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PROJECTIVE EMBEDDING OF PSEUDOCONCAVE SPACES

ALDO ANDREOTTI and YUM-TONG SIU (*)

In [4] Tomassini and the first author investigate the projective embedding of pseudoconcave manifolds (with maximal concavity) and, following an idea of Grauert, prove that a pseudoconcave manifold X of dimension $\geqq 2$ can be embedded as an open subset of a projective algebraic variety if and only if X carries a holomorphic line bundle L such that the graded ring of sections of its powers

$$\mathcal{A}(X, L) = \bigcup_{h=0}^{\infty} \Gamma(X, L^h)$$

separates points and gives local coordinates at each point of X [4, Theorem 2].

In this paper we continue the investigation of projective embeddability in two directions.

Firstly we extend the above result to pseudoconcave normal spaces.

Secondly we show that for the projective embeddability of a pseudoconcave manifold X of dimension $\geqq 3$ it suffices to assume that X carries a holomorphic line bundle L such that $\mathcal{A}(X, L)$ gives local coordinates at each point of X. This is done by using extension techniques. We extend X to a compact complex manifold $\widetilde{X}$ (by the methods of Hironaka [10] and Rossi [13]) and, then, we extend the line bundle L to a holomorphic line bundle $\widetilde{L}$ over $\widetilde{X}$ (by the method of Trautmann [24]). From $\widetilde{L}$ we construct a positive holomorphic line bundle on $\widetilde{X}$ and show that $\widetilde{X}$ is a projective algebraic manifold. It is essential that $\dim X \geqq 3$. A counter-example (which was inspired by a remark of Grauert) is given to show this point.

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At the end of this paper we establish a result concerning the finite generation of $\mathscr{A}(X, L)$. This result on finite generation is independent of the result on finite generation obtained in [4]. A counter-example due to David Prill is included to show this lack of relationship.

All complex spaces and subvarieties in this paper are reduced unless specified otherwise.

§ 1. Pseudoconcavity and pseudoconvexity.

1. A real-valued C^∞ function ψ on an open set D of $\mathbf{C}^n$ is called *strongly q-pseudoconvex* if the hermitian form

$$\mathscr{L}(\varphi) \equiv \sum_{i,j} \frac{\partial^2 \psi}{\partial z_i \, \partial \bar{z}_j} u_i \bar{u}_j$$

has at least $n - q$ positive eigenvalues at each point of D ($z_1, \ldots, z_n$ being the coordinates of $\mathbf{C}^n$).

A real-valued function φ on a complex space X is said to be *strongly q-pseudoconvex* if for every point x_0 of X there exist an open neighborhood U of x, a biholomorphic map Φ of U onto an analytic subset of an open set D of some $\mathbf{C}^n$ and a strongly q-pseudoconvex function ψ on D such that

i) $\varphi = \psi \circ \Phi$,

ii) the closure of the set $\{x \in U \mid \varphi(x) < c\}$ is $\{x \in U \mid \varphi(x) \leqq c\}$ for every $c \in \mathbf{R}$.

DEFINITIONS. (*a*) A complex space X is called *p-convex* if there exists a C^∞ map φ from X to $(-\infty, b)$, where $b \in \mathbf{R} \cup \{+\infty\}$, such that

(i) $\{\varphi \leqq c\}$ is compact for every $c \in (-\infty, b)$,

(ii) for some $b' \in (-\infty, b)$, φ is strongly q-pseudoconvex on $\{\varphi > b'\}$.

(*b*) A complex space X is called *q-concave* if there exists a C^∞ map φ from X to $(a, +\infty)$, where $a \in \{-\infty\} \cup \mathbf{R}$, such that

(i) $\{\varphi \geqq c\}$ is compact for $c \in (a, +\infty)$,

(ii) for some $a' \in (a, +\infty)$, φ is strongly q-pseudoconvex on $\{\varphi < a'\}$.

(*c*) A complex space X is called *(p, q)-convex-concave* if there exists a proper C^∞ map φ from X to (a, b), where $a \in \{-\infty\} \cup \mathbf{R}$ and $b \in \mathbf{R} \cup \{+\infty\}$, such that for some $a < a' < b' < b$, φ is strongly p-pseudoconvex on $\{\varphi > b'\}$ and strongly q-pseudoconvex on $\{\varphi < a'\}$.

In all these cases we call φ an *exhaustion function.*

2. The following proposition is due to Grauert [8, § 2].

PROPOSITION 1.1. *Let X be a 0-convex space. Then there exist a Stein space S, a finite subset A of S, and a proper holomorphic surjection $\gamma: X \to S$ such that*

(i) *$\gamma^{-1}(x)$ is a connected nowhere discrete compact subspace of X for $x \in A$,*

(ii) *$\gamma: X - \gamma^{-1}(A) \to S - A$ is biholomorphic.*

Moreover, if X is normal, S can be chosen to be normal.

This statement can also be viewed as a consequence of the reduction principle of Cartan [6]. In particular we note the following.

COROLLARY. *A 0-convex space without compact positive-dimensional subspaces is Stein.*

The proofs of Propositions 21, 22 and Theorem 15 of [2] yield readily the following:

PROPOSITION 1.2. *Let X be a (p, q)-convex-concave unreduced complex space with exhaustion function φ from X to (a, b).*

Let $a < a' < b' < b$ and suppose that φ is strongly p-pseudoconvex on $\{\varphi > b'\}$ and strongly q pseudoconvex on $\{\varphi < a'\}$.

Let $\mathcal{F}$ be a coherent analytic sheaf on X with prof $\mathcal{F} \geqq r$ on $\{\varphi < a'\}$. Set

$$X_c^d = \{c < \varphi < d\} \quad \text{for} \quad a \leqq c < d \leqq b.$$

Then

(*a*) *the restriction map $H^l(X, \mathcal{F}) \to H\ (X_c^d, \mathcal{F})$ is bijective for $c \in [a, a')$, $d \in (b', b]$ and $p < l < r - q - 1$,*

(*b*) *for $c \in [a, a')$ the restriction map $H^l(X, \mathcal{F}) \to H^l(X_c^b, \mathcal{F})$ is bijective for $l < r - q - 1$ and injective for $l = r - q - 1$.*

The following proposition is adapted from Trautmann [24, (3.1)].

PROPOSITION 1.3. *Let X be a $(0, 0)$-convex-concave unreduced space with exhaustion function φ from X to (a, b) which is strongly 0-pseudoconvex on the whole of X.*

Let $\mathcal{F}$ be a coherent analytic sheaf with prof $\mathcal{F} \geqq 3$ on $\{\varphi < a'\}$ for some $a' \in (a, b)$.

Let $\mathcal{I}$ be a coherent sheaf of ideals on X whose zero-set is disjoint from $\{\varphi < a''\}$ for some $a'' \in (a, b)$. Then the natural map

$$\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}/\mathcal{I}\mathcal{F})$$

is surjective.

PROOF. We can assume $a' = a''$. Set $Y = \{\varphi < a'\}$. From the short exact sequence of sheaves

$$0 \to \mathcal{J}\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{J}\mathcal{F} \to 0$$

we derive the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\Gamma(X, \mathcal{F}) & \xrightarrow{\alpha} & \Gamma(X, \mathcal{F}/\mathcal{J}\mathcal{F}) & \xrightarrow{\beta} & H^1(X, \mathcal{J}\mathcal{F}) & \xrightarrow{\gamma} & H^1(X, \mathcal{F}) \\
 & & & & \downarrow \lambda & & \downarrow \mu \\
 & & & & H^1(Y, \mathcal{J}\mathcal{F}) & \xrightarrow{\nu} & H^1(Y, \mathcal{F})
\end{array}$$

Now ν is an isomorphism, because on Y, $\mathcal{J} = \mathcal{O}$ and hence $\mathcal{J}\mathcal{F} = \mathcal{F}$. Since prof $\mathcal{F} \geqq 3$ on Y, by proposition 1.2 both λ and μ are isomorphisms. Hence γ is an isomorphism. Thus $\beta = 0$ and α is surjective.

3. We end this section by the study of (0, 0)-convex-concave spaces X in which prof $\mathcal{O} \geqq 3$, $\mathcal{O}$ being the structure sheaf of X.

LEMMA 1.1. *Let $(X, \mathcal{O})$ be a (0, 0)-convex-concave space with exhaustion function φ from X to (a, b) which is strongly 0-pseudoconvex on the whole of X.*

Let prof $\mathcal{O} \geqq 2$ *on X. Set for $a \leqq c < d \leqq b$, $X_c^d = \{c < \varphi < d\}$. Then for any $f \in \Gamma(X_a^d, \mathcal{O})$ we have* [1]

$$|f(X_a^d)| = |f(X_c^d)|.$$

PROOF. Clearly $|f(X_a^d)| \geqq |f(X_c^d)|$ since $X_c^d \subset X_a^d$. Suppose that $|f(X_a^d)| > |f(X_c^d)|$. Then there exist an $x \in \{\varphi \leqq c\}$ and a real number M such that

$$|f(x)| > M > |f(X_c^d)|.$$

Now for $n \to \infty$, $\left(\dfrac{f}{M}\right)^n \to 0$ uniformly on X_c^d while

$$\lim_{n \to \infty} \left(\frac{f}{M}\right)^n (x) = \infty.$$

This is a contradiction since, by Proposition 1.2, the restriction map

$$\Gamma(X_a^d, \mathcal{O}) \to \Gamma(X_c^d, \mathcal{O}),$$

being bijective and continuous, must be an isomorphism of Fréchet spaces.

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(1) By $|f(Y)|$ we denote $\sup_{y \in Y} |f(y)|$.

PROPOSITION 1.4. *Let $(X, \mathcal{O})$ be a $(0, 0)$-convex-concave complex space with exhaustion function φ from X to (a, b) which is strongly 0-pseudoconvex on the whole of X.*

Suppose prof $\mathcal{O} \geqq 3$ on $\{\varphi < a'\}$ for some $a' \in (a, b)$. Then

(a) the holomorphic functions on X separate points,

(b) the holomorphic functions on X give local coordinates at every point of X,

(c) for every $d \in (a, b)$ we can find $d' \in (a, b)$ such that the holomorphically convex hull of $\{\varphi \leqq d\}$ is contained in $\{\varphi \leqq d'\}$.

PROOF. We denote by $\mathfrak{m}(x)$ the sheaf of ideals defined by the subspace $\{x\}$ constituted by the single point $x \in X$.

(a) Fix $x \neq y$ in X. Set $\mathcal{J} = \mathfrak{m}(x)\,\mathfrak{m}(y)$ and apply Proposition 1.3 with $\mathcal{F} = \mathcal{O}$ we conclude that

$$\Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{O}/\mathcal{J}) = (\mathcal{O}/\mathfrak{m}(x))_x \oplus (\mathcal{O}/\mathfrak{m}(y))_y$$

is a surjective map. Thus holomorphic functions separate points on X.

(b) Fix $x \in X$ and apply Proposition 1.3 to $\mathcal{F} = \mathfrak{m}(x)$ and $\mathcal{J} = \mathfrak{m}(x)$. We get a surjective map

$$\Gamma(X, \mathfrak{m}(x)) \to \Gamma(X, \mathfrak{m}(x)/\mathfrak{m}(x)^2) = (\mathfrak{m}(x)/\mathfrak{m}(x)^2)_x\,.$$

This shows that holomorphic functions give local coordinates at x.

(c) For $d \in (a, b)$ set $K_d = \{\varphi \leqq d\}$. Suppose that the holomorphically convex hull $\widehat{K}_d$ of K_d is not contained in any $K_{d'}$ for $d' \in (a, b)$.

We can then find a sequence of distinct points $\{x_\nu\}_{\nu=1}^{\infty}$ in $\widehat{K}_d$ such that $\varphi(x_\nu) \to b$.

Now $\bigcup_\nu \{x_\nu\}$ is a subspace of X and $\mathcal{J} = \prod_{\nu=1}^{\infty} \mathfrak{m}(x)$ is a sheaf of ideals on X. By the same Proposition 1.3, taking $\mathcal{F} = \mathcal{O}$, we conclude that

$$\Gamma(X, \mathcal{O}) \to \Gamma\left(X, \mathcal{O}/\prod_{\nu=1}^{\infty} \mathfrak{m}(x_\nu)\right) = \prod_{\nu=1}^{\infty} (\mathcal{O}/\mathfrak{m}(x_\nu))_{x_\nu}$$

is a surjective map. Thus in particular there exists on X a holomorphic function f with $\lim |f(x_\nu)| = \infty$. On the other hand f must be bounded on K_d. Indeed, by Lemma 1.1, for $a < c < d < d' < b$ we get

$$|f(K_d)| \leqq \sup_{\{c < \varphi < d'\}} |f| < \infty.$$

This leads to a contradiction since $|f(x_\nu)| \leqq |f(K_d)|$ by the assumption that $x_\nu \in \widehat{K}_d$.

§ 2. Gap-sheaves and p-normalization.

4. *Relative gap-sheaves.* Let $(X, \mathcal{O})$ be an unreduced complex space and let $\mathcal{G} \subset \mathcal{F}$ be analytic sheaves on X.

A subset $M \subset X$ is said to be *thin of dimension p at a point $x_0 \in X$,* if there exist an open neighborhood U_{x_0} of x_0 in X and an analytic set $A \subset U_{x_0}$ such that

$$M \cap U_{x_0} \subset A, \quad \dim_{x_0} A \leqq p.$$

This notion involves only the germ of the set M at x_0.

DEFINITION. Given an integer $n \geqq 0$, the *n^{th} relative gap-sheaf of $\mathcal{G}$ in $\mathcal{F}$,* denoted by $\mathcal{G}_{[n]\mathcal{F}}$, is the analytic subsheaf of $\mathcal{F}$ defined as follows:

$$(\mathcal{G}_{[n]\mathcal{F}})_x = \{s \in \mathcal{F}_x \mid \text{support of } \beta(s) \text{ is thin of dimension } \leqq n \text{ at } x\},$$

where $\beta : \mathcal{F} \to \mathcal{F}/\mathcal{G}$ is the quotient map.

This notion is due to Thimm [23].

Clearly $\mathcal{G} \subset \mathcal{G}_{[n]\mathcal{F}}$. We set

$$E^n(\mathcal{G}, \mathcal{F}) = \{x \in X \mid \mathcal{G}_x \neq (\mathcal{G}_{[n]\mathcal{F}})_x\}.$$

From [23] and [17] we borrow the following propositions:

PROPOSITION 2.1. *If $\mathcal{G}$ and $\mathcal{F}$ are coherent, then, for any n, the sheaf $\mathcal{G}_{[n]\mathcal{F}}$ is coherent and $E^n(\mathcal{G}, \mathcal{F})$ is an analytic set of dimension $\leqq n$ in X.*

PROPOSITION 2.2. *Suppose that $\mathcal{G}$ and $\mathcal{F}$ are coherent. For any $x \in X$, $(\mathcal{G}_{[n]\mathcal{F}})_x$ is the intersection of all primary components of a primary decomposition of $\mathcal{G}_x$ belonging to prime ideals of dimension $> n$.*

PROPOSITION 2.3. *Let $\mathcal{F}$ be coherent, $g \in \mathcal{O}_x$ is a zero divisor for $\mathcal{F}_x$* (i.e. *$\exists f \in \mathcal{F}_x$ such that $f \neq 0$ and $gf = 0$) if and only if g vanishes on some irreducible germ of some $E^n(0, \mathcal{F})$ at x.* (0 denotes the zero sheaf on X.)

5. *Absolute gap-sheaves.* Let $(X, \mathcal{O})$ be an unreduced complex space and let $\mathcal{F}$ be an analytic sheaf on X.

For any open set $U \subset X$ we can consider the group

$$\mathcal{F}^{[n]}(U) = \varinjlim_{A} \Gamma(U - A, \mathcal{F})$$

where A runs over all analytic subsets of U of dimension $\leqq n$.

If $W \subset U$ is open we have a natural restriction map

$$r_W^U : \mathcal{F}^{[n]}(U) \longrightarrow \mathcal{F}^{[n]}(W).$$

We obtain in this way a presheaf on X for any integer $n \geqq 0$.

We define the n^{th} *absolute gap sheaf* of $\mathcal{F}$, denoted by $\mathcal{F}^{[n]}$, as the sheaf associated to the presheaf $\{\mathcal{F}^{[n]}(U), r_W^U\}$.

This notion was introduced in [18] from which we borrow the following proposition.

PROPOSITION 2.4. *Let $\mathcal{F}$ be a coherent sheaf. The sheaf $\mathcal{F}^{[n]}$ is coherent if and only if* $\dim E^{n+1}(0, \mathcal{F}) \leqq n$.

Set

$$S_k(\mathcal{F}) = \{x \in X \mid \operatorname{prof} \mathcal{F}_x \leqq k\}.$$

If $\mathcal{F}$ is a coherent sheaf, $S_k(\mathcal{F})$ is an analytic set of dimension $\leqq k$ in X [16, Satz 4]. Combining the Corollary to Satz III of [15] and Proposition 19 of [20] we get the following proposition.

PROPOSITION 2.5 *Let $\mathcal{F}$ be coherent. Then $\mathcal{F}^{[n]} = \mathcal{F}$ if and only if* $\dim S_{k+2}(\mathcal{F}) \leqq k$ *for* $-1 \leqq k < n$.

(Proposition 2.5 can also be derived from [25, Satz 2].)

6. *p-normalization.* Let $(X, \mathcal{O})$ be a complex space. We say that X is *p-normal at a point* $x \in X$ if $\mathcal{O}_x^{[p]} = \mathcal{O}_x$. We say that X is *p-normal* if $\mathcal{O}^{[p]} = \mathcal{O}$.

This means the following: X is p-normal at x if, given an open neighborhood U of x, an analytic subset A of U of dimension $\leqq p$ and a holomorphic function f on $U - A$ we can find a neighborhood W of x and a holomorphic function $\widehat{f}$ on W such that $f|_{W-A} = \widehat{f}|_{W-A}$.

Making use of Proposition 2.5 we obtain the following criterion for p-normality:

$(X, \mathcal{O})$ is a p-normal space if and only if

$$\dim \{x \in X \mid \operatorname{prof} \mathcal{O}_x \leqq k + 2\} \leqq k$$

for $-1 \leqq k < p$.

In particular $(X, \mathcal{O})$ is 0-normal if and only if $\operatorname{prof} \mathcal{O} \geqq 2$.

If X is an irreducible normal space of dimension n, then X is p-normal for any $p \leqq n - 2$.

The following proposition is adapted from the usual proof of existence of the normalization of a complex space (cf. for instance [12, § 4]).

PROPOSITION 2.6. *Let X be a complex space whose irreducible components all have dimension $\geqq p+2$. Then*

(*a*) *the set A of points of X where X is not p-normal is an analytic subset of X of dimension $\leqq p$,*

(*b*) *there exist a p-normal complex space X' and a proper surjective holomorphic map with finite fibers $\pi : X' \to X$ such that*

(i) *$\pi^{-1}(A)$ is of dimension $\leqq p$ at each point,*

(ii) *$\pi : X' - \pi^{-1}(A) \to X - A$ is biholomorphic*

(iii) *any proper holomorphic surjective map $\omega : Y \to X$ of a p-normal complex space Y onto X and verifying property* (i) *factors through π,* i. e. *$\exists\, \tau : Y \to X'$ holomorphic such that $\omega = \pi \circ \tau$.*

$$\begin{tikzcd} Y \arrow[rr, "\tau"] \arrow[dr, "\omega"'] & & X' \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Clearly the universal property (iii) defines the space X' up to an isomorphism. We will call $\pi : X' \to X$ the *p-normalization* of X.

PROOF OF (*a*). Since every irreducible component of X has dimension $\geqq p+2$, it follows that $0_{[p+1]\,\mathcal{O}} = 0$, where 0 is the zero sheaf. Therefore, $E^{p+1}(0, \mathcal{O}) = \emptyset$ and, by Proposition 2.4, the sheaf $\mathcal{O}^{[p]}$ is coherent.

Now $A = E^p(\mathcal{O}, \mathcal{O}^{[p]})$ and thus, by Proposition 2.1, A is an analytic subset of X of dimension $\leqq p$.

PROOF OF (*b*). Since the p-normalization of a complex space X (if it exists) is unique, we need only to prove its existence for a sufficiently small neighborhood of every point of X.

Let $x_0 \in X$. Since $(\mathcal{O}^{[p]})_{x_0}$ is finitely generated over $\mathcal{O}_{x_0}$ and $\mathcal{O}_{x_0}$ is noetherian, $(\mathcal{O}^{[p]})_{x_0}$ must be integral over $\mathcal{O}_{x_0}$. Thus for some neighborhood U of x_0 in X there exist $g_1, \ldots, g_k \in \Gamma(U, \mathcal{O}^{[p]})$ such that

$$\mathcal{O}^{[p]} = \Sigma_{i=1}^{k}\, \mathcal{O} g_i \qquad \text{on } U$$

and

$$g_i^{r_i} + \Sigma_{j=0}^{r_i - 1} a_{ij}\, g_i^j = 0 \text{ with } a_{ij} \in \Gamma(U, \mathcal{O}). \tag{1}$$

Let $\mathcal{J}$ be the conductor sheaf of $\mathcal{O}$ into $\mathcal{O}^{[p]}$, i. e. the maximal sheaf of ideals $\mathcal{J}$ such that $\mathcal{J}\mathcal{O}^{[p]} \subset \mathcal{O}$. Since $\mathcal{O}^{[p]}$ is coherent, $\mathcal{J}$ is also coherent and the zero set of $\mathcal{I}$ is A. Taking U sufficiently small, we may assume that

$$\mathcal{I} = \Sigma_{l=1}^{m} \mathcal{O} u_l \text{ on } U.$$

with $u_1, \ldots, u_m \in \Gamma(U, \mathcal{I})$.

We set

$$b_{li} = u_l g_i \tag{2}$$

so that $b_{li} \in \Gamma(U, \mathcal{O})$.

We may assume, without loss of generality, that $X = U$. We consider in $X \times \mathbf{C}^k$ the set $\widetilde{X}$ defined by the following equations

$$\begin{cases} w_i^{r_i} + \Sigma_{j=0}^{r_i - 1} \ a_{ij}(x) w_i^j = 0 \\ 1 \leqq i \leqq k \end{cases} \tag{3}$$

and

$$\begin{cases} u_l(x) w_i = b_{li}(x) \\ 1 \leqq i \leqq k \\ 1 \leqq l \leqq m \end{cases} \tag{4}$$

where $(w_1, \ldots, w_k) \in \mathbf{C}^k$ and $x \in X$.

Let X' be the union of those irreducible components of $\widetilde{X}$ which are not contained in $A \times \mathbf{C}^k$.

Let $\pi: X' \to X$ be the map induced by the natural projection $X \times \mathbf{C}^k \to X$.

Since no irreducible component of X' is contained in $A \times \mathbf{C}^k$ it follows that $\pi^{-1}(A)$ is of codimension $\geqq 1$ in each component of X'. Since X' is contained in the set defined by the equations (3) the map π is proper and its fibers are finite. Since A is the set of common zeros of $u_1, \ldots, u_m$, equations (4) and (2) imply that, for $x \in X - A$, $\pi^{-1}(x) = (x, g_1(x), \ldots, g_k(x))$ and, since g_i is holomorphic on $X - A$, $\pi: X' - \pi^{-1}(A) \to X - A$ is an isomorphism. Since π is proper, the image $\pi(X')$ of π is an analytic set containing $X - A$, which is dense in X. Hence π is surjective.

We show now that X' is p-normal. Let $x' \in X'$. Let W be an open neighborhood of x'. Let B be an analytic subset of W of dimension $\leqq p$ and let f be holomorphic on $W - B$. Let $\pi^{-1}(\pi(x')) = \{x_1', \ldots, x_q'\}$ where $x_1' = x'$. Choose a Stein open neighborhood D of $\pi(x')$ and disjoint open neighborhoods D_i' of x_i' in X' such that

$$D_1' \subset W \text{ and } \pi^{-1}(D) = \bigcup_{i=1}^{q} D_i'.$$

Let $C = \pi(B \cap D_1') \cup (A \cap D)$.

Then $f \circ \pi^{-1}_{|D'_1}$ is a holomorphic function on $D - C$ and, since dim $C \leqq p$, it defines an element $h \in \Gamma(D, \mathcal{O}^{[p]})$. Since D is Stein, we can find $t_1, \dots, t_k \in \Gamma(D, \mathcal{O})$ such that

$$h = \Sigma_{i=1}^{k} t_i g_i \quad \text{on } D.$$

Set $\widehat{f} = \Sigma_{i=1}^{k} (t_i \circ \pi)\, w_i$ on D'_1. This is a holomorphic function on D'_1. Because $u_l(x)\, g_i(x) = b_{li}(x) = u_l(x)\, w_i$ and $(u_1(x), \dots, u_m(x)) \neq (0, \dots, 0)$ if $x \notin A$, we have $g_i \circ \pi = w_i$ on $X' - \pi^{-1}(A)$. Hence $\widehat{f} = h \circ \pi = f$ on $D'_1 - \pi^{-1}(C)$. Since each component of X' is of dimension $> p$, the extension $\widehat{f}$ of f is unique. Thus $\widehat{f}$ extends f from $D'_1 - B$ to D'_1. This proves our assertion.

Finally we have to verify that $X' \xrightarrow{\pi} X$ satisfies the universal property (iii). Without loss of generality we may assume that X' is an analytic subset of some open Stein set in a numerical space $\mathbf{C}^N$. Let $z_1, \dots, z_N$ be the coordinate functions. On $Y - \omega^{-1}(A)$ the functions $z_i \circ \pi^{-1} \circ \omega = \xi_i$ are holomorphic. Since $\omega^{-1}(A)$ is of dimension $\leqq p$ and Y is p-normal, these functions extend to holomorphic functions $\widehat{\xi}_i$ on the whole of Y and they define a map $\tau\colon Y \longrightarrow \mathbf{C}^N$. We claim that $\tau(Y) \subset X'$. By construction $\tau(Y - \omega^{-}\ (A)) = X' - \pi^{-1}(A)$. Let now $y_\nu \longrightarrow y \in \omega^{-1}(A)$, $y_\nu \in Y - \omega^{-1}(A)$. If $\tau(y_\nu)$ is not convergent in X', we can find an unbounded holomorphic function f on $\{\tau(y_\nu)\}$. But $f \circ \tau$ on $Y - \omega^{-1}(A)$ extends to a holomorphic function g_f on the whole of Y. Thus $f \circ \tau(y_\nu) = g_f(y_\nu) \longrightarrow g_f(y)$ which is a contradiction.

By construction $\omega \mid Y - \omega^{-1}(A) = \pi \circ \tau$. By continuity we must have $\omega = \pi \circ \tau$ on the whole of Y.

The idea of using gap-sheaves to investigate problems on removable singularities is due to Thimm [22] although p-normalizations are not considered in that paper. The p-normalization X' of X is the same as the partial normalization of X with respect to A introduced in [19, § 3].

§ 3. Stein completion.

7. Let X be a $(0, 0)$-convex-concave complex space with exhaustion function φ from X to (a, b). We suppose that φ is strongly 0-pseudoconvex on the whole space X. As usual we set, for $a \leqq c < d \leqq b$,

$$X_c^d = \{c < \varphi < d\}\,;\ K^d = \{\varphi \leqq d\}.$$

Definition. *A complex space Y is called a Stein completion of X if*

(i) *X is an open subset of Y,*

(ii) *Y is a Stein space,*

(iii) *$K_d \cup (Y - X)$ is compact for any $d \in (a, b)$.*

Proposition 3.1. *Let $(Y, \mathcal{O})$ be a Stein completion of X and let $\mathcal{F}$ be a coherent analytic sheaf on Y with prof $\mathcal{F} \geqq 2$. Then the restriction map*

$$\Gamma(Y, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F})$$

is bijective. In particular, if $(Y, \mathcal{O})$ is 0-normal then $\Gamma(Y, \mathcal{O}) \xrightarrow{\sim} \Gamma(X, \mathcal{O})$.

Proof. Let $c \in (a, b)$ and let $Y^c = X_a^c \cup (Y - X)$. Then Y^c is a Stein completion of X_a^c. Also we have

$$\Gamma(Y, \mathcal{F}) = \varprojlim_c \Gamma(Y^c, \mathcal{F}) \text{ and } \Gamma(X, \mathcal{F}) = \varprojlim_c \Gamma(X_a^c, \mathcal{F}).$$

If we prove that, for any c, $\Gamma(Y^c, \mathcal{F}) \longrightarrow \Gamma(X_a^c, \mathcal{F})$ is bijective, then the same conclusion holds for $\Gamma(Y, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F})$.

We can thus replace Y by Y^c, X by X_a^c and therefore it is not restrive to assume that Y is imbedded in some $\mathbf{C}^n$ as an analytic subset so that on Y we can find a strongly 0-pseudoconvex function ψ such that $\{\psi \leqq d\}$ is compact for any $d \in (-\infty, +\infty)$. Set $Y_d = \{\psi > d\}$ and consider the commutative diagram of restriction maps

$$\begin{tikzcd}
\Gamma(Y, \mathcal{F}) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \Gamma(X, \mathcal{F}) \arrow[d, "\gamma"] \arrow[dl, "\lambda"'] \\
\Gamma(Y_d, \mathcal{F}) \arrow[r, "\mu"] & \Gamma(X_e^b, \mathcal{F})
\end{tikzcd}$$

where $d \in (-\infty, +\infty)$ and $e \in (a, b)$ are so chosen that

$$X \supset Y_d \supset X_e^b .$$

By virtue of Proposition 1.2 (b), β and γ are bijective. Since $\beta = \lambda\alpha$, α must be injective. Since $\gamma = \mu\lambda$, λ must be injective. Thus, given $f \in \Gamma(X, \mathcal{F})$, we can find $g \in \Gamma(Y, \mathcal{F})$ such that $\beta(g) = \lambda(f)$, i.e. $\lambda(f - \alpha(g)) = 0$, and therefore $f = \alpha(g)$. This shows that α is also surjective. Hence, α is bijective.

The last statement follows from the fact that, if $(Y, \mathcal{O})$ is 0-normal, then prof $\mathcal{O} \geqq 2$.

Let S, Z be complex spaces. We denote by Hol (S, Z) the set of all holomorphic maps from S to Z.

COROLLARY 3.1. *Let S be any Stein space.*

(*a*) *If X is 0-normal, then the restriction map*

$$\operatorname{Hol}(X, S) \longrightarrow \operatorname{Hol}(X_c^b, S)$$

is bijective for any $c \in (a, b)$.

(*b*) *If Y is a 0-normal Stein completion of X, then the restriction map*

$$\operatorname{Hol}(Y, S) \longrightarrow \operatorname{Hol}(X, S)$$

is bijective.

PROOF. For $S = \mathbf{C}$ this is the statement of Propositions 1.2 (*b*) and 3.1 for $\mathcal{F} = \mathcal{O}$. From this it follows that the same is true for any S that can be imbedded as a subspace of some $\mathbf{C}^n$.

In the general case we carry out the proof for case (*b*). Case (*a*) is treated in the same way. Now

$$\begin{aligned} \operatorname{Hol}(Y, S) &= \varprojlim_{c} \operatorname{Hol}(Y^c, S) \\ & & c \in (a, b) \\ \operatorname{Hol}(X, S) &= \varprojlim_{c} \operatorname{Hol}(X_a^c, S). \end{aligned}$$

If $f, g \in \operatorname{Hol}(Y, S)$ agree on X, then they agree on X_a^c. Now $f(Y^c)$ and $g(Y^c)$ are relatively compact in S, so we can find an open subset S' imbedded in some $\mathbf{C}^n$ as an analytic subspace, such that $f(Y^c) \cup g(Y^c) \subset S'$. It follows then that $f = g$ on Y^c. This is true for any c. Hence in general $f = g$ on Y, i. e. $\operatorname{Hol}(Y, S) \longrightarrow \operatorname{Hol}(X, S)$ is injective.

Given $f \in \operatorname{Hol}(X, S)$, we claim that, for any $c \in (a, b)$, $f(X_a^c)$ is relatively compact in S. Indeed, if this is not true; there exist a sequence $\{x_\nu\} \subset X_a^c$ and a holomorphic function g on S such that

$$|g(f(x_\nu))| \longrightarrow \infty \text{ as } \nu \longrightarrow \infty .$$

By Lemma 1.1, $|(g \circ f)(X_a^c)| = |(g \circ f)(X_d^c)|$ for $a < d < c$. Therefore $g \circ f$ is bounded on X_a^c. This is a contradiction.

By replacing S with S' which contains $f(X_a^c)$ and is imbeddable as analytic subspace in some $\mathbf{C}^n$, we see that $f \mid X_a^c$ admits a holomorphic extension to

Y^c. Since this is true for any $c \in (a, b)$, it follows that there exists a $g \in \mathrm{Hol}(Y, S)$ such that $g|_X = f$, i. e. $\mathrm{Hol}(Y, S) \to \mathrm{Hol}(X, S)$ is surjective.

COROLLARY 3.2. *If Y_1, Y_2 are two 0-normal Stein completions of X, there exist holomorphic maps $f : Y_1 \to Y_2$ and $g : Y_2 \to Y_1$ such that*

$$f \circ g = id_{Y_2} \text{ and } g \circ f = id_{Y_1},$$

i. e. if X admits a 0-normal completion, then the 0-normal completion is unique up to an isomorphism which is the identity on X.

8. *Existence of Stein completions.* Let $(X, \mathcal{O})$ be a $(0, 0)$-convex-concave complex space with exhaustion function φ from X to (a, b) which is strongly 0-pseudoconvex on the whole space X.

PROPOSITION 3.2. *We suppose that X is 0 normal and that, for some $a' \in (a, b)$, prof $\mathcal{O} \geqq 3$ on $\{\varphi < a'\}$. Then X admits a 0-normal Stein completion.*

PROOF. Let $c \in (a, a')$ and $d \in (a, c)$ and consider the holomorphically convex hull of K_d in X_a^c. By Proposition 1.4 (c) it is contained in $X_a^{d^*}$ for some $d^* \in (d, c)$.

For every point x on $\{\varphi = d^*\}$ we can find $f \in \Gamma(X_a^c, \mathcal{O})$ and an open neighborhood U of x such that

$$|f(U)| > 1 \text{ and } |f(K_d)| < 1.$$

Replacing f by a convenient power of f, we may assume that $|f(K_d)| < 1/2$. Since $\{\varphi = d^*\}$ is compact, we can find a finite number of functions $f_i \in \Gamma(X_a^c, \mathcal{O})$ and a finite number of open sets U_i, for $1 \leqq i \leqq k$, such that

$$\{\varphi = d^*\} \subset \textstyle\bigcup_{i=1}^k U_i, \, |f_i(U_i)| > 1, |f_i(K_d)| < 1/2.$$

By Proposition 1.4 (a) (b) we can find $f_{k+1}, \dots, f_l \in \Gamma(X, \mathcal{O})$ such that $f_{k+1}, \dots, f_l$ separate points and give local coordinates on $\{d \leqq \varphi \leqq d^*\}$. It is not restrictive to assume that (Lemma 1.1) we also have

$$|f_i(K_d)| < 1/2 \text{ for } k + 1 \leqq i \leqq l.$$

Consider the map $\alpha : X_a^c \to \mathbf{C}^l$ defined by $\alpha(x) = (f_1(x), \dots, f_l(x))$. For $0 < \delta \leqq 1$ set

$$P_\delta = \{z \in \mathbf{C}^l \mid |z_i| < \delta \text{ for } 1 \leqq i \leqq l\}.$$

Then

$$K_d \subset \alpha^{-1}(P_{1/2}), \alpha^{-1}(P_1) \cap \{\varphi = d^*\} = \emptyset.$$

Let $G = P_1 - \overline{P}_{1/2}$ and $H = \alpha^{-1}(G) \cap X_d^{d^*}$. For any K compact in G, $\alpha^{-1}(K) \cap H = \alpha^{-1}(K) \cap \{d \leqq \varphi \leqq d^*\}$ is compact. Thus $\alpha \mid H$ is a proper map and $\alpha(H)$ is an analytic subset of G.

Now every irreducible component of H has dimension $(\geqq 3) \geqq 2$ [2, Proposition 4]. By [9, Theorem VII. D.6] we can find $\delta \in [1/2, 1)$ such that $\alpha(H) \cap (P_1 - \overline{P}_\delta)$ can be extended to an analytic subset V of P_1.

Set $E = \alpha^{-1}(\overline{P}_\delta) \cap X_a^{d^*}$. Let $\widetilde{X}$ be the topological space obtained from $X - E$ and V by the following identification;

$$x \in X - E \text{ is identified with } y \in V \text{ iff}$$

$$x \in H - \alpha^{-1}(\overline{P}_\delta) \text{ and } y = \alpha(x).$$

One verifies that $\widetilde{X}$ is a Hausdorff space so that (since the identifications are holomorphic) $\widetilde{X}$ inherits a natural complex structure in which $X - E$ and V are open subsets of $\widetilde{X}$. For $k + 1 \leqq i \leqq l$, f_i can be extended naturally to a holomorphic function $\widetilde{f}_i$ on $\widetilde{X}$ (by setting $\widetilde{f}_i = z_i$ on V).

We claim that $\widetilde{X}$ is a Stein completion of X_c^b. For this it is enough to verify the following conditions (cf. Corollary to Proposition 1.1):

(i) $\{c < \varphi \leqq e\} \cup (\widetilde{X} - X_c^b)$ is compact for $e \in (c, b)$.

(ii) $\widetilde{X}$ has no compact positive-dimensional subspaces.

Now for $e \in (c, b)$ the set $\{c < \varphi \leqq e\} \cup (\widetilde{X} - X_c^b)$ is the union of the following three sets:

$$\{d^* \leqq \varphi \leqq e\}, \; V \cap \overline{P}_\varepsilon, \; \{\varphi \leqq e\} - \alpha^{-1}(P_\varepsilon),$$

where $\varepsilon = \dfrac{1 + \delta}{2} > \delta$. Of these sets the first two are obviously compact and the third is a closed subset of $\{d \leqq \varphi \leqq e\}$. Hence (i) is verified.

Let ϱ be a C^∞ function on $X - E$ with the following properties

$$\varrho = 1 \text{ on } X - (\alpha^{-1}(P_\varepsilon) \cap X_a^{d^*})$$

$$\varrho = 0 \text{ on } (X - E) \cap \alpha^{-1}(P_\eta) \cap X_a^{d^*}$$

for some $\delta < \eta < \varepsilon$.

Let $\lambda(t)$ be a C^∞ increasing convex function defined in (a, b) with $\lambda(t) \longrightarrow +\infty$ for $t \longrightarrow b$.

On $X - E$ we consider the function $\Phi = \varrho\lambda(\varphi)$. It is a C^∞ function and it can be extended to the whole of $\widetilde{X}$ by setting $\Phi = 0$ on $V \cap P_\eta$.

One now verifies that for large $\mu > 0$ the function $\psi = \mu \sum\limits_{k+1}^{l} |\widetilde{f}_i|^2 + \Phi$ is a strongly 0-pseudoconvex function on the whole of $\widetilde{X}$. Therefore the sets $\{\psi \leqq \text{const}\}$ are compact in $\widetilde{X}$. This forbids the existence on $\widetilde{X}$ of any compact analytic set of positive dimension (by the maximum principle).

Going to the 0 normalization we obtain that there exists a 0-normal Stein completion of X_c^b for every $c \in (a, a')$.

But these completions must all coincide by virtue of Corollary 3.2. Therefore this common completion $\widetilde{X}$ is the 0-normal Stein completion of X.

COROLLARY 3.3. *We suppose that X is 0-normal and that for some $a' \in (a, b)$ the part $\{\varphi < a'\}$ is 1-normal. Then X admits a 0-normal Stein completion.*

PROOF. Let $A = \{x \in X \mid \operatorname{prof} \mathcal{O}_x \leqq 2\}$. Since $X_a^{a'}$ is 1-normal, by Proposition 2.5, $A \cap X_a^{a'}$ is a discrete set.

Fix $c \in (a, a')$ and select $d \in (c, a')$ such that $A \cap X_c^d = \emptyset$. Then, on X_c^d, $\operatorname{prof} \mathcal{O} \geqq 3$ and by the previous proposition there exists a 0-normal Stein completion of X_c^b. Again by Corollary 3.2 all these 0-normal Stein completions of the spaces X_c^b, when c varies, must coincide. Thus there exists a 0-normal Stein completion of X.

§ 4. Projective imbedding of normal pseudoconcave spaces.

9. Let $(X, \mathcal{O})$ be a complex space and L a holomorphic line bundle over X. We consider the graded ring

$$\mathscr{A}(X, L) = \bigcup_{h=0}^{\infty} \Gamma(X, L^h).$$

We say that $\mathscr{A}(X, L)$ *separate points* of X if for $x, y \in X$, $x \neq y$ we can find a positive integer $h = h(x, y)$ and two sections $\sigma, \tau \in \Gamma(X, L^h)$ such that

$$\det \begin{pmatrix} \sigma(x) & \sigma(y) \\ \tau(x) & \tau(y) \end{pmatrix} \neq 0.$$

We say that $\mathscr{A}(X, L)$ *gives local coordinates* at a point $x \in X$ if we can find a positive integer $h = h(x)$ and a finite number of sections $\sigma_0, \ldots, \sigma_k \in \Gamma(X, L^h)$ such that $\sigma_0(x) \neq 0$ and

$$\frac{\sigma_i}{\sigma_0} - \left(\frac{\sigma_i}{\sigma_0}\right)(x), \quad \text{for } 1 \leqq i \leqq k,$$

generate the space $\mathfrak{m}_x/\mathfrak{m}_x^2$ over $\mathbf{C}$ where $\mathfrak{m}_x$ is the maximal ideal of $\mathcal{O}_x$.

If X is an open set of a projective algebraic variety and L is the line bundle of the hyperplane section, then $\mathscr{A}(X, L)$ separates points and gives local coordinates on the whole of X.

THEOREM 4.1. *Let X be a 0-concave normal complex space whose non-compact irreducible components have dimension $\geqq 2$. Suppose that X carries a holomorphic line bundle L such that $\mathscr{A}(X, L)$ separates points and gives local coordinates at each point of X. Then X is isomorphic to an open set of a projective algebraic variety.*

PROOF. Let $\varphi : X \to (a, \infty)$ be an exhaustion function on X which is strongly 0-pseudoconvex on $\{\varphi < a'\}$ for some $a' \in (a, \infty)$. We set

$$X_c = \{\varphi > c\}, \quad X_c^d = \{c < \varphi < d\}$$

for $a \leqq c < d < \infty$.

Every subspace of X disjoint from X_c for some $c \in (a, a')$ must be 0-dimensional [4, Lemma 2]. Thus X must have only a finite number of irreducible components. Since X is assumed normal it is not restrictive to assume X irreducible. (If X is compact, $X = X_c$ for any $c \in (a, a')$).

(*a*) For any two points x, y of X, $x \neq y$, we can find $\sigma^{(1)}_{x,y}, \sigma^{(2)}_{x,y} \in \Gamma(X, L^{m(x,y)})$ for some $m(x, y) \geqq 1$ such that

$$\det \begin{pmatrix} \sigma^{(1)}_{x,y}(x) & \sigma^{(1)}_{x,y}(y) \\ \sigma^{(2)}_{x,y}(x) & \sigma^{(2)}_{x,y}(y) \end{pmatrix} \neq 0.$$

Replacing $\sigma^{(1)}_{x,y}$ and $\sigma^{(2)}_{x,y}$ by linear combinations, we can assume

$$\sigma^{(1)}_{x,y}(y) = 0 = \sigma^{(2)}_{x,y}(x).$$

We can find open neighborhoods $W(x, y)$ of x and $W'(x, y)$ of y such that

$$\left| \left(\frac{\sigma^{(2)}_{x,y}}{\sigma^{(1)}_{x,y}} \right)(z) \right| < 1 \qquad \forall\, z \in W(x, y)$$

$$\left| \left(\frac{\sigma^{(1)}_{x,y}}{\sigma^{(2)}_{x,y}} \right)(w) \right| < 1 \qquad \forall\, w \in W'(x, y).$$

Hence for z and w in these neighborhoods we must have, for every $n \geqq 1$,

$$\det \begin{pmatrix} (\sigma^{(1)}_{x,y})^n (z) & (\sigma^{(1)}_{x,y})^n (w) \\ (\sigma^{(2)}_{x,y})^n (z) & (\sigma^{(2)}_{x,y})^n (w) \end{pmatrix} \neq 0.$$

Also for any $z \in X$ we can find an open neighborhood $U(z)$ and holomorphic sections $\tau^{(0)}_z, \dots, \tau^{(l(z))}_z \in \Gamma(X, L^{n(z)})$ for some $n(z) \geqq 1$ such that $\tau^{(0)}_z$ never vanishes on $U(z)$ and

$$w \longmapsto \left(\left(\frac{\tau^{(1)}_z}{\tau^{(0)}_z} \right)(w), \dots, \left(\frac{\tau^{(l(z))}_z}{\tau^{(0)}_z} \right)(w) \right)$$

is a biholomorphic map of $U(z)$ onto a locally closed analytic set of $\mathbf{C}^{l(z)}$.

Now fix $c \in (a, a')$. Since $\overline{X}_c$ is compact, we can find $x_i, y_i, z_j \in X$ $(1 \leqq i \leqq p,\ 1 \leqq j \leqq q)$ such that

$$X_c \subset \textstyle\bigcup_{j=1}^{q} U(z_j),$$

$$X_c \times X_c \subset \left(\textstyle\bigcup_{i=1}^{p} W(x_i, y_i) \times W'(x_i, y_i)\right) \cup \left(\textstyle\bigcup_{j=1}^{q} U(z_j) \times U(z_j)\right).$$

Set $h = (\Pi_{i=1}^{p} m(x_i, y_i))(\Pi_{j=1}^{q} n(z_j))$. Then the sections of $\Gamma(X, L^h)$ give local coordinates and separate points on the whole space X_c.

Let $s_0, \dots, s_k$ be a basis [2] for $\Gamma(X, L^h)$ over $\mathbf{C}$ and let A be the set of common zeros of these sections. Since by construction $A \cap X_c = \emptyset$, A

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[2] Here we use the fact that $\dim_{\mathbf{C}} \Gamma(X, L^h)$ is finite. This is ensured by the pseudoconcavity assumption. Directly, if we set $\alpha_i = h/m(x_i, y_i)$ and $\beta_j = (h/n(z_j)) - 1$, we can take for $s_1, \dots, s_k$ the following sections of $\Gamma(X, L^h)$:

$$(\sigma^{(\mu)}_{x_i, y_i})^{\alpha_i} \quad \text{for } \mu = 1, 2 \text{ and } 1 \leqq i \leqq p,$$

$$(\tau^{(0)}_{z_j})^{\beta_j}\, \tau^{(\nu)}_{z_j} \quad \text{for } 0 \leqq \nu \leqq l(z_j) \text{ and } 1 \leqq j \leqq q.$$

Here $k = 2p - 1 + \Sigma_{j=1}^{q} (l(z_j) + 1)$.

is discrete. Let us consider the holomorphic map

$$F : X - A \longrightarrow \mathbf{P}_k$$

defined by $x \longmapsto [s_0(x), \dots, s_k(x)]$. The map F is one-to-one and holomorphic on X_c. Thus $F(X_c)$ is concave and $F(X-A)$ is contained in an irreducible algebraic variety Z of the same dimension r as X [1, Theorem 6].

Let $\pi : Z^* \longrightarrow Z$ be the normalization of Z. Since X is normal, the map F factors through π, i. e. there exists a holomorphic map $F^* : X - A \longrightarrow Z^*$ such that $F = \pi \circ F^*$.

Moreover, $F^*(X_c)$ is an open subset of Z^* and $F^* \mid X_c$ is an isomorphism onto that open subset. Indeed, Z^* is locally irreducible and $F^*(X_c)$ is a locally closed analytic set of pure dimension r.

(*b*) We now prove that, for $a < c < d < a'$, X_c^d admits a Stein completion. With the previous notations set

$$D = F^*(X_c^d) \cup (Z^* - F^*(X_c)).$$

Since $F^* \mid X_c$ is an isomorphism, it follows that D is 0-convex. Thus by Proposition 1.1 there exist a normal Stein complex space S and a proper holomorphic surjective map $\gamma : D \longrightarrow S$ and a finite set $B \subset S$ such that $\gamma : D - \gamma^{-1}(B) \xrightarrow{\sim} S - B$ and, for $z \in B$, $\gamma^{-1}(z)$ is connected and nowhere discrete.

By [4, Lemma 2], $\gamma^{-1}(B) \cap F^*(X_c^d) = \emptyset$ so that $\gamma \circ F^* \mid X_c^d$ is an isomorphism onto an open subset $(\gamma \circ F^*)(X_c^d)$ of S. Moreover, $S - (\gamma \circ F^*)(X_c^d) = \gamma(Z^* - F^*(X_c))$ is compact. This proves that S is a Stein completion of X_c^d.

(*c*) Keeping d fixed and letting c vary on (a, d) we see that S is a Stein completion for all X_c^d and thus for X_a^d. In particular there exists a holomorphic map

$$\theta : X_a^d \longrightarrow S$$

which extends $\gamma \circ F^* \mid X_c^d$ and maps X_a^d biholomorphically onto an open subset of S.

Outside of the compact set B the imbedding dimension of S is the same as that of Z^* (which is projective algebraic). Thus the imbedding dimension of S is bounded and we realize S as an analytic subspace of some $\mathbf{C}^N$.

Now $\gamma \circ F^* : X_a^d - A \longrightarrow S$ and $\theta : X_a^d \longrightarrow S$ agree on X_c^d and every irreducible component of X_a^d must meet X_c^d. Thus these maps being given by holomorphic functions must agree on the whole space $X_a^d - A$, i. e. θ extends $\gamma \circ F^*$ to X_a^d.

(*d*) Set $C = \theta^{-1}(B)$. This is a finite set since B is finite and θ on X_a^d is an isomorphism.

We want to prove that $A = C$.

Let $G : X - C \to Z^*$ be defined as follows:

$$G = \gamma^{-1} \circ \theta \quad \text{on} \quad X_a^d - C, \qquad G = F^* \quad \text{on} \quad X_c.$$

This map is one-to-one and biholomorphic and agrees with F^* on $X - C - A$.

Now $C \subset A$, otherwise there exists $x \in C$ such that $x \notin A$. Since $\gamma \circ F^* = \theta$ on X_a^d and θ on X_a^d is injective, we have $F^*(x) = \gamma^{-1}\theta(x) \cap F^*(X - A)$ and $\gamma^{-1}\theta(x)$ contains the isolated point $F^*(x)$. This is a contradiction, because $\theta(x) \in B$.

Also $A \subset C$, otherwise there exists $x \in A$ such that $x \notin C$. Then G extends F holomorphically over x. If U is a sufficiently small neighborhood of x, we can find holomorphic functions $u_0, \ldots, u_k$ such that $y \longmapsto [u_0(y), \ldots \ldots, u_k(y)]$ represents the map $\pi \circ G \mid U$. If U is sufficiently small, we can assume $U \cap A = \{x\}$ and $L \mid U$ to be trivial. If we fix a trivialization of $L \mid U$, the sections $s_i \mid U$ are given by holomorphic functions. Since $F \mid U - \{x\}$ coincide with $\pi \circ G \mid U$, there exists a unique nowhere zero holomorphic function v on $U - \{x\}$ such that

$$s_i = v u_i \quad \text{on} \quad U - \{x\} \quad \text{for} \quad 0 \leqq i \leqq k.$$

Now v extends to a holomorphic function $\tilde{v}$ on U (since U is normal of dimension $\geqq 2$). Because at x some $u_i(x) \neq 0$, we must have $\tilde{v}(x) = 0$. This is impossible since $\tilde{v}(x) \in v(U - \{x\})$ and $v(U - \{x\})$ does not contain 0.

As a consequence of the fact that $A = C$, it follows that A is a finite set.

(*e*) Let $c' \in (a, c)$ be such that $A \subset X_{c'}$. We can find a positive integer h' such that the sections of $\Gamma(X, L^{h'})$ do not have common zeros on $X_{c'}$. Then the sections of $\Gamma(X, L^{hh'})$ do not have common zeros on the whole of X ([3]). Repeating the previous argument, we conclude that $A = C = \emptyset$ and G maps X biholomorphically onto an open set of Z^*.

([3]) If $(s_0', \ldots, s_{k'}')$ are sections of $\Gamma(X, L^{h'})$ without common zeros on $X_{c'}$, we take the map given by the following sections of $\Gamma(X, L^{hh'})$:

$$(\sigma_{x_i, y_i}^{(\mu)})^{\alpha_i h'} \quad \text{for} \quad \mu = 1, 2 \quad \text{and} \quad 1 \leqq i \leqq p,$$

$$(\tau_{z_j}^{(0)})^{\beta_j h' + h' - 1}\, \tau_{z_j}^{(\nu)} \quad \text{for} \quad 0 \leqq \nu \leqq n(z_j) \quad \text{and} \quad 1 \leqq j \leqq q,$$

$$(s_0')^h, \ldots, (s_{k'}')^h.$$

REMARK. For any vector space $V \subset \Gamma(X, L^h)$ whose sections have no common zeros we can consider the natural map $\tau_V : X \to \mathbf{P}(V)$ given by the evaluation

$$x \longmapsto [s_0(x), \dots, s_k(x)]$$

where $s_0, \dots, s_k$ is a basis of V over $\mathbf{C}$.

We can choose the integer h and the vector space V in such a way that the minimal algebraic variety Z containing $\tau_V(X)$ is normal so that τ_V is a realization of the map F^*.

Indeed, with the same notations as in the previous proof, we may assume $A = \emptyset$ and $F : X \to Z$ to be holomorphic. Let Z^* be the normalization of Z. We may assume that $Z^* \subset \mathbf{P}_N$ and that the homogeneous coordinates $\xi_0, \dots, \xi_N$ on Z^* are rational functions homogeneous of the same degree $l \geqq 1$ in the homogeneous coordinates $[z_0, \dots, z_k]$ of the general point of Z (cf. Zariski [26]). It is not restrictive to assume that among the ξ_α are all monomials of degree of l in the $z's$. Since each ξ_α is integral over the coordinate ring of Z, it represents a holomorphic section σ_α of $\Gamma(X_c, L^{hl})$. By the concavity assumption $\Gamma(X_c, L^{hl}) \simeq \Gamma(X, L^{hl})$ so that σ_α is a holomorphic section of L^{hl} over the whole of X. It is therefore enough to take as V the space generated in $\Gamma(X, L^{hl})$ by these sections σ_α. We have $(\Gamma(X, L^h))^l \subset V \subset \Gamma(X, L^{hl})$ so that the sections of V have no common zeros.

In particular we deduce the following

COROLLARY 4.1. *The isomorphism $G : X \to Z$ of the previous theorem can be so chosen that Z is a normal projective algebraic variety and that, for some integer $h \geqq 1$, $G^*E = L^h$ where E is the line bundle of the hyperplane section of Z.*

§ 5. Compactification of 0-concave spaces.

10. Given a complex space X, an isomorphism $i : X \to Y$ of X onto an open subset of a compact complex space Y will be called *a compactification* of X.

In what follows we will be concerned only with 0-normal complex spaces admitting 0-normal compactifications.

A compactification $i : X \to Y$ of the 0-normal complex space X into the 0-normal complex space Y will be called *minimal* if for any other compactification $j : X \to Y'$ into a 0-normal complex space Y' we can find a morphism $\tau : Y' \longrightarrow Y$ such that $i = \tau \circ j$:

$$\begin{tikzcd} X \arrow[rr, "j"] \arrow[dr, "i"'] & & Y' \arrow[dl, "\tau"] \\ & Y & \end{tikzcd}$$

Clearly any two minimal 0-normal compactifications of the same space X are isomorphic by an isomorphism which is the identity on X (when X is identified with its image in the compactification).

PROPOSITION 5.1. *Let X be a 0-concave 0-normal complex space. A 0-normal compactification $i : X \longrightarrow Y$ is minimal if and only if no positive dimensional compact complex space is contained in $Y - i(X)$.*

PROOF. Let $\varphi : X \longrightarrow (a, \infty)$ be an exhaustion function for X and let $a' \in (a, \infty)$ be such that φ is strongly 0-pseudoconvex on $\{\varphi < a'\}$.

Set $Z = \{\varphi < a'\}$. Then Z is a (0, 0)-convex-concave 0-normal complex space. We may assume without loss of generality that any irreducible component of X intersects Z. Let $i : X \longrightarrow Y$ be a 0-normal compactification of X. We may assume that no irreducible component of Y is in $Y - i(X)$. Then the space

$$W = (Y - i(X)) \cup i(Z)$$

is a 0-convex complex space. Any compact irreducible positive-dimensional subspace of W must be contained in $Y - i(X)$. Hence W has no compact irreducible components.

Let $\gamma : W \longrightarrow S$ be the reduction of W according to Proposition 1.1 and let S' be the 0 normalization of S. Then γ factors through $\gamma' : W \longrightarrow S'$.

$$\begin{tikzcd} W \arrow[rr, "\gamma'"] \arrow[dr, "\gamma"'] & & S' \arrow[dl, "\tau"] \\ & S & \end{tikzcd}$$

Since Z is 0 normal, $\gamma' \mid i(Z)$ is an isomorphism onto its image.

Pasting together X and S' along Z according to the map $\sigma = \gamma' \circ i \mid Z$, we obtain a new 0-normal compact space

$$Y' = X \cup_\sigma S'$$

and a holomorphic map $\tau : Y \to Y'$ such that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "j"'] & & Y \arrow[dl, "\tau"] \\ & Y' & \end{tikzcd}$$

j being the natural injection of X into Y'.

By Proposition 1.1, Y' has no compact subspaces in $Y' - j(X)$ of dimension $\geqq 1$. It follows that we have constructed a 0-normal compactification $j : X \to Y'$ which is dominated by the given one and satisfies the property of the proposition.

It follows then that any minimal 0-normal compactification of $X, i : X \to Y$, has the property that $Y - i(X)$ does not contain any compact subspace of dimension $\geqq 1$.

Conversely, if $i : X \to Y$ is any 0-normal compactification such that in $Y - i(X)$ there are no compact subspaces of dimension $\geqq 1$, then $W = (Y - i(X)) \cup i(Z)$ is a 0-normal Stein completion of $i(Z)$.

If $j : X \to Y'$ is any other 0-normal compactification of X, then $W' = (Y' - j(X)) \cup j(Z)$ is a 0-convex complex space. Let W'' be the space obtained by suppressing the irreducible compact components of W'. Then W'' has W as 0-normal Stein reduction. Therefore there exists a holomorphic map $\varrho : W' \to W$ such that, on Z, $\varrho \circ j = i$. From this we deduce a holomorphic factorization of i through j, i. e. the minimality of i.

From the proof we deduce in particular the following:

Corollary 5.1. *If X is a 0-concave 0-normal complex space and if X admits a 0-normal compactification, then it admits also a minimal 0-normal compactification.*

Proposition 5.2. *Let X be a 0-concave 0-normal complex space with exhaustion function $\varphi : X \to (a, \infty)$. Suppose that for some $a' \in (a, \infty)$ the set $\{\varphi < a'\}$ is 1-normal. Then X admits a minimal 0-normal compactification.*

Proof. We may assume that φ is strongly 0-pseudoconvex on $\{\varphi < a'\}$. Set $Y = \{\varphi < a'\}$. Then Y is a (0, 0)-convex-concave 1-normal complex space. By Corollary 3.3, Y admits a 0-normal Stein completion Z. By pasting together X and Z along Y, we obtain a compact complex space $\widetilde{X}$ which is a 0-normal compactification of X with respect to the natural inclusion map and which satisfies the requirements of the previous proposition.

REMARKS 1. Instead of the 1-normality of $\{\varphi < a'\}$ one may as well assume that, on $\{\varphi < a'\}$, prof $\mathcal{O} \geqq 3$.

2. This compactification result can be considered as a refinement of a result of Rossi [13].

§ 6. Extension of coherent sheaves.

11. LEMMA 6.1. *Let X be a $(0, 0)$-convex concave complex space with exhaustion function $\varphi : X \longrightarrow (a, b)$ which is strongly 0-pseudoconvex on the whole of X.*

Let $\mathcal{F}$ be a coherent analytic sheaf on X whith prof $\mathcal{F} \geqq 3$ *on* $\{\varphi < a'\}$ *for some $a' \in (a, b)$.*

Then $\Gamma(X, \mathcal{F})$ generates the stalks $\mathcal{F}_x$ for any $x \in X$.

PROOF. Let $\mathcal{J}$ be the ideal sheaf of germs of holomorphic functions vanishing at x. By Proposition 1.3,

$$\Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}/\mathcal{J}\mathcal{F}) = (\mathcal{F}/\mathcal{J}\mathcal{F})_x$$

is surjective. Since $\mathcal{J}_x$ is the maximal ideal of the local ring $\mathcal{O}_x$, by Nakayama's lemma we deduce that $\Gamma(X, \mathcal{F})$ generates $\mathcal{F}_x$.

PROPOSITION 6.1. *Let X be a $(0, 0)$-convex-concave complex space and let $\widetilde{X}$ be a Stein completion of X. Let $\mathcal{F}$ be a coherent analytic sheaf on X such that $\mathcal{F}^{[1]} = \mathcal{F}$. Then there exist a coherent analytic sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{X}$ which extends $\widetilde{\mathcal{F}}$,* i. e. $\widetilde{\mathcal{F}} \mid X = \mathcal{F}$.

PROOF. Let $\varphi : X \longrightarrow (a, b)$ be an exhaustion function and let $a' \in (a, b]$ be such that φ is strongly 0-pseudoconvex on $\{\varphi < a'\}$. We set $X_c^d = \{c < \varphi < d\}$ for $a \leqq c < d \leqq b$.

The set
$$A = \{x \in X \mid \text{prof}\, \mathcal{F}_x \leqq 2\}$$

is discrete by virtue of Proposition 2.5 since $\mathcal{F}^{[1]} = \mathcal{F}$. We choose c, d with $a < c < d < a'$ such that $A \cap X_c^d = \emptyset$. By the previous lemma we can find $s_1', \dots, s_k' \in \Gamma(X_c^{a'}, \mathcal{F})$ generating $\mathcal{F}$ on X_c^d. By Proposition 2.5 prof $\mathcal{F} \geqq 2$ on the whole of X and thus by Proposition 1.2 (*b*) we can extend $s_1', \dots, s_k'$ to global sections $s_1, \dots, s_k \in \Gamma(X_a^{a'}, \mathcal{F})$.

Let $K = \widetilde{X} - X$. For our problem it is sufficient to extend $\mathcal{F} \mid X_a^d$ to $K \cup X_a^d$ by a coherent analytic sheaf. It is not therefore restrictive to replace

X by X_a^d and $\widetilde{X}$ by $K \cup X_a^d$. So we may assume without loss of generality that

(i) $a' = b$,

(ii) $\widetilde{X}$ is a subspace of some number space $\mathbf{C}^N$ (for $N \geqq 4$),

(iii) $s_1, \dots, s_k \in \Gamma(X, \mathcal{F})$ generate $\mathcal{F}_x$ for x in X_c^b.

Let us denote by $\widetilde{\mathcal{F}}$ the trivial extension of $\mathcal{F}$ on $\mathbf{C}^N - K$. Let $\widetilde{s}_1, \dots$ $\dots, \widetilde{s}_k \in \Gamma(\mathbf{C}^N - K, \widetilde{\mathcal{F}})$ be the corresponding trivial extensions of the sections $s_1, \dots, s_k$. Let ${}_N\mathcal{O}$ denote the structure sheaf of $\mathbf{C}^N$ and let $\mathcal{R} \subset {}_N\mathcal{O}^k$ be the relation sheaf of $\widetilde{s}_1, \dots, \widetilde{s}_k$ on $\mathbf{C}^N - K$.

By the assumption $\mathcal{F}^{[1]} = \mathcal{F}$ it follows that, on $\mathbf{C}^N - K$, $\widetilde{\mathcal{F}}^{[1]} = \widetilde{\mathcal{F}}$ and thus by Proposition 2.4, $0_{[2]}\widetilde{\mathcal{F}} = 0$. Hence on $\mathbf{C}^N - K$, $\mathcal{R}^{[2]} = \mathcal{R}$. Again by Proposition 2.5 it follows that the set $\{x \in \mathbf{C}^N - K \mid \operatorname{prof} \mathcal{R}_x \leqq 2\}$ is 0-dimensional.

Set $D_\alpha^\beta = \{(z_1, \dots, z_N) \in \mathbf{C}^N \mid \alpha < \sum_{i=1}^N |z_i|^2 < \beta\}$ for $-\infty \leqq \alpha < \beta \leqq \infty$. We can choose α and β such that

$$K \subset D_{-\infty}^\alpha ; \qquad \operatorname{prof} \mathcal{R} \geqq 3 \text{ on } D_\alpha^\beta .$$

Now, by virtue of Lemma 6.1, for any choice of α', β' with $\alpha < \alpha' < \beta' < \beta$ we can find $t_1, \dots, t_l \in \Gamma(D_\alpha^\beta, \mathcal{R})$ generating $\mathcal{R}$ on $D_{\alpha'}^{\beta'}$. By Proposition 1.2 (*b*) (actually by Hartogs theorem) $t_1, \dots, t_l$ extends uniquely to sections $\widetilde{t}_1, \dots, \widetilde{t}_l \in \Gamma(D_{-\infty}^\beta, {}_N\mathcal{O}^k)$.

Let $\mathcal{J}$ be the sheaf of ideals defined in $\mathbf{C}^N$ by the subspace $\widetilde{X}$. We define a subsheaf $\mathcal{S}$ of ${}_N\mathcal{O}^k$ as follows:

$$\mathcal{S} = \mathcal{R} \text{ on } D_{\alpha'}^\infty .$$

$$\mathcal{S} = \mathcal{J} \cdot {}_N\mathcal{O}^k + \Sigma_{i=1}^l {}_N\mathcal{O}\, \widetilde{t}_i \text{ on } D_{-\infty}^{\beta'} .$$

On $\mathbf{C}^N - K$ the set A where $\mathcal{R} \neq \mathcal{S}$ is an analytic set with

$$A \subset X - D_{\alpha'}^\infty .$$

By [4, Lemma 2], $\dim A \leqq 0$. Hence $\mathcal{S}^{[1]} = \mathcal{R}$ on the whole set $\mathbf{C}^N - K$ (because $\mathcal{R}^{[2]} = \mathcal{R}$ on $\mathbf{C}^N - K$). Also, since $\mathcal{S}$ is a subsheaf of ${}_N\mathcal{O}^k$, $E^2(0, \mathcal{S}) = \emptyset$. Thus by Proposition 2.4, $\mathcal{S}^{[1]}$ is coherent on the whole space $\mathbf{C}^N$.

Let $\mathcal{G} = ({}_N\mathcal{O}^k / \mathcal{S}^{[1]}) \mid \widetilde{X}$. This is a coherent analytic sheaf on $\widetilde{X}$ since its support is concentrated on $\widetilde{X}$. Moreover, $\mathcal{G}$ is a subsheaf of $\mathcal{F}$ on X and agrees with $\mathcal{F}$ on X_c^b because of the assumption (iii).

The set of points where $\mathcal{G} \neq \mathcal{F}$ on X, by [4, Lemma 2], must be a 0-dimensional analytic set. Thus $\mathcal{G}^{[0]} = \mathcal{F}$ on X (since $\mathcal{F} = \mathcal{F}^{[1]}$). Moreover $E^1(0, \mathcal{G})$ must be empty. Thus $\mathcal{G}^{[0]}$ is coherent analytic (Proposition 2.4) and extends $\mathcal{F}$.

The proof of this proposition is a modification of Trautmann's method [24].

COROLLARY 6.1. *Let X be a 0-concave complex space and $i : X \to \widetilde{X}$ a compactification of X such that no positive-dimensional compact subspace is contained in $\widetilde{X} - i(X)$. Let $\mathcal{F}$ be any analytic coherent sheaf on X such that $\mathcal{F}^{[1]} = \mathcal{F}$. Then there exists a coherent analytic sheaf $\widetilde{\mathcal{F}}$ on $\widetilde{X}$ such that $i^* \widetilde{\mathcal{F}} = \mathcal{F}$.*

PROOF. We identify X with $i(X)$ and let $\varphi : X \to (a, \infty)$ be an exhaustion function for X such that on $\{\varphi < a'\}$, for some $a' \in (a, \infty)$, φ is strongly 0-pseudoconvex.

Set $Y = \{\varphi < a'\}$. Y is a $(0, 0)$-convex-concave space and $(\widetilde{X} - X) \cup Y$ is a Stein completion of Y. By the previous proposition $\mathcal{F} \mid Y$ can be extended to a coherent analytic sheaf $\mathcal{G}$ on $(\widetilde{X} - X) \cup Y$. The sheaf $\widetilde{\mathcal{F}}$ is obtained by gluing together $\mathcal{G}$ and $\mathcal{F}$ along Y by $\mathcal{G} \mid Y \simeq \mathcal{F} \mid Y$.

12. *Extension of line bundles.*

(*a*) We need some preliminaries of commutative algebra.

DEFINITIONS.

(*a*) *A ring (commutative with identity) is called prefactorial if every prime ideal of height* 1 *is the radical of a principal ideal.*

(*b*) *A ring is semifactorial if every prime ideal of height* 1 *is a principal ideal.*

A semifactorial ring is prefactorial; a factorization domain is semifactorial.

Also we recall the theorem: a noetherian integral domain which is integrally closed is a unique factorization domain if and only if it is semifactorial. (cf. for instance [14]).

PROPOSITION 6.2. *Let R be a noetherian integral domain, integrally closed, and let I be a proper ideal of R.*

(*a*) *If R is prefactorial, there exists a positive integer k such that the intersection I' of all primary components of I^{pk} (for any p positive) belonging to prime ideals of height $\leqq 1$ is a principal ideal.*

(*b*) *If R is semifactorial (thus a unique factorization domain), I is a principal ideal if and only if every associated prime ideal has height* 1.

PROOF. Let $P_1, \dots, P_l$ be the associated primes of I of height 1. Let $M = R - \bigcup_{i=1}^{l} P_i$ and let R_M be the quotient ring of R with respect to the multiplicative system M. Since R is integrally closed and noetherian, R_M is also integrally closed and noetherian. Since $P_1 R_M, \dots, P_l R_M$ are all the proper prime ideals of R_M and they are all maximal, R_M is a Dedekind domain [27, p. 275, Theorem 13]. Thus, by the very definition of Dedekind domains we have

$$IR_M = P_1^{j_1} \dots P_l^{j_l}$$

for some non-negative integers $j_1, \dots, j_l$. Every j_i is positive, because $IR_M \subset P_i R_M$ for every i.

(*a*) Assume R to be prefactorial. Then P_i is the radical of some principal ideal Ra_i and $P_i^{s_i} \subset Ra_i$ for some positive integer s_i. The factorization of $R_M a_i$ in R_M must have the form $R_M a_i = P_i^{n_i} R_M$ (for $n_i > 0$) because, for $i \neq j$, $P_i^{s_i} R_M \not\subset P_j R_M$. Let $k = n_1, \dots, n_l$. We claim that k satisfy the requirements.

First of all, for $m_i = pkj_i/n_i$ we get

$$(*) \qquad I^{kp} R_M = R_M a_1^{m_1} \dots a_l^{m_l} .$$

Moreover, since $P_1^{m_1 s_1} \dots P_l^{m_l s_l} \subset Ra_1^{m_1} \dots a_l^{m_l}$, every associated prime ideal of $Ra_1^{m_1} \dots a_l^{m_l}$ contains some P_i. Also, since R is integrally closed and noetherian, every associated prime ideal of the principal ideal $Ra_1^{m_1} \dots a_l^{m_l}$ is isolated [27, p. 277, Theorem 14] and of height $\leqq 1$ [27, p. 238, Theorem 29]. Thus every associated prime ideal of $Ra_1^{m_1} \dots a_l^{m_l}$ must be equal to some P_i. By [27, p. 225, Theorem 17] we have

$$Ra_1^{m_1} \dots a_l^{m_l} = R \cap (R_M a_1^{m_1} \dots a_l^{m_l}) .$$

Every associated prime ideal of I^{kp} contains I^{kp} which in turn contains the product of sufficiently high powers of the associated prime ideals of I. Therefore every associated prime ideal of I^{kp} of height $\leqq 1$ contains some associated prime ideal of I and hence equals some P_i. By, [27, p. 225, Theorem 17], $I' = R \cap I^{kp} R_M$.

Intersecting both sides of (*) with R we get

$$I' = Ra_1^{m_1} \dots a_l^{m_l}.$$

Thus I' is principal.

(b) When R is semifactorial, in the previous argument we can take $s_i = n_i = p = 1$ and $I' = I$. The «only if» part is the combination of [27, p. 277, Theorem 14] and [27, p. 238, Theorem 29].

PROPOSITION 6.3. *Let $(X, \mathcal{O})$ be a connected normal complex space of dimension n. Let Y be a non-empty open subset of X. Let L be a holomorphic line bundle over Y and let $\mathcal{L}$ denote the locally free sheaf associated with L. Assume that $\mathcal{L}$ can be extended to all of X by a coherent analytic sheaf $\mathcal{F}$.*

(a) *If $\mathcal{O}_x$ is semifactorial for all $x \in X - Y$, then L can be extended to a holomorphic line bundle over X (i. e. we can take $\mathcal{F}$ locally free)*

(b) *If $\mathcal{O}_x$ is prefactorial for all $x \in X - Y$ and if $X - Y$ is compact and if moreover no compact positive dimensional subspace of X is contained in $X - Y$, then there exists an integer k such that L^k can be extended to a holomorphic line bundle over X.*

PROOF. By factoring out the torsion subsheaf of $\mathcal{F}$, we can assume without loss of generality that $\mathcal{F}$ is torsion-free [1, Proposition 6]. Since X is connected and $\mathcal{F}$ extends $\mathcal{L}$, $\mathcal{F}$ has rank 1 on X [1, pp. 13-14].

(a) Suppose $\mathcal{O}_x$ is semifactorial for $x \in X - Y$. Replacing $\mathcal{F}$ by $\mathcal{F}^{[n-2]}$, we may assume without loss of generality $\mathcal{F} = \mathcal{F}^{[n-2]}$ (Proposition 2.4).

For any $x \in X - Y$ we can find a connected open neighborhood U of x such that $\mathcal{F} | U$ is isomorphic to an ideal-sheaf $\mathcal{I} \subset \mathcal{O}$ with $\mathcal{I} \neq 0$ [1, Proposition 9]. Since $\mathcal{F} = \mathcal{F}^{[n-2]}$, $\mathcal{I}_{[n-2]\,\mathcal{O}} = \mathcal{I}$. By Proposition 2.2, every associated prime ideal of $\mathcal{I}_x$ has height 1 and therefore, by Proposition 6.2 (b), $\mathcal{I}_x$ is a principal ideal. This implies that $\mathcal{F}$ is locally free at x. Since x is arbitrary, $\mathcal{F}$ is locally free everywhere on X, i. e. $\mathcal{F}$ is the associated sheaf of a holomorphic line bundle $\widetilde{L}$ extending L.

(b) Suppose $X - Y$ is compact and that no positive dimensional compact subspaces of X are contained in $X - Y$. Suppose that $\mathcal{O}_x$ is prefactorial for $x \in X - Y$. Consider the analytic set A of points of X where $\mathcal{F}$ is not locally free [1, Proposition 8]. If A is empty, there is nothing to prove. In any case $A \subset X - Y$. So A must be a finite set $\{x_1, \dots, x_l\}$. Select an open neighborhood U of A with l connected components such that $\mathcal{F} | U$ is iso. morphic to an ideal-sheaf $\mathcal{I} \subset \mathcal{O}$ on U. By Proposition 6.2 (a) we can find a positive integer k_i (for $1 \leq i \leq l$) such that for any positive integer p the intersection $I_i(p)$ of all primary components of $(\mathcal{I}_{x_i})^{pk_i}$ belonging to prime ideals of height $\leqq 1$ is a principal ideal. By Proposition 2.2 the ideal $I_i(p)$ equals the stalk of the sheaf $(\mathcal{J}^{pk_i})_{[n-2]\mathcal{O}}$ at x_i. Therefore for any positive integer p the sheaf $(\mathcal{J}^{pk_i})_{[n-2]\mathcal{O}}$ is locally free at x_i and, consequently, for $k = k_1 \dots k_l$ the sheaf $(\mathcal{J}^k)_{[n-2]\mathcal{O}}$ is locally free on U (since, on $U - A, (\mathcal{J}^k)_{[n-2]\mathcal{O}} = \mathcal{J}^k$).

Let $\mathcal{R}$ be the sheaf obtained by tensoring $\mathcal{F}$ with itself k times and let $\mathcal{S}$ be the sheaf obtained from $\mathcal{R}$ by factoring out the torsion subsheaf of $\mathcal{R}$. The sheaf $\mathcal{S}$ satisfies the following conditions:

(i) $\mathcal{S}$ is torsion free.

(ii) $\mathcal{S}$ is locally free on $X - A$.

(iii) $\mathcal{S}$ extends the locally free sheaf associated to L^k from Y to X.

(iv) $\mathcal{S}$ is isomorphic to $\mathcal{J}^k$ on $U - A$.

Gluing together $\mathcal{S}$ on $X - A$ and $(\mathcal{J}^k)_{[n-2]\mathcal{O}}$ on U by the isomorphism (iv), we obtain a locally free sheaf of rank 1 which extends to all of X the sheaf associated with L^k on Y. We note that the sheaf we have obtained is nothing else than the sheaf $\mathcal{S}^{[n-2]}$.

§ 7. Projective imbeddings of manifolds of dimension $\geqq 3$.

13. We have already established a criterion for projective embeddability of 0-concave spaces with Theorem 4.1. There the crucial assumption was the existence on the space of a holomorphic line bundle L such that the associated ring $\mathcal{A}(X, L)$ had the property of separating points and giving local coordinates. In this paragraph we want to show that, for manifolds of dimension $\geqq 3$ the assumption that $\mathcal{A}(X, L)$ separate points can be dropped.

LEMMA 7.1. *Let $f_1, \dots, f_k$ be holomorphic functions defined in a neighborhood U of $0 \in \mathbf{C}^n$. Consider the Levi form at the origin*

$$\mathcal{L}(\varphi) \equiv \Sigma \left(\frac{\partial^2 \varphi}{\partial z_i \, \partial \bar{z}_j} \right)_0 u_i \bar{u}_j$$

of the function $\varphi = \log (1 + \Sigma_{i=1}^k |f_i|^2)$. Then $\mathcal{L}(\varphi)$ is positive semidefinite and it is positive definite if and only if $f_1, \dots, f_k$ gives a set of local coordinates of $\mathbf{C}^n$ at 0.

The proof is a verification based on the identity

$$\partial\cdot\bar{\partial}\log\left(\sum_{i=0}^{k} g_i\bar{g}_i\right)=\frac{1}{\left(\sum_{i=0}^{k} g_i\bar{g}_i\right)^2}\sum_{i<j}\left|\det\begin{pmatrix} g_i & g_j \\ dg_i & dg_j\end{pmatrix}\right|^2,$$

where $g_0, \dots, g_k$ are holomorphic functions with $\sum_{i=0}^{k} g_i\bar{g}_i > 0$.

THEOREM 7.1. *Let X be a 0-concave complex manifold of dimension $n \geqq 3$. Let $\varphi: X \to (a, \infty)$ be an exhaustion function for X which is strongly 0-pseudoconvex on $\{\varphi < a'\}$, where $a' \in (a, \infty)$. We assume that there exists a holomorphic line bundle L over X such that the graded ring $\mathcal{A}(X, L)$ gives local coordinates on $\{\varphi > a''\}$ for some $a'' \in (a, a')$. Then X is isomorphic to an open subset of a projective algebraic variety.*

PROOF. Let $D = \{\varphi > a''\}$. Moving a'' a little toward a' and eventually replacing L by a sufficiently high power L^h of L, we see that without loss of generality we may assume that a finite number of sections $f_1, \dots, f_k \in \Gamma(X, L)$ give local coordinates at each point of D (cf. part (*a*) of the proof of Theorem 4. 1). At each point of D one the sections $f_1, \dots, f_k$ is non-zero, therefore the sheaf $\mathcal{L}$ associated to L is generated on D by $f_1, \dots, f_k$. By Proposition 5.2, X admits a minimal 0-normal compactification $X \subset X_1$. Since the normalization of X_1 is again a minimal 0-normal compactification of X, the space X_1 must be normal. The structure sheaf of X_1 will be denoted by $\mathcal{O}_1$.

Since $n \geqq 3$, $\mathcal{L}^{[1]} = \mathcal{L}$. So, by Corollary 6.1, the sheaf $\mathcal{L}$ can be extended on X_1 by a coherent analytic sheaf $\mathcal{F}$. Factoring out the torsion subsheaf of $\mathcal{F}$, we may assume that $\mathcal{F}$ is torsion-free. Also we may replace $\mathcal{F}$ by $\mathcal{F}^{[1]}$ which is again coherent (Proposition 2.4). By Proposition 2.5, prof $\mathcal{F} \geqq 2$.

Let $Y = \{\varphi < a'\}$ and set $Y_1 = Y \cup (X_1 - X)$. Then Y_1 is a (normal) Stein completion of Y. By Proposition 3.1, the restriction map

$$\Gamma(Y_1, \mathcal{F}) \to \Gamma(Y, \mathcal{F})$$

is bijective, so that each f_i extends uniquely to a section $f_i' \in \Gamma(Y_1, \mathcal{F})$. The sections $f_1', \dots, f_k'$ generate $\mathcal{F}$ on D.

Since the singular set S of X_1 is contained in $X_1 - X$, S must be a finite set $S = \{x_1, \dots, x_l\}$. Here we apply a result of Hironaka-Rossi (Lemma 5 and Corollary 2 to Lemma 5 of [11]) which states the following:

There exist an open neighborhood U_i of x_i and a coherent ideal-sheaf $\mathcal{I}_i$ on U_i such that $\{x_i\}$ is the zero set of $\mathcal{I}_i$ and such that the complex space obtained from U_i by the monoidal transformation with center at $(\{x_i\}, \mathcal{O}_1/\mathcal{I}_i)$ is non-singular.

Let $\mathcal{I}$ be the ideal sheaf on X_1 which agrees with $\mathcal{O}_1$ on $X - S$ and with $\mathcal{I}_i$ on U_i for $1 \leqq i \leqq l$, and let $\Phi : (X_2, \mathcal{O}_2) \longrightarrow (X_1, \mathcal{O}_1)$ be the monoidal tranformation with center $(S, \mathcal{O}_1/\mathcal{I})$.

Let $\mathcal{R}$ the sheaf obtained by factoring out the torsion subsheaf of $\Phi^{-1}(\mathcal{F})$ on X_2. As in Proposition 6.3 (a) the sheaf $\mathcal{S} = \mathcal{R}^{[n-2]}$ is locally free of rank 1 on X_2 and we have a natural sheaf homomorphism

$$\lambda : \Phi^{-1}(\mathcal{F}) \longrightarrow \mathcal{S}, \text{ on } X_2 .$$

Let f_i'' be the unique section of $\Gamma(X_2, \Phi^{-1}(\mathcal{F}))$ which by the natural map

$$\Gamma(X_1, \mathcal{F}) \longrightarrow \Gamma(X_2, \Phi^{-1}(\mathcal{F}))$$

corresponds to the section f_i. Let $f_i''' = \lambda(f_i'')$. Let $\mathcal{T}$ be the subsheaf of $\mathcal{S}$ generated by the sections $f_1''', \ldots, f_k'''$. The sheaf $\mathcal{T}$ agrees with $\mathcal{S}$ on $\Phi^{-1}(D)$.

Let us consider the conductor sheaf $\mathcal{A}$ of $\mathcal{T}$ into $\mathcal{S}$:

$$\mathcal{A}_x = \{\alpha \in (\mathcal{O}_2)_x \mid \alpha\, \mathcal{S}_x \subset \mathcal{T}_x\}.$$

Since $\mathcal{S}$ is locally free, $\mathcal{A}$ is locally isomorphic to $\mathcal{T}$. Let T be the zero set of $\mathcal{A}$. Obviously $T \cap \Phi^{-1}(D) = \emptyset$ since on $\Phi^{-1}(D)$ we have $\mathcal{S} = \mathcal{T}$. Moreover, $\Phi(T)$ must be a finite set; $\Phi(T) = \{y_1, \ldots, y_m\}$.

By the Remark 2 of [11], for every point of X_1 we can find a neighborhood W such that $\Phi^{-1}(W)$ is isomorphic to a subspace of $W \times \mathbf{P}_N$ for some N. By Lemma 7 and Corollary 2 of Lemma 5 of [11] for each y_i in X_1 we can find a neighborhood W_i and an ideal sheaf $\mathcal{J}_i$ on $\Phi^{-1}(W_i)$ with zero set $T \cap \Phi^{-1}(y_i)$ such that;

(i) the monoidal transformation $\Phi_i : V_i \longrightarrow \Phi^{-1}(W_i)$ of $\Phi^{-1}(W_i)$ with center $(T \cap \Phi^{-1}(y_i), \mathcal{O}_2/\mathcal{J}_i)$ is non-singular

(ii) $\Phi_i^{-1}(\mathcal{A})$ is a locally free sheaf on V_i.

We may assume that $W_1, \ldots, W_m$ are mutually disjoint. Let $\mathcal{J}$ be the ideal sheaf on X_2 which agrees with $\mathcal{O}_2$ on $X_2 - T$ and with $\mathcal{J}_i$ on $\Phi^{-1}(W_i)$ and let $\Phi' : X_3 \longrightarrow X_2$ be the monoidal transformation with center $(T, \mathcal{O}_2/\mathcal{J})$. Then X_3 is non-singular and $(\Phi')^{-1}(\mathcal{A})$ is locally free.

Let $\mathcal{B} = (\Phi')^{-1}(\mathcal{T})$. Since $\mathcal{T}$ is locally isomorphic to $\mathcal{A}$, it follows that $\mathcal{B}$ is a locally free sheaf of rank 1 on X_3 and hence is the sheaf of germs of holomorphic sections of a holomorphic line bundle B on X_3.

By the natural map $\Gamma(X_2, \mathcal{T}) \longrightarrow \Gamma(X_3, \mathcal{B})$, to the section f_i''' of $\mathcal{T}$ corresponds a section f_i^* of B. Since $f_1''', \dots, f_k'''$ generate $\mathcal{T}$ on $X_2, f_1^*, \dots, f_k^*$ generate $\mathcal{B}$ on X_3.

By Corollary 2 to Lemma 5 of [11] there exists a coherent sheaf of ideals $\mathcal{K}$ on X_1 with zero set $S \cup \Phi(T)$ such that $\Phi \circ \Phi' : X_3 \longrightarrow X_1$ is the monoidal transformation with center $(S \cup \Phi(T), \mathcal{O}_1/\mathcal{K})$. Let $\Psi = \Phi \circ \Phi'$ and let $\mathcal{C} = \Psi^{-1}(\mathcal{K})$. $\mathcal{C}$ is locally free by construction and it is thus the sheaf of germs of holomorphic sections of a holomorphic line bundle C on X.

Let $G = (X_1 - X) \cup \{\varphi < a'\}$ and let $G' = \Psi^{-1}(G)$. We claim that $\mathcal{A}(G', C)$ gives local coordinates at every point of G'.

Let $x \in G'$ and let $\mathfrak{m}$ be the ideal-sheaf on X_3 defined by $\{x\}$. By [11, Lemma 2] we can find a positive integer h' such that, for $q \geqq 1$,

$$R^q \Psi(\mathfrak{m} \otimes \mathcal{C}^{h'}) = 0, R^q \Psi(\mathfrak{m}^2 \otimes \mathcal{C}^{h'}) = 0,$$

where $R^q\Psi$ denotes the q^h direct image sheaf under Ψ of the sheaf in parenthesis.

Since Ψ is proper, $R^0\Psi(\mathfrak{m} \otimes \mathcal{C}^{h'})$ and $R^0\Psi(\mathfrak{m}^2 \otimes \mathcal{C}^{h'})$ are coherent ([7]) and, since G is Stein, we get

$$H^1(G', \mathfrak{m} \otimes \mathcal{C}^{h'}) = H^1(G, R^0\Psi(\mathfrak{m} \otimes \mathcal{C}^{h'})) = 0$$

$$H^1(G', \mathfrak{m}^2 \otimes \mathcal{C}^{h'}) = H^1(G, R^0\Psi(\mathfrak{m}^2 \otimes \mathcal{C}^{h'})) = 0.$$

From the short exact sequences of sheaves

$$0 \longrightarrow \mathfrak{m} \otimes \mathcal{C}^{h'} \longrightarrow \mathcal{C}^{h'} \longrightarrow \mathcal{C}^{h'}/\mathfrak{m}\mathcal{C}^{h'} \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{m}^2 \otimes \mathcal{C}^{h'} \longrightarrow \mathfrak{m}\mathcal{C}^{h'} \longrightarrow \mathfrak{m}\mathcal{C}^{h'}/\mathfrak{m}^2\mathcal{C}^{h'} \longrightarrow 0$$

and from the corresponding cohomology sequences we conclude that

$$\Gamma(G', \mathcal{C}^{h'}) \longrightarrow (\mathcal{C}^{h'}/\mathfrak{m}\mathcal{C}^{h'})_x$$

$$\Gamma(G', \mathfrak{m}\mathcal{C}^{h'}) \longrightarrow (\mathfrak{m}\mathcal{C}^{h'}/\mathfrak{m}^2\mathcal{C}^{h'})_x$$

are surjective maps. Thus we can find $g_0 \in \Gamma(G', \mathcal{C}^{h'})$ with $g_0(x) \neq 0$ and $g_1, \dots, g_r \in \Gamma(G', \mathfrak{m}\mathcal{C}^{h'})$ such that their images in $(\mathfrak{m}\mathcal{C}^{h'}/\mathfrak{m}^2\mathcal{C}^{h'})_x$ generate this vector space over $\mathbf{C}$. Therefore the images of $g_1/g_0, \dots, g_r/g_0$ in $(\mathfrak{m}/\mathfrak{m}^2)_x$ generate this vector space over $\mathbf{C}$. This implies that $\Gamma(G', \mathcal{C}^{h'})$ gives local coordinates at x.

Let $a^* \in (a'', a')$, let $K = (X_1 - X) \cup \{\varphi \leqq a^*\}$, and let $K' = \Psi^{-1}(K)$. We can find a positive integer h and a finite number of sections $v_0, \dots, v_p$ of C^h over G' such that $v_0, \dots, v_p$ give local coordinates at every point of K', and therefore in an open neighborhood H of K' in G'.

For a holomorphic line bundle Λ, we denote by $|\Lambda|$ the C^∞ real line bundle with transition functions equal to the absolute values of the transition functions of Λ. In particular

$$(|v_0|^2 + \dots + |v_p|^2)^{1/2} \in \Gamma(G', |C^{h'}|)$$

and has the property to be $\neq 0$ on H. By partition of unity we can find a C^∞ section $\sigma \in \Gamma(X_3, |C^{h'}|)$ which agrees with $(|v_0|^2 + \dots + |v_p|^2)^{1/2}$ on an open neighborhood H' of K', and is nowhere zero on X_3. Then

$$\Sigma \frac{\partial^2 \log \sigma}{\partial z_i \, \partial \bar{z}_j} dz_i \wedge d\bar{z}_j$$

is well defined on X_3 and $\partial \bar{\partial} \log \sigma$ is positive definite on H'. Let us consider

$$\tau = (|f_1^*|^2 + \dots + |f_k^*|^2)^{1/2} \in \Gamma(X_3, |B|).$$

By the previous lemma $\partial \bar{\partial} \log \tau$ is positive semidefinite and well defined on X_3 and it is positive definite on $\Psi^{-1}(D)$. Now $X_3 = H' \cup \Psi^{-1}(D)$, and actually $X_3 - H' \subset\subset \Psi^{-1}(D)$. Thus we can find a positive integer q such that

$$\partial \bar{\partial} \log \sigma + q \, \partial \bar{\partial} \log \tau \text{ is positive definite on } X_3 .$$

Therefore, since $\sigma\tau^q \in \Gamma(X_3, |C^h \otimes B^q|)$ and since $\partial \bar{\partial} \log \sigma\tau^q > 0$, the holomorphic line bundle $C^h \otimes B^q$ is positive [8, p. 343]. Therefore X_3 is an algebraic manifold by a theorem of Kodaira (cf. [8, Satz 2]). The biholomorphic map $\Psi^{-1} | X$ sends X isomorphically onto an open subset of X_3.

REMARK. We do not know if this theorem is still valid if we allow X to have singularities. Also a remark analogous to Corollary 4.1 does not follow any more from the proof.

14. *Some examples of non-imbeddability.* We are going to give examples of 0-concave manifolds of dimension 2 which cannot be compactified. This will show the rôle of the assumptions made in Proposition 3.2, Corollary 3.3, and Theorems 4.1 and 7.1.

We will make use of the following criterion for the non-existence of a compactification :

PROPOSITION 7.1. *Let V be a compact connected manifold. Let U be open in V and let $K \subset U$ be compact. We assume that $U - K$ is* (0, 0)*-convex-concave and that U is a Stein completion of $U - K$. Let X be a connected complex manifold and $\pi : X \to V - K$ a holomorphic map making X into a λ-sheeted ramified covering of $V - K$ with a compact ramification set $E \subset V - U$* (4). *If* $\dim_{\mathbf{C}} V \geqq 2$ *and if $V - E$ is simply-connected, then either $\lambda = 1$ or else X cannot be compactified.*

PROOF. Suppose that X is an open subset of some compact complex space $\widetilde{X}$.

Let $Y = (\widetilde{X} - X) \cup \pi^{-1}(U - K)$. Y is a 0-convex open subset of $\widetilde{X}$. Let S be the Stein space obtained by the reduction $\gamma : Y \to S$ of Proposition 1.1. Let A be the finite subset of S where γ^{-1} is not single-valued. Since $\gamma^{-1}(A)$ is nowhere discrete, we must have $\gamma^{-1}(A) \subset \widetilde{X} - X$ [4, Lemma 2].

Let X^* be the compact complex space obtained from X and S by identifying $x \in X$ and $s \in S$ whenever $x \in \pi^{-1}(U - K)$ and $\gamma(x) = s$.

We can regard X as an open subset of X^* so that we can assume without loss of generality that no positive-dimensional subspaces of $\widetilde{X}$ are contained in $\widetilde{X} - X$. Also, replacing $\widetilde{X}$ by its normalization, we may assume $\widetilde{X}$ to be normal and connected (thus irreducible).

In this situation Y is a 0-normal Stein completion of the (0, 0)-convex-concave space $\pi^{-1}(U - K)$. By Corollary 3.1, $\pi \mid \pi^{-1}(U - K)$ can be extended uniquely to a holomorphic map $\sigma : Y \to U$. We claim that $\sigma(\widetilde{X} - X) \subset K$. Let $v \in U - K$ and let $\varphi : U - K \to (a, b)$ be an exhaustion function on $U - K$. Let $W = K \cup \{\varphi < \varphi(v)\}$. Then $(\widetilde{X} - X) \cup \pi^{-1}(W - K)$ is a 0-normal Stein completion of $\pi^{-1}(W - K)$ and $\pi \mid \pi^{-1}(W - K)$ extends to a holomorphic map

$$\tau : (\widetilde{X} - X) \cup \pi^{-1}(W - K) \to W.$$

Since both σ and τ extend $\pi \mid \pi^{-1}(W - K)$, they must agree on $(\widetilde{X} - X) \cup \pi^{-1}(W - K)$. Thus $v \notin \sigma(\widetilde{X} - X)$. The claim is proved. We define a holomorphic map $\widetilde{\pi} : \widetilde{X} \to V$ by setting

$$\widetilde{\pi} \mid X = \pi \quad \text{and} \quad \widetilde{\pi} \mid Y = \sigma.$$

---

(4) Cf. [1, n. 3].

Let B the set of singular points of $\widetilde{X}$. Since $B \subset \widetilde{X} - X$, it must be a finite set. Let C be the set of points of $\widetilde{X} - B$ where $\widetilde{\pi}$ has a vanishing Jacobian. We have $C \subset \widetilde{X} - X$ and C is at most 0-dimensional. Indeed any positive-dimensional irreducible component of C in $\widetilde{X} - B$ has an analytic closure in $\widetilde{X}$ (by the theorem of Remmert-Stein [9, Theorem V. D. 5]).

Since $\widetilde{\pi}(B)$ is a finite set and $\dim_{\mathbf{C}} V \geqq 2$, $V - \widetilde{\pi}(B) - E$ is simply-connected. Also $\widetilde{\pi}(C) - \widetilde{\pi}(B)$ is a 0-dimensional subset of $V - \widetilde{\pi}(B) - E$. So $V - \widetilde{\pi}(C \cup B) - E$ is again simply-connected.

Now $\widetilde{X} - \widetilde{\pi}^{-1}(\widetilde{\pi}(B \cup C)) - \widetilde{\pi}^{-1}(E)$ is a topological covering over the simply-connected space $V - \widetilde{\pi}(B \cup C) - E$. Since $\widetilde{X}$ is irreducible $\widetilde{X} - \widetilde{\pi}^{-1}(\widetilde{\pi}(B \cup C)) - \widetilde{\pi}^{-1}(E)$ is connected. Thus $\lambda = 1$.

15. We now proceed with the construction of the first example. We will denote by $[u_0, u_1]$ homogeneous coordinates in $\mathbf{P}_1$, by $[z_0, z_1, z_2]$ homogeneous coordinates in $\mathbf{P}_2$, and by $[v_0, v_1, v_2, v_3]$ homogeneous coordinates in $\mathbf{P}_3$.

For $\varepsilon \in \mathbf{C}^*$ we consider the non-singular quadric of $\mathbf{P}_3$:

$$V = \{v_3 (v_3 + \varepsilon v_0) = v_1 v_2\}.$$

All non-singular quadrics of $\mathbf{P}_3$ are isomorphic to $\mathbf{P}_1 \times \mathbf{P}_1$. Thus V is connected (and simply-connected).

We define the following functions on $\mathbf{P}_1$:

$$f_1(u) = -\varepsilon \frac{u_0 \bar{u}_1}{u_0 \bar{u}_0 + u_1 \bar{u}_1}$$

$$f_2(u) = \varepsilon \frac{\bar{u}_0 u_1}{u_0 \bar{u}_0 + u_1 \bar{u}_1}$$

$$f_3(u) = -\varepsilon \frac{u_1 \bar{u}_1}{u_0 \bar{u}_0 + u_1 \bar{u}_1}.$$

Let $\alpha : \mathbf{P}_2 - [1, 0, 0] \to \mathbf{P}_1$ be the projection given by $z \longmapsto \alpha(z) = [z_1, z_2]$. We define a non-holomorphic map $\varphi : \mathbf{P}_2 - [1, 0, 0] \to \mathbf{P}_3$ by the following equations:

$$
(1)\qquad \begin{cases} v_0 = z_0^2 \\ v_1 = \dfrac{z_1^2}{2} + f_1(\alpha(z))\, z_0^2 \\ v_2 = \dfrac{z_2^2}{2} + f_2(\alpha(z))\, z_0^2 \\ v_3 = \dfrac{z_1 z_2}{2} + f_3(\alpha(z))\, z_0^2 . \end{cases}
$$

Direct substitution shows that $\operatorname{Im} \varphi \subset V$. The image of φ does not cover all of V and actually $[1, 0, 0, 0] \in V - \operatorname{Im} \varphi$. Indeed, for $v = \varphi(z)$ we have the identity

$$
z_1^2 v_2 - z_2^2 v_1 = \varepsilon\, z_1 z_2 v_0 .
$$

Thus, if $[1, 0, 0, 0] \in \operatorname{Im} \varphi$, then $z_1 z_2 = 0$. But, if $z_1 = 0$, then $\frac{1}{2} z_2^2 = v_2 - v_0 f_2(\alpha(z)) = 0$, and, if $z_2 = 0$, $\frac{z_1^2}{2} = v_1 - v_0 f_1(\alpha(z)) = 0$.
This is impossible, since $z \in \mathbf{P}_2 - [1, 0, 0]$.

Consider the following diagram

$$
(2)\qquad \begin{tikzcd} \mathbf{P}_2 - [1,0,0] \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & V - [1,0,0,0] \arrow[dl, "\beta"] \\ & \mathbf{P}_1 & \end{tikzcd}
$$

where $\beta(v) = \begin{cases} [v_1, v_3] & \text{if } v_3 \neq 0 \\ [v_3 + \varepsilon v_0, v_2] & \text{if } v_3 = 0. \end{cases}$

Direct substitution shows that, if $v = \varphi(z)$, $v_1 z_2 = v_3 z_1$ and $(v_3 + \varepsilon v_0) z_2 = v_2 z_1$. So the diagram is commutative and β is holomorphic.

Let $A = \{v \in V \mid v_1 - f_1(\beta(v))\, v_0 = 0,\ v_2 - f_2(\beta(v))\, v_0 = 0\}$. Note that $[1, 0, 0, 0] \in A$. We want to show that φ makes $\mathbf{P}_2 - [1, 0, 0]$ a two-sheeted ramified differentiable covering of $V - A$ with ramification set $E = \{v \in V \mid v_0 = 0\}$.

From (1) we deduce

$$
(3) \qquad \begin{cases} z_0^2 = v_0 \\[4pt] \dfrac{z_1^2}{2} = v_1 - f_1(\beta(v))\, v_0 \\[4pt] \dfrac{z_2^2}{2} = v_2 - f_2(\beta(v))\, v_0 \\[4pt] \dfrac{z_1 z_2}{2} = v_3 - f_3(\beta(v))\, v_0 . \end{cases}
$$

Now, if $z \in \mathbf{P}_2 - [1, 0, 0]$ and $v \in V$ satisfy (3), we must have $v \in V - A$ and, since

$$
v_1 \frac{z_1 z_2}{2} = v_3 \frac{z_1^2}{2} \quad \text{and} \quad (v_3 + \varepsilon\, v_0) \frac{z_2^2}{2} = v_2 \frac{z_1 z_2}{2} ,
$$

we must have $\alpha(z) = \beta(v)$. Moreover, we get by direct computation

$$
(4) \qquad (v_3 - f_3(\beta(v))\, v_0)^2 = (v_1 - f_1(\beta(v))\, v_0)(v_2 - f_2(\beta(v))\, v_0)
$$

for $v \in V$.

This shows that, if $v_0 \neq 0$, $\varphi^{-1}(v) \subset \{z \in \mathbf{P}_2 \mid z_0 \neq 0\}$ and consists of two distinct points. In fact, the inverse map is given by

$$
\frac{z_1}{z_0} = \sqrt{2\left(\frac{v_1}{v_0} - f_1(\beta(v))\right)}
$$

$$
\frac{z_2}{z_0} = \sqrt{2\left(\frac{v_2}{v_0} - f_2(\beta(v))\right)},
$$

where the signs of the square roots are taken in such a way that their product equals $2\left(\dfrac{v_3}{v_0} - f_3(\beta(v))\right)$.

For $v_0 = 0$, $\varphi^{-1}(v)$ consists of a single point $(0, z_1, z_2) = (0, v_1, v_3) = (0, v_3, v_2)$ (note that $v_3^2 = v_1 v_2$).

Therefore $\mathbf{P}_2 - [1, 0, 0]$ is a two-sheeted ramified differentiable covering of $V - A$ with ramification set E.

We can give now to $\mathbf{P}_2 - [1, 0, 0]$ the complex structure of a two-dimensional complex manifold Z such that $\varphi : Z \longrightarrow V - A$ is holomorphic.

It is obvious, for any point $z^0 \in Z$ with $\varphi(z^0) \notin E$, how to define such a complex structure. If $v^0 = \varphi(z^0) \in E$, then the complex structure will be defined at z as follows: Let w_1, w_2 be local coordinates in a neighborhood U of v^0 such that $U \cap E = \{w_1 = 0\}$. Then we can consider on $\varphi^{-1}(U)$ the unique complex structure on which $\varphi^* w_1$, $\varphi^* w_2$ and $\sqrt{\varphi^* w_1}$ are holomorphic for any determination of the square root.

Let $H = \{v \in \mathbf{P}_3 \mid v_0 = 0\}$. By virtue of (4), $H \cap A = \emptyset$. We identify $\mathbf{P}_3 - H$ with $\mathbf{C}^3$ so that A will be a compact subset of $\mathbf{C}^3$ contained in some ball K of $\mathbf{C}^3$ centered at the origin.

Set $X = \varphi^{-1}(V - K)$ and let $D = \varphi^{-1}(E)$. Since $V - K$ is 0-concave and $E \cap K = \emptyset$, X is 0-concave. Let $L = \{E|_V\}$ be the bundle of the hyperplane section on V and let $L' = \varphi^{-1}(L)$ the corresponding line bundle on X.

Since $\mathscr{A}(V, L)$ gives local coordinates at each point of V, it follows that $\mathscr{A}(X, L')$ give local coordinates at each point of $X - D$.

Now $V - E$ is simply-connected since it is the affine part of a non-parabolic quadric and these all have the same homotopy type of a 2-sphere.

It follows that

(*a*) X is 0-concave but cannot be compactified,

(*b*) $X - D$ is (0, 0)-convex-concave but does not admit any 0-normal Stein completion.

In particular the assumption prof $\mathcal{O} \geqq 3$ in Proposition 3.2 and the assumption of 1-normality in Corollary 3.3 cannot be relaxed.

REMARK. We do not know whether $\mathscr{A}(X, L')$ gives local coordinates over D (indeed we think this fact doubtful). The example given here is the same (apart from a necessary slight change) as the one given in [13].

16. Let us consider in $\mathbf{P}_3$ the Kummer surface $K_2(z)$ corresponding to the complex torus with periods (I, z) where z is a point in the Siegel upper half plane $\mathcal{H}_2$ of rank 2.

For $z \in \mathcal{H}_2$ outside a proper analytic set the Kummer surface is a surface of order 4 with 16 isolated conical points. It is given by parametric equations

$$v_\mu = \theta[\mu](u; z)$$

where $\mu = 0, 1, 2, 3$ and where $\theta[\mu](u; z)$ is a basis for theta functions of second order with periods matrix (I, z).

These parametric equations exhibit a holomorphic map

$$\sigma : T_2(z) \longrightarrow K_2(z)$$

from the torus $T_2(z)$ corresponding to the periods matrix (I, z), identifying $K_2(z)$ to the quotient of $T_2(z)$ by the involution $\tau : u \longrightarrow -u$ changing each points of $T_2(z)$ to its inverse. The 16 double points are the images of the fixed points of τ.

The proof of all these statements can be found for instance in [3, n. 11]. We fix once for all the period matrix z. Let $G(v) = 0$ be the equation of $K_2(z)$. It is given by the vanishing of a homogeneous polynomial $G(v)$ of degree 4 whose partial derivatives $\partial G / \partial v_i$, for $0 \leqq i \leqq 3$, have 16 distinct nontrivial common zeros $p_1, \dots, p_{16}$.

We consider in $\mathbf{R} \times \mathbf{P}_3$ the set

$$\mathcal{V} = \{(t, v) \in \mathbf{R} \times \mathbf{P}_3 \mid t v_0^4 + G(v) = 0\}.$$

If we assume, as it is permitted by a convenient choice of coordinates in $\mathbf{P}_3$, that $\{v_0 = 0\}$ does not contain any of the points p_i, then it follows that $\mathcal{V}$ is manifold. We consider $\mathcal{V}$ via the projection $\omega = pr_{\mathbf{R}} \mid \mathcal{V}$ as a 1-real-parameter family of algebraic surfaces:

$$\omega : \mathcal{V} \to \mathbf{R},$$

$\omega^{-1}(0)$ is the Kummer surface and $\omega^{-1}(t)$ for $t \neq -\left(\dfrac{\partial G/\partial v_0}{4 v_0^3}\right)_{v=p_i}$ is a non-singular algebraic surface.

We select for each point $p = p_i$ a small neighborhood $U = U_i$ with $U \cap \{v_0 = 0\} = \emptyset$ and we can select in U near p holomorphic coordinates such that the holomorphic function

$$\lambda = -\frac{G(v)}{v_0^4}$$

has the form

$$\lambda = w_1^2 + w_2^2 + w_3^2$$

where $(w_1, w_2, w_3) = (w_1^{(i)}, w_2^{(i)}, w_3^{(i)})$ are defined in the ball

$$\{\Sigma \mid w_i \mid^2 < 2\} \subset U.$$

We set $w_i = x_i + \sqrt{-1}\, y_i$. Fix $\varepsilon > 0$ (such that $3\varepsilon < 2$) and consider the set

$$C = \begin{cases} 1 + \varepsilon < \mid w_1 \mid^2 + \mid w_2 \mid^2 + \mid w_3 \mid^2 < 1 + 2\varepsilon \\ w_1^2 + w_2^2 + w_3^2 = 1. \end{cases}$$

The map

$$\Phi : \begin{cases} x \mapsto \dfrac{x}{|x|} \sqrt{|y|^2 + \lambda} \\ y \mapsto y, \end{cases}$$

where $|x| = (\Sigma x_i^2)^{1/2}$, $|y| = (\Sigma y_i^2)^{1/2}$, is a diffeomorphism of C onto the set

$$C_\lambda \equiv \begin{cases} \varepsilon/2 < |y|^2 < \varepsilon \\ w_1^2 + w_2^2 + w_3^2 = \lambda \end{cases}$$

for $\lambda \in \mathbf{R}$ and $|\lambda| < \varepsilon/4$. Note that $\Sigma w_i^2 = \lambda$ implies $|x|^2 = \lambda + |y|^2$. So on C_λ,

$$\varepsilon + \lambda < |w_1|^2 + |w_2|^2 + |w_3|^2 < 2\varepsilon + \lambda.$$

Now we can select $\delta > 0$ so small that

i) for $0 < |\lambda| < \delta$, $\omega^{-1}(\lambda)$ is non-singular,

ii) $\delta < \varepsilon/4$,

and we consider $\mathcal{V}_\delta = \omega^{-1}(-\delta, \delta)$. Over each U the projection $\pi = pr_{\mathbf{P}_8} | \mathcal{V}$ is an isomorphism. We consider

$$\mathcal{W} = \mathcal{V}_\delta - \bigcup \pi^{-1}(U \cap \{\varepsilon/2 > |y|^2\}),$$

where the union is over all 16 $U = U_i$.

From what preceeds it follows that, letting $\omega_{\mathcal{W}} = \omega | \mathcal{W}$, $\omega_{\mathcal{W}} : \mathcal{W} \to \to (-\delta, \delta)$ has the following properties:

(*a*) it is a family of complex analytic non-singular manifolds,

(*b*) this family is differentiably trivial if δ is sufficiently small.

The first assertion is clear. The second follows from [5, Proposition 4] since the mapping $\Phi : C \times (-\delta, \delta) \to \mathcal{W}$ shows that the family is rigid (differentiably) at infinity.

Consider now $\omega_{\mathcal{W}}^{-1}(0)$, this is the Kummer surface from which we have removed 16 connected neighborhoods of the 16 double points.

Let $\mathcal{C} \to \omega_{\mathcal{W}}^{-1}(0)$ be the double covering of $\omega_{\mathcal{W}}^{-1}(0)$ obtained from the torus $T_2(z)$, where $\mathcal{C} = \sigma^{-1}\, \omega_{\mathcal{W}}^{-1}(0)$. This is a connected non-ramified 2-sheeted covering.

By the remark (*b*) differentiably $\mathcal{W} \simeq \omega_{\mathcal{W}}^{-1}(0) \times (-\delta, +\delta)$. Thus $\tilde{\mathcal{W}} = \mathcal{C} \times \times (-\delta, +\delta)$ turns out to be a two-sheeted unramified covering of $\mathcal{W}$ by a map

$$\nu : \tilde{\mathcal{W}} \to \mathcal{W}.$$

To each fiber $\mathcal{C} \times \{t\}$ we can give a complex structure which makes $\nu \mid \mathcal{C} \times \{t\}$ a local isomorphism.

Select $t_0 \in (-\delta, \delta)$ with $t_0 \neq 0$. We take $V = \omega^{-1}(t_0)$ and $X = \nu^{-1} \omega_{\mathcal{W}}^{-1}(t_0)$; and we apply Proposition 7.1 to the natural map $\varrho : X \longrightarrow V$. If we take for U the set $\bigcup_{U=U_i} U \cap V$ and for K the subset $\bigcup_{U=U_i} (U \cap \{|y|^2 \leqq \varepsilon/2\})$ we conclude that X cannot be compactified.

Now, if L is the line bundle of the hyperplane section on V and $L' = \varrho^{-1}(L)$, then, since ϱ is an unramified covering, $\mathcal{A}(X, L')$ gives local coordinates everywhere.

Also $\omega_{\mathcal{W}}^{-1}(t_0)$ is a 0-concave surface, because the neighborhoods we have removed from $V = \omega^{-1}(t_0)$ have a smooth strongly Levi-convex boundary. Therefore X is 0-concave.

This example shows that in Theorem 7.1 the condition on the dimension cannot be removed. By virtue of Theorem 4.1 in the previous example $\mathcal{A}(X, L')$ cannot separate points.

§ 8. Finite generation of the ring $\mathcal{A}(X, F)$.

17. Let $\mathcal{F}$ be a locally free sheaf of rank 1 on an unreduced compact complex space X. On the reduction X' of X, the sheaf induced by $\mathcal{F}$ is the sheaf of germs of holomorphic sections of a holomorphic line bundle F. Let F^* denote the dual bundle of F. We call the sheaf $\mathcal{F}$ *positive* if the bundle space of F^* is 0-convex.

PROPOSITION 8.1. *Let $\mathcal{F}$ be positive on X and let $\mathcal{S}$ be any coherent sheaf on X. There exists an integer $h_0 = h_0(\mathcal{S}, \mathcal{F})$ such that*

$$H^p(X, \mathcal{S} \otimes \mathcal{F}^h) = 0$$

for $p > 0$ and $h \geqq h_0$.

PROOF. The theorem is known if X is reduced (cf. [2] and [8, Korollar on p. 344]).

Let $\mathcal{K}$ be the sheaf of nilpotent elements in the structure sheaf of X. Let $\mathcal{S}_r = \mathcal{S}/\mathcal{K}^r \mathcal{S}$ for $r \geqq 0$. For some $k \geqq 0$ we have (since X is compact) $\mathcal{S} = \mathcal{S}_k$.

Now $\mathcal{K}^r \mathcal{S}/\mathcal{K}^{r+1} \mathcal{S}$ can be regarded as a coherent analytic sheaf on X'. Hence we can find an integer $h_0(r)$ such that

$$H^p(X, (\mathcal{K}^r \mathcal{S}/\mathcal{K}^{r+1} \mathcal{S}) \otimes \mathcal{F}^h) = 0 \text{ for } h \geqq h_0(r) \text{ and } p > 0.$$

From the exact sequence of sheaves

$$0 \longrightarrow (\mathcal{K}^r \mathcal{S}/\mathcal{K}^{r+1} \mathcal{S}) \otimes \mathcal{F}^h \longrightarrow \mathcal{S}_{r+1} \otimes \mathcal{F}^h \longrightarrow \mathcal{S}_r \otimes \mathcal{F}^h \longrightarrow 0$$

we get $H^p(X, \mathcal{S}_{r+1} \otimes \mathcal{F}^h) \simeq H^p(X, \mathcal{S}_r \otimes \mathcal{F}^h)$ for $h \geqq h_0(r)$ and $p \geqq 1$. Thus for h sufficiently large we get, for $p \geqq 1$, $H^p(X, \mathcal{S} \otimes \mathcal{F}^h) \simeq H^p(X, \mathcal{S}_0 \otimes \mathcal{F}^h) = 0$ since $\mathcal{S}_0 = 0$.

LEMMA 8.1. *Let $M = \bigcup_{h=0}^{\infty} M_h$, $N = \bigcup_{h=0}^{\infty} N_h$ be graded* C*-algebras and let $f: M \longrightarrow N$ be a homomorphism of degree zero. We assume that*

i) *for every $h \geqq 0$, M_h and N_h are finite-dimensional,*
ii) *N is finitely generated,*
iii) *for h sufficiently large $(h \geqq h_0)$, $f(M_h) = N_h$, and*
iv) *there exists an $s \in M_k$ (for some $k \geqq 1$) such that for $h \geqq k$,*

$$\operatorname{Ker} f \cap M_h = sM_{h-k}.$$

Then M is also a finitely generated C*-algebra.*

PROOF. For x in M or N we will denote by $d(x)$ its degree.

We select $p \geqq k$ such that $\bigcup_{h=0}^{p} N_h$ generates N, and $f(M_h) = N_h$ for $h \geqq p$.

First we show that $\bigcup_{h=p}^{3p-1} N_h$ generates the subalgebra $\bigcup_{h=p}^{\infty} N_h$ of N.

Let $x \in N_{h'}$ for $h' \geqq p$ so that x is a sum of products $x_1 \dots x_l$ with $d(x_i) \leqq p$. We define $0 = r_0 < r_1 < r_2 < \dots < r_s \leqq l$ by the following conditions:

(i) $$\Sigma_{j=r_i+1}^{r_{i+1}-1} d(x_j) < p,$$

(ii) $$\Sigma_{j=r_i+1}^{r_{i+1}} d(x_j) \geqq p,$$

(iii) $$\Sigma_{j=r_s+1}^{l} d(x_j) < p.$$

Since $d(x) \geqq p$, we must have $s > 0$. Set

$$y_i = \Pi_{j=r_i+1}^{r_{i+1}} x_j \text{ for } 0 \leqq i \leqq s-2,\ y_{s-1} = \Pi_{j=r_{s-1}+1}^{l} x_j.$$

Since $d(x_i) \leqq p$, we have

$$p \leqq d(y_i) < 2p \quad \text{and} \quad p \leqq d(y_{s-1}) < 3p.$$

Now $x_1 \dots x_l = \Pi_{i=0}^{s-1} y_i$ and this proves our contention.

We now show that for $h' \geqq p$,

$$(*) \qquad M_{h'} \subset s\, M_{h'-k} + \text{subalgebra generated by } \textstyle\bigcup_{h=p}^{3p-1} M_h .$$

Let $x \in M_{h'}$ for $h' \geqq p$. By what we have proved before we can write

$$f(x) = \Sigma_{r=1}^{m} y_1^{(r)} \dots y_{l_r}^{(r)}$$

where $p \leqq d(y_i^{(r)}) < 3p$.

Since $f(M_h) = N_h$ for $h \geqq p$, we can find $x_i^{(r)}$ such that $f(x_i^{(r)}) = y_i^{(r)}$ and $d(x_i^{(r)}) = d(y_i^{(r)})$. Therefore

$$x - \Sigma_{r=1}^{m} x_1^{(r)} \dots x_{l_r}^{(r)} \in M_{h'} \cap \operatorname{Ker} f = sM_{h'-k} .$$

This proves $(*)$. By using induction on h', we conclude that $M_{h'}$ is contained in the subalgebra generated by $\bigcup_{h=0}^{3p-1} M_h$ for any $h' \geqq p$. This proves the lemma.

PROPOSITION 8.2. *If $\mathcal{F}$ is a positive locally free sheaf of rank 1 on a compact unreduced space $(X, \mathcal{H})$, then $\bigcup_{h=0}^{\infty} \Gamma(X, \mathcal{F}^h)$ is a finitely generated graded $\mathbf{C}$ algebra.*

PROOF. We can proceed by induction on the dimension n of X since for $n = 0$ the proposition is obviously true.

Let $n > 0$ and let $Z_1, \dots, Z_l$ be the irreducible analytic sets which appear as irreducible components of $E^p(0, \mathcal{H})$ for some p. The number of these sets is finite since, for $p \geqq n$, $E^p(0, \mathcal{H}) = X$.

Take $x_i \in Z_i$ such that $x_1, \dots, x_l$ are all distinct and let $X_1, \dots, X_m$ be the irreducible components of X. Take $y_i \in X_i - \{x_1, \dots, x_l\}$ so that the $y_1, \dots, y_m$ are also all distinct.

For $x \in X$ let $\mathfrak{m}(x)$ be the sheaf of ideals defined by the point x. Let $\mathcal{J} = \Sigma_{i=i}^{l} \mathfrak{m}(x_i) + \Sigma_{j=1}^{m} \mathfrak{m}(y_j)$. By Proposition 8.1 for large k

$$H^1(X, (\mathcal{H}/\mathcal{J}) \otimes \mathcal{F}^k) = 0.$$

Thus we get a surjective map:

$$\Gamma(X, \mathcal{F}^k) \longrightarrow \left(\bigoplus_{i=1}^{l} (\mathcal{F}^k/\mathfrak{m}(x_i)\, \mathcal{F}^k)\right) \oplus \left(\bigoplus_{j=1}^{m} (\mathcal{F}^k/\mathfrak{m}(y_j)\, \mathcal{F}^k)\right).$$

In particular we can find $s \in \Gamma(X, \mathcal{F}^k)$ such that $s_{x_i} \neq 0$ for $1 \leqq i \leqq l$ and $s_{y_j} \neq 0$ for $1 \leqq j \leqq m$. By Proposition 2.3 the sheaf homomorphism defined by multiplication by s:

$$s: \mathcal{F}^{h-k} \longrightarrow \mathcal{F}^{h} \qquad (h \geqq k)$$

is injective.

Let Y be the zero set of s and let $\mathcal{J}$ be the conductor sheaf of $\mathcal{F}^k$ into $\mathcal{H}s$:

$$\mathcal{J}_x = \{\alpha \in \mathcal{H}_x \mid \alpha\,(\mathcal{F}^k)_x \subset \mathcal{H}_x s_x\}.$$

Then $(Y, \mathcal{H}/\mathcal{J})$ is an unreduced compact complex space of dimension $< n$ and $\mathcal{G} = \mathcal{F}/\mathcal{J}\mathcal{F}$ is a positive locally free sheaf on Y of rank 1. By the induction hypothesis $\bigcup_{h=0}^{\infty} \Gamma(Y, \mathcal{G}^h)$ is a finitely generated $\mathbf{C}$-algebra.

Consider now for $h \geqq k$ the exact sequence

$$0 \longrightarrow \mathcal{F}^{h-k} \xrightarrow{s} \mathcal{F}^{h} \xrightarrow{\beta_h} \mathcal{G}^h \longrightarrow 0,$$

where β_h is defined in a natural way by $\mathcal{F}^h/\mathrm{Im}\,(s) \simeq \mathcal{G}^h$. We deduce the exact cohomology sequence:

$$0 \longrightarrow \Gamma(X, \mathcal{F}^{h-k}) \longrightarrow \Gamma(X, \mathcal{F}^{h}) \xrightarrow{\beta_h^*} \Gamma(Y, \mathcal{G}^h) \longrightarrow H^1(X, \mathcal{F}^{h-k}).$$

Now for large h (Proposition 8.1) $H^1(X, \mathcal{F}^{h-k}) = 0$. Moreover $\mathrm{Ker}\,\beta_h^* = s\Gamma(X, \mathcal{F}^{h-k})$ for $h \geqq k$.

Taking $M_h = \Gamma(X, \mathcal{F}^h)$, $N_h = \Gamma(Y, \mathcal{G}^h)$, and $f_h = \beta_h^*$ in the previous lemma, we obtain the contention of this proposition.

For the case of a non-compact space we have the following criterion of finite generation.

THEOREM 8.1. *Let X be a normal irreducible 0-concave complex space of dimension $\geqq 3$. Let $\widetilde{X}$ be the minimal 0-normal compactification of X. Let L be a holomorphic line bundle on X such that for every positive dimensional subvariety A of X there exists an $s \in \Gamma(X, L^h)$ (for some $h = h(x)$) such that s vanishes at some point of A but not identically on A. Assume that at every point of $\widetilde{X} - X$ the local ring is semi-factorial. Then $\mathcal{A}(X, L)$ is a finitely generated $\mathbf{C}$-algebra.*

PROOF. Let $\mathcal{L}$ be the locally free sheaf of rank 1 on X associated to L. Since X is 1-normal, $\mathcal{L}^{[1]} = \mathcal{L}$. Therefore, by Corollary 6.1, $\mathcal{L}$ can be extended to a coherent analytic sheaf on the whole of $\widetilde{X}$.

Since the normalization of $\widetilde{X}$ is again a minimal 0-normal compactification of X (by Proposition 5.1), it follows that $\widetilde{X}$ must be normal. By Proposition 6.3 (*a*), L can be extended to a holomorphic line bundle $\widetilde{L}$ over $\widetilde{X}$.

Let $\varphi : X \longrightarrow (a, \infty)$ be an exhaustion function for X and let $a' \in (a, \infty)$ be such that, on $\{\varphi < a'\}$, φ is strongly 0-pseudoconvex.

Set $Y = \{\varphi < a'\}$ and $\widetilde{Y} = (\widetilde{X} - X) \cup Y$ so that $\widetilde{Y}$ is a Stein completion of the (0, 0)-convex-concave space Y. For the sheaf $\widetilde{\mathcal{L}}$ we can take the associated sheaf to $\widetilde{L}$ so that, because X and $\widetilde{X}$ are 0-normal, prof $\mathcal{L} \geqq 2$ and prof $\widetilde{\mathcal{L}} \geqq 2$. Applying Proposition 3.1, we get for every $h \geqq 0$ that the restriction map

$$\Gamma(\widetilde{Y}, \widetilde{\mathcal{L}}^h) \longrightarrow \Gamma(Y, \mathcal{L}^h)$$

is bijective.

It follows that the restriction map

$$\Gamma(\widetilde{X}, \widetilde{\mathcal{L}}^h) \longrightarrow \Gamma(X, \mathcal{L}^h)$$

is an isomorphism for all $h \geqq 0$. In particular we get

$$(*) \qquad \mathcal{A}(X, L) \simeq \mathcal{A}(\widetilde{X}, \widetilde{L}).$$

We are going to prove that $\widetilde{L}$ is positive by using the criterion of Grauert given by the Lemma on page 347 of [8]. In view of that criterion $\widetilde{L}$ is positive if for any nowhere discrete analytic subset A of $\widetilde{X}$ we can find $h \geqq 1$ and an $s \in \Gamma(\widetilde{X}, \widetilde{L}^h)$ which vanishes somewhere on A but not identically on A.

Now $\widetilde{X}$ being a minimal 0-normal compactification of X, by Proposition 5.1, we must have $A \not\subset \widetilde{X} - X$ so that $A \cap X$ is a positive-dimensional subspace of X. By the assumption there exists $s \in \Gamma(X, L^h)$ vanishing somewhere on $A \cap X$ but not identically on $A \cap X$. The unique extension $\widetilde{s} \in \Gamma(\widetilde{X}, \widetilde{L}^h)$ shows that Grauert criterion is satisfied for $\widetilde{L}$.

It is now enough to apply Proposition 8.2 in view of the isomorphism (*).

COROLLARY. *Let X be irreducible 0-concave of dimension $\geqq 3$ admitting a minimal 0 normal compactification $\widetilde{X}$ with the property that at each point of $\widetilde{X} - X$ the local ring is semi-factorial. If L is a holomorphic line bundle on X such that the graded ring $\mathcal{A}(X, L)$ either separates points or gives local coordinates on X, then $\mathcal{A}(X, L)$ is a finitely generated $\mathbb{C}$-algebra.*

In particular if X is a manifold of dimension $\geqq 3$, connected and 0-concave, whose minimal 0-normal compactification $\widetilde{X}$ is still a manifold and if, moreover, X admits a holomorphic line bundle L as in this corollary, then $\widetilde{X}$ is projective algebraic and $\mathcal{A}(X, L)$ is finitely generated.

REMARK. *If the assumption on the semi-factoriality of the local rings of $\widetilde{X} - X$ is replaced with the assumption of prefactoriality in the preceeding theorem and corollary, the conclusion will be that for a convenient integer $k > 0$ the graded ring $\mathcal{A}(X, L^k)$ is a finitely generated* $\mathbf{C}$*-algebra.*

18. The last theorem and corollary are not very satisfactory criteria for the reason that the type of singularities one expects in the compactification $\widetilde{X}$ of X are (by the very construction of $\widetilde{X}$) the singularities one obtains by the reduction of a normal 0-convex space. These singularities are isolated normal but, apart from that, they do not present any other special feature; in particular we cannot expect their local rings to be, in general, semifactorial.

Another (equally unsatisfactory) criterion for the finite generation of $\mathcal{A}(X, L)$ was given in [4, Corollary to Proposition 10]. Comparing that to the present criterion one may ask the following question.

Let V be an isolated singularity obtained by « blowing down » in a complex manifold a compact analytic subset of codimension $\geqq 2$. Is the local ring of V prefactorial? The answer is negative as it is shown by the following example of David Prill.

19. *The example of Prill.* Let M be a compact Riemann surface and let $\pi: V \to M$ be a holomorphic vector bundle over M whose bundle space V is 0-convex and has fiber dimension $k \geqq 2$. We identify M with the 0-section of V. For instance we can take the sum of k copies of the negative of the line bundle of the hyperplane section of some projective imbedding of M.

By reduction we can find a normal complex Stein space X, a point $x \in X$, and a holomorphic surjection $\alpha: V \to X$ such that $\alpha: V - M \xrightarrow{\sim} X - \{x\}$ while $\alpha^{-1}(x) = M$.

PROPOSITION 8.3. *The local ring of X at x is not prefactorial.*

PROOF. (*a*) Fix a C^∞ hermitian metric on the fibers of V so that, for $v \in V$, $\|v\|$ denotes the length of that vector. Set for $r > 0$

$$V_r = \{v \in V \mid \|v\| < r\},\ V_r' = V_r - M, \pi_r = \pi \mid V_r.$$

We have a commutative diagram

$$\begin{array}{ccc} H^1(M, {}_M\mathcal{O}^*) & \xrightarrow{\delta} & H^2(M, \mathbf{Z}) \\ \downarrow \pi_r^* & & \downarrow \pi_r^* \\ H^1(V_r', {}_V\mathcal{O}^*) & \xrightarrow{\delta} & H^2(V_r', \mathbf{Z}), \end{array}$$

where δ is the map which associate to each line bundle its Chern class and ${}_M\mathcal{O}^*$ and ${}_V\mathcal{O}^*$ are respectively the (multiplicative) sheaves of germs of nowhere zero holomorphic functions on M and V. In the following diagram

$$\begin{tikzcd} H^s(V_r, V_r'; \mathbf{Z}) \arrow[r] & H^s(V_r, \mathbf{Z}) \arrow[r, "\beta"] & H^s(V_r', \mathbf{Z}) \\ H^s(M, \mathbf{Z}) \arrow[u, "\gamma"] \arrow[urr, "\pi^*"'] & & \end{tikzcd}$$

the row is exact, γ induced by π_r is an isomorphism (since V_r is contractible onto the 0-section), β is induced by $V_r' \subset V_r$, and by Thom isomorphism (cf. [21, Lemma 5.7. 16 (*a*)]),

$$H^s(V_r, V_r'; \mathbf{Z}) \simeq H^{s-2k}(M, \mathbf{Z}).$$

In particular for $s = 2$, since $k \geqq 2$, we conclude that β is injective and therefore

$$\pi^* : H^2(M, \mathbf{Z}) \longrightarrow H^2(V_r', \mathbf{Z})$$

is injective. This implies that:

if ξ is a holomorphic line bundle on M with non-zero Chern class, then $\pi_r^{-1}(\xi)$ on V_r' is also a holomorphic line bundle with non-zero Chern class.

(*b*) Let $p \in M$, since M is of dimension 1, $\{p\}$ is of pure codimension 1. Therefore $Z = \pi^{-1}(p)$ is of pure codimension 1 in V and $A = \alpha(Z)$ is of pure codimension 1 in X.

Note that A is locally irreducible at x since, for every r, $Z \cap V_r'$ is connected.

The ideal P associated to A in the local ring of X at x is therefore prime and of height 1.

We will show that P is not the radical of a principal ideal.

Suppose, if possible, that P is the radical of a principal ideal. In some neighborhood U of x, $A = \{y \in U \mid h(y) = 0\}$ with h holomorphic on U. Let $f = h \circ \pi$.

Let r be so small that $\pi(V_r) \subset U$ and let us consider on V_r' the following sheaves: $\mathcal{I} = \mathcal{O}f$ (where $\mathcal{O}$ is the structure sheaf of V) and the sheaf $\mathcal{J}$ of ideals defined by the anlaytic set $Z \cap V_r'$.

Let $z \in Z \cap V_r'$. For some integer $n > 0$ we must have

$$\mathcal{I}_z = \mathcal{J}_z^n.$$

The analytic set in V_r' where $\mathcal{I}$ and $\mathcal{J}^n$ disagree is a proper analytic subset of Z and therefore it is of codimension $\geqq 2$. But $\mathcal{I}$ and $\mathcal{J}$ are both locally free thus $\mathcal{I} = \mathcal{J}^n$ at each point of V_r'.

Now $\mathcal{I} \simeq \mathcal{O}$. So $\mathcal{J}^n \simeq \mathcal{O}$. If ξ is the holomorphic line bundle over M defined by the divisor p, $\pi_r^{-1}(\xi)$ is the holomorphic line bundle associated with the divisor Z on V_r'. By the above remark $\pi_r^{-1}(\xi^n)$ is trivial. But then ξ^n should be trivial on M. This is not the case since $\delta(\xi^n) = \pm n \neq 0$.

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