ set $(D^k_{N_Q} u)(Q + rN_Q, t) = D^k_s (u(Q + sN_Q, t)|_{s=r}$.

$L^p_{2b,1}(\Omega_T)$ denotes the space of functions $u(\bar{x}, t)$ defined on Ω_T for which $D^\alpha_{\bar{x}} u$, $|\alpha| \leq 2b$, and $D_t u \in L^p(\Omega_T)$. We set

$$\| u \|_{L^p_{2b,1}(\Omega_T)} = \sum_{|\alpha| \leq 2b} \| D^\alpha_{\bar{x}} u \|_{L^p(\Omega_T)} + \| D_t u \|_{L^p(\Omega_T)} .$$

$\overset{\circ}{L}{}^p_{2b,1}(\Omega_T)$ denotes those $u \in L^p_{2b,1}(\Omega_T)$ for which $u = \lim_k u_k$ in

$$L^p_{2b,1}(\Omega_T) \text{ with } u_k \in C^\infty_0 (R^{n+1} \times (0, \infty)).$$

$L^p_{2b,1,\mathrm{loc}}(\Omega_T)$ is the space of functions u on Ω_T for which $D^\alpha_{\bar{x}} u$ and $D_t u \in L^p_{\mathrm{loc}}(\Omega_T)$.

We will now list the theorems in this paper which we consider to be the main results. These theorems are proved in sections 4 and 5 and the numbers appearing here by the theorem correspond to the number of the same theorem in the appropriate section.

THEOREM (4.1.2). If $u \in \overset{\circ}{L}{}^p_{2b,1}(\Omega_T)$, $1 < p < \infty$, and satisfies $Lu = 0$ in Ω_T, then

$$\sup_{r < \delta_0} \left[\sum_{|\alpha| \leq 2b-1} r^{|\alpha|} \| D^\alpha_{\bar{x}} u(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j}(D^j_{N_Q} u)(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \right]$$

$$\leq C_\Omega \sum_{j=0}^{b-1} \| \Lambda^{-j}(D^j_{N_Q} u)(Q, t) \|_{L^p(\partial\Omega_T)} .$$

THEOREM (4.1.5) Assume $u \in \bigcap_{1<p<\infty} \overset{\circ}{L}{}^p_{2b,1}(\Omega_T)$ and satisfies $Lu = 0$ in Ω_T. Then u satisfies the inequality of 4.1.2 with $p = \infty$.

THEOREM (4.1.6) Suppose $\Phi_0, \dots, \Phi_{b-1} \in L^p(\partial\Omega_T)$, $1 < p < \infty$. Then there exists a unique $u(\bar{x}, t)$ satisfying:

i) for any subdomain Ω^* with $\bar{\Omega}^* \subset \Omega$, $u \in \overset{\circ}{L}{}^p_{2b,1}(\Omega^*_T)$

ii) $Lu = 0$ in Ω_T

iii) $\lim_{r\to 0+} \Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) = \Phi_j(Q, t)$ in $L^p(\partial\Omega_T)$, $0 \le j \le b-1$.

Moreover this solution satisfies the inequality

$$\sup_{r<\delta_0} \left[\sum_{|\alpha| \le 2b-1} r^{|\alpha|} \| D^\alpha_x u(Q + rN_{Q,t}) \|_{L^p(\partial\Omega_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \right] \le C_\Omega \sum_{j=0}^{b-1} \| \Phi_j \|_{L^p(\partial\Omega_T)}.$$

THEOREM (4.1.7) Suppose $\Phi_0, \dots, \Phi_{b-1} \in C(\partial\Omega \times [0, T])$ with $\Phi_j(Q, 0) = 0$ for $0 \le j \le b-1$. Then there exists unique $u(\bar{x}, t)$ satisfying:

i) for all Ω^*, $\bar{\Omega}^* \subset \Omega$, $u \in \bigcap_{1<p<\infty} \overset{\circ}{L}{}^p_{2b,1}(\Omega^*_T)$

ii) $Lu = 0$ in Ω_T

iii) $\lim_{t\to 0+} \| u(\cdot, t) \|_{L^\infty(\Omega)} = 0$

iv) $\lim_{r\to 0+} \Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) = \Phi_j(Q, t)$ in $L^\infty(\partial\Omega_T)$, $0 \le j \le b-1$.

Moreover, this solution satisfies the inequality of 4.1.6 with $p = \infty$.

THEOREM (4.2.1) Suppose $u \in L^p_{2b,1}(\Omega_T)$, $1 < p < \infty$, and satisfies $Lu = 0$ in Ω_T. There exists a γ, $0 < \gamma < 1$, depending only on L such that

$$\sup_{r<\delta_0} \left[\sum_{|\alpha| \le 2b-1} \| (d_p^{|\alpha|} D^\alpha_x u)(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \right]$$

$$\le C_\Omega \left[\sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q, t) \|_{L^p(\partial\Omega_T)} + \| u(\cdot, 0) \|_{L^p(\Omega)} \right].$$

THEOREM (4.2.2) Suppose $\Phi_0, \dots, \Phi_{b-1} \in L^p(\partial\Omega_T)$, $1 < p < \infty$, and $h(\bar{x}) \in L^p(\Omega)$. Then there exists a unique $u(\bar{x}, t)$ satisfying:

i) $u(\bar{x}, t) \in L^p_{2b,1,\mathrm{loc}}(\Omega_T)$

ii) $Lu = 0$ in Ω_T

iii) $\lim_{r \to 0+} \lim_{\varepsilon \to 0+} \| \Lambda^{-j} D^j_{N_Q}(u(\cdot,\cdot + \varepsilon))(Q + rN_Q, t) - \Phi_j(Q,t) \|_{L^p(\partial\Omega_{T-\varepsilon})} = 0$

iv) for each subdomain Ω^*, $\bar{\Omega}^* \subset \Omega$, $\lim_{t \to 0+} \| u(\bar{x}, t) - h(\bar{x}) \|_{L^p(Q^*)} = 0$.

Moreover with the same γ as in (4.2.1) this solution satisfies the ine quality,

$$\sup_{r<\delta_0} \Big[\sum_{|\alpha| \leq 2b-1} \| (d_p^{|\alpha|} D^\alpha_{\bar{x}} u)(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} +$$

$$\sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \Big] \leq C_\Omega \left[\sum_{j=0}^{b-1} \| \Phi_j \|_{L^p(\partial\Omega_T)} + \| h(\bar{x}) \|_{L^p(\Omega)} \right].$$

THEOREM (4.2.3) Suppose $u \in \bigcap_{1<p<\infty} L^p_{2b,1}(\Omega_T)$ and satisfies $Lu = 0$. Then u satisfies the inequality of 4.2.1. with $p = \infty$. (d_p is to be replaced by d_∞).

THEOREM (4.2.4) Suppose $\Phi_0, \dots, \Phi_{b-1} \in C(\partial\Omega \times [0, T])$ and $h(\bar{x}) \in C(\bar{\Omega})$. Assume $h(Q) = \Phi_0(Q, 0)$, $Q \in \partial\Omega$, and that $\Phi_j(Q, 0) = 0$, $1 \leq j \leq b-1$. Then there exists a unique $u(\bar{x}, t)$ satisfying:

i) $u(\bar{x}, t) \in \bigcap_{1<p<\infty} L^p_{2b,1,\mathrm{loc}}(\Omega_T)$

ii) $Lu = 0$ in Ω_T

iii) $\lim_{r \to 0+} \lim_{\varepsilon \to 0+} \| \Lambda^{-j} D^j_{N_Q}(u(\cdot,\cdot + \varepsilon))(Q + rN_Q, t) -$

$$- \Phi_j(Q, t) \|_{L^\infty(\partial\Omega_{T-\varepsilon})} = 0 \qquad (0 \leq j \leq b-1)$$

iv) $\lim_{t \to 0+} \| u(\bar{x}, t) - h(\bar{x}) \|_{L^\infty(\Omega)} = 0$.

Moreover this solution satisfies the inequality of 4.2.2 with $p = \infty$.

In the final two results to be stated here the operator $\mathcal{E} = \sum_{|\alpha| \leq 2b} a_\alpha(\bar{x}) D^\alpha_{\bar{x}}$ is assumed to be strongly elliptic in Ω, i.e. $\mathcal{E} - D_t$ is parabolic in the Petrovsky sense. The coefficients are assumed to be bounded and measurable for all α and for $|\alpha| = 2b$ Hölder continuous in $\bar{\Omega}$.

THEOREM (5.1). Suppose $u \in L^p_{2b}(\Omega)$, $1 < p < \infty$, and satisfies $\mathcal{E}u = 0$ in Ω. Then

$$\sup_{r<\delta_0} \Big[\sum_{|\alpha| \leq 2b-1} r^{|\alpha|} \| D^\alpha_x u(Q + rN_Q) \|_{L^p(\partial\Omega)} + \sum_{j=0}^{b-1} \| G_j D^j_{N_Q} u(Q + rN_Q) \|_{L^p(\partial\Omega)} \Big]$$

$$\leq C_\Omega \left[\sum_{j=0}^{b-1} \| G_j D^j_{N_Q} u(Q) \|_{L^p(\partial\Omega)} + \| u \|_{L^1(\Omega)} \right].$$

THEOREM (5.3) Suppose $u \in \bigcap_{1<p<\infty} L^p_{2b}(\Omega)$ satisfies $\mathcal{E}u = 0$ in Ω. Then u satisfies the inequality in (5.1) with $p = \infty$.

3. Estimates for the Half-Space.

For any $\delta \geq 0$ we set $R^{n+1}_\delta = R^n \times (\delta, \infty)$. For R^{n+1}_0 we will also use the notation R^{n+1}_+. A point in R^{n+1}_+ will be generally denoted by (x, y) where $x \in R^n$ and $y > 0$. For $1 < p < \infty$, $L^p_{2b,1}(R^{n+1}_\delta \times (0, T))$ denotes the space of function $u(x, y, t)$ for which $D^\alpha_{x,y} u$, $|\alpha| \leq 2b$, and $D_t u$ belong to $L^p(R^{n+1}_\delta \times (0, T))$. Again we define

$$\| u \|_{L^p_{2b,1}(R^{n+1}_\delta \times (0,T))} = \sum_{|\alpha| \leq 2b} \| D^\alpha_{x,y} u \|_{L^p(R^{n+1}_\delta \times (0,T))} + \| D_t u \|_{L^p(R^{n+1}_\delta \times (0,T))}.$$

$\overset{\circ}{L}{}^p_{2b,1}(R^{n+1}_\delta \times (0, T))$ denotes the space of functions $u \in L^p_{2b,1}(R^{n+1}_\delta \times (0, T))$ such that $u = \lim_k u_k$, the limit taken in the space $L^p_{2b,1}(R^{n+1}_\delta \times (0, T))$, with $u_k \in C^\infty_0(R^{n+1} \times (0, \infty))$.

The parabolic operator we consider now has the form $L = \sum_{|\alpha| \leq 2b} a_\alpha(x, y, t) \cdot D^\alpha_{x,y} - D_t$ and again we assume

$$\operatorname{Re} \Big(\sum_{|\alpha| = 2b} a_\alpha(x, y, t)(i\xi)^\alpha \Big) < - C |\xi|^{2b},$$

$(\xi \neq 0)$, with $C > 0$ and independent of (x, y, t).

For a given number γ, $0 < \gamma < 1$, and for $1 < p \leq \infty$, set $d_p(x, y, t) = \min(y, t^{\gamma_p/2b})$ where again $\gamma_\infty = 1$ and for $1 < p < \infty$, γ_p is any number satisfying $1 - 1/p < \gamma_p < \dfrac{2b}{2b - \gamma}(1 - 1/p)$.

3.1. *$Lu = f$, $u(x, y, 0) = 0$ and «almost» zero data at $y = 0$.*

A. The parametrix.

Let

$$f \in L^{p_0}(R^{n+1}_+ \times (0, T)) \, (1 < p_0 < \infty).$$

We will construct a function u_1 satisfying:

(a) $$u_1 \in \overset{\circ}{L}{}^{p_0}_{2b,1}(R^{n+1}_+ \times (0,T))$$

(b) $$Lu_1(x,y,t) = f(x,y,t), \quad y > 0$$

(c) $$\lim_{y\to 0+} (D^k_y \Lambda^{-k} u_1(x,y,t)) = \lim_{y \to 0+} D^k_y \left(\int_0^t \int_{R^n} \Lambda^{-k}(x-z, t-s) u_1(z,y,s)\, dz\, ds \right)$$

exists in $L^{p_0}(S_T)$, $0 \le k \le b-1$.

(d) For

$$1 < p \le \infty, \sup_{y>0} \| D^k_y \Lambda^{-k} u_1(\cdot, y, \cdot) \|_{L^p(S_T)} \le \omega(T) \sup_{y>0} \| d_p^{2b-\gamma} f(\cdot, y, \cdot) \|_{L^p(S_T)}$$

where $\omega(T) \to 0$ as $T \to 0^+$. (Condition (d) expresses the meaning of «almost» zero data at $y = 0$).

We will begin by constructing a parametrix for a fundamental solution to the above problem. To do this we first consider a homogeneous parabolic operator $L = \sum_{|\alpha| = 2b} a_\alpha D^\alpha_{x,y} - D_t$ with constant coefficients, and we begin our development by considering the function $G_0(x,y,v,t)$ which as a function of (x,y,t) satisfies

1) $$G(\cdot,\cdot,v,\cdot) \in \overset{\circ}{L}{}^{p}_{2b,1}(R^n \times (\delta, \infty) \times (0,T)) \text{ for each } \delta > 0.$$

2) $$L\, G_0(x,y,v,t) = 0 \text{ for } y > 0.$$

3) For $k = 0, \dots, b-1$,

$$\lim_{y\to 0+} \Lambda^{-k} D^k_y G_0(x,y,v,t) = \Lambda^{-k}(D^k_y \Gamma(\cdot, -v, \cdot))(x,t) \text{ in } L^p(S_T)$$

$$(\Gamma(x,y,t) = \mathcal{F}_\xi (\exp(\sum_{|\alpha|=2b} a_\alpha (i\xi)^\alpha) t)(x,y)).$$

Here Λ^{-k} is taken in the variables (x,t). We will set

$$\Lambda^{-k}(D^k_y \Gamma(\cdot, -v, \cdot))(x,t) = \Lambda^{-k} D^k_y \Gamma(x, -v, t).$$

In [5] it is shown that there is a $b \times b$ matrix of parabolic singular integral operators, (T_{kj}), for which

$$G_0(x,y,v,t)=\sum_{k,j=0}^{b-1}\int_0^t\int_{R^n}\Lambda^{2b-1-k}\,D_y^k\,\Gamma(x-w,y,t-r)\cdot$$

$$T_{kj}(\Lambda^{-j}D_y^j\Gamma(\cdot,-v,\cdot))(w,r)\,dw\,dr.$$

For $j=0,\dots,b-1$ set

$$g_{j,b}(x,t)=\lim_{y\to 0^+}(\Lambda^{-j}D_y^j(\lim_{v\to 0^+}D_v^b(\Gamma(x,y-v,t)-G_0(x,y,v,t))).$$

The inner limit, i.e. the limit in v, is a point-wise limit, and the limit in y exists in L^p for some $p>1$. Now define;

$$G_b(x,y,v,t)=\sum_{k,j=0}^{b-1}\int_0^t\int_{R^n}\Lambda^{2b-1-k}\,D_y^k\,\Gamma(x-w,y,t-r)\,T_{k,j}\,(g_{j,b}*D_y^{2b-1}\,\Gamma(-v)\,(w,r)\,dw\,dr.$$

By induction on l, $0\leq l\leq b-2$, and for $j=0,\dots,b-1$, set

$$g_{j,b+l}(x,t)=\lim_{y\to 0^+}\Big(\Lambda^{-j}D_y^j\Big(\lim_{v\to 0^+}D_v^{b+l}\Big[\Gamma(x,y-v,t)-G_0(x,y,v,t)-$$

$$-\sum_{k=0}^{l-1}G_{b+k}(x,y,v,t)\frac{v^{b+k}}{(b+k)!}\Big]\Big)\Big)$$

and

$$G_{b+l}(x,y,v,t)=\sum_{k,j=0}^{b-1}\int_0^t\int_{R^n}\Lambda^{2b-1-k}D_y^k\,\Gamma(x-w,y,t-r)\,T_{kj}\,(g_{j,b+l}*D_y^{2b-1}\Gamma(-v)(w,r)\,dwdr$$

if $b=1$ we take $G_{b+l}=0$).

The proof the following lemma appears in the appendix.

LEMMA 3.1.1. For $y>0$ and $v>0$.

$$|\,D_{x,y}^{\alpha}\,D_v^k(G_{b+l}(x,y,v,t))\,|\leq\frac{\psi\left(\frac{|x|}{t^{1/2b}}\right)\psi\left(\frac{y}{t^{1/2p}}\right)\psi\left(\frac{v}{t^{1/2b}}\right)}{t^{(n+1+|\alpha|+k+b+l)/2b}}.$$

In this estimate and whenever used $\psi(r)$ is a function of the form e^{-cr}, $r\geq 0$, and depends only on the parameter of parabolicity and on the num. ber $\max_{|\alpha|=2b}|a_\alpha|$.

Set

$$(3.1.2) \quad \ldots R(x,y,v,t) = \Gamma(x,y-v,t) - G_0(x,y,v,t) - \sum_{l=0}^{b-2} G_{(b+l)}(x,y,v,t)\frac{v^{b+l}}{(b+l)!}\,.$$

Lemma 3.1.3. For $0 \leq k \leq 2b-2$; $D_v^k(R(x,y,v,t))|_{v=0} = 0$. Also for

$$y > 0,\ v > 0,\ |D_{x,y}^{\alpha} D_v^k R(x,y,v,t)| \leq \psi\left(\frac{|x|}{t^{1/2b}}\right)\psi\left(\frac{|y-v|}{t^{1/2b}}\right) t^{-\frac{n+1}{2b}}\,.$$

Proof. Suppose first that $k \geq b$. Then

$$D_v^k R(x,y,0,t) = \lim_{v\to 0+} D_v^k \Bigg[\Gamma(x,y-v,t) - G_0(x,y,v,t) -$$

$$- \sum_{b+l<k} G_{b+l}(x,y,v,t)\frac{v^{b+l}}{(b+l)!}\Bigg] - \lim_{v\to 0+} G_k(x,y,v,t).$$

Each limit on the right side of the above equality is a solution of the equation, $L(u)(x,y,t) = (\sum_{|\alpha|=2b} a_\alpha D_{x,y}^{\alpha} u - D_t u)(x,y,t) = 0$ for $y > 0$. Moreover from the formula for G_{b+l} and by the definition of $g_{j,b+l}$ we see that $\lim_{y\to 0+} \Lambda^{-r} D_y^r(D_v^k R(x,y,0,t)) = 0$, $0 \leq r \leq b-1$, in $L^p(S_T)$ for some p sufficiently near one ($\infty > p > 1$). Hence the uniqueness theorem for constant coefficients proved in [5], shows that $D_v^k R(x,y,0,t) \equiv 0$. In the case $k < b$ we have

$$D_v^k R(x,y,0,t) = (-1)^k \Bigg[D_y^k \Gamma(x,y,t) -$$

$$\sum_{j,l=0}^{b-1} \int_0^t\int_{R^n} \Lambda^{2b-1-l} D_y^l\,\Gamma(x-w,y,t-r)\,T_{l,j}\big(\lim_{v\to 0+} \Lambda^{-j} D_y^{j+k}\,\Gamma(-v)(w,r)\,dw\,dr\Bigg].$$

Once again by the uniqueness theorem in [5] we see that the function inside the brackets in identically zero.

The estimates follow from 3.1.1 and the known estimates on Γ (see [7]).

We now consider a general parabolic operator $L = \sum_{|\alpha|\leq 2b} a_\alpha(x,y,t) D_{x,y}^{\alpha} - D_t$. We denote by $R_{z,u,s}(x,y,v,t)$ the function 3.1.2 constructed for the operator $L_{z,u,s} = \sum_{|\alpha|=2b} a_\alpha(z,u,s) D_{x,y}^{\alpha} - D_t$.

B. *Estimates for the solution of* (3.1).

LEMMA 3.1.4. For $f \in L^p(R^{n+1}_+ \times (0, T))$ set

$$J(f)(x, y, t) = \int_0^t \int_{R^{n+1}_+} L_{x,y,t}(R_{z,v,s}(x-z, y, v, t-s)) f(z, v, s)\, dz\, dv\, ds.$$

Then for $1 < p \leq \infty$

$$\operatorname*{Sup}_{y>0} \| d_p^{2b-\gamma}(\cdot, y, \cdot)(Jf)(\cdot, y, \cdot)\|_{L^p(S_T)} \leq \omega(T) \sup_{y>0} \| d_p^{2b-\gamma}(\cdot, y, \cdot) f(\cdot, y, \cdot)\|_{L^p(S_T)}$$

where $\omega(T) \to 0$ as $T \to 0$. (See § 2 for definition of γ and d_p).

PROOF: We will first prove the inequality with d_p replaced by y and then with d_p seplaced by $t^{\gamma p/2b}$.

$$Jf(x, y, t) = \int_0^{y/2} + \int_{y/2}^{2y} + \int_{2y}^{\infty} \left[\int_0^t \int_{R^n} L_{x,y,t} R_{z,v,s}(x-z, y, v, t-s) f(z, v, s)\, dz\, ds \right] dv =$$

$$= J_1 f + J_2 f + J_3 f.$$

Using the fact that $D_v^l R_{z,u,s}(x, y, v, t) = 0$ at $v = 0$ for $0 \leq l \leq 2b-2$ and from the estimates in Lemma 3.1.3 we see that

$$\| J_1 f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq \omega(T) \left(\int_0^{y/2} \| f(\cdot, v, \cdot)\|_{L^p(S_T)} v^{2b-1} dv \right) \left(\int_0^T \frac{\psi\left(\frac{y}{s^{1/2b}}\right)}{s^2} ds \right) \leq$$

$$\leq \omega(T) \frac{1}{y^{2b}} \int_0^{y/2} \| f(\cdot, v, \cdot)\|_{L^p(S_T)} v^{2b-1}\, dv.$$

Hence

$$y^{2b-\gamma} \| J_1 f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq \omega(T) \frac{1}{y^\gamma} \int_0^{y/2} \frac{\| f(\cdot, v, \cdot)\|_{L^p(S_T)} v^{2b-\gamma}}{v^{1-\gamma}} dv \leq$$

$$\leq \omega(T) \sup_{v>0} [v^{2b-\gamma} \| f(\cdot, v, \cdot)\|_{L^p(S_T)}].$$

Using only the estimates of lemma 3.1.3

$$\| J_2 f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq \int_{y/2}^{2y} \| f(\cdot, v, \cdot)\|_{L^p(S_T)} \left(\int_0^T \frac{\psi\left(\frac{|y-v|}{s^{1/2b}}\right)}{s^{1+(1-u)/2b}} ds \right) dv \text{ where } u > 0,$$

depends only on the Hölder continuity of the coefficients of $L_{x,y,t}$. Hence

$$y^{2b-\gamma} \| J_2 f(\cdot, y, \cdot) \|_{L^p(S_T)} \leq$$

$$\leq \int_{y/2}^{2y} \left(\int_0^T \frac{\psi\left(\frac{|v|}{s^{1/2b}}\right)}{s^{1+(1-u)/2b}} \, ds \right) \left(\frac{y}{v}\right)^{2b-\gamma} v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p(S_T)} \, dv \leq$$

$$\leq c \left(\int_0^T \int_{-\infty}^{\infty} \frac{\psi\left(\frac{|v|}{s^{1/2b}}\right)}{s^{1+(1-\mu)/2b}} \, ds \right) \sup_{v>0} [v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p(S_T)} \leq$$

$$\leq \omega(T) \sup_{v>0} [v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p(S_T)}].$$

For J_3 we use once again the estimates from Lemma 3.1.3 to obtain that

$$\| J_3 f(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \int_{2y}^{\infty} \| f(\cdot, v, \cdot) \|_{L^p(S_T)} \left(\int_0^{\infty} \frac{\psi\left(\frac{|v|}{s^{1/2b}}\right)}{s^{1+1/2b}} \, ds \right) dv.$$

Hence

$$y^{2b-\gamma} \| J_3 f(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \, y^{2b-\gamma} \int_{2y}^{\infty} \frac{1}{v^{2b-\gamma+1}} v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p} \, dv.$$

So the estimate for $J_3 f$ and hence for Jf follows with d_p replaced by y. We will now show the estimate with d_p replaced by $t^{\gamma_p/2b}$.

$$\| Jf(\cdot, y, t) \|_{L^p(R^n)} \leq \int_0^t (t-s)^{-1-(1-\mu)/2b} \int_0^{\infty} \psi\left(\frac{|y-v|}{(t-s)^{1/2b}}\right) \| f(\cdot, v, s) \|_{L^p(R^n)} \, dv \, ds.$$

Since for any $r > 0$ $\psi(u) \leq C_r \, u^{-r} \psi_1(u)$ we have

$$\| Jf(\cdot, y, t) \|_{L^p(R^n)} \leq \int_0^{\infty} \frac{\psi_1\left(\frac{|y-v|}{T^{1/2b}}\right)}{|y-v|^r} \left(\int_0^t (t-s)^{-1-(1-\mu-r)/2b} \| f(\cdot, v, s) \|_{L^p(R^n)} \, ds \right) dv.$$

Choosing $r < 1$ so that $r + \mu > 1$, and using Hardy's lemma ([11, I]) we see that

$$\| t^{\gamma_p\left(\frac{2b-\gamma}{2b}\right)} Jf(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \sup_{y>0} \| t^{\gamma_p\left(\frac{2b-\gamma}{2b}\right)} f(\cdot, y, \cdot) \|_{L^p(S_T)}$$

provided $\gamma_p \dfrac{2b-\gamma}{2b} + \dfrac{1}{p} < 1$, which we have assumed.

To obtain the estimate for $d_p(x, y, t)$ we write

$$f(x, y, t) = f(x, y, t)\, X_{(y<t^{\gamma_p/2b})}(x, y, t) + $$
$$+ f(x, y, t)\, X_{(y \geq t^{\gamma_p/2b})}(x, y, t) = f_1 + f_2,$$

where X_E denotes the characteristic function of E. Now we observe that

$$\| d_p^{2b-\gamma} Jf_1(\cdot, y, \cdot) \|_{L^p(S_T)} \leq y^{2b-\gamma} \| Jf_1(\cdot, y, \cdot) \|_{L^p(S_T)} \leq$$

$$\leq \omega(T) \sup_{y>0} y^{2b-\gamma} \| f_1(\cdot, y, \cdot) \|_{L^p(S_T)} \leq w(T) \sup_{y>0} \| d_p^{2b-\gamma} f(\cdot, y, \cdot) \|_{L^p(S_T)}.$$

A similar inequality holds for Jf_2.

Because of the Hölder continuity of the highest order coefficients of L, it is easy to see that J maps $L^p(R_+^{n+1} \times (0, T))$ into itself continuously for $1 \leq p \leq \infty$, and $\| Jf\chi_{(a,b)} \|_{L^p(R_+^{n+1} \times (a,b))} \leq \omega(b-a) \| f\chi_{(a,b)} \|_{L^p(R_+^{n+1} \times (a,b))}$ where $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0_+$. (Actually this last inequality holds when the coefficients of order $2b$ are only uniformly continuous in $\overline{R_+^{n+1}} \times [0, T]$ and for $1 < p < \infty$). Moreover J has the additional property that if $f = 0$ for $s < a$ then $Jf = 0$ for $s < a$. It follows that $(I - J)$ is invertible over $L^p(R_+^{n+1} \times (0, T))$. In fact if $f \in L^p(R_+^{n+1} \times (0, T))$ in order to find $g \in L^p(R_+^{n+1} \times (0, T))$ such that $(I - J)g = f$ we proceed as follows: write $[0, T] = \bigcup_{k=1}^{M_T} [a_k, b_k]$ with $0 = a_1 < b_1 = a_2 < b_2 = a_3 < b_3 \ldots$. We choose $[a_k, b_k]$ so that for g with support $\subset R_+^{n+1} \times [a_k, b_k]$, $\| Jg \|_{L^p(R_+^{n+1} \times [a_k, b_k])} \leq \leq \frac{1}{2} \| g \|_{L^p(R_+^{n+1} \times [a_k, b_k])}$. We can find g_1 with support $\subset R_+^{n+1} \times [a_1, b_1]$ such that $(I - J)g_1 = f$ on $R_+^{n+1} \times [a_1, b]$. We can find g_2 with support $\subset R_+^{n+1} \times [a_2, b_2]$ such that $(I - J)g_2 = f - (I - J)g_1$ on $R_+^{n+1} \times [a_2, b_2]$. In general we choose g_k with support $\subset R_+^{n+1} \times [a_k, b_k]$ such that $(I - J)g_k = = f - \sum_{l=1}^{k-1} (I - J)g_l$ on $R_+^{n+1}[a_k, b_k]$. The function $g = \sum_{k=1}^{M_T} g_k$ has the property that $(I - J)g = f$.

Lemma (3.1.5). Suppose $f \in L^p(R_+^{n+1} \times (0, T))$, $1 < p \leq \infty$. Then

$$\| d_p^{2b-\gamma} (I - J)^{-1} f(\cdot, y, \cdot) \|_{L^p(S_T)} \leq C \sup_{y>0} \| d_p^{2b-\gamma} f(\cdot, y, \cdot) \|_{L^p(S_T)} .$$

Proof: For simplicity let us assume that we are able to write $(I - J)^{-1} f = g = g_1 + g_2$ as described above. Then $R^n \times [0, T] = R^n \times \times [a_1, b_1] \cup R^n \times [a_2, b_2] = S_1 + S_2$.

$$\| d_p^{2b-\gamma} g(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \| d_p^{2b-\gamma} g_1(\cdot, y, \cdot) \|_{L^p(S_1)} + \| d_p^{2b-\gamma} g_2(\cdot, y, \cdot) \|_{L^p(S_2)} .$$

On S_1, $g_1 = (I - J)^{-1} f = \sum_{k=0}^{\infty} J_k f$. Hence by 3.1.4 if $(b_2 - a_2)$ is small $\| d_p^{2b-\gamma} g_1(\cdot, y, \cdot) \|_{L^p(S_1)} \leq C \sup_{y>0} \| d_p^{2b-\gamma} f(\cdot, y, \cdot) \|_{L^p(S_1)}$. On S_2, $g_2 = (I - J)^{-1} (f X_{S_2}) - - (I - J)^{-1} (I - J) g_1 X_{S_2})$. Hence if $(b_2 - a_2)$ is small enough,

$$\| d_p^{2b-\gamma} g_2(\cdot, y, \cdot) \|_{L^p(S_2)} \leq$$

$$\leq C \sup_{y>0} [\, \| d_p^{2b-\gamma} f(\cdot, y, \cdot) \|_{L^p(S_2)} + \| d_p^{2b-\gamma} (I - J) g_1(\cdot, y, \cdot) \|_{L^p(S_2)}] \leq$$

$$\leq C \sup_{y>0} [\, \| d_p^{2b-\gamma} f(\cdot, y, \cdot) \|_{L^p(S_2)} + \| d_p^{2b-\gamma} (I - J) g_1(\cdot, y, \cdot) \|_{L^p(S_T)}].$$

Again using 3.1.4 and remembering that $\operatorname{suppt} g_1 \subset R_+^{n+1} \times [a_1, b_1]$,

$$\| d_p^{2b-\gamma} (I - J) g_1(\cdot, y, \cdot) \|_{L^p(S_T)} \leq$$

$$\leq C \sup_{y>0} \| d_p^{2b-\gamma} g_1(\cdot, y, \cdot) \|_{L^p(S_1)} \leq C \sup_{y>0} \| d_p^{2b-\gamma} f(\cdot, y, \cdot) \|_{L^p(S_1)} .$$

This finishes the proof of 3.1.5.

Lemma (3.1.6). For $f \in L^p(R_+^{n+1} \times (0, T))$, $1 < p \leq \infty$, set

$$u(x, y, t) = \int_0^t \int_{R_+^{n+1}} R_{z, v, s}(x - z, y, v, t - s) f(z, v, s)\, dz\, dv\, ds.$$

Then

$$\sup_{y>0} \sum_{\substack{|\alpha| \leq 2b-1 \\ 0 \leq k \leq b-1}} \| d_p^{|\alpha|} D_{x,y}^{\alpha} (D_y^k \Lambda^{-k} u)(\cdot, y, \cdot) \|_{L^p(S_T)} +$$

$$+ \| (d_p^{2b-1} D_t \Lambda^{-1} u)(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \sup_{y>0} \| (d_p^{2b-\gamma} f)(\cdot, y, \cdot) \|_{L^p(S_T)} .$$

PROOF: As in the proof of lemma (3.1.4) it is sufficient to prove our estimate first with d_p replaced by y and then with it replaced by $t^{\gamma_p/2b}$. We will do the case $k=0$. The other k's and the term $D_t \Lambda^{-1} u$ are handled in exactly the same manner.

For $|\alpha| > 0$

$$D^{\alpha}_{x,y}\, u(x, y, t) =$$

$$= \int_0^{y/2} + \int_{y/2}^{2y} + \int_{2y}^{\infty} \left[\int_0^t \int_{R^n} D^{\alpha}_{x,y}\, R_{z,v,s}(x-z, y, v, t-s)\, f(z, v, s)\, dz\, ds \right] dv =$$

$$= u^{\alpha}_1(x, y, t) + u^{\alpha}_2(x, y, t) + u^{\alpha}_3(x, y, t).$$

For u^{α}_1 we use the fact that $D^l_v D^{\alpha}_{x,y} R_{z,u,s}(x, y, v, t)|_{v=0} = 0$ for $0 \leq l \leq$ $\leq 2b-2$ (see 3.1.3) and the estimates of 3.1.3 to obtain

$$\| u^{\alpha}_1(\cdot, y, \cdot) \|_{L^p(S_T)} \leq c \int_0^{y/2} v^{2b-1} \| f(\cdot, v, \cdot) \|_{L^p(S_T)} \left[\int_0^t \frac{\psi\left(\frac{y}{s^{1/2b}}\right)}{s^{1+|\alpha|/2b}}\, ds \right] dv \leq$$

$$\leq \omega(T) \int_0^{y/2} v^{2b-1} \| f(\cdot, v, \cdot) \|_{L^p(S_T)} \left[\int_0^T \frac{\psi\left(\frac{y}{s^{1/2b}}\right)}{s^{1+(|\alpha|+\gamma)/2b}}\, ds \right] dv \leq$$

$$\leq \omega(T)\, y^{-|\alpha|-\gamma} \int_0^{y/2} v^{2b-1} \| f(\cdot, v, \cdot) \|_{L^p(S_T)}\, dv.$$

Hence

$$y^{|\alpha|} \| u^{\alpha}_1(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \sup_{v>0} v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p(S_T)}.$$

For u^{α}_2 we again use the estimates of 3.1.3 and that $D^l_v D^{\alpha}_{x,y} R_{z,u,s}$ at $v=0$ equals 0 for $0 \leq l \leq 2b - |\alpha| - 2$ if $|\alpha| \leq 2b-2$ to obtain

$$\| u^{\alpha}_2(\cdot, y, \cdot) \|_{L^p(S_T)} \leq c \int_0^1 dl \int_{y/2}^{2y} v^{2b-1-|\alpha|} \| f(\cdot, v, \cdot) \|_{L^p(S_T)} \left[\int_0^T \frac{\psi\left(\frac{|y-lv|}{s^{1/2b}}\right)}{s}\, ds \right] dv$$

$$\leq \omega(T) \int_0^1 dl \int_{y/2}^{2y} \frac{v^{2b-1-|\alpha|}}{|y-lv|^{\gamma}} \| f(\cdot, v, \cdot) \|_{L^p(S_T)}$$

$$\leq \omega(T) \sup_{v>0} v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p(S_T)}.$$

In like manner as for u_2^α we have

$$\| u_3^\alpha(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \int_0^1 dl \int_{2y}^\infty \| f(\cdot, v, \cdot) \|_{L^p(S_T)} v^{2b-1-|\alpha|} \frac{1}{|y - lv|^\gamma} dv.$$

Hence

$$y^{|\alpha|} \| u_3^\alpha(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) y^{|\alpha|} \left(\int_{2y}^\infty v^{-|\alpha|-1+\gamma} \int_0^1 \frac{dl}{|y - lv|^\gamma} dv \right) \sup_{v>0} v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p(S_T)}.$$

Now observe that

$$\int_{2y}^\infty v^{-|\alpha|-1+\gamma} \left(\int_0^1 |y - lv|^{-\gamma} dl \right) dv \leq c y^{-|\alpha|}.$$

We have proved the estimate in the case $k = 0$ and $|\alpha| > 0$ with d_p replaced by y. To obtain the estimate with d_p replaced by $t^{\gamma_p/2b}$ we first note that for $|\alpha| \geq 0$

$$\| D_{x,y}^\alpha u(\cdot, y, t) \|_{L^p(R^n)} \leq \int_0^t \int_0^\infty \frac{\psi\left(\frac{|y - v|}{(t - s)^{1/2b}}\right)}{(t - s)^{(|\alpha|+1)/2b}} \| f(\cdot, v, s) \|_{L^p(R^n)} dv\, ds.$$

We multiply both sides by $t^{|\alpha| \gamma_p/2b}$ and proceed as in the similar situation in lemma 3.1.4.

To finish the case $k = 0$ we must show that

$$\| u(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \sup_{y>0} y^{2b-\gamma} \| f(\cdot, y, \cdot) \|_{L^p(S_T)}.$$

Again we write

$$u(x, y, t) = \int_0^{y/2} + \int_{y/2}^{2y} + \int_{2y}^\infty \left[\int_0^t \int_{R^n} R_{z,v,s}(x - z, y, v, t - s) f(z, v, s)\, dz\, ds \right] dv$$

$$= u_1 + u_2 + u_3.$$

u_1 and u_2 are handled exactly in the corresponding case for $|\alpha| > 0$. For $u_3(x, y, t)$ we consider two cases, first when $y \geq 1$ and then when $y < 1$.

If $y > 1$,

$$\| u_3(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \int_2^\infty \| f(\cdot, y, \cdot) \|_{L^p(S_T)}$$

$$\leq \omega(T) \left(\int_2^\infty \frac{1}{v^{2b-\gamma}} \, dv \right) \sup_{v>0} v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p(S_T)} .$$

If $y < 1$ we first write

$$u_3(x, y, t) = \int_{2y}^{2} + \int_2^\infty \left[\int_0^t \int_{R^n} R_{z, v, s}(x - z, y, v, t - s) f(z, v, s) \, dz \, ds \right] dv = u_{3,1} + u_{3,2} .$$

The estimate for $u_{3,2}$ follows the same argument for u_3 when $y \geq 1$. Once again

$$\| u_{3,1}(\cdot, y, \cdot) \|_{L^p(S_T)} \leq c \int_0^1 dl \int_{2y}^2 v^{2b-1} \| f(\cdot, v, \cdot) \|_{L^p(S_T)} \left(\int_0^T \frac{\psi\left(\frac{|y - lv|}{s^{1/2b}} \right)}{s} ds \right) dv$$

$$\leq c\omega(T) [\sup_{v>0} v^{2b-\gamma} \| f(\cdot, y, \cdot) \|_{L^p(S_T)}] \int_0^1 \int_{2y}^2 \frac{dv}{v^{1-\gamma}} \left(\int_0^T \frac{\psi\left(\frac{|y - lv|}{s^{1/2b}} \right)}{s^{1+\gamma'/2b}} \, ds \right)$$

where $0 < \gamma' < \gamma$.

Hence

$$\| u_{3,1}(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) [\sup_{v>0} v^{2b-\gamma} \| f(\cdot, v, \cdot) \|_{L^p(S_T)}] \int_{2y}^2 \frac{dv}{v^{1-\gamma}} \left(\int_0^1 \frac{dl}{|y - lv|^{\gamma'}} \right) .$$

Now note that since $0 < \gamma' < \gamma$ the integral in the above inequality is bounded by a constant independent of y for $y < 1$. The case for $k = 0$ is now complete.

Recall now from 3.1.4 the operator

$$Jf(x, y, t) = \int_0^t \int_{R_+^{n+1}} L_{x, y, t} R_{z, v, s}(x - z, y, v, t - s) f(z, v, s) \, dz \, dv \, ds.$$

Theorem (3.1.7). Suppose $f \in L^{p_0}(R_+^{n+1} \times (0, T)$ for some p_0, $1 < p_0 < \infty$, and set

$$u(x, y, t) = -\int_0^t \int_{R_+^{n+1}} R_{z, v, s}(x - z, y, v, t - s)(I - J)^{-1} f(z, v, s)\, dz\, dv\, ds.$$

Then i) $u \in \mathring{L}^{p_0}{}_{2b,1}(R_+^{n+1} \times (0, T))$ and $Lu = f$,

ii) for $1 < p \leq \infty$,

$$\sum_{\substack{|\alpha| \leq 2b-1 \\ 0 \leq k \leq b-1}} \| d_p^{|\alpha|} D_{x,y}^\alpha (\Lambda^{-k} D_y^k u)(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \sup_{y>0} \| d_p^{2b-\gamma} f(\cdot, y, \cdot) \|_{L^p(S_T)}$$

iii) for $0 \leq k \leq b - 1$, $\lim_{y \to 0+} (\Lambda^{-k} D_y^k u)(x, y, t)$ exists in $L^{p_0}(S_T)$.

The function $\omega(T)$ depends only on the bounds of a_α, the Hölder continuity of the highest order coefficients, and on the parameter of parabolicity of $L = \sum_{|\alpha| \leq 2b} a_\alpha(x, y, t) D_{x,y}^\alpha - D_t$.

Proof: Set

$$W_1(x, y, t) = \int_0^t \int_{R_+^{n+1}} \Gamma_{z, v, s}(x - z, y - v, t - s)\, g(z, v, s)\, dz\, dv\, ds,$$

$$W_2(x, y, t) = \int_0^t \int_{R_+^{n+1}} \left[G_0^{(z, v, s)}(x - z, y, v, t - s) + \sum_{l=0}^{2b-2} G_{b+l}^{(z, v, s)}(x - z, y, v, t - s) \cdot \frac{v^{b+l}}{(b+l)!} \right] g(z, v, s)\, dz\, dv\, ds$$

where $g \in L^{p_0}(R_+^{n+1} \times (0, T))$, $1 < p_0 < \infty$. It is known (see [3]) that $W_1 \in \mathring{L}^{p_0}{}_{2b,1}(R_+^{n+1} \times (0, T))$ and

$$\sum_{|\alpha| \leq 2b} \| D_{x,y}^\alpha W_1 \|_{L^{p_0}(R_+^{n+1} \times (0, T)} + \| D_t W_1 \|_{L^{p_0}(R_+^{n+1} \times (0, T))} \leq C_{p_0} \| g \|_{L^{p_0}(R_+^{n+1} \times (0, T))}.$$

Moreover $L W_1 = -g + J_1(g)$ where

$$J_1(g) = \int_0^t \int_{R_+^{n+1}} L_{x, y, t} \Gamma_{z, v, s}(x - z, y - v, t - s)\, g(z, v, s)\, dz\, dv\, ds.$$

From the estimates of 3.1.1 we have for $|\alpha| < 2b$,

$$\| D^{\alpha}_{x,y} W_2(\cdot, \cdot, t) \|_{L^{p_0}(R^{n+1}_+)} \leq C \int_0^t (t-s)^{-|\alpha|/2b} \| g(\cdot, \cdot, s) \|_{L^{p_0}(R^{n+1}_+)} ds$$

and hence

$$\| D^{\alpha}_{x,y} W_2 \|_{L^{p_0}(R^{n+1}_+ \times (0,T))} \leq C \| g \|_{L^{p_0}(R^{n+1}_+ \times (0,T))}.$$

For $|\alpha| = 2b$ and again using the estimates of 3.1.1 we have

$$\| D^{\alpha}_{x,y} W_2(\cdot, y, \cdot) \|_{L^{p_0}(S_T)} \leq C \int_0^{\infty} \left(\int_0^{\infty} \frac{\psi\left(\frac{y+v}{s^{1/2b}}\right)}{s^{1+1/2b}} ds \right) \| g(\cdot, v, \cdot) \|_{L^{p_0}(S_T)} dv.$$

Hence

$$\| D^{\alpha}_{x,y} W_2(\cdot, y, \cdot) \|_{L^{p_0}(S_T)} \leq C \int_0^{\infty} \frac{1}{(y+v)} \| g(\cdot, v, \cdot) \|_{L^{p_0}(S_T)}.$$

By Hardy's lemma,

$$\| D^{\alpha}_{x,y} W_2 \|_{L^{p_0}(R^{n+1}_+ \times (0,T))} \leq C \| g \|_{L^{p_0}(R^{n+1}_+ \times (0,T))}.$$

A similar estimate holds for $D_t W_2$. Also $L W_2 = J_2 g$ where

$$J_2(g)(x, y, t) = \int_0^t \int_{R^{n+1}_+} L_{x,y,t} \Big[G_0^{(z,v,s)}(x-z, y, v, t-s) +$$

$$+ \sum_{l=0}^{2b-2} G_{b+l}^{(z,v,s)}(x-z, y, v, t-s) \frac{v^{b+l}}{(b+l)!} \Big] g(z, v, s)\, dz\, dv\, ds.$$

From the above discussion of W_1 and W_2 we conclude that since $J = J_1 + J_2$, $u \in \mathring{L}^{p_0}_{2b,1}(R^{n+1}_+ \times (0, T))$ and $Lu = (I-J)^{-1} f - J(I-J)^{-1} f = (I-J)(I-J)^{-1} f = f$. The estimates in lemmas 3.1.5 and 3.1.6 immediately imply part (ii). Since $u \in \mathring{L}^{p_0}_{2b,1}(R^{n+1}_+ \times (0, T))$ it is easy to see that $D^k_y u(x, y, t)$ converges in $L^{p_0}(S_T)$ as $y \to 0+$, $0 \leq k \leq b-1$. Hence $\Lambda^{-k} D^k_y u(x, y, t)$ converges in $L^{p_0}(S_T)$ as $y \to 0+$.

3.2. *The case $Lu = 0$, $u(x, y, 0) = 0$, and given lateral data.*

For a given *function* $\Phi(x, t) \in L^p(S_T)$ and *for* $j = 0, \dots, b-1$ set

$$(T_j \Phi(x, y, t) = \int_0^t \int_{R^n} \Lambda^{2b-j-1} D_y^j \Gamma_{z, 0, s}(x - z, y, t - s) \Phi(z, s)\, dz\, ds,$$

and

(3.2.1) $\dots u_j(x, y, t) = (T_j \Phi)(x, y, t) +$

$$+ \int_0^t \int_{R_+^{n+1}} R_{z, v, s}(x - z, y, v, t - s)(I - J)^{-1}(LT_j \Phi)(z, v, s)\, dz\, dv\, ds$$

(Again $L = \sum_{|\alpha| \leq 2b} a_\alpha(x, y, t) D_{x,y}^\alpha - D_t$).

THEOREM (3.2.2). Assume $\Phi \in L^{p_0}(S_T)$, $1 < p_0 < \infty$. Then

1) for each $\delta > 0$ $u_j \in \mathring{L}_{2b,1}^{p_0}(R^n \times (\delta, \infty) \times (0, T))$ and $Lu_j = 0$ in $R_+^{n+1} \times (0, T)$

2) $\sum_{\substack{|\alpha| \leq 2b-1 \\ 0 \leq k \leq b-1}} y^{|\alpha|} \| D_{x,y}^\alpha (\Lambda^{-k} D_y^k u_j)(\cdot, y, \cdot) \|_{L^{p_0}(S_T)} \leq C_{p_0} \| \Phi \|_{L^{p_0}(S_T)}$ and

3) $\lim_{y \to 0+} \Lambda^{-k} D_y^k u_j(x, y, t) = (S_{k,j} + J_{k,j})(\Phi)$ in $L^{p_0}(S_T)$, when $J_{k,j} \in \mathcal{J}$ and $S_{k,j}$ is the p.s.i.o whose symbol is given *by*

$$\sigma(S_{k,j})(z, s; x, t) = (|x|^{2b} - it)^{\frac{2b-j-k-1}{2b}} \oint \frac{(-i\zeta)^{k+j}}{A_{z, o, s}(ix, i\zeta) + it}\, d\zeta.$$

(The contour integral is taken over a closed contour in the lower-half-plane enclosing all roots (in ζ) in this plane of $A_{z, o, s}(ix, i\zeta) + it = 0$). C_{p_0} depends on the Hölder norm of the coefficients of highest order and on the parameter of parabolicity of L.

PROOF: We will first assume that any derivative of the coefficients of L is bounded in $\overline{R_+^{n+1}} \times [0, T]$ and that $\Phi \in C_0^\infty(R^n \times (0, \infty))$. In this case it is easy to see that $u_j \in \mathring{L}_{2b,1}^{p_0}(R_+^{n+1} \times (0, T))$. From theorem (3.1.7) we have $Lu_j = L(T_j \Phi) - L(T_j \Phi) = 0 \ (y > 0)$. Moreover for $0 \leq k \leq b - 1$

$$\sum_{|\alpha| \leq 2b-1} y^{|\alpha|} \| D^{\alpha}_{x,y} (\Lambda^{-k} D^k_y (u_j - T_j(\Phi))(\cdot, y, \cdot) \|_{L^{p_0}(S_T)} \leq$$

$$\leq \omega(T) \sup_{y>0} \| y^{2b-\gamma} (LT_j \Phi)(\cdot, y, \cdot) \|_{L^{p_0}(S_T)}$$

$$| L(T_j \Phi)(x, y, t) | \leq \int_0^t \int_{R^n} \frac{\psi\left(\frac{|x-z|}{(t-s)^{1/2b}}\right) \psi\left(\frac{y}{(t-s)^{1/2b}}\right)}{(t-s)^{\frac{n}{2b}+2-\frac{u}{2b}}} | \Phi(z, s) | \, dz \, ds$$

where $u > 0$ depends on the Hölder exponent of the coefficients of order $2b$. Hence by taking $\gamma \leqq u$ we have $\sup_{y>0} \| y^{2b-\gamma} (LT_j \Phi)(\cdot, y, \cdot) \|_{L^{p_0}(S_T)} \leq \leq C_{p_0} \| \Phi \|_{L^{p_0}(S_T)}$. Finally we see that $\Lambda^{-k} D^k_y (u_j - T_j \Phi)(x, y, t) \to J^{(1)}_{k,j} \Phi$ as $y \to 0+$ in $L^{p_0}(S_T)$ where $J^{(1)}_{k,j} \in \mathcal{J}$. In [5] it was shown that $\Lambda^{-k} D^k_y (T_j \Phi)(x,y,t)$ converges in $L^{p_0}(S_T)$ as $y \to 0+$ to $(S_{k,j} + J^{(2)}_{k,j}) \Phi$ where $J^{(2)}_{k,j} \in \mathcal{J}$ and $S_{k,j}$ is the p.s.i.o described in (3).

To obtain theorem 3.2.2 when the coefficients are only Hölder continuous in $R^{n+1}_+ \times [0, T]$ and $\Phi \in C^\infty_0(R^n \times (0, \infty))$ we consider sequences $\{a^{(m)}_\alpha\}$, where for $|\alpha| = 2b$, $a^{(m)}_\alpha$ belongs to the same Hölder class as a_α with norm bounded by the norm of a_α, and $a^{(m)}_\alpha \to a_\alpha$ pointwise and boundedly as $m \to \infty$. Set $A^{(m)}(z, u, s, \xi) = \sum_{|\alpha|=2b} a^{(m)}_\alpha (z, u, s)(i\xi)^\alpha$

$$\Gamma^{(m)}_{z,u,s}(x, y, t) = \mathcal{F}_\xi (\exp A^{(m)}(z, u, s; \xi) t)(x, y),$$

$$T^{(m)}_j(\Phi) = \int_0^t \int_{R^n} \Lambda^{2b-j-1} D^j_y \Gamma^{(m)}_{z,0,s}(x - z, y, t - s) \, \Phi(z, s) \, dz \, ds,$$

and

$$u^m_j(x, y, t) = T^m_j(\Phi) + \int_0^t \int_{R^n} R_{z,v,s}(x - z, y, v, t - s)(I - J)^{-1} \cdot$$
$$\cdot (LT^m_j(\Phi))(z, v, s) \, dz \, dv \, ds.$$

It is easy to see that $LT^{(m)}_j(\Phi) \in L^{p_0}(R^{n+1}_+ \times (0, T))$ and that $u^{(m)}_j \in \mathring{L}^{p_0}_{2b,1}(R^{n+1}_+ \times (0, T))$. Moreover $Lu^{(m)}_j = 0$ for $y > 0$. Now let $\varphi(y) \in C^\infty[0, \infty)$

with $\varphi(y) = 0$ for y near zero and equal to 1 for y near infinity. Then

$$\| u_j^m \varphi \|_{L^{p_0}_{2b,1}(R^{n+1}_+ \times (0,T))} \leq C \| L(u_j^{(m)} \varphi) \|_{L^{p_0}(R^{n+1}_+ \times (0,T))} .$$

From the estimates in part (2) of 3.2.2 for u_j^m we conclude that this last norm is bounded independent of m. Hence $\{u_j^m\}_m$ converges weakly in $L^{p_0}_{2b,1}(R^n \times (\delta, \infty) \times (0, T))$ to $u_j(x, y, t) \in \overset{\circ}{L}{}^{p_0}_{2b,1}(R^n \times (\delta, \infty) \times (0, T))$ for each $\delta > 0$. Clearly $Lu_j = 0$. Since $T_j^{(m)}(\Phi) \to T_j(\Phi)$ in $L^{p_0}(S_T)$ and since $\sup_{y>0} y^{2b-\gamma} \| L(T_j^{(m)} \Phi - T_j \Phi)(\cdot, y, \cdot) \|_{L^{p_0}(S_T)} \to 0$ as $m \to \infty$, it follows that u_j has the representation 3.2.1. By approximating $L^{p_0}(S_T)$ with $C_0^\infty(R^n \times (0, \infty))$, the general result follows for $\Phi \in L^{p_0}(S_T)$.

The matrix of symbols $(\sigma(S_{k,j})(z, s; x, t))$ has an inverse for each $(z, s; x, t)$ with $(x, t) \neq (0, 0)$, and we can write $(\sigma(S_{k,j})(z, s; x, t))^{-1} = (\sigma(T_{k,j})$ $z, s; x, t)$ where $T_{k,j}$ is a p.s.i.o (see [8]). Now set $g_k = \sum_{j=0}^{b-1} T_{k,j} \Phi_j$ $(k = 0, \ldots$ $\ldots, b-1)$ where $\Phi_0 \ldots, \Phi_{b-1}$ are given functions in $L^{p_0}(S_T)$. Finally set

$$(u_k(x, y, t) = T_k(g_k)(x, y, t) + \int_0^t \int_{R^{n+1}_+} R_{z,v,s}(x-z, y, v, t-s)(I-J)^{-1}(L(T_k g_k))(z, v, s) \cdot$$

$dz\, dv\, ds$ and $u = \sum_{k=0}^{b-1} u_k$.

THEOREM (3.2.3). For each $\delta > 0$ $u \in L^{p_0}_{2b,1}(R^n \times (\delta, \infty) \times (0, T))$, $1 < p_0 < \infty$, and $Lu = 0$ in $R^{n+1}_+ \times (0, T)$. Moreover,

a) for $1 < p \leq \infty$,
$$\sum_{\substack{|\alpha| \leq 2b-1 \\ k \leq b-1}} y^{|\alpha|} \| D^\alpha_{x,y} \Lambda^{-k} D^k_y u(\cdot, y, \cdot) \|_{L^p(S_T)} \leq C_p \sum_{k=0}^{b-1} \| \Phi_k \|_{L^p(S_T)} .$$

C_p depends on p, the Hölder norm of the coefficients of L of highest order, and the parameter of parabolicity.

b) $$\lim_{y \to 0} \Lambda^{-k} D^k_y u(x, y, t) = (I + J_k) \Phi_k \text{ in } L^{p_0}(S_T) \text{ with } J_k \in \mathcal{J}.$$

PROOF: All the statements in 3.2.3 are corollaries of 3.2.2 except for the estimate in (a) with $p = \infty$. For $0 < |\alpha| \leq 2b - 1$ one can check that

$$y^{|\alpha|} \| D^\alpha_{x,y} T_k(g_k)(\cdot, y, \cdot) \|_{L^\infty(S_T)} \leq \left(y^{|\alpha|} \int_0^\infty \psi \frac{\left(\frac{y}{s^{1/2b}}\right)}{s^{1+|\alpha|/2b}} ds \right) \sum_{j-0}^{b-1} \| \Phi_j \|_{L^\infty(S_T)}$$
$$\leq C \sum_{j-0}^{b-1} \| \Phi_j \|_{L^\infty(S_T)} .$$

What is then left to show is that for $0 \leq l \leq b-1$

$$\left| \Lambda^{-l} D_y^l \left(\sum_{k=0}^{b-1} T_k (g_k) \right) (x, y, t \right| \leq C \sum_{j=0}^{b-1} \| \Phi_j \|_{L^\infty (S_T)} .$$

We first write

$$T_{k,j} \Phi (x, t) = A_{k,j} (x, t) \Phi (x, t) + \int_0^t \int_{R^n} T_{k,j} (x, t; x-z, t-s) \Phi (z, s) \, dz \, ds.$$

Then

$$\Lambda^{-l} D_y^l (T_k g_k) (x, y, t) =$$

$$= \sum_{j-0}^{b-1} \int_0^t \int_{R^n} \Lambda^{2b-1-k-l} D_y^{l+k} \Gamma_{x,0,t} (x-z, y, t-s) [A_{k,j} (x, t) \Phi_j (z, s) +$$

$$+ \int_0^s \int_{R^n} T_{k,j} (x, t; z-w, s-r) \Phi_j (w, r) \, dw \, dr] \, dz \, ds + \sum_{j=0}^{b-1} J_{k,j}^{(l)} (\Phi_j) (x, y, t).$$

Let

$$T_{k,j}^{(M,\tau)} (\Phi_j) (z, s) = A_{k,j} (M, \tau) \Phi_j (z, s) + \int_0^s \int_{R^n} T_{k,j} (M, \tau; z-w, s-r) \Phi ; (w, r) \, dw \, dr.$$

Hence

$$\Lambda^{-l} D_y^l \left(\sum_{k-0}^{b-1} T_k g_k \right) (x, y, t) = \sum_{k,j=0}^{b-1} J_{k,j}^{(l)} (\Phi_j) (x, y, t) +$$

$$+ \sum_{k,j=0}^{b-1} \int_0^t \int_{R^n} \Lambda^{2b-1-k-l} D_y^{l+k} \Gamma_{x,0,t} (x-z, y, t-s) (T_{k,j}^{(x,t)}) (\Phi_j) (z, s) \, dz \, ds.$$

Now set

$$u_{l,j}^{(M,\tau)} (x, y, t) = \sum_{k=0}^{b-1} \int_0^t \int_{R^n} \Lambda^{2b-1-k-l} D_y^{l+k} \Gamma_{M,0,\tau} (x-z, y, t-s) (T_{k,j}^{(M,\tau)}) (\Phi_i) (z, s) \, dz \, ds.$$

The estimate we seek for $p = \infty$ will be complete once we show that $| u_{l,j}^{(M,\tau)} (x, y, t) | \leq C \| \Phi_j \|_{L^\infty (S_T)}$ where C is independent of (M, τ) and $y > 0$.

We leave the proof of this to the Appendix (A.2.1). This completes the proof of 3.2.3.

(3.3) *The initial-value problem.*

In this section we will briefly review some known results for the initial value problem,

$$Lu(x,y,t) = [\sum_{|\alpha| \leq 2b} a_\alpha(x,y,t) D^\alpha_{x,y} - D_t] u(x,y,t) = 0, (x,y) \in R^{n+1}, u(x,y,0) = g(x,y).$$

To construct a solution we use a fundamental solution for this problem constructed in the manner of Pogorzelski in [9[. We denote this function by $W(x, y, t: z, v, s)$ and it has the following properties:

1) $$W(x, y, t; z, v, s) = \Gamma_{z,v,s}(x - z, y - v, t - s) + \int_0^t \int_{R^{n+1}} \Gamma_{M,u,r}(x - M, y - u, t - r)\, \Phi(M, u, r; z, v, s)\, dM\, du\, dr,$$

and for $t > s$ $\Phi(x,y,t; z, v, s)$ satisfies the integral identity

$$L_{x,y,t}[\Gamma_{z,v,s}(x - z, y - v, t - s)] = -\Phi(x, y, t; z, v, s) + \int_s^t \int_{R^{n+1}} L_{x,y,t}[\Gamma_{M,u,r}(x - M, y - u, t - r)]\, \Phi(M, u, r; z, v, s)\, dM\, du\, dr.$$

Here

$$L_{x,y,t} = \sum_{|\alpha| \leq 2b} a_\alpha(x, y, t)\, D^\alpha_{x,y} - D_t\,.$$

2) For $t > s$, $L_{x,y,t}(W(x, y, t; z, v, s)) = 0$.

3) For $|\alpha| \leq 2b - 1$ and $t > s$,

$$|D^\alpha_{x,y} W(x, y, t; z, v, s)| \leq \frac{\psi\left(\frac{|x - z|}{(t - s)^{2/3b}}\right) \psi\left(\frac{|y - v|}{(t - s)^{1/2b}}\right)}{(t - s)^{(n+1)/2b + |\alpha|/2b}},$$

and

$$|\Phi(x,y,t; z,v,s) \leq \frac{\varphi\left(\frac{|x - z|}{(t - s)^{1/2b}}\right) \psi\left(\frac{|y - v|}{(t - s)^{1/2b}}\right)}{(t - s)^{(n+1)/2b+1-\mu/2b}}, \; 0 < \mu \leq 1.$$

(μ depends on the Hölder continuity of the highest order coefficients of L).

Now set

$$(3.3.1) \qquad \ldots u(x, y, t) = \int_{R^{n+1}} W(x, y, t; z, v, 0)\, g(z, v)\, dz\, dv \cdot$$

$$= \int_{R^{n+1}} \Gamma_{z, v, 0}(x - z, y - v, t)\, g(z, v)\, dz\, dv + \int_0^t \int_{R^{n+1}} \Gamma_{M, v, r}(x - z, y - v, t - r)\, \Phi(g)(z, v, r)\, dz\, dv\, dr$$

where

$$\Phi(g)(x, y, t) = \int_{R^{n+1}} \Phi(x, y, t; M, u, 0)\, g(M, u)\, dM\, du.$$

THEOREM (3.2.2). If $g \in L^p(R^{n+1})$, $1 < p < \infty$, then

4) $u(x, y, t) \in L^p_{2b, 1}(R^{n+1} \times (a, T))$ $\forall\, a,\ 0 < a < T.$

5) $Lu = 0$

6) for $1 < p \leq \infty$,

$$\sum_{\substack{|\alpha| \leq 2b-1 \\ k \leq b-1}} \| t^{\gamma_p |\alpha|/2b} D^{\alpha}_{x, y} (\Lambda^{-k} D^k_y u)(\cdot, y, \cdot) \|_{L^p(S_T)} \leq C \| g \|_{L^p(R^{n+1})} \cdot$$

(C depends only on the Hölder continuity of the coefficients of order $2b$ of L, on the bounds of all the coefficients, and on the parameter of parabolicity).

7) for $1 < p < \infty$, $\lim_{y \to +0} \Lambda^{-k} D^k_y u(x, y, t)$ exists in $L^p(S_T)$. The limit exists pointwise if $p = \infty$.

8) If $1 < p < \infty$, $\| u(\cdot, \cdot, t) - g(\cdot, \cdot) \|_{L^p(R^{n+1})} \to 0$ as $t \to 0+$.

If g is bounded and uniformly continuous in R^{n+1} then

$$\| u(\cdot, \cdot, t) - g(\cdot, \cdot) \|_{L^\infty(R^{n+1})} \to 0 \text{ as } t \to 0+ .$$

PROOF: 4) and 5): The estimate in three together with the results in [3] imply that for p_1 near one if $g \in L^{p_1}(R^{n+1})$ $u \in L^{p_1}_{2b, 1}(R^{n+1} \times (a, T))$ and $Lu = 0$. Now if $g \in L^p(R^{n+1})$ we take $g_k \in C^\infty_0(R^{n+1})$, $g_k \to g$ in L^p as $k \to \infty$. The first estimate in 3 (for $|\alpha| = 0$) implies that for

$$u_k = \int_{R^{n+1}} W(x, y, t; z, v)\, g_k(z, v)\, dz\, dv,\ \| u_k \|_{L^p(R^{n+1})} \leq C \| g \|_{L^p(R^{n+1})}.$$

By multiplying u_k by a fixed function $\varphi(t) \in C_0^\infty(0, \infty)$ we see that the sequence $\{u_k\}$ is bounded in $L^p{}_{2b,1}(R^{n+1} \times (a, T))\ \forall\ a,\ 0 < a < T$. In fact since $\varphi(t)\, u_k(x, y, t) \in \mathring{L}^{p_1}{}_{2b,1}(R^{n+1} \times (0, T))$, the results in [4] show that $\varphi u_k = \int_0^t \int_{R^{n+1}} \Gamma(x - z, y - v, t - s)\, T(L(\varphi u_k))(z, v, s)\, dz\, dv\, ds$ with T a bounded operator on $L^p(R^{n+1} \times (0, T))$ and Γ the usual fundamental solution of the operator $(-)^b \Delta^{2b} + D_t$. Hence

$$\| \varphi u_k \|_{L^p{}_{2b,1}(R^{n+1} \times (0,T))} \leq C \| \varphi' u_k \|_{L^p(R^{n+1} \times (0,T))} \leq C \| g_k \|_{L^p(R^{n+1})} .$$

Hence u_k converges weakly in $L^p_{2b,1}(R^{n+1} \times (a, T))\ \forall\ a, 0, < a < T$, to a function, which must be u since u_k converges to u in $L^p(R^{n+1} \times (a, T))$.

To show 6-8 we first write

$$\Lambda^{-k} D_y^k u(x, y, t) = \int_{R^{n+1}} \Lambda^{-k} D_y^k \Gamma_{z,v,\sigma})(x - z, y - v, t)\, g(z, v)\, dz\, dv +$$

$$+ \int_0^t \int_{R^{n+1}} \Lambda^{-k} D_y^k \Gamma_{M,v,r}(x - M, y - v, t - r)\, \Phi(g)(M, v, r)\, dM\, dv\, dr.$$

where

$$\Phi(g)(x, y, t) = \int_{R^{n+1}} \Phi(x, y, t; M, v, 0)\, g(M, v)\, dM\, dv.$$

The first term in the above equality is easily seen to satisfy the desired estimate. Denoting the second term by $u_2(x, y, t)$ we have

$$\| D^a_{x,y}(\Lambda^{-k} D_y^k u_2)(\cdot, y, t) \|_{L^p(R^n)} \leq \int_0^t (t - s)^{-(|a| + 1/p)1/2b} \| \Phi(g)(\cdot, \cdot, s) \|_{L^p(R^{n+1})}\, ds.$$

From the conditions on γ_p and γ we see that

$$\| t^{\gamma_p |a|/2b} D^a_{x,y}(\Lambda^{-k} D^k_y u_2)(\cdot, y, \cdot) \|_{L^p(S_T)} \leq C \| s^{\gamma_p (2b - \gamma)/2b}\, \Phi(g) \|_{L^p(R^{n+1} \times (0,T))} .$$

Using the estimate (3) for $\Phi(x, y, t;\ z, v, 0)$,

$$\| \Phi(g)(\cdot, \cdot, t) \|_{L^p(R^{n+1})} \leq \frac{C}{t^{1 - \mu/2b}} \| g \|_{L^p(R^{n+1})}$$

and since for $0 < \gamma \leq \mu$, $1 - \mu/2b - (2b - \gamma/2b)\gamma_p < 1/p$, we conclude the desired estimate for $u_2(x, y, t)$.

Again the first term in the expression for $\Lambda^{-k} D_y^k u(x, y, t)$ is easily seen to converge in $L^p(S_T)$, $1 < p < \infty$, as $y \to 0+$. For $g \in C_0^\infty(R^{n+1})$ it is not difficult to see that $\Lambda^{-k} D_y^k u_2$ converges pointwise as $y \to 0+$ and since it is bounded, uniformly in y, by a function in $L^p(S_T)$ it follows that $\Lambda^{-k} D_y^k u_2$ converges in $L^p(S_T)$ as $y \to 0+$ when g is smooth. By a density argument the L^p-convergence follows for any $g \in L^p(R^{n+1})$, $1 < p < \infty$. The pointwise convergence of $(\Lambda^{-k} D_y^k u)(x, y, t)$ when $p = \infty$ is immediate by Lebesque's Dominated convergence theorem.

(3.4) *General problem in the half space.*

We will now construct a solution to the following initial-boundary-value problem. Fix a number p, $1 < p < \infty$. Assume we are given functions $f(x, y, t)$, $\Phi_0(x, t), \ldots, \Phi_{b-1}(x, t)$, and $g(x, y)$ with $f \in L^p(R_+^{n+1} \times (0, T))$, $\Phi_j \in L^p(S_T)$, $j = 0, \ldots, b-1$, and $g(x, y) \in L^p(R_+^{n+1})$. Find $u(x, y, t)$ such that,

$$
\begin{aligned}
&1)\ u \in L^p_{2b, 1, \mathrm{loc}}(R_+^{n+1} \times (0, T)) \text{ and } Lu = f \text{ for } y > 0 \\
&2)\ \text{for } j = 0, \ldots, b-1,\ \| \Lambda^{-j} D_y^j u(\cdot, y, \cdot) - \Phi_j(\cdot, \cdot) \|_{L^p(S_T)} \to 0 \text{ as } y \to 0. \\
&3)\ \| u(\cdot, \cdot, t) - g(\cdot, \cdot) \|_{L^p(R^{n+1})} \to 0 \text{ as } t \to 0+ \text{ for any } \delta > 0.
\end{aligned}
\tag{3.4.1}
$$

We recall that $L = \sum_{|\alpha| \leq 2b} a_\alpha(x, y, t) D_{x,y}^\alpha - D_t$ and that a_α is bounded for all α and uniformly Hölder continuous for $|\alpha| = 2b$.

Set

$$
\begin{aligned}
u_1(x, y, t) &= \int_0^t \int_{R_+^{n+1}} R_{z, v, s}(x - z, y, v, t - s)(I - J)^{-1} f(z, v, s)\, dz\, dv\, ds \\
u_2(x, y, t) &= \int_{R_+^{n+1}} W(x, y, t; z, v, 0)\, g(z, v)\, dz\, dv
\end{aligned}
\tag{3.4.2}
$$

$$
u_3(x, y, t) = \sum_{k=0}^{b-1} \Big[T_k(g_k)(x, y, t) + \\
+ \int_0^t \int_{R_+^{n+1}} R_{z, v, s}(x - z, y, v, t - s)(I - J)^{-1}(L(T_k g_k))\, dz\, dv\, ds \Big]
$$

where

$$g_k(x,t) = \sum_{j=0}^{b-1} T_{k,j}(I - J_j)^{-1}(\Phi_j - \Lambda^{-j}(D_y^j(u_1 + u_2))(\cdot, 0, \cdot))(x,t).$$

THEOREM (3.4.3). Assume the coefficients of $L = \sum_{|\alpha| \leq 2b} a_\alpha(x,y,t) D_{x,y}^\alpha - D_t$ are bounded for all α and uniformly Hölder continuous for $|\alpha| = 2b$. The function $u = u_1 + u_2 + u_3$ described in 3.4.2 is a solution of the problem 3.4.1 in $R_+^{n+1} \times (0, T)$. Moreover there is a γ, $0 < \gamma \leq 1$, and constants depending only on the structure of L such that for $1 < p \leq \infty$,

$$\sum_{\substack{|\alpha| \leq 2b-1 \\ k \leq b-1}} \| d_p^{|\alpha|} D_{x,y}^\alpha (\Lambda^{-k} D_y^k u)(\cdot, y, \cdot)\|_{L^p(S_T)} \leq$$

$$\omega(T) \sup_{y>0} \| d_p^{2b-\gamma} Lu(\cdot, y, \cdot)\|_{L^p(S_T)} + C\left[\sum_{j=0}^{b-1} \|\Phi_j\|_{L^p(S_T)} + \|g\|_{L^p(R_+^{n+1})}\right].$$

PROOF: That u is a solution follows from 3.1.7, 3.2.3, and 3.3.2. Also from these results the left side of the above inequality is bounded by

$$\omega(T) \sup_{y>0} \left[\| d_p^{2b-\gamma} Lu(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{k=0}^{b-1} \| d_p^{2b-\gamma} L(T_k g_k)(\cdot, y, \cdot)\|_{L^p(S_T)}\right] +$$

$$C\left[\sum_{j=0}^{b-1} \|\Phi_j(x,t) - \Lambda^{-j} D_y^j (u_1 + u_2)(x, 0, t)\|_{L^p(S_T)} + \|g\|_{L^p(R_+^{n+1})}\right].$$

Now

$$\|\Lambda^{-j} D_y^j (u_1 + u_2)(\cdot, 0, \cdot)\|_{L^p(S_T)} \leq$$

$$\omega(T) \sup_{y>0} \| d_p^{2b-\gamma} Lu(\cdot, y, \cdot)\|_{L^p(S_T)} + C\|g\|_{L^p(R_+^{n+1})}.$$

It is not too difficult to see that

$$|L(T_k g_k)(x,y,t)| \leq$$

$$\sum_{j=0}^{b-1} \int_0^t \int_{R^n} \frac{\psi\left(\frac{x-z}{(t-s)^{1/2b}}\right) \psi\left(\frac{y}{(t-s)^{1/2b}}\right)}{(t-s)^{n/2b+2-\mu/2b}} [|\Phi_j(z,s)| + |\Lambda^{-j} D_y^j (u_1 + u_2)(z,0,s)|]\, dz\, ds$$

where $\mu > 0$. Therefore

$$y^{2b-\gamma} \| d_p^{2b-\gamma} L(T_k g_k)(\cdot, y, \cdot) \|_{L^p(S_T)} \leq$$

$$C \sum_{j=0}^{b-1} (\| \Phi_j \|_{L^p(S_T)} + \| \Lambda^{-j} D_y^j (u_1 + u_2)(\cdot, 0, \cdot) \|_{L^p(S_T)}).$$

The inequality now follows.

THEOREM (3.4.4). Suppose $u \in L^p{}_{2b,1}(R_+^{n+1} \times (0, T))$, $1 < p < \infty$. Then u has the representation $u = u_1 + u_2 + u_3$ with u_1, u_2, u_2 given by (3.4.2).

PROOF: When the coefficients of L are $C^\infty(\overline{R_+^{n+1}} \times [0, T])$ and when $u \in C_0^\infty(R^{n+2})$ then $u = u_1 + u_2 + u_3$, for in this case u and $(u_1 + u_2 + u_3) \in L_{2b,1}^p(R_+^{n+1} \times (0,T))$(for some $p > 1$) and their difference satisfies the initial-Dirichlet-problem with homogeneous data. To obtain the result for general $u \in L^p{}_{2b,1}$ we first approximate u in $L^p{}_{2b,1}(R_+^{n+1} \times (0, T))$ by $u_n \in C_0^\infty(R^{n+2})$. We also regularize the coefficients of L denoting the new operator by L^n. Then $u_n = \overset{1}{u}_n + \overset{2}{u}_n + \overset{3}{u}_n$. Since $L^n u_n \to Lu$ in $L^p(R_+^{n+1} \times (0, T))$ and since $(\Lambda^{-j} D_y^j u_n)(x, 0, t) \to \Lambda^{-j}(D_y^j u)(x, 0, t)$ in $L^p(R^n \times (0, T))$, it easily follows that $\overset{i}{u}_n \to u_i$ $(i = 1, 2, 3)$ pointwise a. e. in $R_n^{n+1} \times (0, T)$. Since $u_n \to u$ in $L^p{}_{2b,1}(R_+^{n+1} \times (0, T))$ it follows that $u = u_1 + u_2 + u_3$.

THEOREM (3.4.5). Suppose $u \in \overset{\circ}{L}{}^{p_1}{}_{2b,1}(R_+^{n+1} \times (0, T))$ for some p_1, $1 < p_1 < \infty$. Then under the assumptions of 3.4.3, for $1 < p \leq \infty$,

$$\sum_{\substack{|\alpha| \leq 2b-1 \\ k \leq b-1}} y^{|\alpha|} \| D_{x,y}^\alpha (\Lambda^{-k} D_y^k u(\cdot, y, \cdot) \|_{L^p(S_T)} \leq \omega(T) \sup_{y>0} y^{2b-y} \| Lu(\cdot, y, \cdot) \|_{L^p(S_T)}$$

$$+ C \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j u(\cdot, 0, \cdot) \|_{L^p(S_T)}.$$

PROOF: From 3.4.4 u is of the form $u_1 + u_3$. The proofs of 3.1.7 and 3.2.3 now give the above estimate.

(3.5) *Estimates away from* $t = 0$.

THEOREM (3.5.1). For each a, $0 < a < T$, any function

$$u \in L^{p_0}{}_{2b,1}(R_+^{n+1} \times (0, T)), \quad 1 < p_0 < \infty,$$

satisfies the following inequality over $S_{a,T} = R^n \times (a, T)$ for $1 < p \leq \infty$.

$$\sum_{|\alpha| \leq 2b-1} \| (d_p^{|\alpha|} D_{x,y}^{\alpha} u)(\cdot, y, \cdot) \|_{L^p(S_{a,T})} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j u(\cdot, y, \cdot) \|_{L^p(S_{a,T})} \leq$$

$$\omega(T) \sup_{y>0} \| (d_p^{2b-\gamma} Lu)(\cdot, y, \cdot) \|_{L^p(S_T)} +$$

$$C \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j u(\cdot, 0, \cdot) \|_{L^p(S_T)} + C_a \| u(\cdot, \cdot, 0) \|_{L^1(R_+^{n+1})} .$$

Proof: Again we write $u = u_1 + u_2 + u_3$ as in 3.4.2. For u_1 the left side of the inequality is bounded by $\omega(T) \sup_{y>0} \| (d_p^{2b-\gamma} Lu)(\cdot, y, \cdot) \|_{L^p(S_T)}$. Recall that

$$u_3(x, y, t) = \sum_{k=0}^{b-1} T_k(g_k) +$$

$$+ \int_0^t \int_{R_+^{n+1}} R_{z,v,s}(x - z, y, v, t - s)(I - J)^{-1}(L(T_k g_k))(z, v, s)\, dz\, dv\, ds$$

where

$$g_k(x, t) = \sum_{j=0}^{b-1} T_{k,j}(I - J_j)^{-1}(\Lambda^{-j} D_y^j(u - u_1 - u_2)(\cdot, 0, \cdot))(x, t).$$

Set

$$g_{k,1}(x, t) = \sum_{j=0}^{b-1} T_{k,j}(I - J_j)^{-1}(\Lambda^{-j} D_y^j(u - u_1)(\cdot, 0, \cdot)(x, t)$$

and $g_k(x, t) = g_{k,1}(x, t) - g_{k,2}(x, t)$. We correspondingly write $u_3 = u_{3,1} - u_{3,2}$. Hence $u_2 + u_3 = u_{3,1} + (u_2 - u_{3,2})$. For $u_{3,1}$ the left side of the inequality is bounded by

$$C_a \sum_{j=0}^{b-1} [\| \Lambda^{-j} D_y^j u(\cdot, 0, \cdot) \|_{L^p(S_T)} + \| \Lambda^{-j} D_y^j u_1(\cdot, 0, \cdot) \|_{L^p(S_T)}]$$

$$\leq C_a \left[\sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j u(\cdot, 0, \cdot) \|_{L^p(S_T)} + \omega(T) \sup_{y>0} \| d_p^{2b-\gamma}(\cdot, y, \cdot) Lu(\cdot, y, \cdot) \|_{L^p(S_T)} \right].$$

$$(u_2 - u_{3,2})(x, y, t) = \int_{R_+^{n+1}} g(z, v)\, [W(x, y, t; z, v, 0) - W_1(x, y, t; z, v)]\, dz\, dv,$$

where

$$W_1(x, y, t; z, v) = \sum_{k=0}^{b-1} T_k(g_{k,2}(\cdot, \cdot; z, v))(x, y, t) +$$

$$+ \int_0^t \int_{R_+^{n+1}} R_{M,u,s}(x-M, y, u, t-s)(I-J)^{-1} L(T_k(g_{k,2}(\cdot, \cdot; z, v)))(M, u, s)\, dM\, du\, ds$$

and

$$g_{k,2}(x, t; z, v) = \sum_{j=0}^{b-1} T_{k,j}(I-J_j)^{-1}(\Lambda^{-j} D_y^j W(\cdot, 0, \cdot; z, v, 0))(x, t).$$

The formula for $W_1(x, y, t; z, v)$ shows that for $(z, v) \in R_+^{n+1}$ fixed, $W_1(x, y, t; z, v)$ satisfies (see 3.2.3):

1) $W_1(x, y, t; z, v) \in \mathring{L}^p_{2b,1}(R^n \times (\delta, \infty) \times (0, T))$ for each $\delta > 0$ and for all p, $1 < p < \infty$.

2) $L_{x,y,t} W_1(x, y, t; z, v) = (\sum_{|\alpha| \leq 2b} a_\alpha(x, y, t) D^\alpha_{x,y} - D_t) W_1(x, y, t; z, v) = 0, y > 0.$

3) $\Lambda^{-k} D_y^k W_1(\cdot, 0, \cdot; z, v)(x, t) = \Lambda^{-k} D_y^k W(\cdot, 0, \cdot; z, v, 0)(x, t), \ 0 \leq k \leq b-1.$

LEMMA. Set

$$V(x, y, t) = [W(x, y, t; z, v, 0) - W_1(x, y, t; z, v)].$$

For $1 < p \leq \infty$,

$$\sum_{|\alpha| \leq 2b-1} \| d_p^{|\alpha|} D^\alpha_{x,y} V(\cdot, y, \cdot) \|_{L^p(S_{a,T})} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j V(\cdot, y, \cdot) \|_{L^p(S_{a,T})} \leq C_a,$$

with C_a independent of z, v, y.

PROOF: Let $\varphi_a(t)$ denote a C^∞-function such that $\varphi_a(t) = 0$ for $t \leq a/4$ and $= 1$ for $t \geq a/2$. For $1 \leq k \leq b-1$,

$$\| \Lambda^{-k}(D_y^k V(\cdot, y, \cdot) \|_{L^p(S_{a,T})} \leq \| \Lambda^{-k} D_y^k (V(\varphi_a)(\cdot, y, \cdot) \|_{L^p(S_T)} +$$

$$+ \| \Lambda^{-k} D_y^k V(1-\varphi_a)(\cdot, y, \cdot) \|_{L^p(S_{a,T})}.$$

Since $(1-\varphi_a) = 0$ for $t \geq a/2$ it is not difficult to show that

$$\| \Lambda^{-k} D_y^k V(1-\varphi_a)(\cdot, y, \cdot) \|_{L^p(S_{a,T})} \leq C_a \| \Lambda^{-k} D_y^k V(\cdot, y, \cdot) \|_{L^q(S_T)}$$

with $q > 1$ but as near to 1 as we wish. For q near 1

$$\| \Lambda^{-k} D_y^k W(\cdot, y, \cdot; z, v, 0) \|_{L^q(S_T)} \le C,$$

C independent of (z, v) and y. Using (3.4.5)

$$\| \Lambda^{-k} D_y^k W_1(\cdot, y, \cdot; z, v) \|_{L^q(S_T)} \le$$

$$C \lim_{\delta \to 0} \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j W_1(\cdot, \delta, \cdot; z, v) \|_{L^q(S_T)} \le C \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j W(\cdot, 0, \cdot; z, v, 0) \|_{L^q(S_T)}$$

$$\le C, \quad C \text{ independent of } z, v.$$

From these observations we see that the left side of the inequality in the lemma is bounded by

$$(3.5.2) \qquad \sum_{|\alpha| \le 2b-1} \| d_p^{|\alpha|} D_{x,y}^\alpha (V\varphi_a)(\cdot, y, \cdot) \|_{L^p(S_T)} + $$

$$+ \sum_{k=1}^{b-1} \| \Lambda^{-k} D_y^k (V\varphi_a)(\cdot, y, \cdot) \|_{L^p(S_T)} + C_a .$$

For $1 < p_0 < \infty$ $V\varphi_a \in \mathring{L}^{p_0}_{2b,1}(R^n \times (\delta, \infty) \times (0, T))$ $\forall \delta > 0$ and $\Lambda^{-j} D_y^j (V\varphi_a)(x, y, t)$ converges in $L^{p_0}(S_T)$ to

$$\lim_{y \to 0} \int_0^t \int_{R^n} \Lambda^{-j}(x - w, t - s)[\varphi_a(s) - \varphi_a(t)] D_y^j V(w, y, s)\, dw\, ds.$$

Writing $D_y^j V = \varphi_{a/2} D_y^j V + (1 - \varphi_{a/2}) D_y^j V$ we can see that the L^p-norm over $S_{a/2, T}$ of this last function is bounded by

$$C_a + C_a \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j (V\varphi_{a/2})(\cdot, y, \cdot) \|_{L^{p_1}(S_T)}$$

where $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{r_1} - 1$ and r_1 is any number > 1 for which $|\Lambda^{-j}(x, t)| t \in L^{r_1}(S_T)$. Using this and 3.4.3 it follows that the above norm on $V\varphi_a$ is smaller than

$$C_a [\sup_{y>0} \| d_p^{2b-\gamma} V(\cdot, y, \cdot) \varphi_a'(\cdot) \|_{L^p(S_T)} +$$

$$\lim_{\delta \to 0+} \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j (V\varphi_{a/2})(\cdot, \delta, \cdot) \|_{L^{p_1}(S_T)}] + C_a .$$

$$\| V(\cdot, y, \cdot)\varphi'_a \|_{L^p(S_T)} \leq C_a \| (V C_{a/2})(\cdot, y, \cdot) \|_{L^p(S_T)} \leq C_a \| \Lambda^{2b-1}(V C_{a/2})(\cdot, y, \cdot) \|_{L^{p_2}(S_T)}$$

where $\frac{1}{p} = -1 + \frac{1}{p_2} + \frac{1}{r_2}$ and $r_2 > 1$ is any number such that $\Lambda^{-2b+1}(x, t) \in L^{r_2}(S_T)$.

Set $d_{p_2}(y) = \min(y, T^{\gamma p_2/2b})$. Then from 3.4.3

$$d_{p_2}^{2b-1}(y) \| \Lambda^{2b-1}(V\varphi_{a/2})(\cdot, y, \cdot) \|_{L^{p_2}(S_T)} \leq$$

$$C_a \| d_{p_2}^{2b-1}(\cdot, y, \cdot) \Lambda^{2b-1}(V\varphi_{a/2})(\cdot, y, \cdot) \|_{L^{p_2}(S_T)} \leq$$

$$C_a [\sup_{y>0} \| d_{p_2}^{2b-\gamma}(\cdot, y, \cdot)(V\varphi_{a/2})(\cdot, y, \cdot) \|_{L^{p_2}(S_T)} +$$

$$+ \lim_{y \to 0+} \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j (V\varphi_{a/2})(\cdot, y, \cdot) \|_{L^{p_2}(S_T)}].$$

We may choose $r_1 = r_2$ and so $p_1 = p_2$.

What we have shown is that

$$\sup_{y>0} \left[\sum_{|\alpha| \leq 2b-1} \| [d_p^{|\alpha|} D_{x,y}^\alpha (V\varphi_a)](\cdot, y, \cdot) \|_{L^p(S_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j (V\varphi_a)(\cdot, y, \cdot) \|_{L^p(S_T)} \right]$$

$\leq$ constant depending only on a plus a constant times the same expression with p replaced by p_1; and a replaced by $a/2$. This means that once we have found a p, $1 < p < \infty$, for which 3.5.2 is bounded independent of z, v, it then holds for all finite $p > 1$ and also by the above reduction for $p = \infty$.

For p near 1, (3.5.2) with V replaced by $W(x, y, t; z, v, 0)$ is bounded by a constant independent of z, v, y.

Now

$$\sum_{|\alpha| \leq 2b-1} \| d_p^{|\alpha|} D_{x,y}^\alpha W_1 C_a(\cdot, y, \cdot; z, v) \|_{L^p(S_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j (W_1 \varphi_a)(\cdot, y, \cdot; z, v) \|_{L^p(S_T)}$$

$$\leq C_a \left[\sum_{|\alpha| \leq 2b-1} \| d_p^{|\alpha|} D_{x,y}^\alpha (W_1 C_a)(\cdot, y, \cdot; z, v) \|_{L^p(S_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j W_1(\cdot, y, \cdot; z, v) \|_{L^p(S_T)} \right]$$

$$\leq C_a \lim_{y \to 0+} \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j W_1(\cdot, y, \cdot\,; z, v) \|_{L^p(S_T)}$$

$$\leq C_a \sum_{j=0}^{b-1} \| \Lambda^{-j} D_y^j W(\cdot, 0, \cdot\,; z, v, 0) \|_{L^p(S_T)} \leq C_a ,$$

independent of z, v provided p is near 1. This concludes the proof of 3.5.1.

In the next result $\mathcal{E} = \sum_{|\alpha| \leq 2b} a_\alpha(x, y) D_{x,y}^\alpha$ denotes a strongly elliptic operator in R_+^{n+1}, i.e. $\operatorname{Re}\left(\sum_{|\alpha| = 2b} a_\alpha(x, y)(i\xi)^\alpha\right) < -C|\xi|^{2b}$, $C > 0$ and independent of (x, y). The coefficients are assumed bounded and measurable for all α and uniformly Hölder continuous in $\overline{R_+^{n+1}}$ for $|\alpha| = 2b$. For $1 < p \leq \infty$ again set $d_p(y) = \min(y, T^{\gamma p/2b})$.

Theorem (3.5.3). Suppose $u \in L_{2b}^{p_0}(R_+^{n+1})$ for some p_0, $1 < p_0 < \infty$. Then for $1 < p \leq \infty$,

$$\sup_{y>0}\left[\sum_{|\alpha| \leq 2b-1} d_p^{|\alpha|}(y) \|D_{x,y}^\alpha u(\cdot, y)\|_{L^p(R^n)} + \sum_{j=0}^{b-1} \| G_j(D_y^j u)(\cdot, y) \|_{L^p(R^n)}\right]$$

$$\leq C\left[\sup_{y>0} d_p(y)^{2b-\gamma} \| \mathcal{E} u(\cdot, y) \|_{L^p(R^n)} + \sum_{j=0}^{b-1} \| G_j(D_y^j u(\cdot, 0)) \|_{L^p(R^n)} + \|u\|_{L^1(R_+^{n+1})}\right]$$

Proof: Set $L = \mathcal{E} - D_t$. L is parabolic and by 3.5.1,

$$\left[\sum_{|\alpha| \leq 2b-1} \| D_{x,y}^\alpha u(\cdot, y) \|_{L^p(R^n)} \left(\int_{T/2}^{T} [\min(y, t^{\gamma p/2b})]^{|\alpha| p}\, dt\right)^{1/p}\right.$$

$$\left. + \sum_{j=0}^{b-1} \| \Lambda^{-j}(D_y^j u)(\cdot, y, \cdot) \|_{L^p(S_{T/2,\,T})}\right]$$

$$\leq C \sup_{y>0} \| \mathcal{E} u(\cdot, y) \|_{L^p(R^n)} \left(\int_0^T [\min(y, t^{\gamma p/2b})]^{(2b-\gamma)p}\, dt\right)^{1/p}$$

$$+ C \sum_{j=1}^{b} \| \Lambda^{-j} D_y^j u(\cdot, 0, \cdot) \|_{L^p(S_T)} + C_T \| u \|_{L^1(R_+^{n+1})} .$$

Recall $d_p(y) = \min(y, T^{\gamma p/2b})$. It is easy to check that there exists a constant

$C > 0$ such that

$$\frac{1}{C}\, d_p(y)^{|\alpha|}\, T^{1/p} \leq \left(\int_{T/2}^{T} [\min (y, t^{\gamma p/2b})]^{|\alpha| p} \right)^{1/p} \leq C d_p(y)^{|\alpha|}\, T^{1/p}.$$

The same inequality holds when $|\alpha|$ is replaced by $2b - \gamma$ and the integral from $T/2$ to T by the integral from 0 to T. Therefore

$$\sup_{y>0} \left[\sum_{|\alpha| \leq 2b-1} d_p(y)^{|\alpha|} \| D^{\alpha}_{x,y} u(\cdot, y) \|_{L^p(R^n)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_y u(\cdot, y, \cdot) \|_{L^p(S_{T/2,\,T})} \right]$$

$$\leq C_T \left[\sup_{y>0} d_p(y)^{2b-\gamma} \| \mathcal{E} u(\cdot, y) \|_{L^p(R^n)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_y u(\cdot, 0) \|_{L^p(S_T)} + \| u \|_{L^1(R^{n+1}_+)} \right].$$

To conclude the proof we note that in the appendix (A.2) it is proved that there are constants C and C_T such that for $f(x) \in L^p(R^n)$, $1 \leq p \leq \infty$,

$$\| \Lambda^{-k}(f)(\cdot,\cdot) \|_{L^p(S_T)} \leq C \| G_k(f)(\cdot) \|_{L^p(R^n)} \leq C_T \| \Lambda^{-k}(f)(\cdot,\cdot) \|_{L^p(S_{T/2,\,T})}.$$

4. Parabolic estimates and the initial-Dirichlet problem.

(4.1) *Case of zero initial data.*

Throughout the remainder of this paper Ω will denote a bounded domain in R^{n+1} whose boundary, $\partial\Omega$, is assumed to be in the class C^{2b+1}. We recall from section 1.2 that there is a positive number δ_0 such that each point $\bar{x} \in \Omega$ with $d(\bar{x}) = \text{dist.}(\bar{x}, \partial\Omega) < 4\delta_0$ can be uniquely written as $\bar{x} = rN_Q + Q$ with $Q \in \partial\Omega$, $0 < r < 4\delta_0$, and N_Q denoting the unit inner-normal at Q. We set $\Omega_T = \Omega \times (0, T)$ and $\partial\Omega_T = \partial\Omega \times (0, T)$.

THEOREM (4.1.1). As usual we assume the coefficients of L are bounded and measurable for all α and uniformly Hölder continuous in $\bar{\Omega}_T$ for $|\alpha| = 2b$. If $u \in \overset{\circ}{L}{}^p_{2b,1}(\Omega_T)$ with $1 < p < \infty$ and satisfies $Lu = 0$ in Ω_T, then

$$\sup_{r<\delta_0} \left[\sum_{|\alpha| \leq 2b-1} r^{|\alpha|} \| D^{\alpha}_{x} u(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} + \right.$$

$$\left. \sum_{k=0}^{b-1} \| \Lambda^{-k} (D^k_{N_Q} u)(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \right] \leq C \sum_{k=0}^{b-1} \| \Lambda^{-k} (D^k_{N_Q} u)(Q, t) \|_{L^p(\partial\Omega_T)}$$

$(D^k_{N_Q} u(Q + rN_Q) = D^k_s(u(Q + sN_Q))(r))$.

PROOF: We recall the finite open covering $\{U_i\}$ of $\bar{\Omega} \setminus \Omega_{\delta_0}$ ($\Omega_\delta = \{\bar{x} \in \Omega : d(\bar{x}) > \delta\}$) described in section 1.2 the partition of unity, $\{\varphi_i\}$, subordinate to it, the diffeomorphism $F_i : B_i^+ \to U_i$ ($B_i^+ = \{(x, y) \in R_+^{n+1} : |x|^2 + y^2 < r_i\}$), and finally the sequence, $\{\psi_i\}$, with the property that $\psi_i \in C_0^\infty(U_i)$ and $\psi_i \equiv 1$ in a neighborhood of the support of φ_i. In $\Omega \setminus \Omega_{\delta_0}$, $u(\bar{x}, t) = \sum_i (u\varphi_i)(\bar{x}, t)$. Set $u_i = u\varphi_i$.

The left side of the inequality in 4.1.1 applied to u_i is dominated by a constant times

$$\sum_i \sup_{y>0} \Big[\sum_{|\alpha| \leq 2b-1} y^{|\alpha|} \| D^\alpha_{x,y}(u_i \circ F_i)(\cdot, y, \cdot) \|_{L^p(S_T)} + \sum \| \Lambda^{-k} D^k_y (u_i \circ F_i)(\cdot, y, \cdot) \|_{L^p(S_T)} \Big]. \tag{4.1.2}$$

The function $u_i \circ F_i(x,y,t)$ belongs to $\mathring{L}^p_{2b,1}(R^{n+1}_+ \times (0, T))$. Also there exists a parabolic operator $L^i = \sum_{|\alpha| \leq 2b} a^i_\alpha(x, y, t) D^\alpha_{x,y} - D_t$ satisfying

i) a^i_α are bounded and measurable in $R^{n+1}_+ \times (0, T)$, and for $|\alpha| = 2b$ are uniformly Hölder continuous in $\overline{R^{n+1}_+} \times [0, T]$,

ii) $$L^i(u \circ F_i) \circ F_i^{-1}(\bar{x}, t) = Lu(\bar{x}, t), \; \bar{x} \in U_i \cap \Omega.$$

From theorem 3.4.5 there is a γ, $0 < \gamma < 1$, such that 4.1.2 is bounded by

$$C \sum_i \left[\omega(T) \sup_{y>0} y^{2b-\gamma} \| L^i(u_i \circ F_i)(\cdot, y, \cdot) \|_{L^p(S_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_y (u_i \circ F_i)(\cdot, 0, \cdot) \|_{L^p(S_T)} \right].$$

Now

$$\Lambda^{-j} D^j_y (u_i \circ F_i) = \psi_i \Lambda^{-j} [\varphi_i D^j_y (u \circ F_i)] + (1 - \psi_i) \Lambda^{-j} [\varphi_i D^j_y (u \circ F_i)]$$
$$\sum_{0 < l \leq j} \Lambda^{-j} [D^l_y \varphi_i D^{j-l}_y (u \circ F_i)].$$

Hence 4.1.2 is bounded by a constant times

$$\sum_i \Big[\omega(T) \sup_{r < 2\delta_0} r^{2b-\gamma} \| L(u_i)(Q + rN_Q, t) \|_{L^p(\partial \Omega_T)} +$$
$$\sum_{j=0}^{b-1} \| \Lambda^{-j-1} D^j_y (u_i \circ F_i)(x, 0, t) \|_{L^p(S_T)} \Big] + \sum_{j=0}^{b-1} \| \Lambda^{-j} (D^j_{N_Q} u)(Q, t) \|_{L^p(\partial \Omega_T)}.$$

Now set $\| u \|_{T,p}$ equal to the left side of the inequality in theorem (4.1.1).

Since $0 < \gamma < 1$ by picking a positive number δ small enough the above inequality implies that

$$(4.1.3) \qquad \ldots \| u \|_{T,p} \leq C \Big[\omega(T) \sup_{2\delta_0 > r > \delta} \sum_{|\alpha| \leq 2b-1} \| D^\alpha u(Q + rN_Q, t) \|_{L^p(\partial \Omega_T)} +$$

$$+ \sum_{j=0}^{b-} \| \Lambda^{-i} D^j_{N_Q} u(Q, t) \|_{L^p(\partial \Omega_T)} + \sum_i \sum_{j=0}^{b-1} \| \Lambda^{-j-1} D^j_y (u_i \circ F_i)(x, 0, t) \|_{L^p(S_T)} \Big].$$

Set

$$\| u \|_{B,p} = \sup_{2\delta_0 > r > \delta} \sum_{|\alpha| \leq 2b-1} \| D^\alpha u(Q + rN_{Q,t}) \|_{L^p(\partial \Omega_T)} +$$

$$+ \sup_{y > 0} \sum_i \sum_{j=0}^{b=1} \| \Lambda^{-1-j} D^j_y (u_i \circ F_i)(x, y, t) \|_{L^p(S_T)}.$$

At this point we will point out that we have also shown that 4.1.3 remains true with $p = \infty$. This observation will be used in theorem 4.1.5.

Lemma. Given $\varepsilon > 0$ there exists a constant C_ε such that for all $u \in \mathring{L}^p_{2b,1}(\Omega_T)$ with $Lu = 0$,

$$\| u \|_{B,p} \leq \varepsilon \| u \|_{T,p} + C_\varepsilon \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q, t) \|_{L^p(\partial \Omega_T)} \qquad 1 < p < \infty).$$

Proof: If not then there exists $\varepsilon_0 > 0$ and a sequence $\{u_k\}$, $u_k \in \mathring{L}^p_{2b,1}(\Omega_T)$ such that

$$Lu_k = 0, \ \| u_k \|_{B,p} = 1, \text{ and } 1 > \varepsilon_0 \| u_k \|_{T,p} + k \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u_k(Q, t) \|_{L^p(\partial \Omega_T)}.$$

In our new notation the inequality 4.1.3 implies that for $u \in \mathring{L}^p_{2b,1}(\Omega_T)$, with $Lu = 0$,

$$(4.1.4) \qquad \ldots \| u \|_{T,p} \leq C \Big(\omega(T) \| u \|_{B,p} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q}(u) \|_{L^p(\partial \Omega_T)} \Big).$$

Now, $\| u_k \|_{T,p} \leq \dfrac{1}{\varepsilon_0}$. We claim that in any domain Ω^* with $\bar{\Omega}^* \subset \Omega$ the sequence $\{u_k\}$ is bounded in $L^p_{2b,1}(\Omega^*_T)$. In fact if $\varphi(\bar{x}) \in C_0^\infty(\Omega)$ with $\varphi \equiv 1$ on Ω_{δ_0} then

$$\| u_k \varphi \|_{L^p_{2b,1}(\Omega_T)} \leq C \| L(u_k \varphi) \|_{L^p_{2b,1}(\Omega_T)} \leq C_\varphi \| u_k \|_{T,p}.$$

Hence a subsequence, which we again denote by $\{u_k\}$, converges weakly in $L^p_{2b,1}(\Omega^*_T)$, Ω^* any domain with $\bar{\Omega}^* \subset \Omega$, to a function $u \in \mathring{L}^p_{2b,1}(\Omega^*_T)$.

Clearly $Lu = 0$. Moreover by 4.1.4

$$\| u_k - u_l \|_{T,p} \leq C\left(\| u_k - u_l \|_{B,p} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q}(u_k - u_l) \|_{L^p(\partial\Omega_T)} \right).$$

The second term converges to zero as $k, l \to \infty$ since $\lim_{k\to\infty} \| \Lambda^{-j} D^j_{N_Q}(u_k) \|_{L^p(\partial\Omega_T)} = 0$. For the moment let us assume $\| u_k - u_l \|_{B,p} \to 0$ as $k, l \to \infty$. We will prove this at the end of the discussion. Then $\| u_k - u_l \|_{T,p} \to 0$ as $k, l \to \infty$. This immediately implies that $\| u_k - u \|_{T,p} \to 0$ as $k \to \infty$ and that for $0 \leq j \leq b-1$, $\lim_{r\to 0^+} \Lambda^{-j}(D^j_{N_Q} u)(Q + rN_Q, t)$ exists in $L^p(\partial\Omega_T)$ and equals $\lim_{k\to\infty} \Lambda^{-j}(D^j_{N_Q} u_k)(Q, t)$ which exists since $\| u_k - u_l \|_{T,p} \to 0$ as $k, l \to \infty$. Because of the presence of $\omega(T)$ in 4.1.3 it is not difficult to see that 4.1.3 implies that there exists T_0 such that for $T \leq T_0$, the inequality in (4.1.1) holds. In fact taking $\varphi(\bar{x}) \in C_0^\infty(\Omega)$ with $\varphi \equiv 1$ in Ω_δ we have that

$$\sup_{2\delta_0 > r > \delta} \sum_{|\alpha| \leq 2b-1} \| D^\alpha u(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} =$$

$$= \sup_{2\delta_0 > r > \delta} \sum_{|\alpha| \leq 2b-1} \| D^\alpha (u\varphi)(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \leq C \| L(u\varphi) \|_{L^p(\Omega_T)} \leq C \| u \|_{T,p}.$$

Also

$$\sum_{j=0}^{b-1} \| \Lambda^{-1-j} D^j_y (u_i \circ F_i)(\cdot, 0, \cdot) \|_{L^p(S_T)} \leq \omega(T) \| \Lambda^{-j} D^j_y (u_i \circ F_i)(\cdot, 0, \cdot) \|_{L^p(S_T)}.$$

By taking T small enough the last two inequalities imply that 4.1.2 is bounded by

$$C \sum_i \left[\omega(T) \sup r^{2b-\gamma} \| L(u_i)(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q}(u) \|_{L^p(\partial\Omega_T)} \right].$$

This implies that for $T \leq T_0$, 4.1.3 holds without the last summation. Finally for $T \leq T_0$ $\| u \|_{T,p} \leq C \sum_{j=1}^{b} \| \Lambda^{-j} D^j_{N_Q} u(Q, t) \|_{L^p(\partial\Omega_T)}$. This implies that for the above u, which is the weak limit of $\{u_k\}$, $\| u \|_{T_0,p} = 0$ and so $u \equiv 0$ in $(\Omega \setminus \bar{\Omega}_{\delta_0})_{T_0}$. But then $u \in \mathring{L}^p_{2b,1}(\Omega_{T_0})$ and $u(\bar{x}, t)$ is zero in a neighborhood of the boundary $(t \leq T_0)$. Hence $u \equiv 0$ in Ω_{T_0}. Now consider the function $u_{T_0} = u(\bar{x}, t + T_0)$. Once again $\| u_{T_0} \|_{T_0,p} \leq C \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q}(u_{T_0}) \|_{L^p(\partial\Omega_{T_0})}$.

Since $u=0$ for $t<T_0$, $\Lambda^{-j}D^j_{N_Q}(u_{T_0})(Q,t)=\Lambda^{-j}D^j_{N_Q}(u)(Q,t+T_0)$. Hence

$$\| \Lambda^{-j}D^j_{N_Q}(u_{T_0})\|_{L^p(\partial\Omega_{T_0})} \leqq \| \Lambda^{-j}D^j_{N_Q}(u)\|_{L^p(\partial\Omega_{2T_0})} \leq$$

$$\leq \lim_{k\to\infty} \| \Lambda^{-j}(D^j_{N_Q}(u_k))\|_{L^p(\partial\Omega_{2T_0})} = 0.$$

So $u_{T_0}\equiv 0$ in $\Omega_{T_0}\to u\equiv 0$ in Ω_{2T_0}. Hence $u\equiv 0$ in Ω_T. It will follow from the proof of the fact that $\| u_k-u_l\|_{B,p}\to 0$ as $k,\ l\to\infty$ that $\| u_k\|_{B,p}\to \| u\|_{B,p}$ as $k\to\infty$. Hence $\| u\|_{B,p}=1$, a contradiction.

With the lemma, of course, we conclude the proof of the inequality in (4.1.1). For the completion of the lemma we need to show:

SUBLEMMA: If $\| u_k\|_{T,p}\leq C$ and $\| u_k\|_{B,p}\leq C$ for all k, then there is a subsequence, again called $\{u_k\}$, such that $\| u_k-u_l\|_{B,p}\to 0$ as $k,l\to\infty$.

PROOF: First consider $\sup\limits_{2\delta_0>r>\delta} \| D^\alpha_x (u_k-u_l)(Q+rN_Q,t)\|_{L^p(\partial\Omega_T)}$.

Take $\varphi\in C_0^\infty(\Omega)$, $\varphi\equiv 1$ in Ω_δ. The above norm is bounded by

$$C\,\| (u_k-u_l)\varphi\|_{L^p_{2b,1}(\Omega_T)} \leq C\,\| L[(u_k-u_l)\varphi]\|_{L^p(\Omega_T)} \leq$$

$$\leq C \sum_{|\alpha|\leq 2b-1} \| D^\alpha(u_k-u_l)\|_{L^p(\Omega^\varphi_T)}$$

where $\overline{\Omega^\varphi}\subset\Omega$. Since $\| u_k\|_{L^p_{2b,1}(\Omega^\varphi_T)}\leq C$, C independent of k, a subsequence can be chosen so that the above sum tends to zero as k and l tend to ∞. Now consider $\sup\limits_{y>0} \sum\limits_i \sum\limits_{j=0}^{b-1} \| \Lambda^{-j-1} D^j_y(\varphi_j u_k\circ F_i)(\cdot,y,\cdot)\|_{L^p(S_T)}$. As a function of y the above norm is continuous in $[0,\infty)$ and is zero for $y\geq R$, R independent of k. So there exists $y_k\in[0,R]$ such that

$$\sup_{y>0} \| \Lambda^{-j-1} D^j_y(\varphi_i u_k\circ F_i)(\cdot,y,\cdot)\|_{L^p(S_T)}$$

is attained at y_k. Set $f_k(x,t)=\Lambda^{-j}D^j_y(\varphi_i u_k\circ F_i)(x,y_k,t)$. The function $\varphi_i u_k\circ F_i$ has support contained in D_T where D is a bounded open set in R^n which does not depend on k. Let $\psi(x)\in C_0^\infty(R^n)$ with $\psi\equiv 1$ in a neighborhood of D. Now $f_k=\psi f_k+(1-\psi)f_k=g_k+h_k$. Since $\| u_k\|_{T,p}$ and $\| u_k\|_{B,p}$ are bounded in k, $\| f_k\|_{L^p(S_T)}\leq C$ independent of k.

CLAIM: A subsequence of $\Lambda^{-1}(h_k)$ converges in $L^p(S_T)$.

Since $\|f_k\|_{L^p(S_T)} \leq C$, independent of k, a subsequence of the $\{f_k\}$ converges weakly in $L^p(S_T)$. We again call this subsequence f_k. It is not difficult to see that this implies that the sequence $\{h_k\}$ is a bounded sequence in $L^p(S_T)$ and $L^\infty(S_T)$ and that $h_k \to h$ pointwise in S_T. By Young's inequality $\|\Lambda^{-1}(h_k - h)\|_{L^{p-\delta}(S_T)} \leq C$ for some $\delta > 0$ and small.

$$\int_{S_T} |\Lambda^{-1}(h_k - h)|^p = \int_{\{|\Lambda^{-1}(h_k-h)| > \varepsilon} |\Lambda^{-1}(h_k - h)|^p + \int_{\{|\Lambda^{-1}(h_k-h)| \leq \varepsilon\}} |\Lambda^{-1}(h_k - h)|^p .$$

The last term is bounded by $\varepsilon^\delta \|\Lambda^{-1}(h_k - h)\|^{p-\delta}_{L^{p-\delta}(S_T)} \leq C\,\varepsilon^\delta$, C independent of k. The first term is bounded by

$$\|\Lambda^{-1}(h_k - h)\|^p_{L^\infty(S_T)} \left|\{|\Lambda^{-1}(h_k - h)| > \varepsilon\}\right|.$$

For ε fixed this converges to 0 as $k \to \infty$. Hence we have shown that $\|\Lambda^{-1}(h_k - h)\|_{L^p(S_T)} \to 0$ as $k \to 0$.

CLAIM: A subsequence of $\Lambda^{-1}(g_k)$ converges in $L^p(S_T)$.

The support of g_k is contained in $A_T = A \times (0, T)$ where A is a bounded open subset of R^n. Again a subsequence, which we call $\{g_k\}$, converges weakly in $L^p(S_T)$. Let $\theta(x) \in C_0^\infty(R^n)$ with $\theta(x) = 1$ on a neighborhood of A. The sequence $\theta(x)\Lambda^{-1}(g_k)(x, t)$ converges in $L^p(S_T)$. On the other hand $(1 - \theta)\Lambda^{-1}(g_k)$ converges pointwise and is bounded by

$$|1 - \theta(x)| \int_0^t \int_A |\Lambda^{-1}(x - y, t - s)|^{p'}\, dy\, ds)^{1/p'} \|g_k\|_{L^p(S_T)}, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

This in turn is bounded by $e^{-c|x|}$ since $1 - \theta(x) = 0$ in a neighborhood of A. Hence by Lebesque's dominated convergence theorem $(1 - \theta(x))\Lambda^{-1}(g_k)$ converges in $L^p(S_T)$.

We have now completed the proof of 4.1.1.

THEOREM (4.1.5) Assume $u \in \bigcap_{1<p<\infty} \overset{\circ}{L}{}^p_{2b,1}(\Omega_T)$ and satisfies $Lu = 0$. Then

$$\sup_{r<\delta_0} \Big[\sum_{|\alpha| \leq 2b-1} r^{|\alpha|} \|D_x^\alpha u(Q + rN_Q, t)\|_{L^\infty(\partial\Omega_T)} +$$

$$+ \sum_{k=0}^{b-1} \|\Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t)\|_{L^\infty(\partial\Omega_T)}\Big] \leq C \sum_{k=0}^{b-1} \|\Lambda^{-k} D^k_{N_Q} u(Q, t)\|_{L^\infty(\partial\Omega_r)}.$$

PROOF: We recall from the observation early in the proof of 4.1.1 that 4.1.3 or what is the same that 4.1.4 holds with $p = \infty$. (u needed only to belong to $\overset{\circ}{L}{}^{p_1}_{2b,1}(\Omega_T)$ for some p_1, $1 < p_1 < \infty$). This means that

$$\| u \|_{T,\infty} \leq C \left(\| u \|_{B,\infty} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q,t) \|_{L^\infty(\partial\Omega_T)} \right).$$

However for q large ($1 < q < \infty$),

$$\sup_{2\delta_0 > r > \delta} \sum_{|\alpha| \leq 2b-1} \| (D^{\alpha}_x u)(Q + rN_Q, t) \|_{L^\infty(\partial\Omega_T)} \leq C \| u\varphi \|_{L^q_{2b,1}(\Omega_T)}$$

where $\varphi \in C^\infty_0(\Omega)$ and $\varphi \equiv 1$ on Ω_δ. Once again

$$\| u\varphi \|_{L^q_{2b,1}(\Omega_T)} \leq C \| L(u\varphi) \|_{L^q(\Omega_T)} \leq C \| u \|_{T,q}.$$

Using 4.1.1,

$$\| u \|_{T,q} \leq C \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q,t) \|_{L^q(\partial\Omega_T)}.$$

For $1 \leq j \leq b-1$,

$$\Lambda^{-j} D^j_y (u_i \circ F_i) = \Lambda^{-j} ((\varphi_i \circ F_i) D^j_y (u \circ F_i)) + \sum_{0 < l \leq j} \Lambda^{-j} [D^l_y (\varphi_i \circ F_i) D^{j-l}_y (u \circ F_i)].$$

Hence

$$\sum_i \sum_{j=0}^{b-1} \| \Lambda^{-1-j} D^j_y (u_i \circ F_i)(x,0,t) \|_{L^\infty(S_T)} \leq C_\Omega \left[\sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q,t) \|_{L^\infty(\partial\Omega_T)} + \sum_i \sum_{j=0}^{b-1} \| \Lambda^{-1-j} D^j_y (u_i \circ F_i) \|_{L^q(S_T)} \right]$$

where q is taken to be sufficiently large. Since the last summation on the right side is bounded by $\| u \|_{B,q}$, it follows from the lemma in 4.1.1 that for any $\varepsilon > 0$

$$\| u \|_{B,q} \leq \varepsilon \| u \|_{T,q} + C_\varepsilon \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q,t) \|_{L^q(\partial\Omega_T)}$$

$$\leq \varepsilon \| u \|_{T,\infty} + C_\varepsilon \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q,t) \|_{L^\infty(\partial\Omega_T)}.$$

THEOREM (4.16) Suppose $\Phi_0, \ldots, \Phi_{b-1} \in L^p(\partial\Omega_T)$, $1 < p < \infty$. Then there exists a unique $u(\bar{x}, t)$ satisfying:

i) for any subdomain Ω^* with $\bar{\Omega}^* \subset \Omega$, $u \in \mathring{L}^p_{2b,1}(\Omega^*_T)$

ii) $Lu = 0$ in Ω_T

iii) $\lim\limits_{r \to 0+} (\Lambda^{-j} D^j_{N_Q} u)(Q + rN_Q, t) = \Phi_j(Q, t)$ in $L^p(\partial\Omega_T)$.

Moreover this solution satisfies the inequality

$$\sup_{r<\delta_0} \left[\sum_{|\alpha| \leq 2b-1} r^{|\alpha|} \| D^\alpha_{\bar{x}} u(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \right]$$

$$\leq C_{p,\Omega} \sum_{j=0}^{b-1} \| \Phi_j \|_{L^p(\partial\Omega_T)}.$$

PROOF :

Existence. Let $\{\Phi^{(k)}_j(Q,t)\}$ be a sequence of functions each in $C^\infty_0(\partial\Omega \times (0, \infty))$ such that $\Phi^{(k)}_j \to \Phi_j$ in $L^p(\partial\Omega_T)$ as $k \to \infty$. We can find $u_k(\bar{x}, t) \in \mathring{L}^p_{2b,1}(\Omega_T)$ satisfying (ii) and (iii) with Φ_j replaced by $\Phi^{(k)}_j$. (See [10]. Actually $u_k \in \mathring{L}^p_{2b,1}(Q_T)$ for all p, $1 < p < \infty$). Take any function $\psi(\bar{x}) \in C^\infty_0(\Omega)$. Then

$$\| \psi(\bar{x}) u_k(\bar{x}, t) \|_{L^p_{2b,1}(\Omega_T)} \leq C \sum_{\substack{|\beta| > 0 \\ |\beta| + |\gamma| \leq 2b}} \| D^\beta_{\bar{x}} \psi)(D^\gamma_{\bar{x}} u_k) \|_{L^p(\Omega_T)}.$$

This last estimate together with 4.1.1 shows that in any subdomain Ω^* with $\overline{\Omega^*} \subset \Omega$ the sequence $\{u_k\}$ is a Cauchy sequence in $L^p_{2b,1}(\Omega^*_T)$. We let $u(\bar{x}, t)$ denote that function which is the limit of u_k in $L^p_{2b,1}(\Omega^*_T)$ for any Ω^* ($\overline{\Omega^*} \subset \Omega$).

Clearly $u(\bar{x}, t)$ satisfies (ii). To see that condition (iii) is satisfied we first observe from 4.1.1 that

$$\sup_{r<\delta_0} \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} (u_k - u_m)(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \leq C \sum_{j=0}^{b-1} \| \Phi^{(k)}_j - \Phi^{(m)}_j \|_{L^p(\partial\Omega_T)}.$$

Now letting $m \to \infty$ we conclude that

$$\sup_{r<\delta_0} \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} (u_k - u)(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \to 0 \text{ as } k \to \infty.$$

It is easy to see now that $\Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) \to \Phi_j(Q, t)$ in $L^p(\partial\Omega_T)$ as $r \to 0+$.

Uniqueness. This is immediate consequence of the inequality in 4.1.1 applied to u in the cylinder $(\Omega_\delta)_T$ for δ sufficiently small, and then letting $\delta \to 0+$.

In much the same manner using 4.1.5 instead of 4.1.1, one obtains the theorem

THEOREM (4.1.7). Suppose $\Phi_0, \dots, \Phi_{b-1} \in C(\partial\Omega \times [0,T])$ with $\Phi_j(Q,0) = 0$ for $0 \leq j \leq b-1$. Then there exists unique $u(\bar{x}, t)$ satisfying:

i) for all $\Omega^*, \bar{\Omega}^* \subset \Omega$, $u \in \bigcap_{1<p<\infty} \overset{\circ}{L}{}^p(\Omega_T^*)$

ii) $Lu = 0$ in Ω_T

iii) $\lim_{t \to 0+} \| u(\cdot, t) \|_{L^\infty(\Omega)} = 0$

iv) $\lim_{r \to 0+} \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) - \Phi_j(Q,t) \|_{L^\infty(\partial\Omega_T)} = 0.$

Moreover this solution satisfies the inequality

$$\sup_{r<\delta_0} \Big[\sum_{|\alpha| \leq 2b-1} r^{|\alpha|} \| D^\alpha_x u(Q + rN_Q, t) \|_{L^\infty(\partial\Omega_T)} + \\ + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u(Q + rN_Q, t) \|_{L^\infty(\partial Q_T)} \Big] \leq C_\Omega \sum_{j=0}^{b-1} \| \Phi_j \|_{L^\infty(\partial\Omega_T)}.$$

PROOF: As we pointed out above the existence of u satisfying i, ii, and iv follows the exact lines of proof as 4.1.6 using 4.1.5 instead of 4.1.1. To check condition (iii) we observe first that the indicated argument shows that

$$\| u(\cdot, t) \|_{L^\infty(\Omega \setminus \Omega_{\delta_0})} \leq \| u(\cdot, t) - u_k(\cdot, t) \|_{L^\infty(\Omega \setminus \Omega_{\delta_0})} + \| u_k(\cdot, t) \|_{L^\infty(\Omega \setminus \Omega_{\delta_0})},$$

and this implies $\| u(\cdot, t) \|_{L^\infty(\Omega \setminus \Omega_{\delta_0})} \to 0$ as $t \to 0+$. Since

$u \in \bigcap_{1<p<\infty} \overset{\circ}{L}{}^p_{2b,1}((\Omega_{\delta_0})_T)$ it follows that $\lim_{t \to 0+} \| u(\cdot, t) \|_{L^\infty(\Omega_{\delta_0})} = 0$.

(4.2) *Non-zero initial data*

THEOREM (4.2.1). Suppose $u \in L^p_{2b,1}(\Omega_T)$, $1 < p < \infty$, satisfies $Lu = 0$ in Ω_T. There exists γ, $0 < \gamma < 1$, depending only on L, such that

$$\sup_{r<\delta_0}\left[\sum_{|\alpha|\leq 2b-1}\| d_p^{|\alpha|}(Q+rN_Q,t)D_{\bar{x}}^{\alpha}\, u(Q+rN_Q,t)\|_{L^p(\partial\Omega\times(0,T))}+\right.$$

$$\left.+\sum_{j-1}^{b}\|\Lambda^{-j}D_{N_Q}^{j}u(Q+rN_Q,t)\|_{L^p(\partial\Omega\times(0,T))}\right]\leq$$

$$\leq C_{p,\Omega}\left[\sum_{j=1}^{b}\|\Lambda^{-j}D_{N_Q}^{j}u(Q,t)\|_{L^p(\partial\Omega\times(0,T))}+\|u(\bar{x},0)\|_{L^p(\Omega)}\right].$$

PROOF: We may assume the coefficients of L are defined in all of $R^{n+1}\times[0,T]$ are bounded and measurable and that the coefficients of highest order are uniformly Hölder continuous. We let $W(\bar{x},t;M,s)$ denote a fundamental solution for the initial-value problem of L as constructed in section (3.3). More precisely:

$$\mathrm{W}(\bar{x},t;M,s)=\Gamma_{M,s}(\bar{x}-M,t-s)+\int_0^t\int_{R^{n+1}}\Gamma_{w,r}(\bar{x}-w,t-r)\,\Phi(w,r;M,s)\,dw\,dr$$

where for $t>s$ $\Phi(w,r;M,s)$ satisfies the integral identity

$$L_{\bar{x},t}\Gamma_{M,s}(\bar{x},M;t-s)=-\Phi(\bar{x},t;M,s)+$$

$$+\int_s^t\int_{R^{n+1}}L_{\bar{x},t}(\Gamma_{w,r}(\bar{x}-w,t-r))\,\Phi(w,r;M,s)\,dw\,dr.$$

Here $L_{\bar{x},t}=\sum_{|\alpha|\leq 2b}a_\alpha(\bar{x},t)\,D_{\bar{x}}^{\alpha}-D_t$.

We let $W_\alpha^p(\Omega)$, $0<\alpha<1$, $1<p<\infty$, denote the space of functions $f(\bar{x})$ defined on Ω such that $f\in L^p(\Omega)$ and $\int_\Omega\int_\Omega\frac{|f(\bar{y})-f(\bar{x})|^p}{|\bar{x}-\bar{y}|^{n+1+\alpha p}}\,d\bar{x}\,d\bar{y}<\infty$. Since $u\in L_{2b,1}^p(\Omega_T)$ it belongs to $L_{2b,1}^q(\Omega_T)$ for $1<q\leq p$. It is known that $u(\bar{x},0)\in W_{q,\frac{2b}{q'}}(\Omega)$, $\frac{1}{q}+\frac{1}{q'}=1$. Since we may assume q is near 1 and hence $\frac{2b}{q'}<1$, it is not difficult to see that we can extend $u(\bar{x},0)$ to a function having compact support in R^{n+1}, belonging to $W_{q,\frac{2b}{q'}}(R^{n+1})$, and whose L^p-norm over R^{n+1} is dominated by a constant times the L^p-norm over Ω of $u(\bar{x},0)$. We will denote the extension also by $u(\bar{x},0)$.

Now set $u_1(\bar{x}, t) = \int_{R^{n+1}} W(\bar{x}, t; M, 0)\, u(M, 0)\, dM$. For q near 1, $u_1 \in L^q_{2b,1}$ $(R^{n+1} \times (0, T))$. Clearly $Lu_1(\bar{x}, t) = 0$ and

$$\| u_1(\cdot, t) - u(\cdot, 0) \|_{L^q(\Omega)} \to 0 \text{ as } t \to 0.$$

The function $u - u_1 \in \mathring{L}^q_{2b,1}(\Omega \times (0, T))$. We leave it to the reader to verify that there is a $\gamma, 0 < \gamma < 1$ such that

$$\sup_{r<\delta_0} \left[\sum_{|\alpha| \le 2b-1} \| t^{|\alpha|/2b\, \gamma_p} D^\alpha_x u_1(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} + \right.$$

$$\left. + \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q} u_1(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \right] \le$$

$\le C_\Omega \| u(\bar{x}, 0) \|_{L^p(\Omega)}$ and that $\lim_{r \to 0+} \Lambda^{-j} D^j_{N_Q} u_1(Q + rN_Q, t)$ exists in $L^p(\partial\Omega_T)$.

With u given in $L^p_{2b,1}(\Omega_T)$ the latter limit also exists for u. Since $1 < q \le$ $\le p$ $\lim_{r \to 0+} \Lambda^{-j} D^j_{N_Q}(u - u_1)(Q + rN_{Q,t})$ exists in $L^p \cap L^q(\partial\Omega_T)$.

So by the uniqueness of 4.1.6, $u - u_1 = u_0$ satisfies the condition of 4.1.6 for p. The estimate in 4.2.1 follows from the estimate for u_0 given in 4.1.6. and from the above estimates on u_1.

THEOREM (4.2.2) Suppose $\Phi_0, \dots \Phi_{b-1}$ are given function in $L^p(\partial\Omega_T)$, $1 < p < \infty$, and $h(\bar{x}) \in L^p(\Omega)$ Then there is a unique $u(\bar{x}, t)$ satisfying,

a) $$u(\bar{x}, t) \in L^p_{2b,1,\text{loc}}(\Omega_T)$$

b) $$Lu = 0 \text{ in } \Omega_T$$

c) $$\lim_{r \to 0+} \lim_{\varepsilon \to 0+} \sum_{j=0}^{b-1} \| \Lambda^{-j} D^j_{N_Q}(u(\cdot, \cdot + \varepsilon))(Q + rN_Q, t) - $$
$$- \Phi_j(Q, t) \|_{L^p(\partial\Omega_{T-\varepsilon})} = 0$$

d) for each subdomain Ω^*, $\overline{\Omega^*} \subset \Omega$, $\lim_{t \to 0+} \| u(\bar{x}, t) - h(\bar{x}) \|_{L^p(\Omega^*)} = 0$.

Moreover, for the same γ as in 4.2.1, this solution satisfies the inequality

$$\sup_{r<\delta_0}\left[\sum_{|\alpha|\leq 2b-1}\|(d_p^{|\alpha|} D_x^\alpha u)(Q+rN_Q,t)\|_{L^p(\partial\Omega_T)}+\right.$$

$$\left.+\sum_{j=0}^{b-1}\|\Lambda^{-j} D_{N_Q}^j(u)(Q+rN_Q,t)\|_{L^p(\partial\Omega_T)}\right]\leq$$

$$\leq C_\Omega\left[\sum_{j=0}^{b-1}\|\Lambda^{-j} D_{N_Q}^j u(Q,t)\|_{L^p(\partial\Omega_T)}+\|h(\bar{x})\|_{L^p(\Omega)}\right].$$

Proof :

Existence: By extension we may again assume that the coefficients of L are defined in all of $R^{n+1}\times[0,T]$, are bounded and measurable there, and that those of highest order are uniformly Hölder continuous. Let $W(\bar{x},t,M,s)$ again denote a fundamental solution as described in the proof of 4.1.7.

Let $\{h_k(\bar{x})\}$ denote a sequence of functions with $h_k\in C_0^\infty(\Omega)$ and $h_k\longrightarrow h$ in $L^p(\Omega)$ as $k\longrightarrow\infty$. Set $u_{1,k}(\bar{x},t)=\int_\Omega W(\bar{x},t;M,0)\,h_k(M)\,dM$. By theorem 4 1.4 there exists a unique $u_{0,k}$ satisfying conditions (i)-(iii) of 4.1.6 with Φ_j replaced by $\Phi_j(Q,t)-\Lambda^{-j}D_{N_Q}^j u_{1,k}(Q,t)$. Set $u_k=u_{0,k}+u_{1,k}$. For p_1 near 1, $tu_{1,k}\in \mathring{L}_{2b,1}^{p_1}(R^{n+1}\times(0,T))$ and since $L(u_{1,k})=0$, $tu_{1,k}(\bar{x},t)=$ $=\int_0^t\int_{R^{n+1}}\Lambda^{-2b}(\bar{x}-M,t-s)(T(u_{1,k})(M,s)\,dM\,ds$ where T is a bounded operator on $L^p(R_T^{n+1})$, and $\Lambda^{-2b}(\bar{x},t)=\mathcal{F}_\xi(e^{-|\xi|^{2b}t})(\bar{x})$ (see [4]). Since $u_{1,k}\rightarrow u_1=$ $=\int_\Omega W(\bar{x},t;M,0)\,h(M)\,dM$ in $L^p(R^{n+1}\times(0,T))$ as $k\longrightarrow\infty$ we conclude that $u_{1,k}$ converges in $L_{2b,1}^p(R^{n+1}\times(a,T))$ for all a, $0<a<T$, to $u_1(\bar{x},t)$. Moreover

$$\sup_{r<\delta_0}\sum_{j=0}^{b-1}\|\Lambda^{-j}D_{N_Q}^j(u_{1,k}-u_{1,l})(Q+rN_Q,t)\|_{L^p(\partial\Omega_T)}\leq C\|h_k-h_l\|_{L^p(\Omega)}.$$

Hence for each $0\leq r<\delta_0$, $\Lambda^{-j}D_{N_Q}^j(u_{1,k})(Q+rM_Q,t)$ converges in $L^p(\partial\Omega_T)$ as $k\rightarrow\infty$ for $0\leq j\leq b-1$. Also this limit equals $\Lambda^{-j}(D_{N_Q}^j u_1)(Q+rN_{Q,t})$ which we will shortly see belongs to $L^p(\partial\Omega\times(0,T))$ and which itself equals $\lim_{\varepsilon\to 0}X_{(0,T-\varepsilon)}\Lambda^{-j}D_{N_Q}^j(u_1(P+rN_P,s+\varepsilon))(Q,t)$, the limit taken in $L^p(\partial\Omega_T)$ .We accept this last statement for the moment.

As in the proof of 4.1.6. the sequence $\{u_{0,k}\}$ converges in $\overset{\circ}{L}{}^p_{2b,1}(\Omega_T^*)$ for each Ω^*, $\Omega^* \subset \bar{\Omega}$. Moreover for $r < \delta_0$,

$$\sum_{|\alpha| \leq 2b-1} \| d^{|\alpha|}(Q + rN_Q) D_x^\alpha u_{0,k}(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} +$$

$$+ \sum_{j=0}^{b-1} \| \Lambda^{-j} D_{N_Q}^j u_{0,k}(Q + rN_Q, t) \|_{L^p(\partial\Omega_T)} \leq$$

$$\leq C \sum_{j=1}^{b} \| \Phi_j - \Lambda^{-j} D_{N_Q}^j u_{1,k} \|_{L^p(\partial\Omega \times (0,T))} \leq$$

$$\leq C \left[\sum_{j=1}^{b} \| \Phi_j \|_{L^p(\partial\Omega \times (0,T))} + \| h_k \|_{L^p(\Omega)} \right].$$

Hence the same inequality holds for u_0 with h_k replaced by h.

Now set $u = u_0 + u_1$, $u \in L^p{}_{2b,1,\mathrm{loc}}(\Omega_T)$ and

$$\| \Lambda^{-j} D_{N_Q}^j u_0(Q + rN_Q, t) - (\Phi_j - \Lambda^{-j} D_{N_Q}^j u_1)(Q, t) \|_{L^p(\partial\Omega_T)} \to 0 \text{ as } r \to 0+.$$

From this fact and the above statements on u_1 we see that u satisfies condition (c).

For the function u_1 we already noted that $\| u_1(\bar{x}, t) - h(\bar{x}) \|_{L^p(\Omega)} \to 0$ as $t \to 0+$. However for the function $u_0(\bar{x}, t)$ we can only say that for each Ω^*, $\bar{\Omega}^* \subset \Omega$, $\lim\limits_{t \to 0+} \| u_0(\bar{x}, t) \|_{L^p(\Omega^*)} = 0$.

This completes the existence proof of theorem 4.2.2 together with the estimate except that we must show now that for $j \leq b - 1$,

$$\Lambda^{-j} D_{N_Q}^j u_1(Q + rN_Q, t) \in L^p(\partial\Omega \times (0, T))$$

$$\text{and} = \lim_{\varepsilon \to 0} \Lambda^{-j} D_{N_Q}^j (u_1(\cdot, \cdot + \varepsilon))(Q + rN_Q, t) \chi_{(0, T-\varepsilon)}(t),$$

the limit taken in $L^p(\partial\Omega \times (0, T))$. This follows from the following lemma.

LEMMA. Suppose $f(Q, t) \in L^p(\partial\Omega \times (a, T))$ $\forall a$, $0 < a < T$, and for some $k < 2b$

$$f(Q, t)\, t^{\gamma_p k/2b} \in L^p(\partial\Omega \times (0, T)), \quad 1 < p \leq \infty.$$

$$\left(\text{Recall } 1 - \frac{1}{p} < \gamma_p < \frac{2b}{2b - \gamma}\left(1 - \frac{1}{p}\right) \text{ if } 1 < p < \infty \text{ and } \gamma_\infty = 1 \right).$$

Then $\Lambda^{-k} f \in L^p(\partial\Omega \times (0,T))$ and $\| \Lambda^{-k} f \|_{L^p(\partial\Omega\times(0,T))} \leq C \| t^{\gamma_p k/2b} f \|_{L^p(\partial\Omega\times(0,T))}$. Also for $1 < p < \infty$, $\Lambda^{-k} f = \lim_{\varepsilon\to 0} \Lambda^{-k}(f(P, s+\varepsilon))(Q,t) X_{(0,T-\varepsilon)}(t)$ in $L^p(\partial\Omega \times (0,T))$.

Proof: By use of a partition of unity to show $\Lambda^{-k} f \in L^p(\partial\Omega \times (0,T))$ it suffices to show this in the case of $S_T = R^n \times (0,T)$. So now $f = f(x,t) \in L^p(S_{a,T})$, for $0 < a < T$, and $t^{\gamma_p k/2b} f \in L^p(S_T)$.

$$\| \Lambda^{-k} f(\cdot, t) \|_{L^p(R^n)} \leq \int_0^t \frac{1}{(t-s)^{1-k/2b}} \| f(\cdot, s) \|_{L^p(R^n)} \, ds$$

$$\leq \int_0^t \frac{1}{(t-s)^{1-k/2b} \, s^{\gamma_p k/2b}} \left(s^{\gamma_p k/2b} \| f(\cdot, s) \|_{L^p(R^n)} \right) ds.$$

From the conditions on γ_p it is easy to see by Hardy's lemma that

$$\| \Lambda^{-k} f \|_{L^p(S_T)} \leq \| t^{\gamma_p k/2b} f \|_{L^p(S_T)}.$$

For the second part of the lemma we observe that for $t < T - \varepsilon$, $\Lambda^{-k}(f(P, s+\varepsilon))(Q,t) = (\Lambda^{-k} f)(Q, t+\varepsilon) + R_\varepsilon(f)(Q,t)$ where

$$R_\varepsilon(f)(Q,t) = \sum_i \psi_i(Q) \int_0^\varepsilon \int_{R^n} \Lambda^{-k}(x-z, t+\varepsilon-s)(\varphi_i f \circ F_i)(z,s) \, dz \, ds \quad (x = F_i^{-1}(Q)).$$

Now

$$\| R_\varepsilon(f)(Q,t) \|_{L^p(\partial\Omega_{T-\varepsilon})}$$

$$\leq C \sum_i \int_0^\varepsilon \left(\left(\int_0^{T-\varepsilon} \frac{dt}{(t+\varepsilon-s)^{(1-k/2b)p}} \right)^{1/p} \| (\varphi_i f) \circ F_i(\cdot, s) \|_{L^p(R^n)} \right) ds$$

$$\leq C \sum_i \int_0^\varepsilon \frac{1}{(\varepsilon - s)^{1-1/p-k/2b} \, s^{\gamma_p k/2b}} \left(s^{\gamma_p k/2b} \| (\varphi_i f \circ F_i)(\cdot, s) \|_{L^p(R^n)} \right) ds.$$

Now by Holder's inequality and the condition on γ_p, $\| R_\varepsilon(f) \|_{L^p(\partial\Omega\times(0,T-\varepsilon))} \to 0$ as $\varepsilon \to 0+$.

This completes the proof of the lemma. The application to $u_1(\bar{x}, t)$ is immediate once one observes that

$$\sum_{|\alpha| \leq 2b-1} \| t^{\gamma_p |\alpha|/2b} D_{\bar{x}}^{\alpha} u_1(Q + rN_Q, t) \|_{L^p(\partial\Omega \times (0,T))} \leq C \| h \|_{L^p(\Omega)} .$$

Uniqueness. We consider the problem as described in $a - d$ with zero data. In the cylinder $\Omega_\delta \times (0, T - \varepsilon)$, $u(\bar{x}, t + \varepsilon) \in L^p{}_{2b,1}$. From 4.2.1

$$\sup_{\delta < r < \delta_0} \| u(Q + rN_Q, t + \varepsilon) \|_{L^p(\partial\Omega \times (0, T-\varepsilon))}$$

$$\leq C \left[\sum_{j=0}^{b-1} \| \Lambda^{-j} D_{N_Q}^j (u(\cdot, \cdot + \varepsilon))(Q + \delta N_Q, t) \|_{L^p(\partial\Omega_{T-\varepsilon})} + \| \bar{u}(x, \varepsilon) \|_{L^p(\Omega_\delta)} \right] .$$

Now let $\varepsilon \to 0+$ and then $\delta \to 0+$, and we conclude that $u \equiv 0$ in $\Omega'_{\delta_0} \cap \Omega) \times (0, T)$ ($\Omega'_{\delta_0} =$ complement of Ω_{δ_0}). By extending u to be zero outside Ω we then have $u \in L^p{}_{2b,1}(R^{n+1} \times (a, T))$ for all a, $0 < a < T$ and satisfying the conditions $Lu = 0$ in $R^{n+1} \times (0, T)$ and $\| u(\cdot, t) \|_{L^p(\Omega)} \to 0$ as $t \to 0$. So for $0 < t < T - \varepsilon$, $u(\bar{x}, t + \varepsilon) = \int_{R^{n+1}} W(\bar{x}, t + \varepsilon; M, 0) u(M, \varepsilon) dM$. Letting $\varepsilon \to 0$ we see that $u \equiv 0$ in $R^{n+1} \times (0, T)$.

THEOREM (4.2.3). Suppose $u \in \bigcap_{1 < p < \infty} L^p{}_{2b,1}(\Omega_T)$ and satisfies $Lu = 0$. Then

$$\sup_{r < \delta_0} \Bigg[\sum_{|\alpha| \leq 2b-1} \| (d_\infty^{|\alpha|} D_{\bar{x}}^{\alpha} u)(Q + rN_Q, t) \|_{L^\infty(\partial\Omega_T)}$$

$$+ \sum_{j=0}^{b-1} \| \Lambda^{-j} D_{N_Q}^j u(Q + rN_Q, t) \|_{L^\infty(\partial\Omega_T)} \Bigg]$$

$$\leq C_\Omega \left[\sum_{j=0}^{b-1} \| \Lambda^{-j} D_{N_Q}^j u(Q, t) \|_{L^\infty(\partial\Omega_T)} + \| u(\bar{x}, 0) \|_{L^\infty(\Omega)} \right] .$$

PROOF: From the uniqueness part of 4.2.2 and from the existence proof in that theorem, $u(\bar{x}, t) = u_0(\bar{x}, t) + u_1(\bar{x}, t)$ where $u_1(\bar{x}, t) = \int_\Omega W(\bar{x}, t; M, 0) u(M, 0) dM$. It is not difficult to see that u_1 satisfies the above estimate with right-hand side including only $\| u(\bar{x}, 0) \|_{L^\infty(\Omega)}$. The function $u_0(\bar{x}, t)$ satisfies the conditions of (4.1.5) on $(\Omega_\delta)_T$ for each δ, $0 <$

$< \delta < \delta_0$. We now apply the estimate in 4.1.5 to u_0 with Ω_T replaced by $(\Omega_\delta)_T$ and then let $\delta \to 0+$. Since for $0 \le j \le b-1$, $\Lambda^{-j} D^j_{N_Q} u_0 (Q + \delta N_Q, t) = \Lambda^{-j} D^j_{N_Q} (u - u_1)(Q + \delta N_Q, t)$ we have that the L^∞-norm of the left side over $\partial\Omega_T$ is dominated by $C_\Omega [\| \Lambda^{-j} D^j_{N_Q} u(Q + \delta N_Q, t)\|_{L^\infty(\partial\Omega_T)} + \| u(\bar{x}, 0)\|_{L^\infty(\Omega)}]$. Since $u \in \bigcap_{1<p<\infty} L^p_{2b,1}(\Omega_T)$ the first term converges as $\delta \to 0+$ to $\| \Lambda^{-j} D^j_{N_Q} u(Q, t)\|_{L^\infty(\partial\Omega_T)}$. This completes the proof of 4.2.3.

Theorem (4.2.4). Suppose $\Phi_0, \dots, \Phi_{b-1}$ are functions belonging to $C(\partial\Omega \times [0, T])$ and $h(\bar{x})$ belongs to $C(\bar{\Omega})$. Assume $h(Q) = \Phi_0(Q, 0)$, $Q \in \partial\Omega$, and that $\Phi_k(Q, 0) = 0$, $1 \le k \le b-1$. Then there exists a unique $u(\bar{x}, t)$ satisfying:

a) $u(\bar{x}, t) \in \bigcap_{1<p<\infty} L^p_{2b,1,\mathrm{loc}}(\Omega_T)$ and $Lu = 0$ in Ω_T,

b) $\lim_{r\to 0} \lim_{\varepsilon \to 0} \| \Lambda^{-k} D^k_{N_Q}(u(\cdot, \cdot + \varepsilon))(Q + rN_Q, t) - \Phi_k(Q, t)\|_{L^\infty(\partial\Omega_{T-\varepsilon})} = 0.$

$0 \le k \le b-1,$

c) $\lim_{t\to 0+} \| u(\bar{x}, t) - h(\bar{x})\|_{L^\infty(\Omega)} = 0.$

Moreover this solution satisfies the inequality,

$$\sup_{r<\delta_0}\left[\sum_{|\alpha|\le 2b-1} \| (d_\infty^{|\alpha|} D^\alpha_{\bar{x}} u)(Q + rN_Q, t)\|_{L^\infty(\partial\Omega_T)} + \sum_{k=0}^{b-1} \| \Lambda^{-k} D^k_{N_Q} u(Q + rN_Q, t)\|_{L^\infty(\partial\Omega_T)}\right] \le C_\Omega \left[\| h\|_{L^\infty(\Omega)} + \sum_{k=0}^{b-1} \| \Phi_k\|_{L^\infty(\partial\Omega_T)}\right].$$

Proof: The uniqueness follows from the uniqueness in 4.2.2 for if (*a*)-(*c*) of 4.2.3 hold then (*a*)-(*d*) of 4.2.2 hold for any p, $1 < p < \infty$.

We begin the existence proof by again extending the coefficients of L to all of $R^{n+1} \times [0, T]$ as in 4.2.2, and we again let $W(\bar{x}, t; M, s)$ denote the fundamental solution described there. We extend $h(\bar{x})$ so that $h(\bar{x}) \in C_0(R^{n+1})$.

Set $u_1(\bar{x}, t) = \int_{R^{n+1}} W(\bar{x}, t; M, 0) h(M)\, dM$. $u_1(\bar{x}, t)$ is continuous on $R^{n+1} \times [0, T]$ and $u_1(\bar{x}, 0) = h(\bar{x})$. Since $u_1(\bar{x}, t)$ is uniformly continuous on $\bar{\Omega} \times [0, T]$, $\| u_1(\cdot, t) - h(\cdot)\|_{L^\infty(\Omega)} \to 0$ as $t \to 0+$.

For $\bar{x} \in \Omega'_{\delta_0} \cap \bar{\Omega}$ ($\Omega'_{\delta_0} = E^{n+1} \setminus \Omega_{\delta_0}$) and for $1 \leq k \leq b-1$ we can write

$$\Lambda^{-k} D^k_{N_Q} u_1(\bar{x}, t) = \int_{E^{n+1}} h(M)\, W_k(\bar{x}, t; M, 0)$$

where

$$W_k(\bar{x}, t; M, 0) = \Lambda^{-k}(D^k_r(W(P + rNp, s; M, 0))(Q, t),\ \bar{x} = Q + rN_Q .$$

The following properties of $W_k(\bar{x}, t; M, o)$ are easily verified. ($1 \leq k \leq b-1$)

i) $| W_k(\bar{x}, t; M, 0) | \leq \psi\left(\dfrac{|\bar{x} - M|}{t^{1/2b}}\right) t^{-(n+1)/2b}$

ii) $\left| \int_{E^{n+1}} W_k(\bar{x}, t; M, 0)\, dM \right| \to 0$ as $t \to 0+$ uniformly for $\bar{x} \in \Omega'_{\delta_0} \cap \bar{\Omega}$.

With properties i and ii it follows that for $1 \leq k \leq b-1$, $|\Lambda^{-k} D^k_{N_Q} u_1(\bar{x}, t)| \to 0$ as $t \to 0+$ uniformly for $\bar{x} \in \Omega'_{\delta_0} \cap \bar{\Omega}$.

Now set $\psi_k(Q, t) = \Lambda^{-k} D^k_{N_Q} u_1(Q, t)$, $0 \leq k \leq b-1$. From Theorem (4.1.7) there exists $u_0(\bar{x}, t)$ satisfying the conclusion of (4.1.7) with data $\Phi_k(Q, t) - \psi_k(Q, t)$, $k = 0, \dots, b-1$. (Note $\Phi_0(Q, 0) = h(Q) = \psi_0(Q, 0)$). The function $u(\bar{x}, t) = u_0(\bar{x}, t) + u_1(\bar{x}, t)$ is our desired function.

5. The elliptic estimate.

In this section we let $\mathcal{E} = \sum_{|\alpha| \leq 2b} a_\alpha(\bar{x}) D^\alpha_{\bar{x}}$ and we assume that $\mathcal{E}$ is strongly elliptic, i. e. $\mathcal{E} - D_t$ is parabolic in the sense of Petrovsky. About the coefficients we again assume that a_α is bounded and measurable in Ω and that for $|\alpha| = 2b$ a_α is Hölder continuous in $\bar{\Omega}$.

THEOREM 5.1. Suppose $u \in L^p_{2b}(\Omega)$, $1 < p < \infty$, satisfies $\mathcal{E}u = 0$. Then

$$\sup_{r < \delta_0} \left[\sum_{|\alpha| \leq 2b-1} r^{|\alpha|} \| D^\alpha_{\bar{x}} u(Q + rN_Q) \|_{L^p(\partial\Omega)} + \sum_{j=0}^{b-1} \| G_j(D^j_{N_Q} u)(Q + rN_Q) \|_{L^p(\partial\Omega)} \right]$$

$$\leq C_\Omega \left[\sum_{j=0}^{b-1} \| G_j(D^j_{N_Q} u)(Q) \|_{L^p(\partial\Omega)} + \| u \|_{L^1(\Omega)} \right].$$

PROOF: The proof follows the same line of argument as 4.1.1 except that the half space estimate used is (3.5.3). If we proceed in this manner we arrive at the conclusion that the left side of 5.1 above is bounded by a constant times the right side plus the term

$$(5.2)\quad \ldots C_\Omega \Big[\sup_{2\delta_0 > r > \delta} \sum_{|\alpha| \leq 2b-1} \| D^\alpha_{\bar{x}} u(Q + rN_Q) \|_{L^p(\partial\Omega)} + \\ + \sum_i \sum_{j=0}^{b-1} \| G_{1+j} D^j_y (u_i \circ F_i)(x, 0) \|_{L^p(R^n)} \Big] .$$

$$(u_i = \varphi_i u).$$

Now suppose $\varphi(\bar{x}) \in C_0^\infty(\Omega)$ with $\varphi \equiv 1$ in Ω_{δ_0}. Then since $\mathcal{E}u = 0$

$$\sup_{2\delta_0 > r > \delta} \sum_{|\alpha| \leq 2b-1} \| D^\alpha_{\bar{x}} u(Q + rN_Q) \|_{L^p(\partial\Omega)}$$

$$= \sup_{r < 2\delta_0} \sum_{|\alpha| \leq 2b-1} \| D^\alpha_{\bar{x}} (\varphi u)(Q + rN_Q) \|_{L^p(\partial\Omega)} \leq$$

$$\leq C \| \varphi u \|_{L^p_{2b}(\Omega)} \leq C [\| \mathcal{E}(\varphi u) \|_{L^p(\Omega)} + \| u \|_{L^1(\Omega)}] \leq \quad \text{(see [2])}.$$

$$\leq C \epsilon \sup_{r < \delta_0} \sum_{|\alpha| \leq 2b-1} r^{|\beta|} \| D^\beta_{\bar{x}} u(Q + rN_Q) \|_{L^p(\partial\Omega)} + C_\varepsilon \| u \|_{L^1(\Omega)}$$

Denote by $\| u \|_{\Omega, p}$ the left side of (5.1), and set

$$\| u \|_{B, p} = \sup_{y > 0} \sum_i \sum_{j=0}^{b-1} \| G_{1+j} D^j_y (u_i \circ F_i)(x, y) \|_{L^p(R^n)} .$$

We have shown that

$$\| u \|_{\Omega, p} \leq C_\Omega \Big[\sum_{j=0}^{b-1} \| G_j D^j_{N_Q} u(Q) \|_{L^p(\partial\Omega)} + \| u \|_{L^1(\Omega)} + \| u \|_{B, p} \Big] .$$

LEMMA. Given $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $u \in L^p_{2b}(\Omega)$ with $\mathcal{E}u = 0$ in Ω we have $\| u \|_{B, p} \leq \varepsilon \| u \|_{\Omega, p} + C_\varepsilon \| u \|_{L^1(\Omega)}$.

PROOF: Assume that the lemma is false. Then there exists $\varepsilon_0 > 0$ and a sequence of functions $\{u_k\} \subset L^p_{2b}(\Omega)$ such that $Lu_k = 0$ in Ω and $1 \geq \geq \varepsilon_0 \| u_k \|_{\Omega, p} + k \| u_k \|_{L^1(\Omega)}$.

Take $\varphi(\bar{x}) \in C_0^\infty(\Omega)$ with $\varphi \equiv 1$ in Ω_{δ_0}. Then

$$\| u_k \varphi \|_{L^p_{2b}(\Omega)} \leq C [\| \mathcal{E}(u_k \varphi) \|_{L^p(\Omega)} + \| u_k \|_{L^1(\Omega)}] \qquad \text{(see [2])}.$$

$$\leq C_{\delta_0} [\| u_k \|_{\Omega, p} + \| u_k \|_{L^1(\Omega)}] \leq C, \qquad \text{independent of } k.$$

Hence a subsequence of the u_k which we again call u_k converges weakly in $L^p{}_{2b}(\Omega^*)$ for all Ω^* with $\bar{\Omega}^* \subset \Omega$ to a function $u(\bar{x})$. For each Ω^* as above, $\| u_k \|_{L^1(\Omega^*)} \to \| u \|_{L^1(\Omega^*)}$. So $\| u \|_{L^1(\Omega^*)} = 0$ and hence $u \equiv 0$ in Ω. But in the same manner as in 4.1.1 we can show that if $\| u_k \|_{\Omega,p} + \| u_k \|_{B,p}$ is bounded independent of k then there is a subsequence, u_k, such that $\| u_k - u_l \|_{B,p} \to 0$. However this implies $\| u_k - u \|_{B,p} \to 0$ as $k \to \infty$ and hence $\| u \|_{B,p} = 1$, a contradiction since $u \equiv 0$ in Ω.

THEOREM (5.3). Suppose $u \in \bigcap_{1<p<\infty} L^p{}_{2b}(\Omega)$ satisfies $\mathcal{E}u = 0$ in Ω. Then

$$\sup_{r<\delta_0} \left[\sum_{|\alpha| \leq 2b-1} r^{|\alpha|} \| D^\alpha u(Q + rN_Q) \|_{L^\infty(\partial\Omega)} + \sum_{j=0}^{b-1} \| G_j D^j_{N_Q} u(Q + rN_Q) \|_{L^\infty(\partial\Omega)} \right]$$

$$\leq C_\Omega \left[\sum_{j=0}^{b-1} \| G_j D^j_{N_Q} u(Q) \|_{L^\infty(\partial\Omega)} + \| u \|_{L^1(\Omega)} \right].$$

PROOF: We first observe that 5.2 holds when $p = \infty$ and then to finish we proceed as in 4.1.5.

APPENDIX

(A.1). *Estimates on the parametrix.*

Let $L = \sum_{|\alpha|=2b} a_\alpha D^\alpha_{x,y} - D_t$ be a parabolic operator with constant coefficients. Set $A(\xi) = \sum_{|\alpha|=2b} a_\alpha (i\xi)^\alpha$ and $\Gamma(x, y, t) = \mathcal{F}_\xi (e^{A(\xi)t})(x, y)$. Finally set $k(x, y, v, t) = \int_0^t \int_{R^n} \Lambda^{2b-1-k} D^k_y \Gamma(x - z, y, t - s)\, T(\Lambda^{-j} D^j_y \Gamma)(z, -v, s)\, dz\, ds$ where T is a (translation-invariant) p. s. i. o. and $0 \leq j \leq b-1$, $0 \leq k \leq b-1$.

From known estimates on the derivatives of Γ (see [7]) it is tedious but not difficult to show,

THEOREM (A.1.1). For $y > 0$ and $v > 0$,

$$| D^\alpha_{x,y,v} D^l_t K(x,y,v,t) | \leq \psi\left(\frac{|x|}{t^{1/2b}}\right) \psi\left(\frac{y}{t^{1/2b}}\right) \psi\left(\frac{v}{t^{1/2b}}\right) t^{-\frac{n+1+|\alpha|}{2b} - l}.$$

Recall that

$$G_0(x,y,v,t) = \sum_{k,j=0}^{b-1} \int_0^t \int_{R^n} \Lambda^{2b-1-k} D_y^k \Gamma(x-z,y,t-s)\, T_{k,j}(\Lambda^{-j} D_y^j \Gamma)(z,-v,s)\,dz\,ds.$$

An immediate corollary of A.1.1 is.

THEOREM (A.1.2) Estimate in A.1.1 holds for $G_0(x, y, v, t)$.

Recall now that for $0 \leq j \leq b-1$,

$$g_{j,b}(x,t) = \lim_{y \to 0+} \Lambda^{-j} D_y^j [\lim_{v \to 0+} D_v^b (\Gamma(x, y-v, t) - G_0(x,y,v,t))].$$

THEOREM (A.1.3).

$$| D_x^\alpha D_t^l g_{j,b}(x,t) | \leq \psi\left(\frac{|x|}{t^{1/2b}}\right) t^{-\frac{n+1+b+|\alpha|}{2b} - l}.$$

PROOF: The function $\lim_{y \to 0+} \Lambda^{-j} D_y^j [\lim_{v \to 0} D_v^b (\Gamma(x, -v, t)]$ satisfies the estimates of (A.1.3). From the formula for G_0,

$$\lim_{y \to 0+} \Lambda^{-j} D_y^j (\lim_{v \to 0+} D_v^b\, G_0(x,y,v,t) = \lim_{v \to 0+} (-1)^b \Lambda^{-j} D_y^{j+b} \Gamma(x, -v, t)$$

and hence satisfies the estimate in (A.1.3).

Now that A.1.3 is established it is straightforward to prove.

THEOREM (A.1.4). For $v > 0$,

$$| D_{x,v}^\alpha D_t^l [g_{j,b} * D_y^{2b-1} \Gamma(\cdot, -v, \cdot)](x,t) | \leq \psi\left(\frac{|x|}{t^{1/2b}}\right) \psi\left(\frac{v}{t^{1/2b}}\right) t^{-\frac{n+1+b+|\alpha|}{2b} - l}.$$

Now recall that

$$G_b(x,y,v,t) = \sum_{k,j=0}^{b-1} \int_0^t \int_{R^n} \Lambda^{2b-1-k} D_y^k \Gamma(x-z,y,t-r) T_{k,j}(g_{j,b} *$$

$$* D_y^{2b-1} \Gamma(\cdot, -v, \cdot))(z,s)\,dz\,ds.$$

From A.1.4 we have.

THEOREM (A.1.5) For $y > 0$, $v > 0$,

$$| D_{x,y,v}^\alpha D_t^l G_b(x,y,v,t) | \leq \psi\left(\frac{|x|}{t^{1/2b}}\right) \psi\left(\frac{y}{t^{1/2b}}\right) \psi\left(\frac{v}{t^{1/2b}}\right) t^{-\left(\frac{n+1+b+|\alpha|}{2b} + l\right)}.$$

Finally recall that by induction on $l, 0 \leq l \leq 2b-2$, and for $j = 0,\dots,b-1$, we defined

$$g_{j,b+l}(x,t) = \lim_{y\to 0+} \Lambda^{-j} D_y^j \lim_{v\to 0+} D_v^{b+l}\Big(\Gamma(x, y-v, t) -$$

$$- G_0(x,y,v,t) - \sum_{k=0}^{l-1} G_{b+k}(x,y,v,t)\frac{v^{b+k}}{(b+k)!}\Big)$$

and

$$G_{b+l}(x,y,v,t) = \sum_{k,j=0}^{b-1} \int_0^t \int_{R^n} \Lambda^{2b-1-k} D_y^k \Gamma(x-z, y, t-r)\, T_{k,j}(g_{j,b+l} *$$

$$* D_y^{2b-1}\Gamma(\cdot, -v, \cdot))(z,s)\, dz\, ds.$$

THEOREM (A.1.6). The estimates in A.1.3. and A.1.5 with b replaced by $b+l$ hold for $g_{j,b+l}(x,t)$ and $G_{b+l}(x,y,v,t)$ respectively.

PROOF: We proceed by induction on l. A.1.3 and A.1.5 form the case $l=0$.

$$g_{j,b+l} = C \lim_{y\to 0+} \Lambda^{-j} D_y^{b+l+j}\Gamma(x,y,t) - \lim_{y\to 0+} \Lambda^{-j} D_y^j \lim_{v\to 0+} D_v^{b+l} G_0(x,y,v,t)$$

$$- \sum_{k=0}^{l-1} \lim_{y\to 0+} \Lambda^{-j} D_y^j \lim_{v\to 0} D_v^{l-k} G_{b+k}(x,y,v,t)$$

$$= C \lim_{y\to 0} \Lambda^{-j} D_y^{b+l+j}\Gamma(x,y,t) - C \lim_{v\to 0} \Lambda^{-j} D_y^{b+j+l}\Gamma(x,-v,t)$$

$$- \sum_{k=0}^{l-1} \lim_{v\to 0+} D_v^{l-k}[g_{j,b+k} * D_y^{2b-1}\Gamma(-v)(x,t)].$$

So the estimates follow for $g_{j,b+l}(x,t)$. Same argument will prove that A.1.4 now holds with b everywhere replaced by $b+l$. With this established the desired estimate for $G_{b+l}(x,y,v,t)$ follows.

(A.2) *Poisson kernels.*

In this section, as in (A.1), we consider a constant coefficient parabolic operator $L = \sum_{|\alpha|=2b} a_\alpha D_{x,y}^\alpha - D_t$. We will use the fundamental solution $\Gamma(x,y,t)$ as defined in (A.1).

Set

$$u(x,y,t) = \sum_{k=0}^{b-1} \int_0^t \int_{R^n} \Lambda^{2b-1-k-l} D_y^{l+k}\, \Gamma(x-z, y, t-s)\, T_{k,j}(\Phi_j)(z,s)\, dz\, ds$$

where $T_{k,j}$ is the p. s. i. o defined through the symbol relation,

$$(\sigma(T_{k,j})(x,t))^{-1} = (\sigma(S_{k,j})(x,t))$$

and

$$\sigma(S_{k,j}(x,t)) = (|x|^{2b} - it)^{(2b-j-k-1)/2b} \oint \frac{(-i\zeta)^{k+j}}{A(ix, i\zeta) + it} d\zeta \qquad \text{(see 3.2.2).}$$

Here $0 \leq k \leq b-1$ and $0 \leq j \leq b-1$.

THEOREM (A.2.1). $|u(x,y,t)| \leq C \| \Phi_j \|_{L^\infty(S_T)}$, C independent of $y > 0$.

PROOF: May assume $\Phi_j \in C_0^\infty(R^n \times (0,\infty))$.

$$\mathcal{F}_{z,s}(u(z,y,s))(x,t) = H(x,y,t)\,\mathcal{F}(\Phi_j)(x,t)$$

where

$$H(x,y,t) = \sum_{k=0}^{b-1} (|x|^{2b} - it)^{\frac{2b-1-k-l}{2b}} \oint \frac{(-i\zeta)^{l+k} e^{-iy\zeta}}{A(ix, i\zeta) + it} ds\, \sigma(T_{k,j})(x,t)$$

(A.2.1) will be complete once we show the following:

LEMMA (A.2.2). $H(x,y,t) = \mathcal{F}_{z,s}(k(z,y,s))$ with $\int_{-\infty}^{\infty} \int_{R^n} |(k(x,y,t)| \, dx\, dt \leq C$, C independent of $y > 0$.

PROOF:

$$\int |k(x,y,t)|\, dx\, dt = \int_{|x|+|t|^{1/2b} < y} |k(x,y,t)|\, dx\, dt + \int_{|x|+|t|^{1/2b} > y} |k(x,y,t)|\, dx\, dt$$
$$= I + II.$$

$I \leq y^{n+2b} \int |H(x,y,t)|\, dx\, dt$. It is not difficult to see that $|H(x,y,t)| \leq \leq Ae^{-c(|x|+|t|^{1/2b})y}$. Hence $I \leq C$ independent of $y > 0$.

To handle the second integral let us first assume that we can find numbers p and k satisfying $1 < p \leq 2$, $k = 2bm$, m a positive, integer, and $\frac{n+2b}{p} < k < \frac{n+2b}{p} + 1$. Then

$$II \leq \left(\int_{|x|+|t|^{1/2b} > y} |x| + |t|^{1/2b} \right)^{-kp} \Bigg)^{1/p} \| (|x| + |t|^{1/2b})^k k(x,y,t) \|_{L^{p'}(R^{n+1})}, \; \frac{1}{p} + \frac{1}{p'} = 1.$$

So by Riesz's Theorem [11,II]

$$II \leq C y^{\frac{n+2b}{p} - k} \Big[\sum_{|\alpha| = 2bm} \| D_x^\alpha H(x, y, t) \|_{L^p(R^{n+1})} + D_t^m H(x, y, t) \|_{L^p(R^{n+1})} \Big].$$

We will show that $\| D_x^\alpha H(\cdot, y, \cdot) \|_{L^p(R^{n+1})} \leq C y^{k - \frac{n+2b}{p}}$, $|\alpha| = k$. The term involving differentiation in t is handled in the same manner. Now

$$\Big(\int_{|x| + |t|^{1/2b} > 1/y} | D_x^\alpha H(x, y, t) |^p \, dx \, dt)^{1/p} \leq C \Big(\int_{|x| + |t|^{1/2b} > 1/y} (|x| + |t|^{1/2b})^{-kp} \Big)^{1/p} \leq$$

$$\leq C y^{k - \frac{n+2b}{p}}. \qquad \text{(Recall } kp > n + 2b).$$

Since $H(x, 0, t) = \delta_{l,j}$, we have

$$D_x^\alpha H(x, y, t) = \sum_{k=0}^{b-1} D_x^\alpha \Big((|x|^{2b} - it)^{\frac{2b-1-j-l}{2b}} \oint \frac{(-i\zeta)^{l+j} [e^{-iy\zeta} - 1]}{A(ix, i\zeta) + it} ds \, \sigma(T_{k,j})(x, t) \Big).$$

Hence if $|x| + |t|^{1/2b} < y^{-1}$, then $| D_x^\alpha H(x, y, t) | \leq C y (|x| + |t|^{1/2b})^{-k+1}$ and therefore

$$\Big(\int_{|x| + |t|^{1/2b} < 1/y} | D_x^\alpha H(x, y, t) |^p \, dx \, dt \Big)^{1/p} \leq C y \, y^{k-1-\frac{n+2b}{p}} \text{ since } k - 1 < \frac{n+2b}{p}.$$

We conclude that $II \leq C$, independent of $y > 0$.

Our proof is complete once we show the existence of $k = 2bm$ and p, $1 < p \leq 2$, satisfying $\frac{n + 2b}{p} < k < \frac{n + 2b}{p} + 1$.

First write $n = 2br + s$, $0 \leq s < 2b$. We want to find m and p such that $\frac{(r+1) + \frac{s}{2b}}{p} < m < \frac{(r+1) + \frac{s}{2b}}{p} + \frac{1}{2b}$.

Take $m = (r+1)$. Set $p_1 = \frac{r + 1 + \frac{s}{2b}}{r+1}$. Then $1 \leq p_1 < 2$ and $\frac{r+1+\frac{s}{2b}}{p_1} = r + 1 < \frac{r + 1 + \frac{s}{2b}}{p_1} + \frac{1}{2b}$. Now take $p = p_1 + \varepsilon$ for $\varepsilon > 0$ and small enough so that $\frac{r + 1 + \frac{s}{2b}}{p} < r + 1 < \frac{r + 1 + \frac{s}{2b}}{p} + \frac{1}{2b}$.

(A.3). *Relation between* G_k and Λ^{-k}.

THEOREM (A.3.1). Suppose $f(x) \in C_0^\infty(R^n)$. Then for $1 \leq p \leq \infty$, $\| \Lambda^{-k}(f)(x,t) \|_{L^p(S_T)} \leq C_T \| G_k(f) \|_{L^p(R^n)}$. (For $T \leq a$, $C_T \leq C_a$).

PROOF: The result is obvious for $k = 0$. For $k \geq 1$,

$$\mathcal{F}_z(\Lambda^{-k}(f)(z,t))(x) = \left(\int_0^t \frac{e^{-|x|^{2b} s}}{s^{1-k/2b}} \, ds \right) \mathcal{F}(f)(x).$$

Let $\varphi(x) \in C_0^\infty(R^n)$ with $\varphi(x) = 1$ for $|x| \leq 1$ and $= 0$ for $|x| \geq 2$.

$$\mathcal{F}_z(\Lambda^{-k}(f))\, z, t))(x) = \left[(1 + |x|^2)^{k/2} \int_0^t \frac{e^{-|x|^{2b} s}}{s^{1-k/2b}} \, ds \right] \varphi(x)\, \mathcal{F}(G_k f)(x) +$$

$$+ \left[(1 + |x|^2)^{k/2} \int_0^t \frac{e^{-|x|^{2b} s}}{s^{1-k/2b}} \, ds \right] (1 - \varphi(x))\, \mathcal{F}(G_k(f))(x) =$$

$$= H_1(x,t) + H_2(x,t).$$

It is easy to check that $H_1(x,t) = \mathcal{F}_z(k_1(z,t))(x)$ with $\int_{R^n} |k_1(x,t)|\, dx \leq C_T\, (t \leq T)$.

Now set $\psi(x) = 1 - \varphi(x)$. Then

$$H_2(x,t) = C \left(1 + \frac{1}{|x|^2} \right)^{k/2} \psi(x) - \left(1 + \frac{1}{|x|^2} \right)^{k/2} \psi(x) \int_{t|x|^{2b}}^\infty \frac{e^{-s}}{s^{1-k/2b}} \, ds.$$

$$H_2(x,t) = C \left[\left(1 + \frac{1}{|x|^2} \right)^{k/2} \psi(x) - 1 \right] + C - \left[\left(1 + \frac{1}{|x|^2} \right)^{k/2} \psi(x) - 1 \right] \int_{t|x|^{2b}}^\infty \frac{e^{-s}}{s^{1-k/2b}} \, ds -$$

$$- \int_{t|x|^{2b}}^\infty \frac{e^{-s}}{s^{1-k/2b}} \, ds.$$

The function $\left[\left(1 + \frac{1}{|x|^2} \right)^{k/2} \psi(x) - 1 \right] = \mathcal{F}(k)(x)$ where $k(x) \in L^1(R^n)$.

Set

$$H_3(x) = \int_{|x|^{2b}}^\infty \frac{e^{-s}}{s^{1-k/2b}} \, ds. \qquad \mathcal{F}_z(H_3(zt^{1/2b}))(x) = t^{-n/2b}\, \mathcal{F}(H_3) \left(\frac{x}{t} 1/2b \right).$$

So to show that $H_3(xt^{1/2b}) = \mathcal{F}_x(k_3(x,t))$ with $\int_{R^n} |k_3(x,t)|\,dx \leq C_T$, it is sufficient to show that $H_3(x) = \mathcal{F}(k_3)(x)$ with $\int_{R^n} |k_3(x)|\,dx < \infty$. Since $H_3 \in L^1$, $\int_{|x|\leq 1} |k_3|\,dx \leq 1$. By Riesz's theorem, for $1 < p \leq 2$,

$$\int_{|x|\geq 1} |k_3|\,dx \leq C\left(\int_{|x|\geq 1} \frac{1}{|x|^{np}}\right)^{1/p}\Big(\sum_{|\alpha|=n} \| D^\alpha H_3 \|_{L^p(R^n)}\Big).$$

For $|x| \geq 1$ $\quad |D^\alpha H_3(x)| \leq e^{-c|x|^{2b}}$, and since $|\alpha| = n$,

for $|x| < 1$ $\quad |D^\alpha H_3(x)| \leq c\,|x|^{k-n}$. Hence for $|\alpha| = n$,

$D^\alpha H_3 \in L^p$ for p near 1.

THEOREM (A.3.2). Suppose $f \in C_0^\infty(R^n)$. Then for $1 \leq p \leq \infty$, $\| G_k(f) \|_{L^p(R^n)} \leq C_{a,T} \| \Lambda^{-k}(f) \|_{L^p(S_{a,T})}$ where $0 < a < T$. ($S_{a,T} = R^n \times (a, T)$).

PROOF: Again may assume $k > 0$.

$$\mathcal{F}(G_k(f))(x) = \frac{1}{(1+|x|^2)^{k/2}}\mathcal{F}(f)(x) = \frac{1}{H(x,t)(1+|x|^2)^{k/2}}\mathcal{F}_z(\Lambda^{-k}(f)(z,t))(x)$$

where

$$H(x,t) = \int_0^t \frac{e^{-|x|^{2b}s}}{s^{1-k/2b}}\,ds = |x|^{-k}\int_0^{t|x|^{2b}} \frac{e^{-s}}{s^{1-k/2b}}\,ds.$$

Set

$$\lim_{|x|\to\infty} (H(x,t)(1+|x|^2)^{k/2}) = \int_0^\infty \frac{e^{-s}}{s^{1-k/2b}}\,ds = C > 0.$$

It is not difficult to check that for $0 < a \leq t \leq T$, $H(x,t) \geq C_{a,T}$ when $|x| \leq 1$ and that $|D_x^\alpha H(x,t)| \leq C_{\alpha,a,T}(1+|x|^2)^{(k+|\alpha|)/2}$. Now set $G(x,t) = [H(x,t)(1+|x|^2)^{k/2}]^{-1}$. For each $t \in [a,T]$ $G(x,t) \in C^\infty(R^n)$ and $\forall t \in [a,T]$

$$|D_x^\alpha G(x,t)| \leq C_{\alpha,a,T}(1+|x|^2)^{-|\alpha|/2}.$$

Let $\varphi(x) \in C_0^\infty(R^n)$ with $\varphi(x) = 1$ for $|x| \leq 1$ and set $\psi(x) = (1-\varphi(x))$. Then $G(x,t) = \varphi(x)\,G(x,t) + \psi(x)\,G(x,t) = G_1 + G_2$. $G_1(x,t) = \mathcal{F}_z(k_1(z,t))(x)$

and $\int_{R^n} |k_1(x,t)|\, dx \leq C_{a,T}$ provided $0 < a \leq t \leq T$. $G_2(x,t)$ vanishes for $|x| \leq 1$ and $\lim_{|x| \to \infty} G_2(x,t) = 1/c$.

CLAIM. $G_2(x,t) - 1/c = \mathcal{F}(k_2(z,t))(x)$ where $\int_{R^n} |k_2(x,t)|\, dx \leq C_{a,T}$ provided $0 < a \leq t \leq T$.

PROOF: For $|x| \geq 1$, $|k_2(x,t)| \leq \frac{C_{a,T}}{|x|^{n+1}}$ $(a \leq t \leq T)$, and so $\int_{R^n} |k_2(x,t)|\, dx \leq$ $\leq C_{a,T}$. By Riesz's theorem for $1 < p \leq 2$,

$$\int_{|x| \leq 1} |k_2(x,t)|\, dx \leq \left(\int_{|x| \leq 1} \frac{1}{|x|^{(n-1)p}} \right)^{1/p} \sum_{|\alpha| = n-1} \| D_x^\alpha G_2(\cdot, t) \|_{L^p(R^n)} .$$

For $|x| \leq 1$ and $|\alpha| = n-1$, $|D_x^\alpha G_2(x,t)| \leq C_{a,T} |x|^{-n+1}$ and for $|x| > 1$, $|D_x^\alpha [G_2(x,t) - 1/c]| \leq C_{a,T} |x|^{-n}$. $\left(\text{To see this write } G_2(x,t) = \frac{|x|^k}{(1 + |x|^2)^{k/2}} \cdot \left(\int_0^{t|x|^{2b}} \frac{e^{-s}}{s^{1-k/2b}}\, ds \right)^{-1} \right)$. Hence $\sum_{|\alpha| = n-1} \| D_x^\alpha G_2(\cdot, t) \|_{L^p(R^n)} \leq C_{a,T}$ provided p is near 1.

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