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CLUSTER DIRECTIONS OF EUCLIDEAN SETS

by JACK G. CEDER

An uncountable subset A of Euclidean 3-space, E^3, (or of E^2) is said to have a *cluster direction* L at x provided $x \in A$ and L is a line containing x such that whenever S is a sphere centered at x and K is an open convex cone which has vertex x and contains a half-ray of L, the set $A \cap S \cap K$ is uncountable. The main result of this paper, Theorem 3, asserts that a countable set C can be deleted from each uncountable subset A of E^3 (and hence E^2 too) so that at each point of $A - C$ there is a cluster direction. This result extends and generalizes several results found in [1], [3] and [4] and, for instance, immediately implies that each convex body has at most countably many corners. We also apply this basic result to the graphs of real-valued functions of two variables and show that the points at which there are cluster directions in all directions are numerous in the sense of category and measure.

First we establish the following conventions and definitions. All functions considered in the sequel will be real-valued and, moreover, a function will be identified with its graph. We say that a point x of an uncountable planar set A is *a* (*bilateral*) *condensation point of* A if each open square (resp. each vertical half of an open square) centered at x contains uncountably many points of A. Finally, the *left derived set* $D_L(f, x)$ and the *right-derived set* $D_R(f, x)$ of a function f (defined on a set of reals) at x are defined to be the sets of all possible sequential limits (as extended real numbers) of the difference quotient $\dfrac{f(y) - f(x)}{y - x}$ as y approaches x from the left and the right respectively.

Theorem 3 generalizes the fact (see Lemma 1 of [3] that a countable set may be deleted from each function which has an uncountable domain of reals so that each point of the new function is a bilateral point of

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condensation of that function. Moreover, Theorem 3 also strengthens the following refinement of a result due to Bagemihl [1], which will be used in the sequel.

THEOREM 1. (Ceder [3]) Let f be a function defined on an uncountable set A of real numbers. Then there exists a countable set C such that if g is the restriction of f to $A - C$, then for each $x \in A - C$

$$D_L(g, x) \cap D_R(g, x) \neq \emptyset$$

and each point of g is a bilateral point of condensation of g.

If g is a function each point of which is a bilateral point of condensation of g and if $\lambda \in D_L(g, x) \cap D_R(g, x)$ and λ is neither $+\infty$ nor $-\infty$, then obviously the line of slope λ which passes through $(x, g(x))$ is a cluster direction of g. However, if $+\infty$ or $-\infty$ is the only common point of the left and right-derived sets then the vertical line through $(x, g(x))$ may not be a cluster direction of g. For example, if g is given by: $g(x) = 0$ when $x < 0$, $g(x) = 2$ when $x > 0$ and $g(0) = 1$, then $+\infty \in D_L(g, x) \cap D_R(g, x)$ yet g has no cluster direction at 0.

As the first step leading to the proof of Theorem 3 we have the following lemma.

LEMMA 1. Let f be a function defined on an uncountable set A of the reals. Then there exists a function $g \subseteq f$ with uncountable domain such that at each point of g there is a cluster direction.

PROOF: If the range of f is countable, then there exists an uncountable subset D of A on which f is constant. In this case let B consist of the bilateral condensation points of D. Then the restriction of f to B will be the desired function g.

If, on the other hand, the range of f is uncountable we can find an uncountable subset of A on which f is one-to-one. So let us assume without loss of generality that f is one-to-one on A itself. First of all, apply Theorem 1 to f to obtain an $h \subseteq f$ with uncountable domain H such that $D_L(h, x) \cap D_R(h, x) = \emptyset$ for each $x \in H$. Next apply Theorem 1 to the function h^{-1} to obtain a function $k \subseteq h^{-1}$ with uncountable domain K such that $D_L(k, y) \cap D_R(k, y) \neq \emptyset$ for all $y \in K$.

We consider $g = k^{-1}$ and let $(x, y) \in g$. Suppose that there is no cluster direction of g at (x, y). Then clearly there is no cluster direction of k at (y, x) This means that $D_L(k, y) \cap D_R(k, y) \subseteq \{+\infty, -\infty\}$. But this in turn implies that h has no cluster direction at x, which is a contradiction. Thus g is the desired function.

Next we prove the main result for the planar case. As a consequence of this, Lemma 1 ean be strengthened to assert that the set difference $f - g$ is actually countable.

THEOREM 2. Let A be an uncountable subset of the plane. Then there exists a subset B of A such that $A - B$ is countable and at each point of B there is a cluster direction.

PROOF: Let $\mathcal{G}$ consist of all uncountable subsets G of A for which G has a cluster direction at each of its points. First we show that $\mathcal{G}$ is non-empty. Denote by π the projection mapping onto the x-axis. If $\pi(A)$ is countable we may find an $x_0 \in \pi(A)$ for which $\pi^{-1}(A)$ is uncountable. Letting G consist of the bilateral condensation points of $\pi^{-1}(x_0)$ we have $g \in \mathcal{G}$. On the other hand, if $\pi(A)$ is uncountable let F be a choice function for the family $\{\pi^{-1}(x) : x \in \pi(A)\}$ Now apply Lemma 1 to the function F to obtain $G \subseteq F$ for which $G \in \mathcal{G}$. Hence, in either case $\mathcal{G}$ is non-empty.

By Zorn's Lemma there will exist a maximal disjoint subfamily $\mathcal{M}$ af $\mathcal{G}$. That is, if $\mathcal{B} \subseteq \mathcal{G}$ and each two distinct members of $\mathcal{B}$ have empty intersection, then $\mathcal{B} \subseteq \mathcal{M}$. Define $B = \cup \mathcal{M}$. If $A - B$ were uncountable then the proof of the fact that $\mathcal{G} \neq \emptyset$ yelds an $H \subseteq A - B$ for wich $H \in \mathcal{G}$. Then $\mathcal{M} \cup \{H\}$ would belong to $\mathcal{G}$, which contradicts the maximality of $\mathcal{M}$. Therefore $A - B$ is countable and since $B \in \mathcal{G}$ the theorem is proved.

When specialized to a function having an uncountable domain, Theorem 2 asserts that there is a cluster direction at all but a countable number of points. Accordingly, Theorem 2 implies Theorem 1 and also the fact [3] that each function with an uncountable real domain has, except for countably many points a bilateral condensation point at each of its points. As another immediate consequence of Theorem 2 we have

COROLLARY 1. A planar convex body can have at most countably many corners.

Surprisingly, it can happen that a function has only « one » cluster direction. For example, choose f to be a strictly increasing function on $[0, 1]$ such that $f' = 0$ a. e.. Let $A = \{x : f'(x) = 0\}$. Then A is uncountable and the restriction of f to A, g, is one to-one on A and g has a zero derivative everywhere in A. Therefore, g has only horizontal cluster directions.

Now we proceed to extend the planar case, Theorem 2, to the spatial case.

THEOREM 3. Let A be an uncountable subset of E^3. Then there exist a subset B of A such that $A - B$ is countable and at each point of B there is a cluster direction.

PROOF: Since the non-condensation points of A are countable (see Lemma 2 of [4]), we can assume without loss of generality that each point of A is a point of condensation. We divide the proof into two cases.

Case I: Suppose that there is no uncountable subset of A which is coplanar.

Let $\{R_m\}_{m=1}^{\infty}$ denote the sequence of all rational planes through the origin (that is, a plane one of whose sets of direction numbers is a rational triple) Let $S_m(x)$ denote the open sphere of radius $1/m$ about x. Let C_m^n be the set of all points $x \in A$ for which there exists a rational plane R_n and sphere $S_m(x)$ such that $S_m(x) \cap R_n$ is countable and one of the two components of $S_m(x) - R_n$ has empty intersection with A. Put $C = \bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty} C_m^n$. Clearly each C_m^n can not have a point of condensation. Therefore each C_m^n is countable and C itself is countable.

Now put $A' - A - C$ and denote by p_m the projection mapping onto the plane R_m. Let $V = \{V_k\}_{k=1}^{\infty}$ be a countable basis for the Euclidean topology in E^3. To each set of the forn $p_m(V_k \cap A')$, which is either empty or uncountable, we may apply Theorem 2 to obtain $p_m(V_k \cap A') = B_k^m \cup D_k^m$ where D_k^m is countable and at each point of B_k^m there is a cluster direction in R_m. Next put $B = A' - \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} D_k^m$.

Then B has the following properties: (1) $A - B$ is countable; (2) each point of B is a condensation point; (3) $B \cap C = \varnothing$; (4) there is no uncountable subset of B which is coplanar; and (5) for any $x \in B$ and open set $V(x)$ and rational plane R, the projection of $B \cap V(x)$ has a cluster direction in R at the projection image of x.

If K is any open convex cone with vertex x, we say that K is a B-cone if for each sphere S centered at x we have $K \cap S \cap B \neq \varnothing$ (or equivalently $K \cap S \cap B$ is uncountable by (2)).

Choose any $x \in B$ and let P_1, P_2 and P_3 be the tranlates of the coordinate planes which pass through x. Consider the four cones which are the components of $E^3 - P_1 - P_2$. Among these four cones there will exist a pair consisting of « opposite » B cones. If this were not the case, it is easy to see that we must have (1) $x \in C$ or (2) there exist some sphere $S(x)$ whose projection upon P_3 has no cluster direction at x. Denote these two opposite B cones by K_1 and K_1^*. Next consider the four cones which are the components of $(K_1 \cup K_1^*) - P_3$. By the same argument there must exist a pair of opposite B-cones among these four cones. Denote these cones by K_2 and K_2^* where $K_2 \subseteq K_1$.

Continuing in this fashion by induction we will obtain sequences of cones $\{K_n\}_{n=1}^{\infty}$ and $\{K_n^*\}_{n=1}^{\infty}$ such that (*a*) $K_{n+1} \subseteq K_n$ and $K_{n+1}^* \subseteq K_n^*$ for each n; (*b*) K_n and K_n^* are opposite B-cones at x; (*c*) if U denotes the surface of the unit sphere centered at x, $\{\overline{K}_n \cap U\}_{n=1}^{\infty}$ and $\{\overline{K}_n^* \cap U\}_{n=1}^{\infty}$ are nested sequences of closed sets in U whose diameters tend to zero as $n \to \infty$.

By property (*c*) we can find a line L through x such that

$$\bigcap_{n=1}^{\infty} (K_n \cup \{x\} \cup K_n^*) = L.$$

By property (*b*) it is obvious that L is a cluster direction of B at x. This finishes the proof of Case I.

Case II: Now consider the case of an arbitrary uncountable set A. Let $\mathcal{L}$ be the family of all uncountable subsets D of A such that there exists a plane P such that $D \subseteq P$ and at each point of D there is a cluster direction. Assuming Case I does not hold, choose by Zorn's Lemma $\mathcal{M}$ to be a maximal disjoint subfamily of $\mathcal{L}$ and put $N = A - \cup\mathcal{M}$. Clearly each point of $\cup\mathcal{M}$ has a cluster direction. In case N is countable, $\cup\mathcal{M}$ will serve as the desired set B. On the other hand, when N is uncountable N has, by the maximality of $\mathcal{M}$, the property that it contains no uncountable coplanar subset. Therefore we may apply Case I to obtain a $C \subseteq N$ such that $N - C$ is countable and each point of C has a cluster direction. Now put $B = C \cup (\cup\mathcal{M})$ to get the desired set B. This completes the proof of the theorem.

As an immediate consequence of Theorem 3 we have

COROLLARY 2. A convex body in E^3 can have at most countably many corners.

In sets of second category or of positive Lebesgue outer measure in E^2 or E^3 it turns out that there are many points such that *each* line through the point is a cluster direction. More precisely we have the following result.

THEOREM 4. Let A be an uncountable subset of E^2 (or E^3). Then the subset of A consisting of all points which do not have cluster directions in all directions through the point is a subset of a first category, null F_σ-set in E^2 (resp. E^3).

PROOF: We carry out the proof only for the E^3 case and note that the E^2 case is similar. Since the non-condensation points of A are countable we may without loss of generality assume that each point of A is a condensation point. Let R and R^+ denote the set of all rational numbers and the set of all positive rational uumbers respectively. Let $\mathcal{J} = R \times R \times \times R - \{(0, 0, 0)\}$. Let C consist of all points in A at which at least one direction is not a cluster direction. For $T \in \mathcal{J}$, $\theta \in R^+ \cap (0, \pi)$ and $r \in R^+$ let $C(T, \theta, r)$ denote the set of all points $z \in A$ such that $A \cap S_r(z) \cap K(T, \theta, r) = \emptyset$ where $S_r(z)$ is the open sphere of radius r centered at z and $K(T, \theta, r)$ is the right circular cone with vertex z, with axis having T as a set of direction numbers and with θ as its angle. Clearly.

$$C = \cup \{C(T, \theta, r) : T \in \mathcal{J}, r \in R^+, \theta \in R^+ \cap (0, \pi)\}.$$

To complete the proof it suffices to show that the closure of each $C(T, \theta, r)$ is a nowhere dense, null set. First of all, it is obvious that each $C(T, \theta, r)$ is nowhere dense. Let us now show that any $C(T, \theta, r)$ has measure zero. Without loss of generality we may assume that $T = (0, 0, 1)$ and $r > 1$. Let B be the closure of $C(T, \theta, r)$. Note that if $z \in B$, then $B \cap K(T, \theta, r) \cap \cap S_r(z) = \emptyset$.

We will now that B has Lebesgue measure zero. First enumerate the set of all closed intervals with consecutive integral endpoints as $\{I_n\}_{n=1}^{\infty}$. Now put $B_n = \{z \in B : z_3 \in I_n\}$. It will suffice to show that each B_n has measure zero. Choose any n. Then clearly B_n is the graph of some function f with some domain $D \subseteq E^2$ and range contained in I_n. Since (the graph of) f is closed its projection D upon the xy-plane will be a K_σ set. It also follows that $f^{-1}(F)$ is a $F_{\sigma\delta}$ set for each closed $F \subseteq I_h$. Therefore, f is a Borel measurable function of class 2 relative to the K_σ set D. Consequently f can be extended to a Borel measurable function g of class 2 with domain all of E^2 (see p. 341 of [7]). Hence, if is a subset of a Lebesgue measurable function g. However, the graph of a Lebesgue measurable function has measure zero, as can be seen quickly by considering sequences of simple functions approaching from above and below. Hence, f or B_n has measure zero. This complete the proof of the theorem.

A different proof of Theorem 4 minus the F_σ stipulation can be based upon Hunter's proof [6] of a result of Young [9] which asserts that for a real valued function f with domain a planar set B there exists a first category null set A such that for each $x \in B - A$ the cluster set of f at x is the intersection of the sectoral cluster sets of f at x. (See [6]).

If we specialize the set A in the above Theorem to be a function from E^2 into E^1, then Theorem 4 doesn't give much information. For example, the graph of a continuous function is itself a first category null set. A much more natural question to consider is whether at a given point there is a cluster direction in each vertical plane through the point. We conjecture that for all functions from E^2 to E^1 the domain of the set of points through which there exists a vertical plane containing no cluster direction is indeed a first category null set in the plane. However we are only able to prove it for continuous functions as shown by Theorem 5 below. If this conjecture is true it implies easily Young's theorem.

Before proving Theorem 5 we need the following lemma of Marcus [8], a direct proof of which may be patterned after that of Theorem 7.

LEMMA 2. (Marcus [8]) . If G is a planar second category set having the property of Baire, then there exists a horizontal line L such that $L \cap G$ is residual in $L \cap G$.

Lemma 2 is not valid for arbitrary second category sets as shown by the following example.

EXAMPLE 1. There exists a second category set in the plane which intersects each line in at most two points. To show the existence of such a set, let $\mathcal{C}$ be the collection of all perfect sets which are contained in some circular arc. Let $\mathcal{C}$ be well ordered by the ordinal c so that $\mathcal{C} = \{C_\alpha\}_{\alpha < c}$. Given a planar set A let $L(A)$ denote the set of all points collinear with a pair of points of A. Pick $z_0 \in C_0$ arbitrarily. By induction suppose $\beta < c$ and we have chosen z_α for each $\alpha < \beta$ such that $z_\alpha \in C_\alpha$ and no three points of $\{z_\xi\}_{\xi < \alpha}$ are collinear. Since the set $C_\beta - L(\{z_\alpha : \alpha < \beta\})$ is non-empty we can pick a point z_β in it. Clearly no three points of $\{z_\alpha : \alpha \leq \beta\}$ are collinear. Hence, letting $Z = \{z_\alpha\}_{\alpha < c}$ we have that Z intersects each member of $\mathcal{C}$ and no three points of Z are collinear.

If Z were of the first category, then Z would be a subset of a first category F_σ set, whose complement B is a residual G_δ which contains no perfect subset lying in a circular arc. Consider the conformal mapping F given by $F(z) = (l - i) + i/z$. It maps each circle tangent to the y-axis at the origin into a horizontal line. If $H = E^2 - \{z : z_1 = 0\}$, then F restricted to H is a homeomorphism of H onto $E^2 - \{z : z_2 = 0\}$. Therefore F maps the residual G_δ, $B \cap H$, onto a residual G_δ. Applying Lemma 2 there exists a horizontal line L which intersects $F(B \cap H)$ in an uncountable set. Therefore $B \cap H$ intersects the circular arc $F^{-1}(L)$ in an uncountable Borel set. Consequently, there exists a $C \in \mathcal{C}$ such that $C \subseteq F^{-1}(L) \cap B \cap H \subseteq B$. This is a contradiction so that Z must be of the second category.

THEOREM 5. Let f be a continuous function from E^2 into E^1. Then there exists a first category, null $G_{\delta\sigma}$ set C of E^2 such that for each $x \notin C$

and vertical plane P passing through $(x, f(x))$ f has a cluster direction lying in P at the point $(x, f(x))$.

PROOF. If z is a point of f for which there exists a vertical plane Q through z having no cluster direction lying in Q at z, then we can find a rational vertical plane P through z for which there is no cluster direction at z in Q. (If all rational planes through z had cluster directions, one could find a sequence of planes $P_n \longrightarrow Q$ with P_n having a cluster direction λ_n and $\lambda_n \longrightarrow$ some λ. Then clearly λ is a cluster direction in Q).

If P is any vertical rational plane passing through the origin, let P_z be the translate of P which passes through z. For P a vertical rationai plane and $z \in f$ let S denote the unit circle in P_z centered at z and let S_R and S_L be the right and left, closed, vertical half circles of S (right and left are determined relative to the plane P). Define $D_R(f, P_z)$ to be the set of all $\lambda \in S_R$ such that the line in P_z passing through z and λ is a cluster direction for the set $f \cup I$ where I is the interval joining z to $-\lambda$. Similarly we define $D_L(f, P_z)$.

From the continuity of f $D_R(f, P_z)$ and $D_L(f, P_z)$ will be closed intervals contained in S_R and S_L, respectively. Moreover, let $-D_L(f, P_z) = \{-\lambda : \lambda \in D_L(f, P_z)\}$. If f has no cluster direction in P_z at z then $-D_L(f, P_z) \cap D_R(f, P_z) = \emptyset$.

Now let C be the set of all points $x \in E^2$ such that there is no cluster direction at $(x, f(x))$ in some vertical plane through $(x, f(x))$. For P a rational vertical plane, let $C(P)$ consist of those $x \in C$ such that P_z contains no cluster direction at $z = (x, f(x))$. Clearly $C = \cup\{C(P) : P$ a rational vertical plane$\}$. If $x \in C(P)$ and $z = (x, f(x))$ we can find disjoint closed rational intervals I and J in S_R such that $D_R(f, P_z) \subseteq I$ and $-D_L(f, P_z) \subseteq J$. For a pair of disjoint closed rational intervals, (I, J) in S_R let $C(P, I, J)$ denote the set of all $x \in C(P)$ for which $D_R(f, P_z) \subseteq I$ and $-D_L(f, P_z) \subseteq J$ where $z = (x, f(x))$. Since $C(P)$ is the union of the countable family of such $C(P, I, J)$ it will suffice to show that each $C(P, I, J)$ is a first category, null G_δ set in E^2.

Let us consider a specific $C(P, I, J)$ and suppose $I = [a, b]$ and $J = [c, d]$ where $a < b$ and $c < d$ in the positive orientation in S_R. For n a positive integer define

$$W_n^1 = \left\{ z_1 : \text{there exists } e \text{ such that } e \leq a - \frac{1}{n} \right\}$$

$$\text{and } e \in D_R(f, P_z)$$

$$W_n^2 = \left\{ z_1 : \text{ there exists } e \text{ such that } b + \frac{1}{n} \leq e \text{ and } e \in D_R(f, P_z) \right\}$$

$$W_n^3 = \left\{ z_1 : \text{ there exists } e \text{ such that } e \leq c - \frac{1}{n} \text{ and } e \in -D_L(f, P_z) \right\}$$

and

$$W_n^4 = \left\{ z_1 : \text{ there exists } e \text{ such that } d + \frac{1}{n} \leq e \text{ and } e \in D_L(f, P_z) \right\}$$

Clearly each W_i^n is a closed set disjoint from $C(P, I, J)$. Moreover, it is not difficult to see that

$$C(P, I, J) \subseteq \left(\bigcup_{i=1}^{4} \bigcup_{n=1}^{\infty} W_n^i \right) \cap C(P, I, J).$$

From this it follows that $C(P, I, J)$ is a G_δ set.

If $C(P, I, J)$ were dense in some disc D, then each W_n^i would be nowhere dense in D and $C(P, I, J)$ would be a residual G_δ in the open set D. Now applying Lemma 2 we have for some z $P_z \cap D \cap C(P, I, J)$ is of second category (hence uncountable) in the line $P_z \cap D$. Next apply Theorem 2 to the function f restricted to $P_z \cap D$ to obtain a cluster direction at some point $(w, f(w))$ for $w \in P_z \cap D \cap C(P, I, J)$. This is a contradiction so that $C(P, I, J)$ is nowhere dense.

Now we proceed to show that $C(P, I, J)$ is of measure zero. For this we need the following lemma, whose proof immediately follows from Fubini's Theorem: Lemma: Let B be a Lebesgue measurable subset of E^2. If for each horizontal line L, $L \cap B$ has zero measure in L, then B has measure zero. By this lemma, if the G_δ set $C(P, I, J)$ had measure zero there would exist a plane P_z such that $P_z \cap C(P, I, J)$ is uncountable. Now applying Theorem 2 as above we obtain a contradiction.

This finishes the proof of the theorem.

We can also consider cluster directions in a more restricted sense. For example, we could stipulate that the cluster direction L is also a cluster direction for $A \cap L$ or for $P \cap P$ for some plane P containing L. Obviously the analogues of Theorem 2 and 3 do not hold for these notions. However, we do have the following result.

THEOREM 6. (Bruckner and Rosenfeld [2]). Let B be a Lebesgue measurable subset of the plane. Then for almost all points x of B, L is a cluster direction for $B \cap L$ for almost all lines L passing through x.

Example 1 shows that there is no analogue of Theorem 6 for arbitrary second category sets. However, for second category sets which have the property of Baire we have the following theorem.

THEOREM 7. Let B be a planar set of second category having the property of Baire. Then there exists a first category set A such that for each $x \in B - A$, there exists a set D everywhere of second category in the unit circle C_x centered at x such that each line L passing through x and a point of D is a sluster direction for $L \cap B$.

PROOF. Clearly it suffices to prove the theorem for a set B residual in some open set, which, without loss of generality, we may assume to be the whole plane. Let $E^2 - B = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a no-where closed set. Let $\{V_k\}_{k=1}^{\infty}$ be an enumeration of all the rational open intervals in $(0, +\infty)$. We shall show that for each $x \in B$ there exists a set D everywhere of second category in C_x such that for each line L passing through x and a point of D $L \cap B$ is everywhere of second category in L.

Assume the contrary, so that there exists an $x \in B$, an arc J of C_x and a first category set $A \subseteq J$ such that for each line L passing through x and a point of $J - A$, $L \cap B$ is of first category in some subinterval of L. Let L_y denote the open half ray emanating from x and passing through $y \in C_x$. For positive integers n and k let $C_{n,k}$ consist of all points $y \in C$ for which $V_k \subseteq F_n$ where V_k is considered in L_y. Obviously, we have $J - - A = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} C_{n,k}$, whence it follows that some $C_{n,k}$ is dense in some subinterval I of J. Since F_n is closed, F_n contains the open « sector » determined by V_k and the two rays through x and the endpoints of I. This contradicts the fact that F_n is no-where dense. Therefore the theorem is proved.

It is unknown whether Theorem 7 can be improved to assert that D is residual.

It is also unknown whether the analogue of Theorem 5 is valid for cluster directions in the restricted sense: that is, given a continuous function F from E^2 into E^1 the set of all $x \in E^2$ for which there exists a vertical plane P through $(x, F(x))$ such that $F \cap P$ has no cluster direction at $(x, F(x))$ is a first category null set. This proposition, however, is not valid for non-continuous functions as shown by

EXAMPLE 2. There exists a function F from E^2 into E^1 such that through each $z \in F$ there exists a vertical plane P containing no cluster direction of $F \cap P$. Let $\{L_n\}_{n=1}^{\infty}$ be an enumeration of all rational lines through the origin and let $\mathcal{L}$ be the familly of all translates of the members of the sequence $\{L_n\}_{n=1}^{\infty}$. According to a result of Davies [5] there exists a one-to-one function g from E^2 into $\mathcal{L}$ such that $x \in g(x)$ for all $x \in E^2$. Let $A_n = \{x \in E^2 : g(x) = L_n + x\}$. Define $F(x) = 2n$ whenever $x \in A_n$. Then clearly for each $z = (x, F(x))$ $F \cap P$ has no cluster direction where P is the plane $(L_n + x) \times E^1$ and $x \in A_n$. On the other hand it is clear that for each vertical plane P through $z \in F$ the horizontal line in P through z is a cluster direction for F.

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