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ON THE ASYMPTOTIC BEHAVIOR OF RESOLVENT KERNELS, SPECTRAL FUNCTIONS AND EIGENVALUES OF SEMI-ELLIPTIC SYSTEMS

YAKAR KANNAI (*)

Introduction.

This paper deals with the asymptotic properties of resolvent kernels, spectral functions and eigenvalues of systems of semi-elliptic differential operators.

The asymptotic behavior of spectral functions was first investigated by Carleman [11] for a class of second order elliptic operators. Carleman showed that this behavior is closely related to the asymptotic properties of the resolvent kernels of such operators. Later the problem was studied by many authors for more general elliptic operators (for references see Agmon [3] and Bergendal [8]). Denote by $e(x, y; t)$ the spectral function of a self-adjoint realization of a positive elliptic operator of order m, defined on an open set $\Omega \subset R^n$ (x, y are points in Ω, t is a real number). Gårding [16] proved that

$$e(x, x; t) \underset{t \to \infty}{=} c(x)\, t^{\frac{n}{m}} + o\left(t^{\frac{n}{m}}\right), \tag{0.1}$$

where $c(x)$ depends on the coefficients of the operator. Gårding has also shown that if the differential operator has constant coefficients then the

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remainder term $o(t^{\frac{n}{m}})$ in (0.1) may be replaced by the term $0(t^{\frac{n-1}{m}})$. In some special cases of operators which have discrete spectrm it was shown ([13], [14], [15]) that one can integrate (0.1) over Ω and get

$$\sum_{\lambda_i \leq t} 1 \underset{t\to\infty}{=} ct^{\frac{n}{m}} + o(t^{\frac{n}{m}}), \tag{0.2}$$

where $\{\lambda_i\}$ is the sequence of eigenvalues, each repeated according to its multiplicity. Remainder estimates in the asymptotic formulas (0.1) for operators with variable coefficients were known in special cases only (Avakumovic [6]).

Agmon ([1], [2], [3]) developed a powerful method for the study of resolvent kernels which makes it possible to deduce asymptotic formulas such as (0.1) and (0.2) in very general situations. He also found (in [1] and in other unpublished works) that by a close investigation of resolvents in the complex plane one can derive the asymptotic formula

$$e(x, x; t) = ct^{\frac{n}{m}} + 0(t^{\frac{n-\theta}{m}}), \tag{0.3}$$

where θ is any positive number less than $\frac{1}{3}$ in the general case and less than $\frac{2}{3}$ if the principal part of the operator has constant coefficients.

In a joint paper of Agmon and the author [4] it was proved that (0.3) holds for general semi-bounded elliptic operators with any $\theta < \frac{1}{2}$, and that if the principal part has constant coefficients then (0.3) holds with any $\theta < 1$ (actually a somewhat more general result was proved in [4]). Identical results for the remainder were also obtained by Hörmander [11] using a different method. (Very recently, Hörmander proved that (0.3) holds in the general case with $\theta = 1$; his method would not, however, yield easily results for general semi elliptic systems).

In a recent work [5], Agmon developed asymptotic formulas with remainder estimates for the eigenvalues, extending the remainder estimates of [4]. Agmon also obtained results for elliptic systems and removed the assumption of semi-boundedness.

F. Browder [9] derived a formula similar to (0.1) for semi-elliptic differential operators. In [9] and [10] Browder obtained results similar to those of [3]; in particular, he proved the formula (0.2) for the elliptic case. Let us note here that, unlike the situation in the elliptic case, the highest order of differentiation which appear in a semi-elliptic operator varies with the direction; precise definitions are given in section 4 of the present paper and in [9]. (See also [7], [12], [17] and [26]).

Substantially, the present paper is an extension of [4] for semi-elliptic systems. The first chapter treats, as a natural preliminary study, the function spaces appropriate for the treatment of semi-elliptic systems and integral operators acting in these function spaces. The proof of the kernel theorem is similar to the proof of Agmon's kernel theorem [3] and to Agmon's unpublished proof of his matrix kernel theorem. In the first section of chapter two (section 4) the semi elliptic operators are defined and elementary properties of resolvent kernels and spectral functions associated with such operators are briefly discussed (this section corresponds to sections 1 and 2 in [4]). In section 5 it is shown that asymptotic formulas with remainder estimates for spectral functions follow from an accurate asymptotic expansion theorem (in the complex plane) for the resolvent kernels. The rest of chapter two is devoted to the proof of this expansion theorem. In section 6 relevant properties of fundamental solutions and related kernels are discussed. Since we are dealing here with systems of operators and not with single (scalar) operators, commutation does not necessarily lower the « order » of an operator; this difficulty exists also in the case of elliptic systems. The commutator technique, introduced and described in [4], is sharpened in section 7 so that it yields results also for systems. Some localization and comparison lemmas (some weaker versions of which suffice in the elliptic case) are proved in section 8. Using these tools we finish the proof of the asymptotic expansion theorem in sections 9. Thus, section 5, 6, 7 and 9 of the present paper are extension of sectionss 3, 4, 5 and 6 of [4], respectively.

The kernel theorems (of section 3) used in the semi-elliptic case do not yield sufficiently strong estimates *up to the boundary* for the spectral function. Consequently, one cannot obtain results on the distribution of eigenvalues by integration of the asymptotic formula for the spectral function (analog of (0. 1)) over Ω, as is possible in the elliptic case ([3], [5]), without further justification. In special cases one can use some of the older methods — of Gårding [14], [15], or of Ehrling [13]. In other cases the situation improves if some conditions on the geometry of Ω and on the « global » (up to the boundary) regularity of functions in the range of the resolvent are imposed. This global regularity is also connected with the geometry of Ω. The problem of estimating near the boundary is discussed in section 10 — the first section of Chapter Three. In section 11 the application to the eigenvalue distribution is made. The analog of formula (0.2) which is obtained is new even in the case of a single semi-elliptic operator. It is curious that all the new difficulties appear already in the derivation of the analog of (0.2) whereas the extension of the remainder estimates of [5] (for the eigenvalue distribution) to the semi-elliptic case presents no serious additional difficulties.

In the semi-elliptic case terms which do not belong to the principal part of the operator may influence the asymptotic expansion of the resolvent kernel in a manner which has no analog in the elliptic case. This, and the fact that we are dealing with systems of operators, makes several arguments and theorems appear less transparent than in the case of a single elliptic operator.

Several arguments have been sketched briefly in the present paper, if they are identical to the proof of known elliptic theorems. This has been done in order to keep the size of the paper under control.

I would like to thank Professor Agmon for his encouragement and advice and for making me acquainted with his ideas during all the stages of the preparation of this work. I would like to thank also Professor L. Hörmander for several very helpful suggestions, and Dr. R. C. Lacher or a topological discussion related to section 10.

CHAPTER ONE

THE FUNCTION SPACES AND INTEGRAL OPERATORS OPERATING IN THEM

1. Notation and definitions.

Let Ω be an open set in real space R^n with generic point $x=(x_1,\dots,x_n)$. We shall deal with p-vector functions:

$$u(x)=\begin{pmatrix} u_1(x) \\ \vdots \\ u_p(x) \end{pmatrix} \tag{1.1}$$

where the components $u_j(x)$ of $u(x)$ are complex valued functions defined on Ω. We denote by u (unless otherwise explicitly specified) a column vector; $u=(u_1,\dots,u_p)^{\sim}$ ($M^{\sim}$ denoting the transpose of a matrix M). We set:

$$|u(x)|=[\Sigma_{j=1}^{p}|u_j(x)|^2]^{\frac{1}{2}} \tag{1.2}$$

and we denote by $L_2(\Omega)^p$ the space of vector functions (1.1) having components $u_j(x)$ in $L_2(\Omega)$, $1\le j\le p$ (i.e., $L_2(\Omega)^p$ is the direct sum of p copies of $L_2(\Omega)$). Thus $L_2(\Omega)^p$ is a Hilbert space with the scalar product

$$(u,v)_{L_2(\Omega)^p}=\Sigma_{j=1}^{p}(u_j,v_j)_{L_2(\Omega)}\,. \tag{1.3}$$

Denote the $L_2(\Omega)^p$-norm of u by $\|u\|_{0,\Omega}$. The subclass of vector functions defined on Ω and belonging locally to $(L_2)^p$ will be denoted by $L_2^{\mathrm{loc}}(\Omega)^p$. For scalar functions we shall use also the notation $L_2(\Omega)\,(L_2^{\mathrm{loc}}(\Omega))$ instead of $L_2(\Omega)^1\,(L_2^{\mathrm{loc}}(\Omega)^1)$.

We use the standard notation for differentiation:

$$D_j=-i\frac{\partial}{\partial x_j},\; i=\sqrt{-1},\; D=(D_1,\dots,D_n);$$

for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \qquad D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}.$$

Set $D^\alpha u = (D^\alpha u_1, \dots, D^\alpha u_p)^\sim$.

Let $\mathsf{m} = (m_1, \dots, m_n)$ be a multi-index with *positive* components. We denote by $|\alpha : \mathsf{m}|$ the number $\Sigma_{i=1}^n \alpha_i/m_i$. If $m_i = 1$ for $1 \leq i \leq n$ then $|\alpha ; \mathsf{m}| = \alpha_1 + \dots + \alpha_n = |\alpha|$ coincides with the usual « length » of α. Otherwise it is a « reduced length».

We denote by $H_{\mathsf{m}}(\Omega)^p$ the subclass of vector functions $u \in L_2(\Omega)^p$ with distribution derivatives $D_j^i u \in L_2(\Omega)^p$ for all $i \leq m_j$, $1 \leq j \leq n$. $H_{\mathsf{m}}^{\mathrm{loc}}(\Omega)^p$ is defined to be the class of functions defined on Ω and belonging locally to H_{m}. In $H_{\mathsf{m}}(\Omega)^p$ we introduce the norm

$$\| u \|_{\mathsf{m}, \Omega} = \left\{ \int_\Omega [\, | u(x) |^2 + \Sigma_{j=1}^n | (D_j^{m_j} u)(x) |^2]\, dx \right\}^{\frac{1}{2}}. \tag{1.5}$$

Under this norm $H_{\mathsf{m}}(\Omega)^p$ is a Hilbert space.

In the special case $\Omega = R^n$ we consider also the spaces $H_{s\mathsf{m}}$ where s is an arbitray real number. These spaces are defined by means of Fourier transforms. As usual, we denote by $\widehat{u}(\xi)$ the Fourier transform of u,

$$\widehat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{-i\xi \cdot x} u(x)\, dx \tag{1.6}$$

where $\xi = \xi_1, \dots, \xi_n$, $\xi \cdot x = \xi_1 x_1 + \dots + \xi_n x_n$. The $s\mathsf{m}$-norm of u is defined by

$$\| u \|_{s\mathsf{m}, R^n} = \left[\int_{R^n} | \widehat{u}(\xi) |^2 (1 + \Sigma_{j=1}^n | \xi_j |^{2m_j})^s\, d\xi \right]^{\frac{1}{2}}. \tag{1.7}$$

If $s \geq 0$ then the subclass of functions $u \in L_2(R^n)^p$ with $\| u \|_{s\mathsf{m}, R^n} < \infty$ is a Hilbert space under the norm (1.7). This space will be denoted by $H_{s\mathsf{m}}(R^n)^p$. If $s < 0$ then $H_{s\mathsf{m}}(R^n)^p$ denotes the Hilbert space which is the $\| \;\|_{s\mathsf{m}}$-completion of $L_2(R^n)^p$. Obviously,

$$\| u \|_{0\mathsf{m}} = \| u \|_{0, R^n}$$

and

$$\| u \|_{1\,m} = \| u \|_{m,\,R^n}\,.$$

We note for further use the interpolation inequality

$$(1.8) \qquad \| u \|_{s_2 m,\,R^n} \leq (\| u \|_{s_1 m,\,R^n})^{\frac{s_3-s_2}{s_3-s_1}} (\| u \|_{s_3 m,\,R^n})^{\frac{s_2-s_1}{s_3-s_1}},$$

$$s_1 < s_2 < s_3\,, \quad u \in H_{s_3 m}(R^n)^p\,,$$

which follows immediately from Holder's inequality.

Let Ω be an open subset of R^n. In addition to (1.5) we shall use the following norms:

$$(1.9) \qquad \| u \|_{m_j,\,j,\,\Omega} = \left\{ \int_\Omega [\,| u(x) |^2 + | (D_j^{m_j} u)(x) |^2]\, dx \right\}^{\frac{1}{2}},$$

and semi-norms:

$$(1.10) \qquad | u |_{m_j,\,j,\,\Omega} = \left[\int_\Omega | (D_j^{m_j} u)(x) |^2\, dx \right]^{\frac{1}{2}}$$

for $1 \leq j \leq m$. In the special case $\Omega = R^n$ we shall also use the semi-norms (for $s \geq 0$)

$$(1.11) \qquad | u |_{sm_j,\,j,\,R^n} = \left[\int_{R^n} | \xi_j |^{2sm_j}\, | \hat{u}(\xi) |^2\, d\xi \right]^{\frac{1}{2}}.$$

Let T be a bounded linear operator in $L_2(\Omega)^p$ such that the range of T is contained in $H_m(\Omega)^p$. By the closed graph theorem T is also bounded when considered as a linear transformation from $L_2(\Omega)^p$ into $H_m(\Omega)^p$. The norm of T when considered as an operator: $L_2 \to H_m$ will be denoted by

$$(1.12) \qquad \| T \|_{m,\,\Omega} = \sup_{\| f \|_{0,\,\Omega} = 1} \| Tf \|_{m,\,\Omega}\,.$$

The norms $\| T \|_{m_j,\,j,\,\Omega}$ and the semi-norms $| T |_{m_j,\,j,\,\Omega}$, $1 \leq j \leq n$, are defined similarly. If $\Omega = R^n$ we use also the norm

$$(1.13) \qquad \| T \|_{sm,\,R^n} = \sup_{\| f \|_{0,\,R^n} = 1} \| Tf \|_{sm,\,R^n}$$

and the semi-norms $| T |_{sm_j, j, R^n}$ for $1 \leq j \leq n$. Let Ω, Ω_1 be open subsets of R^n and let $T: L_2(\Omega)^p \longrightarrow H_{\mathrm{m}}(\Omega_1)^p$. We set:

$$\| T \|_{0, \Omega_1, \Omega} = \sup_{\| f \|_{0, \Omega} = 1} \| Tf \|_{0, \Omega_1} \tag{1.14}$$

$$\| T \|_{m, \Omega_1, \Omega} = \sup_{\| f \|_{0, \Omega} = 1} \| Tf \|_{\mathrm{m}, \Omega_1}. \tag{1.15}$$

We remark finally that a bounded linear operator T in $L_2(\Omega)^p$ may be represented by a $p \times p$ matrix $(T_{i,j})$ of bounded linear operators in $L_2(\Omega)$. This matrix is determined by the equation

$$(Tf, g)_{L_2(\Omega)^p} = \Sigma_{i,j=1}^{p} (T_{i,j} f_j, g_i)_{L_2(\Omega)}. \tag{1.16}$$

Clearly,

$$(T^*)_{i,j} = (T_{j,i})^*. \tag{1.17}$$

2. Some properties of the $H_{\mathrm{m}}(\Omega)^p$ spaces.

In this section we shall discuss several simple properties of the $H_{\mathrm{m}}(\Omega)^p$ spaces which will be needed later. We shall also begin with the proof of the kernel theorem in this section.

We shall use the following simple extension property in order to be able to treat $H_{\mathrm{m}}(\Omega)^p$ space in case $\Omega \neq R^n$.

LEMMA 2.1: Let Σ be an n-dimensional box (i. e. a rectangular parallelepiped), the edges of which are parallel to the coordinate axes. Then there exists a bounded linear transformation $V: L_2(\Sigma)^p \longrightarrow L_2(R^n)^p$ such that Vf is an extension of f and V is a bounded transformation of $H_{\mathrm{m}}(\Sigma)^p$ into $H_{\mathrm{m}}(R^n)^p$. Moreover there exists a constant C (depending on Σ and on m) such that

$$| Vf |_{m_j, j, R^n} \leq C \| f \|_{m_j, j, \Sigma}, \quad 1 \leq j \leq n. \tag{2.1}$$

This lemma is essentially well known [26]. One may use the same construction as in [3] in order to prove the lemma, since in every stage (of extending across a planar face of a box) an inequality similar to (2.1) is satisfied by our assumption on Σ. It is obvious that it is impossible to replace in the right hand side of (2.1) norm by semi-norm $| \quad |_{m_j, j, \Sigma}$.

We shall need in the sequel a somewhat more delicate dimensional argument than the one that suffices in [3]. Let $r_1, \dots, r_n$ be positive numbers. Define a unitary operator $U(r_1, \dots, r_n): L_2(R^n)^p \longrightarrow L_2(R^n)^p$ by

$$(2.2)\qquad [U(r_1, \dots, r_n) f](x_1, \dots, x_n) = (r_1 \dots r_n)^{\frac{1}{2}} f(r_1 x_1, \dots, r_n x_n).$$

Then

$$(2.3)\qquad [U(r_1, \dots, r_n)]^* = U(r_1^{-1}, \dots, r_n^{-1})$$

and

$$(2.4)\qquad | U(r_1, \dots, r_n) f |_{sm_j, j, R^n} = r_j^{sm_j} | f |_{sm_j, j, R^n}.$$

Let

$$\Omega_1 = \{x; 0 \leq x_i \leq r_i a_i, 1 \leq i \leq n\}, \Omega_2 = \{x; 0 \leq x_i \leq a_i, 1 \leq i \leq n\}.$$

The transformation $U(r_1, \dots, r_n; \Omega_1) L_2: (\Omega_1)^p \longrightarrow L_2(\Omega_2)^p$ defined by

$$(2.5)\qquad [U(r_1, \dots, r_n; \Omega_1) f](x_1, \dots, x_n) = (r_1 \dots r_n)^{\frac{1}{2}} f(r_1 x_1, \dots r_n x_n)$$

is isometric. Its inverse is the transformation $U(r_1^{-1}, \dots, r_n^{-1}; \Omega_2)$. If $D^\alpha f \in L_2(\Omega_1)^p$ for a multi-index α, then $D^\alpha U(r_1, \dots, r_n; \Omega_1) f \in L_2(\Omega_2)^p$ and

$$(2.6)\qquad \| D^\alpha U(r_1, \dots, r_n; \Omega_1) f \|_{0, \Omega_2} = r^\alpha \| D^\alpha f \|_{0, \Omega_1}.$$

In particular it follows from (2.6) that

$$(2.7)\qquad | U(r_1, \dots, r_n; \Omega_1) f |_{m_j, j, \Omega_2} = r_j^{m_j} | f |_{m_j j, \Omega_1}.$$

LEMMA 2.2: If Ω is an open subset of R^n and if $u \in H_{\mathsf{m}}(\Omega)^p$ then $D^\alpha u \in L_2^{\text{loc}}(\Omega)^p$ for every multi-index α satisfyng $|\alpha : \mathsf{m}| \leq 1$. If either $\Omega = R^n$ or if Ω is an n-dimensional box, the edges of which are parallel to the coordinate axes then the assumption $u \in H_{\mathsf{m}}(\Omega)^p$ implies $D^\alpha u \in L_2(\Omega)^p$ for $|\alpha : \mathsf{m}| \leq 1$. If in the latter case the lengths of the edges of Ω are $r_1, \dots, r_n$, then (for $|\alpha : \mathsf{m}| \leq 1$)

$$(2.8)\quad \| D^\alpha u \|_{0, \Omega} \leq C r^{-\alpha} [\| u \|_{0, \Omega} + (\| u \|_{0, \Omega})^{1 - |\alpha : \mathsf{m}|} (\| u \|_{\mathsf{m}, \Omega})^{|\alpha : \mathsf{m}|} (\Sigma_{j=1}^n r_j^{m_j})^{|\alpha : \mathsf{m}|}],$$

where the constant C depends only on m.

PROOF: The assertion of the lemma for the case that $\Omega = R^n$ iswell known [26] and it follows easily from formula (4. 1. 15) of [18] and Hölder's inequality (using of course Fourier transform), that

$$\| D^\alpha u \|_{0, R^n} \leq C (\| u \|_{\mathsf{m}, R^n})^{|\alpha : \mathsf{m}|} (\| u \|_{0, R^n})^{1-|\alpha : \mathsf{m}|} \tag{2.9}$$

for $u \in H_{\mathsf{m}} (R^n)^p$ and $|\alpha : \mathsf{m}| \leq 1$.

Set $\Sigma = \{x : 0 \leq x_i \leq 1, 1 \leq i \leq n\}$. It follows from the extension lemma 2.1 and from (2.9) that there exists a constant C, depending only on m, such that the inequality

$$\|D^\alpha v \|_{0, \Sigma} \leq C (\| v \|_{\mathsf{m}, \Sigma})^{|\alpha : \mathsf{m}|} (\| v \|_{0, \Sigma})^{1-|\alpha : \mathsf{m}|} \tag{2.10}$$

holds for every $v \in H_{\mathsf{m}}(\Sigma)^p$ and $|\alpha : \mathsf{m}| \leq 1$.

Let now Ω be an n-dimensional box, the edges of which are parallel to the coordinate axes and their lengths are $r_1, \ldots, r_n$. We may assume that $\Omega = \{x : 0 \leq x_i \leq r_i, 1 \leq i \leq n\}$. The tranformation (2.5) $U(r_1, \ldots, r_n; \Omega): L_2(\Omega)^p \longrightarrow L_2(\Sigma)^p$ is an isometry, and (2.6) and (2.7) hold. For $u \in H_{\mathsf{m}}(\Omega)^p$ set $v = U(r_1, \ldots, r_n; \Omega)\, u$. Then $v \in H_{\mathsf{m}}(\Sigma)^p$. Combining (2.10), (2.6), (2.7) with the obvious estimate

$$\| v \|_{\mathsf{m}, \Sigma} \leq \| v \|_{0, \Sigma} + \Sigma_{j=1}^n | v |_{m_j, j, \Sigma}$$

we get (2.8).

If Ω is an arbitrary open-subset of R^n, $x_0 \in \Omega$, there exists an n-dimensional box $U \subset \Omega$ (the edges of which are parallel to the coordinate axes) containing x_0 in its interior. If $u \in H_m(\Omega)^p$ then $u \in H_m(U)^p$, so that $D^\alpha u \in L_2(U)^p$. Hence $D^\alpha u \in L_2^{\mathrm{loc}}(\Omega)^p$.

REMARK: If Ω satisfies the assumption of the Aronszaju-Smith coerciveness theorems [2], then $D^\alpha u \in L_2(\Omega)^p$ for $|\alpha| \leq \min_{1 \leq j \leq n} m_j$.

The following is a «Sobolev's lemma».

LEMMA 2.3: Let m be a multi-index and let s be a positive number, $2s > \Sigma_{j=1}^n \dfrac{1}{m_j}$. Let $u \in H_{s\mathsf{m}}(R^n)^p$. Then u is (equal almost everywhere to) a continuous bounded function satisfying

$$| u(x) | \leq C (\| u \|_{0, R^n})^{1-\Sigma_{j=1}^n \frac{1}{2sm_j}} (\| u \|_{s\mathsf{m}, R^n})^{\Sigma_{j=1}^n \frac{1}{2sm_j}}. \tag{2.11}$$

Moreover u is Hölder continuous of order μ.

$$|u(x) - u(x')| \leq C\,|x - x'|^{\mu}\, \|u\|_{sm,\, R^n} \tag{2.12}$$

for every $x, x' \in R^n$. Here $0 < \mu < 1$ and C depend only on m and on s.

PROOF: For every multi-index α set

$$e_{\alpha}(\lambda) = \int\limits_{[1+\Sigma_{j=1}^{n} |\xi_j|^{2m_j}]^{\frac{1}{2}} < \lambda} \xi^{2\alpha}\, d\xi. \tag{2.13}$$

Then it follows from a simple «homogeneity» argument that

$$e_{\alpha}(\lambda) \underset{\lambda \to \infty}{\backsim} C\, \lambda^{\Sigma_{j=1}^{n} \frac{1}{m_j} + 2|\alpha : \mathsf{m}}\tag{2.14}$$

C depends on m and α.

According to Fourier integral formula,

$$u(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{R^n} e^{ix\cdot\xi}\, \widehat{u}(\xi)\, d\xi. \tag{2.15}$$

Hence by Cauchy-Schwarz inequality and (1.7),

$$|u(x)| \leq C \int\limits_{R^n} |\widehat{u}(\xi)|\, d\xi \leq C\, \|u\|_{sm,\, R^n} \left[\int\limits_{R^n} \left(1 + \Sigma_{j=1}^{n} |\xi_j|^{2m_j}\right)^{-s} d\xi\right]^{\frac{1}{2}}.$$

But

$$\int\limits_{R^n} (1 + \Sigma_{j=1}^{n} |\xi_j|^{2m_j})^{-s}\, d\xi = \int\limits_{0}^{\infty} \lambda^{-2s}\, de_{(0,\dots,0)}(\lambda). \tag{2.16}$$

From (2.14) it follows, using integration by parts and the assumption $2s > \Sigma \frac{1}{m_j}$, that the integrals in (2.16) converge. Therefore u may be redefined on a null set so that (2.15) is true for all $x \in R^n$ (compare also theorem (2.27) in [18]). After this correction u is continuous and the estimate

$$|u(x)| \leq C\, \|u\|_{sm,\, R^n} \tag{2.17}$$

holds for every $x \in R^n$.

We deduce the estimate (2.11) from (2.17) using dimensional argument. Set $v = U(r_1, \dots, r_n) u$. Combining (2.2), (2.4), and (2.17) we find that

$$(2.18) \qquad \sup_{x \in R^n} | u(x) | = (r_1 \dots r_n)^{-\frac{1}{2}} \sup_{x \in R^n} | v(x) | \leq$$

$$\leq C (r_1 \dots r_n)^{\frac{1}{2}} [\| u \|_0 + \Sigma_{j=1}^n r^{sm_j} | u |_{sm_j, j, R^n}].$$

We may assume without loss of generality that u is not identically zero. Choosing

$$r_j = (\| u \|_{0, R^n} / | u |_{sm_j, j, R^n})^{\frac{1}{sm_j}}, \qquad 1 \leq j \leq n,$$

we obtain (2.11) from (2.18).

It remains to prove the Holder continuity of the function u. Let $x, x' \in R^n$. Then

$$| u(x) - u(x') | \leq C \int_{R^n} | e^{ix \cdot \xi} - e^{ix' \cdot \xi} | | \widehat{u}(\xi) | \, d\xi \leq$$

$$\leq C \| u \|_{sm, R^n} \left[\int_{R^n} [e^{ix \cdot \xi} - e^{ix' \cdot \xi}]^2 (1 + \Sigma_{j=1}^n | \xi_j |^{2m_j})^{-s} \, d\xi \right]^{\frac{1}{2}}.$$

Let t be a sufficiently large positive number. Then

$$(2.19) \qquad \int_{R^n} | e^{ix \cdot \xi} - e^{ix' \cdot \xi} |^2 (1 + \Sigma_{j=1}^n | \xi_j |^{2m_j})^{-s} \, d\xi \leq$$

$$\leq \int_{(1 + \Sigma_{j=1}^n | \xi_j |^{2m_j})^{\frac{1}{2}} \leq t} | e^{ix \cdot \xi} - e^{ix' \cdot \xi} |^2 \, d\xi + 4 \int_{(1 + \Sigma_{j=1}^n | \xi_j |^{2m_j})^{\frac{1}{2}} \geq t} (1 + \Sigma_{j=1}^n | \xi_j |^{2m_j})^{-s} \, d\xi.$$

But by definition (2.13), the second term in the right hand side of (2.19) is equal to $\int_t^\infty \lambda^{-2s} \, de_{(0, \dots, 0)}(\lambda)$, which is (by (2.19) and integration by parts) $0 (t^{\Sigma_{j=1}^n \frac{1}{m_j} - 2s})$ as $t \to \infty$. Also

$$| e^{ix \cdot \xi} - e^{ix' \cdot \xi} | \leq | x - x' | | \xi |.$$

Hence

$$(2.20) \qquad \int\limits_{(1+\Sigma_{j=1}^n |\xi_j|^{2m_j})^{\frac{1}{2}} \leq t} | e^{ix\cdot\xi} - e^{ix'\cdot\xi} |^2 \, d\xi \leq$$

$$\leq | x - x' |^2 [e_{(1,0,\ldots,0)}(t) + \ldots + e_{(0,\ldots,0,1)}(t)] = 0 (| x - x' |^2 t^d),$$

where

$$d = \max_{1 \leq i \leq n} \left[\Sigma_{j=1}^n \frac{1}{m_j} + \frac{2}{m_i} \right].$$

Choosing

$$t = | x - x' |^{-2/\left(2s+d-\Sigma_{j=1}^n \frac{1}{m_j}\right)}$$

we find by (2.19) and (2.20) that

$$| u(x) - u(x') | \leq C \| u \|_{sm, R^n} | x - x' |^{\mu}$$

where $\mu = 1 - d/\left(2s + d - \Sigma_{j=1}^n \frac{1}{m_j}\right)$, which proves (2.12). (The inequality (2.12) is obviously true by (2.17) if $| x - x' |$ is not small).

The following is a generalization of a part of Lemma 2.1 of [3] (Browder-Maurin's kernel lemma).

LEMMA 2.4: Let T be a bounded linear operator in $L_2(R^n)^p$. Suppose that the range of T is contained in $H_{sm}(R^n)^p$ where $\Sigma_{j=1}^n \frac{1}{m_j} < 2s$. Then there exists a $p \times p$ matrix $K(x, y) = (K_{i,j}(x, y))$ $i, j = 1, \ldots, p$ of kernels such that for every $f \in L_2(R^n)^p$

$$(2.21) \qquad Tf = \int\limits_{R^n} K(x, y) f(y) \, dy,$$

(Kf stands for the action of a matrix on a column vector). The kernels $K_{i,j}(x, y)$ on $R^n \times R^n$, $(1 \leq i, j \leq p)$ have the following properties:

(i) For each fixed $x_0 \in R^n$ the function $K(x_0, y) \in L_2(R^n)$.

(ii) The function $K_{i,j}(x, \cdot)$ from R^n to $L_2(R^n)$ is uniformly continuous in R^n.

(iii) The following estimate holds:

$$(2.22) \qquad \left(\int_{R^n} | K_{i,j}(x, y) |^2 \, dy\right)^{\frac{1}{2}} \leq C (\| T \|_{s\mathbf{m}, R^n})^{\Sigma_{t=1}^n \frac{1}{2sm_t}} (\| T \|_{0, R^n})^{1-\Sigma_{t=1}^n \frac{1}{2sm_t}}$$

where C is a constant depending only on $\mathbf{m}$ and s.

It is obvious (considering the matrix representation of T introduced at the end of section 1) that it suffices to prove lemma 2.4 for the special case $p = 1$. In this scalar case the proof is entirely similar to the proof of Lemma 2.1 in [3] (since we have already proved a « Sobolev's lemma » in our case), only one has to substitute $s\mathbf{m}, R^n$ for m, Ω, and instead of $\frac{n}{2m}$ one has to set $\Sigma_{t=1}^n \frac{1}{2sm_t}$.

We recall that it is usual to denote by S the class of C^∞ functions in R^n which together with their derivatives die down faster than any power of $|x|$ at infinity. It is well-known that the Fourier transform maps S onto itself and that S is dense in $H_{s\mathbf{m}}(R^n)^p$ for every real s. For any complex number z introduce the operator L^z which acts on functions in S in the following manner:

$$(2.23) \qquad L^z u = (2\pi)^{-\frac{n}{2}} \int_{R^n} \widehat{u}(\xi) (1 + \Sigma_{j=1}^n | \xi_j |^{2m_j})^{\frac{z}{2}} e^{i\xi \cdot x} \, d\xi .$$

It is immediate that L^z maps S onto itself and that if $\operatorname{Re} z = \sigma$ then for every real t,

$$(2.24) \qquad \| L^z u \|_{t\mathbf{m}, R^n} \leq \| u \|_{(t+\sigma)\mathbf{m}, R^n} .$$

It is also clear that a necessary and sufficient condition for a function $f \in L_2(R^n)^p$ to belong to $H_{s\mathbf{m}}(R^n)^p$ is that

$$(2.25) \qquad N = \sup_{\substack{u \in S \\ \| u \|_{0, R^n} = 1}} |(f, L^z u)_{L_2(R^n)^p}| < \infty$$

and if $N < \infty$ then $\| f \|_{s\mathbf{m}, R^n} = N$.

The following lemma generalizes lemma 2.2 of [3].

LEMMA 2.5: Let T be a bounded linear operator in $L_2(R^n)^p$ satisfying the conditions of lemma 2.4. Suppose in addition that

$$\| Tf \|_{s\mathbf{m},\, R^n} \leq C \, \| f \|_{-s\mathbf{m},\, R^n} \tag{2.26}$$

for all $f \in L_2(R^n)^p$, C a constant. Then T is an integral operator with a matrix kernel $K(x, y) = (K_{i,j}(x, y))$ possessing the properties of Lemma 2.4. Moreover every $K_{i,j}(x, y)$ is a bounded and a uniformly continuous function on $R^n \times R^n$ and the following estimate holds:

$$| K_{i,j}(x, y) | \leq \gamma C \qquad 1 \leq i, j \leq p \tag{2.27}$$

where γ is a constant depending only on $\mathbf{m}$ and s.

As in the proof of Lemma 2.4, it suffices also here to treat only the case $p = 1$. For this case the proof coincides with the proof of Lemma 2.2 in [3], except for the substitution of L^s instead of Λ^m, $s\mathbf{m}$, R^n instead of m, E_n; $-s\mathbf{m}$, R^n instead of $-m$, E_n.

3. A class of integral operators with a bounded matrix kernel.

In this section we shall obtain the main theorems of the first chapter of the present paper. These theorems play a basic role in our study of resolvent kernels and spectral functions. The following theorem is a partial generalization of Theorem 3.1 of [3] (and of Agmon's matrix kernel theorem).

THEOREM 3.1: Let Ω be either R^n or an n-dimensional box whose edges are parallel to the coordinate axes. Let T be a bounded linear operator in $L_2(\Omega)^p$. Suppose that the range of T and the range of its adjoint T^* are contained in $H_{\mathbf{m}}(\Omega)^p$ where $\mathbf{m}$ is a multi-index with positive components such that $\Sigma_{j=1}^n \frac{1}{m_j} < 1$. Then there exists a $p \times p$ matrix $K(x,y) = (K_{i,j}(x,y))$ $1 \leq i, j \leq p$, where the $K_{i,j}(x, y)$ are continuous and bounded scalar kernels on $\Omega \times \Omega$, $K_{i,j}(x, y)$ belongs to $L_2(\Omega)$ as a function of y for x fixed and as a function of x for y fixed, and

$$Tf = \int_\Omega K(x, y) f(y) \, dy, \qquad f \in L_2(\Omega)^p, \tag{3.1}$$

(Kf stands for matrix multiplication). Moreover, if $\Omega = R^n$ then there exists a constant C depending only on m such that

$$(3.2)\qquad |K_{i,j}(x,y)| \leq C\left(\|T\|_{\mathsf{m},\Omega} + \|T^*\|_{\mathsf{m},\Omega}\right)^{\Sigma_{t=1}^n \frac{1}{m_t}} \left(\|T\|_{0,\Omega}\right)^{1-\Sigma_{t=1}^n \frac{1}{m_t}}$$

$$1 \leq i, j \leq p.$$

If Ω is an n-dimensional box whose edges are parallel to the coordinate axes and their lengths are $r_1, \dots, r_n$, then

$$(3.3)\qquad |K_{i,j}(x,y)| \leq C \prod_{t=1}^{n} \left(\|T\|_{\mathsf{m},\Omega} + \|T^*\|_{\mathsf{m},\Omega} + r_t^{-m_t}\|T\|_{0,\Omega}\right)^{\frac{1}{m_t}} \cdot$$

$$\cdot \left(\|T\|_{0,\Omega}\right)^{1-\Sigma_{t=1}^n \frac{1}{m_t}}, \qquad 1 \leq i, j \leq p.$$

REMARKS: i) For most applications it suffices that (3.2) is valid for boxes satisfying the conditions of the theorem, with a consant C which depends, not only on m, but also on the length of the edges of the box. For other purposes the exact dependence, as given by (3.3) is needed.

ii) If $\Omega = R^n$ it is possible to replace m by $s\,\mathsf{m}$ $(s > 0)$ and to assume that $\Sigma_{t=1}^n \frac{1}{m_t} < s$. The conclusion of Theorem 3.1 will hold with suitable modifications. We shall not use this extension in the sequel.

PROOF: Let us consider first the case $\Omega = R^n$. Then T satisfies the conditions of lemma 2.5 $\left(\text{with } s \text{ replaced by } \frac{s}{2}\right)$. The proof of this fact is similar to the first part of the proof of Theorem 3.1 in [3]. One considers the analytic function $F(z)$ defined by

$$F(z) = (TL^z u, L^{1-\bar{z}} v)_{L_2(R^n)^p}$$

where u and v are two fixed (arbitrary) vector functions in S and L^z is the operator defined by (2.23). Applying Hadamard's three lines theorem to F and the three lines $\mathrm{Re}\, z = 0$, $\mathrm{Re}\, z = \frac{1}{2}$ and $\mathrm{Re}\, z = 1$ ($\mathrm{Re}\, z = 0$, $\mathrm{Re}\, z = \frac{s}{2}$, $\mathrm{Re}\, z = s$ in the case described in Remark ii), using (2.24), (2.25), and setting $u = L^{-\frac{1}{2}} f$, we find (in complete analogy to [3]) that the constant

C appearing in the assumptions of lemma 2.5 may be estimated by

$$C \leq \frac{1}{2} (\| T \|_{\mathrm{m},\, R^n} + \| T^* \|_{\mathrm{m},\, R^n}). \tag{3.4}$$

Hence there exists a matrix $K(x, y) = (K_{i,j}(x, y))$ of bounded and uniformly continuous scalar kernels such that $K_{i,j}(x, y)$ belongs to $L_2(R^n)$ as a function of y for x fixed and (3.1) holds. Since T^* is represented by $\overline{(K_{j,i}(y,x))}$, $K_{i,j}(x, y)$ also belongs to $L_2(R^n)$ as a function of x for y fixed. From Lemma (2.5) (estimate (2.37)) and (3.4) it follows that

$$| K_{i,j}(x, y) | \leq \frac{\gamma}{2} (\| T \|_{\mathrm{m},\, R^n} + \| T^* \|_{\mathrm{m},\, R^n}) \leq \tag{3.5}$$

$$\leq \gamma_1 (\| T \|_{0\, R^n} + \Sigma_{t=1}^{n} | T |_{m_t,\, t,\, R^n} + \Sigma_{t=1}^{n} | T^* |_{m_t,\, t,\, R^n})\ 1 \leq i, j \leq p,$$

where γ_1 depends only on m.

We obtain (3.2) from (3.5) using a somewhat more delicate dimensional argument than the one used in [3]. Without loss of generally assume $p = 1$. For arbitrary positive numbers $r_1, \dots, r_n$ denote

$$T_{r_1, \dots, r_n} = U(r_1, \dots, r_n)\, T U(r_1^{-1}, \dots, r_n^{-1}) \tag{3.6}$$

where $U(r_1, \dots, r_n)$ is the unitary operator defined by (2.2). Then $T_{r_1, \dots, r_n}$ is an integral operator with a kernel

$$K_{r_1, \dots, r_n}(x_1, \dots, x_n ; y_1, \dots, y_n) = r_1 \dots r_n\, K(r_1 x_1, \dots, r_n x_n ; r_1 y_1, \dots, r_n y_n). \tag{3.7}$$

Applying the inequality (3.5) for the operator $T_{r_1, \dots, r_n}$ and its kernel $K_{r_1, \dots, r_n}$ (and using (2.3), (2.4)) we find that

$$| K_{r_1, \dots, r_n}(x, y) | \leq \tag{3.8}$$

$$\leq \gamma_1 [\Sigma_{t=1}^{n} r_t^{m_t} (| T |_{m_t,\, t,\, R^n} + | T^* |_{m_t,\, t,\, R^n}) + \| T \|_{0,\, R^n}].$$

Assuming without loss of generality that T is not the zero operator, we choose

$$r_t = \left[\frac{\| T \|_{0,\, R^n}}{| T |_{m_t,\, t,\, R^n} + | T^* |_{m_t,\, t,\, R^n}} \right]^{\frac{1}{m_t}}$$

and we obtain (3.2) from (3.7) and (3.8), for the case $\Omega = R^n$. As a matter of fact, we obtain the (slightly stronger) estimate

$$(3.9) \qquad |K(x,y)| \leq$$
$$\leq C \prod_{t=1}^{n} (|T|_{m_t, t, R^n} + |T^*|_{m_t, t, R^n})^{\frac{1}{m_t}} (\|T\|_{0, R^n})^{1-\Sigma_{t=1}^{n} \frac{1}{m_t}},$$

where C depends only on $\mathbf{m}$.

Let Σ be the box $\{x : 0 \leq x_i \leq 1, 1 \leq i \leq n\}$. Let V be the extension operator $L_2(\Sigma) \rightarrow L_2(R^n)$ whose existence is guaranteed by Lemma 2.1. Set $T_0 = VTV^*$ where $V^* : L_2(R^n) \rightarrow L_2(\Sigma)$ is the adjoint transformation of V. Using (2.1) and (3.9) we find that T_0 is an integral operator with a continuous and bounded kernel $K_0(x,y)$ satisfying

$$|K_0(x,y)| \leq C \prod_{t=1}^{n} (\|T\|_{m_t, t, \Sigma} + \|T^*\|_{m_t, t, \Sigma})^{\frac{1}{m_t}} (\|T\|_{0, \Sigma})^{1-\Sigma_{t=1}^{n} \frac{1}{m_t}}.$$

It is easy to see (as in [3]) that for $x, y \in \Sigma \times \Sigma$ it is possible to define $K(x,y) = K_0(x,y)$ and to show that K is the kernel of T, and to obtain the estimate

$$(3.10) \qquad |K(x,y)| \leq C \prod_{t=1}^{n} (|T|_{m_t, t, \Sigma} + |T^*|_{m_t, t, \Sigma} + \|T\|_{0, \Sigma})^{\frac{1}{m_t}} \cdot$$
$$\cdot (\|T\|_{0, \Sigma})^{1-\Sigma_{t=1}^{n} \frac{1}{m_t}},$$

where C depends solely on $\mathbf{m}$.

Let now Ω be a box satisfying the assumptions of the theorem and let the lengths of the edges of Ω be $r_1, \ldots, r_n$; without loss of generality assume that $\Omega = \{x : 0 \leq x_i \leq r_i, 1 \leq i \leq n\}$. Define an operator $\widetilde{T}_{r_1, \ldots, r_n}$ in $L_2(\Sigma)$ by

$$(3.11) \qquad \widetilde{T}_{r_1, \ldots, r_n} = U(r_1, \ldots, r_n; \Omega)\, T U(r_1^{-1}, \ldots, r_n^{-1}; \Sigma)$$

where $U(r_1, \ldots, r_n; \Omega) : L_2(\Omega) \rightarrow L_2(\Sigma)$ is the isometry defined by (2.5) and $U(r_1^{-1}, \ldots, r_n^{-1}; \Sigma)$ is its inverse. If follows from (2.7) and (3.10) that $\widetilde{T}_{r_1, \ldots, r_n}$ is an integral operator with a continuous and bounded kernel

$\widetilde{K}_{r_1, \dots, r_n}(x, y)$ $(x, y \in \Sigma \times \Sigma)$ and

$$(3.11) \qquad |\widetilde{K}_{r_1, \dots, r_n}(x, y)| \leq C \prod_{t=1}^{n} [r_t^{m_t}(|T^*|_{m_t, t, \Omega} + |T|_{m_t, t, \Omega}) + \|T\|_{0, \Omega}]^{\frac{1}{m_t}} \cdot$$

$$\cdot (\|T\|_{0, \Omega})^{1 - \Sigma_{t=1}^{n} \frac{1}{m_t}} .$$

Hence T is also an integral operator (in $L_2(\Omega)$) with a continuous kernel $K(x, y)$ $(x, y \in \Omega \times \Omega)$ given by $K(x_1, \dots, x_n; y_1, \dots, y_n) = (r_1 \dots r_n)^{-1} \cdot \cdot \widetilde{K}_{r_1, \dots, r_n}(r_1^{-1} x_1, \dots, r_n^{-1} x_n; r_1^{-1} y_1, \dots, r_n^{-1} y_n)$ and by (3.11),

$$(3.12) \qquad |K(x, y)| \leq C \prod_{t=1}^{n} [|T|_{m_t, t, \Omega} + |T^*|_{m_t, t, \Omega} + r_t^{-m_t} \|T\|_{0, \Omega}]^{\frac{1}{m_t}} \cdot$$

$$\cdot (\|T\|_{0, \Omega})^{1 - \Sigma_{t=1}^{n} \frac{1}{m_t}} .$$

The estimate (3.3) follows immediately from (3.12).

REMARK: If we assume $\min_{1 \leq j \leq n} m_j > n$ then it follows directly from [3] that T is an integral operator with a continuous and bounded kernel so that it is possible to start the proof with the dimensional argument. The arguments at the end of section 2 and at the beginning of the proof of Theorem 3.1 are needed for the general case where not all the m_j are greater than n. (These arguments, as noted above, are actually slight modifications of arguments of [3] and of an unpublished work on systems by Agmon).

We shall need occasionally a theorem which is true for n-dimensional manifolds Ω that are more general than the domains considered in Theorem 3.1. We shall use the notations and terminology of [18], sections 1.8 and 2.6. The manifolds considered will have a positive C^∞ density dx, kept fixed throughout. By $L_2(\Omega)^p$ we denote the space of vector functions whose components are square summable with respect to dx.

THEOREM 3.2: Let Ω be a n-dimensional C^∞ manifold. Let T be a bounded linear operator in $L_2(\Omega)^p$ satisfying the following condition: there exists a complete set of C^∞ coordinate systems $\varkappa$ such that if u is either in the range of T or in the range of T^* then the composite function $u \circ \varkappa^{-1} \in H_{\mathsf{m}}(\widetilde{\Omega}_\varkappa)^p$. If $\Sigma_{t=1}^n \frac{1}{m_t} < 1$ then there exists a matrix kernel $K(x,y) = (K_{i,j}(x,y))$, $1 \leq i, j \leq p$, where the $K_{i,j}(x,y)$ are bounded scalar kernels which are continuous in the interior of $\Omega \times \Omega$, $K_{i,j}(x,y)$ belongs to $L_2(\Omega)$ as a function of y for fixed x and as a function of x for fixed y and

$$Tf = \int_\Omega K(x,y) f(y)\, dy \qquad f \in L_2(\Omega)^p, \tag{3.13}$$

(Kf denotes matrix multiplications).

Denote by $T_\varkappa$ the operator $L_2(\Omega)^p \to H_{\mathsf{m}}(\widetilde{\Omega}_\varkappa)^p$ defined by $T_\varkappa f = (Tf) \circ \varkappa^{-1}$. If $\Sigma \subset \widetilde{\Omega}_\varkappa$ is an n-dimensional box obtained by translation from the box $\{x : 0 \leq x_i \leq r_i,\ 1 \leq i \leq n\}$, then

$$| K_{i,j}(x,y) | \leq C(\| T_\varkappa \|_{0,\Sigma,\Omega})^{1-\Sigma_{t=1}^n \frac{1}{m_t}} \cdot \prod_{t=1}^n (\| T_\varkappa \|_{\mathsf{m},\Sigma,\Omega} + \| (T^*)_\varkappa \|_{\mathsf{m},\Sigma,\Omega} + r_t^{-m_t} \| T \|_{0,\Sigma,\Omega})^{\frac{1}{m_t}} \tag{3.14}$$

for $x, y \in \varkappa^{-1}(\Sigma)$, $1 \leq i, j \leq p$, C is a constant depending only on m and $\varkappa$. (We use the obvious extensions, in the case where Ω is a manifold of the definitions (1.14) and (1.15)).

REMARKS: i) If $\Omega \subset R^n$ and $\varkappa$ is the identity, the theorem has a much simpler form. It is compulsory to consider general $\varkappa$ even if $\Omega \subset R^n$, since the H_{m} spaces are not invariant even under very simple coordinate transformations.

ii) Sometimes it suffices to have instead of (3.14) the weaker estimate

$$| K_{i,j}(x,y) | \leq C(\| T_\varkappa \|_{\mathsf{m},\Sigma,\Omega} + \| (T^*)_\varkappa \|_{\mathsf{m},\Sigma,\Omega})^{\Sigma_{t=1}^n \frac{1}{m_t}} \cdot (\| T_\varkappa \|_{0,\Sigma,\Omega})^{1-\Sigma_{t=1}^n \frac{1}{m_t}} \tag{3.15}$$

valid for $x, y \in \varkappa^{-1}(\Sigma)$, $1 \leq i, j \leq p$, with a constant C which depends on the dimensions of Σ in addition to its dependence on m.

PROOF. The proof of (3.13) is very similar to the proof of Theorem 2.1 bis in [4]. One must only replace the sets Ω_j of that proof by unions of n dimensional boxes whose edges are parallel to the coordinate axes which are contained in $\widetilde{\Omega}_\varkappa$. We only have to prove the estimate (3.14).

Define an « extension » operator $E: L_2(\Sigma)^p \longrightarrow L_2(\Omega)^p$ by

$$(Ef)(x) = \begin{cases} f \circ \varkappa(x) & \text{if} \quad x \in \varkappa^{-1}(\Sigma) \\ 0 & \text{otherwise,} \end{cases} \tag{3.16}$$

and define a « restriction » $R: L_2(\Omega)^p \longrightarrow L_2(\Sigma)^p$ by

$$Rf = f \circ \varkappa^{-1}. \tag{3.17}$$

Then $T_\varkappa = TE$. Consider the operator $T_0: L_2(\Sigma)^p \longrightarrow L_2(\Sigma)^p$ defined by $T_0 = RTE$. Let $L^\varkappa$ represent the given density on Ω with respect to the coordinate system $\varkappa$ (see [18], p. 28). A simple calculation shows that the $L_2(\Sigma)^p$ adjoint of T_0 is $L^\varkappa RT^* E \dfrac{1}{L^\varkappa \circ \varkappa} = L^\varkappa (T^*)_\varkappa E \dfrac{1}{L^\varkappa \circ \varkappa}$. We assumed that $L^\varkappa$ is positive. Hence

$$\| T_0 \|_{0, \Sigma} \leq C \| T \|_{0, \Sigma, \Omega},$$

$$\| T_0 \|_{m, \Sigma} \leq C \| T_\varkappa \|_{m, \Sigma, \Omega},$$

$$\| (T_0)^* \|_{m, \Sigma} \leq C \| (T^*)_\varkappa \|_{m, \Sigma, \Omega},$$

C depending on $L^\varkappa$. Applying (3.3) we obtain (3.14) for the kernel of T_0; this kernel coincides with $K(x, y)$ for $x, y \in \varkappa^{-1}(\Sigma)$.

An operator $T: L_2(\Omega)^p \longrightarrow L_2(\Omega)^p$ represented by (3.1) where $K = (K_{i,j}(x, y))$, $1 \leq i, j \leq p$ is a continuous matrix kernel such that $K_{i,j}(x, y)$ is in $L_2(\Omega)$ as a function of y for x fixed will be called « an integral operator with a continuous matrix kernel ». In the sequel we shall denote by the same letter an operator and its matrix kernel.

REMARK: For $\Omega = R^n$ it is possible to define spaces, more general than the H_{sm} spaces, defined by the norm

$$\| u \|_{k_s, R^n} = \left[\int_{R^n} | \widehat{u}(\xi) |^2 k^{2s}(\xi) \, d\xi \right]^{1/2}$$

where k is a weight function which is a (fractional) power of a hypoelliptic polynomial. One can prove « Sobolev's lemma » for these spaces using a theorem of Nilsson [22] and it is possible to find classes of integral operators with continuous kernels. However, one can obtain only an extension of the inequality (2.17), and it is impossible to have an analog of (2.11) nor is it possible to extend (3.2) to the case of general k's. The dimensional arguments do not carry over to that case.

CHAPTER TWO

ASYMPTOTIC PROPERTIES OF KERNELS ASSOCIATED WITH SEMI-ELLIPTIC OPERATORS

4. Semi-elliptic differential operators and integral operators connected with them.

Due to the non-invariant character of semi elliptic operators, it seems worthwhile to introduce the concept of semi-ellipticity by two steps.

Consider, to begin with, a linear differential operator $\mathcal{A}$ acting on p-vector valued functions defined on an open set $\Omega \subset R^n$. We assume that $\mathcal{A}$ is of the form:

(4.1) $$\mathcal{A} = \mathcal{A}(x, D) = \Sigma \mathcal{A}_\alpha(x) D^\alpha$$

where

$$\mathcal{A}_\alpha(x) = (\mathcal{A}_{\alpha, i, j}(x)) \qquad 1 \leq i, j \leq p$$

is a $p \times p$ matrix whose entries $\mathcal{A}_\alpha^{ij}(x)$ are complex C^∞ functions defined in Ω, and where $\mathcal{A}_\alpha \equiv 0$ for all but a finite number of multi-indices α.

We shall also write

(4.2) $$\mathcal{A}(x, D) = (\mathcal{A}_{i, j}(x, D)) \qquad 1 \leq i, j \leq p$$

where $\mathcal{A}_{i, j}(x, D)$ are scalar differential operators. Let now $\mathsf{m} = (m_1, \dots, m_n)$ be a multi-index having positive components. The *reduced order*, $w(A)$, of a scalar linear differential operator $A = \Sigma A_\alpha(x) D^\alpha$ is (temporarily) defined by

$$w(A) = \max_{A_\alpha(x) \neq 0} | \alpha ; \mathsf{m} | .$$

(Clearly $w(A)$ depends on m). Note that if $m_k = 1$, $1 \leq k \leq n$, then $w(A)$ coincides with the (usual) order $o(A)$ of A. The reduced order of a matrix operator $\mathcal{A}$ is defined by

$$w(\mathcal{A}) = \max_{1 \leq i, j \leq p} w(\mathcal{A}_{i, j}).$$

The semi-principal part of $\mathcal{A}$ is by definition the operator:

$$\mathcal{A}' = \mathcal{A}'(x, D) = \sum_{|\alpha ; \mathsf{m}| = w(\mathcal{A})} \mathcal{A}_\alpha(x) D^\alpha = \Big(\sum_{|\alpha ; \mathsf{m}| = w(\mathcal{A})} \mathcal{A}_{\alpha, i, j}(x) D^\alpha \Big),$$

$$1 \leq i, j \leq p.$$

Recall that $\mathcal{A}$ is elliptic at x^0 if the determinant of the characteristic matrix of the principal part of $\mathcal{A}$ does not vanish. The operator $\mathcal{A}(x, D)$ is said to be semi-elliptic in the restricted sense at x^0 if

$$\det \mathcal{A}'(x^0, \xi) = \det \Big(\sum_{|\alpha ; \mathsf{m}| = w(\mathcal{A})} \mathcal{A}_{\alpha, i, j}(x^0) \xi^\alpha \Big) \neq 0$$

for all real $\xi = (\xi_1, \dots, \xi_n) \neq 0$.

This definition of semi-ellipticity in the restricted sense generalizes the definition of semi-ellipticity given in [18] for scalar operators with constant coefficients. Several authors (e. g. [7], [12], [17]) call these operators quasi-elliptic.

We note that whereas the (usual) order of a differential operator is independent of the coordinate system and ellipticity is invariant under non-singular coordinate transformations, even very simple coordinate transformations may make it impossible to recognize the semi-principal part of a differential operator, let alone its semi-elliptic character. We would also like to consider semi-elliptic operators defined on manifolds.

Using a generalized definition of semi-ellipticity given by F. Browder [9], we shall be able to overcome these difficulties.

We recall that a differential operator (with C^∞ coefficients) in a C^∞ manifold Ω is a linear mapping $\mathcal{A}$ of $C^\infty(\Omega)^p$ into itself, for which to every coordinate system $\varkappa$ there exists a differential operator $\mathcal{A}_\varkappa$ such that $(\mathcal{A}u) \circ \varkappa^{-1} = \mathcal{A}_\varkappa (u \circ \varkappa^{-1})$ in $\tilde{\Omega}_k$, if $u \in C^\infty(\Omega)^p$. The operator $\mathcal{A}$ is said to be semi-elliptic (in the extended sense) in Ω if there exists a *complete* set $\mathcal{F}$ of coordinate systems such that for each $\varkappa \in \mathcal{F}$, the operator $\mathcal{A}_\varkappa$ is semi-elliptic in the restricted sense at every point of $\tilde{\Omega}_\varkappa$. Such a family $\mathcal{F}$ will be called « a complete family belonging to $\mathcal{A}$ ». The reduced order (in the extended sense) of the differential operator $\mathcal{A}$ defined in a manifold Ω, with respect to m and $\mathcal{F}$, is defined to be $\sup_{\varkappa \in \mathcal{F}} w(\mathcal{A}_\varkappa)$. The reduced order is independent of $\mathcal{F}$ only if all the m_i are equal, $1 \leq i \leq n$.

We emphasize that the manifold Ω may be occassionally a subset of R^n, but nevertheless the extended and the restricted senses are not the same.

EXAMPLE: The operator $\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial \theta}$, defined by polar coordinate (r, θ) in a plane annulus is semi-elliptic there in the extended sense but not in the restricted sense (when written in the usual (x_1, x_2) plane coordinates).

Let us note that if the reduced order of the semi-elliptic operator $\mathcal{A}$ with respect to m is w, then the vector $w\mathsf{m}$ has integral components (see also [7]). Hence $\mathcal{A}$ has reduced order 1 with respect to the multi-index $w\mathsf{m}$.

Let $\mathcal{A}$ be semi-elliptic in the restricted sense, with respect to m, of reduced order w. It follows from lemma 2.2 that $\mathcal{A}$ acts on functions $u \in H^{\mathrm{loc}}_{w\mathsf{m}}(\Omega)^p$ and that the mapping $u \to \mathcal{A}u$ is linear mapping from $H^{\mathrm{loc}}_{w\mathsf{m}}(\Omega)^p$ into $L_2^{\mathrm{loc}}(\Omega)^p$.

It is well known ([9], [18], [26]) that semi elliptic (in the extended sense) differential operators are hypoelliptic, i. e. if $f \in C^\infty(\Omega)^p$ and $\mathcal{A}u = f$ (in distribution sense), then $u \in C^\infty(\Omega)^p$.

We denote by $C^\infty_*(R^n)$ the class of functions $u \in C^\infty(R^n)$ such that u and all its derivatives are bounded on R^n. Let the coefficients of the differential operator $\mathcal{A}$ be matrices, the entries of which are in $C^\infty_*(R^n)$ and let the reduced order of $\mathcal{A}$ with respect to m be w. For every real s and for every $u \in H_{(s+w)\mathsf{m}}(R^n)^p$,

$$\| \mathcal{A}u \|_{s\mathsf{m},\, R^n} \leq C \| u \|_{(s+w)\mathsf{m},\, R^n} \tag{4.3}$$

where C depends only on s, w, m and on a common bound for the coefficients of $\mathcal{A}$ and their derivatives up to a certain order depending on s, w and m (compare Theorem 2. 2. 5 in [18]).

Let $\mathcal{A}$ be a linear differential operator defined on a manifold Ω. The formal adjoint $\mathcal{A}^*$ of $\mathcal{A}$ is the differential operator satisfying

$$(\mathcal{A}u, v)_{L_2(\Omega)^p} = (u, \mathcal{A}^* v)_{L_2(\Omega)^p}$$

for every $u, v \in C_0^\infty(\Omega)^p$ (class of infinitely differential p-vector functions with compact support in Ω). The operator $\mathcal{A}$ is said to be formally self-adjoint if $\mathcal{A} = \mathcal{A}^* \cdot \mathcal{A}$ is formally semi-bounded from below in Ω if

$$(\mathcal{A}u, u)_{L_2(\Omega)^p} \geq c\,(u, u)_{L_2(\Omega)^p}$$

for all $u \in C_0^\infty(\Omega)^p$, with c a real constant. If $c \geq 0$ then $\mathcal{A}$ is said to be positive. Note that if $\mathcal{A}$ is semi-elliptic in the extended sense in Ω and is formally semi bounded from below in Ω, a complete family $\mathcal{F}$ belonging to $\mathcal{A}$ may be chosen, such that for each $\varkappa \in \mathcal{F}$, the matrix $\mathcal{A}'_\varkappa(\varkappa(x), \xi)$ is positive definite [8] and $L^{\varkappa}$ (the function representing locally the given density on Ω) is positive (in $\widetilde{\Omega}^{\varkappa}$).

We denote by $\widetilde{\mathscr{A}}$ a self-adjoint realization of $\mathscr{A}$ in $L_2(\Omega)^p$. That is, $\widetilde{\mathscr{A}}$ is a self adjoint operator in the Hilbert space $L_2(\Omega)^p$ with domain of definition $D_{\widetilde{\mathscr{A}}}$ such that any $u \in D_{\widetilde{\mathscr{A}}}$ is a solution in the *distribution* sense (a weak solution) of the differential equation:

$$\mathscr{A}(x, D)\, u = \widetilde{\mathscr{A}} u. \tag{4.4}$$

Let $\mathbf{m}$ be a multi-index with positive components. If $\mathscr{A}$ is semi elliptic in the restricted sense with respect to $\mathbf{m}$ in $\Omega\ (\subset R^n)$ and $w(\mathscr{A}) = 1$ (as one may assume without loss of generality) it follows from (1.4) according to well-known interior regularity results for semi-elliptic operators ([9], [17], [18], [26]) that $D_{\widetilde{\mathscr{A}}} \subset H_{\mathbf{m}}^{\mathrm{loc}}(\Omega)^p$. Moreover, if $\mathscr{A}$ is semi-elliptic in the extended sense on the manifold Ω, then $u \in D_{\widetilde{\mathscr{A}}}$ implies $u \circ \varkappa^{-1} \in H_{\mathbf{m}}^{\mathrm{loc}}(\widetilde{\Omega}_{\varkappa})^p$ for every $\varkappa$ in the complete family belonging to $\mathscr{A}$. More generally, since $\widetilde{\mathscr{A}}^k$ is a realization of $\mathscr{A}^k$,

$$D_{\widetilde{\mathscr{A}}^k} \subset H_{k\mathbf{m}}^{\mathrm{loc}}(\widetilde{\Omega}_k)^p \ \text{ for } k = 1, 2, \ldots \tag{4.5}$$

Let now $R_\lambda = (\widetilde{\mathscr{A}} - \lambda)^{-1}$ be the resolvent of $\widetilde{\mathscr{A}}$ defined for every complex λ not in the spectrum of $\widetilde{\mathscr{A}}$. Then range (R_λ) = range $(R_\lambda^*) = D_{\widetilde{\mathscr{A}}}$, so that if $\Sigma_{t=1}^{n} \dfrac{1}{m_t} < 1$ it follows from (4.5) and Theorem 3.2 that R_λ is an integral operator with a continuous matrix kernel $R_\lambda(x, y)$. We shall refer to $R_\lambda(x, y)$ as the resolvent kernel of $\widetilde{\mathscr{A}}$.

Next assume that $\mathscr{A}$ is formally semi bounded from below and that $\widetilde{\mathscr{A}}$ is also semi bounded from below, and let $\mathbf{m}$ be a multi-index with positive components, $\mathscr{A}$ semi-elliptic in the extended sense with respect to $\mathbf{m}$. Let $\{E_t\}$ be the spectral resolution of $\widetilde{\mathscr{A}}$ which we normalize so that it is continuous to the left. It is well known ([8], [19]; it may be also proved easily using the kernels theorems in the same way as in [4]) that E_t is an integral operator with a continuous (actually C^∞) matrix kernel. Since E_t is self adjoint it follows, using (1.17) that the matrix kernel of E_t is hermitian. This matrix $E_t(x, y) = (E_{t,i,j}(x, y))$, $1 \le i, j \le p$ is called ([8]) the *spectral function* of $\widetilde{\mathscr{A}}$.

It is well known ([8], [16]) and easily seen that not only is $E_{t,i,i}(x, x)$ real monotone non decreasing function of t, but also the function

$$\sigma(t) = E_{t,i,i}(x, x) \mid z \mid^2 + 2 \, Re \, [E_{t,i,j}(x, x) \, z] + E_{t,j,j}(x) \tag{4.6}$$

is real monotone non-decreasing function of t for all (fixed) complex z, with x, i, j fixed.

Suppose that $\widetilde{\mathcal{A}}$ is semi-bounded from below (and w. l. o. g $\widetilde{\mathcal{A}}$ is positive) and that $\Sigma_{t=1}^{n} \frac{1}{m_t} < 1$. In this case both the resolvent kernel and the spectral function exist, and the following relation holds:

$$R_{\lambda,i,j}(x, y) = \int_0^\infty (t - \lambda)^{-1} \, dE_{t,i,j}(x, y) \tag{4.7}$$

for $1 \leq i, j \leq p$, where the Stieltjes integral converges absolutely. Formula (4.7) is well known for elliptc systems (see e. g. [8]). A simple proof of (4.7) can be given with the aid of the theorems of section 3, in the same way as in [4].

If $\widetilde{\mathcal{A}}$ has a compact resolvent then the spectrum of $\widetilde{\mathcal{A}}$ consists of a discrete set of eigenvalues. Let $\{\lambda_k\}$ be the sequence of eigenvalues, each repeated according to its multiplicity, and let $\{\varphi_k(x)\}$ be the corresponding sequence of normalized eigenfunctions. Denote by $\overline{\varphi}_k$ the vector function $(\overline{\varphi}_{k,1}, \ldots, \overline{\varphi}_{k,p})^\sim$ where $\overline{\varphi}_{k,j}$ is the complex conjugate of $\varphi_{k,j}$. The spectral function in this case is given by

$$E_t(x, y) = \Sigma_{\lambda_k < t} \, \varphi_k(x) \, \overline{\varphi}_k(y)^\sim . \tag{4.8}$$

By orthonormality it follows that

$$\sum_{\lambda_k < t} 1 = \Sigma_{i=1}^{p} \int E_{t,i,i}(x, x) \, dx. \tag{4.9}$$

One has to use formula (4.9) if one wants to get information about the asymptotic behavior of eigenvalues from information about the spectral function.

5. The main theorems.

The key result of this chapter (and one of the main results of this apaper) is n asymptotic expansion theorem for resolvent kernels of semi-elliptic operators. We introduce the following notations (in order to be able to formulate this theorem for the general semi-elliptic case).

Let $\mathsf{m} = (m_1, \ldots, m_n)$ be a multi-index with positive components. Set $b = b(\mathsf{m}) = \min_{1 \leq i \leq n} \frac{1}{m_i}$, and let $a = a(\mathsf{m})$ denote the reciprocal of the lowest common multiple of $m_1, \ldots, m_n$. Set $Q = \{q \text{ real}; \, q \geq b, \, \frac{q}{a}$ is a natrual number). The function $u \in C^\infty(\Omega)$ has a zero of type q with respect to m at a point $x \in \Omega$ where $q \in Q \cap \{+\infty\}$ if u and all its derivatives $D^\alpha u$ with $|\alpha; \mathsf{m}| < q$ vanish at x. If $u(x) \neq 0$ we say that u has a zero of type $q = 0$ at x.

Let $\mathcal{A}$ be a differential operator defined on $\Omega \subset R^n$, $w(\mathcal{A}) = w$,

$$\mathcal{A} = \sum_{|\alpha; \mathsf{m}| \leq w} \mathcal{A}_\alpha(x) D^\alpha$$

Denote by $q_0 = q_0(x^0)$ the maximal element in $\{q \in Q \cup \{+\infty\}$; all the entries of the matrices $\mathcal{A}'_\alpha(x) - \mathcal{A}'_\alpha(x^0)$ have a zero of type q at $x^0\}$ and by $q_i = q_i(x^0)\,(i > 0)$ denote the maximal element of $\{q \in Q \cup \{0\} \cup \{+\infty\}$; all the entries of the matrices $\mathcal{A}_\alpha(x)$ have a zero of type q at x^0 for $|\alpha; \mathsf{m}| = w - ia\}$. We associate with $\mathcal{A}$ a rational number $\theta(x^0)$ defined by

$$\theta(x^0) = \min_{0 \leq i \leq w/a} \frac{q_i + ia}{q_i + b} \tag{5.1}$$

where we agree that $\frac{q_i + ia}{q_i + b} = 1$ if $q_i = \infty$. If $\mathcal{A}$ is semi-elliptic in the extended sense on a manifold Ω, we define a number $\theta(x)$ for $x \in \Omega_\varkappa$ where $\varkappa$ is in a complete family belonging to $\mathcal{A}$ to be the number $\theta(\varkappa(x))$ associated with $\mathcal{A}_\varkappa$.

We denote by $d(\lambda)$ (for a complex number λ) the distance of λ from the positive axis ($d(\lambda) = |\lambda|$ if $\operatorname{Re}\lambda \leq 0$, $d(\lambda) = |\operatorname{Im}\lambda|$ if $\operatorname{Re}\lambda > 0$).

THEOREM 5.1: Let Ω be a manifold, let $\tilde{\mathcal{A}}$ be a positive self-adjoint operator in $L_2(\Omega)^p$ which is a realization of a formally self-adjoint differential operator $\mathcal{A}$. Assume that $\mathcal{A}$ is semi-elliptic in the extended sense in Ω (with respect to m), $w(\mathcal{A}) = w$, and $\Sigma_{h=1}^n \frac{1}{m_h} < w$. Let $\mathcal{F}$ be a complete family of coordinate systems belonging to $\mathcal{A}$ such that for each $\varkappa \in \mathcal{F}$ the matrix $\mathcal{A}'_\varkappa(\varkappa(x), \xi)$ is positive definite and $L^\varkappa$ is positive in $\tilde{\Omega}_\varkappa$. Let $R_\lambda(x, y)$ be the resolvent kernel of $\tilde{\mathcal{A}}$. Then $R_\lambda(x, x)$ has an asymptotic expansion of the form:

$$(5.2)\qquad R_\lambda(x,x) \backsim (-\lambda)^{\frac{1}{w}\sum_{h=1}^{n}\frac{1}{m_h}-1}\ \Sigma_{j=0}^{\infty}\, C_j(x)\,(-\lambda)^{-ja/w}$$

valid for $\lambda \to \infty$ in the region $|\lambda| \geq 1, d(\lambda) \geq |\lambda|^{1-\theta(x)\frac{b}{w}+\varepsilon}$ where ε is any given positive number and $\theta(x)$ is the number associated with $\mathscr{A}$ according to (5.1), uniformly in x in every compact subset of Ω. That is, for any integer $N \geq 1$,

$$(5.2')\qquad (-\lambda)^{1-\frac{1}{w}\sum_{h=1}^{n}\frac{1}{m_h}} R_{\lambda,s,t}(x,x) - \Sigma_{j=0}^{N-1}\, C_{j,s,t}(x,x)\,(-\lambda)^{-ja/w}\,| \leq$$

$$\leq \text{Const.}\,|\lambda|^{-Na/w} \qquad 1 \leq s, t \leq p$$

for $|\lambda| \geq 1, d(\lambda) \geq |\lambda|^{1-\theta(x)b/w+\varepsilon}$ where the constant in (5.2′) depends on N and ε but is indipendent of x in any compact subset of Ω. In these formulas $(-\lambda)^{-ja/w}$ stands for the branch of the power which is positive on the negative axis while $C_j(x)$ are certain $p \times p$ matrices of C^∞ functions on Ω depending only on the differential operator $\mathscr{A}$ (and on the family $\mathscr{F}$). In particular,

$$(5.3)\qquad C_0(x) = (2\pi)^{-n}\,(L^\varkappa(\varkappa(x)))^{-1}\int_{R^n} [\mathscr{A}'_\varkappa(\varkappa(x),\xi) + I]^{-1}\,d\xi$$

where $[\mathscr{A}'_\varkappa + I]^{-1}$ is the inverse matrix of $\mathscr{A}'_\varkappa + I$ (I — the $p \times p$ identity matrix). Moreover, $C_i(x) = 0$ for all $0 < j < b/a$ satisfying $q_i > 0$ for $0 \leq i \leq j$.

This theorem is an extension of Theorem 3.1 in [4] which deals with a single elliptic operator. Before going on, it is worthwhile to elucidate several points related to Theorem 5.1 by means of examples.

In the elliptic case $\mathrm{m} = (1, \dots, 1)$, so that $a = b = 1$ and w is the ordinary order of the operator (usually designated by m), $\theta(x)$ is just $\frac{p}{p+1}$ $(p = q_0)$ as in [4]. If the largest m_i is divisible by all the other m_i (as is the case for the heat operator) then $a = b$ and again only q_0 is interesting, hence $\theta(x) = \frac{q_0}{q_0+b} \geq \frac{1}{2}$ in the general case and $\theta(x) = 1$ if the semi-principal part of $\mathscr{A}$ has constant coefficient. If, however, the largest m_i is not divisible by all the other m_i then $a \neq b$, e. g. $\mathrm{m} = (4,6)$; then $a = \frac{1}{12}$ whereas $b = \frac{1}{6}$. Differential operators which have reduced order no larger than

$w - b$, are, in a sense, unimportant (compare [22]); they are dominated (in the sense of [18]) by $\det \mathcal{A}'$ and (which is crucial) they are *weaker* than $| \operatorname{grad}_\xi \det \mathcal{A}'(\xi) |$ (this is not a polynomial, but a symbol of a pseudo-differential operator). On the other hand, differential operators of reduced order w_1 with $w - a < w_1 < w$, while dominated by $\mathcal{A}'$ are not weaker than $| \operatorname{grad}_\xi \det \mathcal{A}'(\xi) |$, so that their influence (if occurring in $\mathcal{A}$) is not negligible. In the above example, the (single) operator $f(x_1, x_2) D_1^3 D_2$ is dominated by $D_1^4 + D_2^6$ and the uperator $\mathcal{A} = D_1^4 + D_2^6 + f(x_1, x_2) D_1^3 D_2$ is semi-elliptic, but properties of f will influence the results of Theorem 5.1, since $w(D_1^3 D_2) = \frac{11}{12} > 1 - b$. This fact made the formulation of the main theorems cumbersome.

It was pointed out to the author by L. Hörmander that similar « in between » terms may occur even in the elliptic case, if one permits the inclusion of pseudo-differential terms. Then an operator of (usual) order m may contain terms of order $m - t$ with $0 < t < 1$; these terms have no « negligible » influence on the asymptotic expansion of the resolvent kernel since their order is greater than $m - 1$.

We shall demonstrate now how Theorem 5.1 together with a tauberian theorem of Malliavin [21], yields the estimates for the remainder in the asymptotic formula for the spectral function. A simple proof of Malliavin's theorem is due to Pleijel [23] who also gave a slight extension of the theorem. It is the following:

Tauberian theorem: Let $\sigma(t)$ be a non-decreasing function for $t \geq 0$ such that $\int_0^\infty \frac{d\sigma(t)}{1+t} < +\infty$. Suppose that

$$\int_0^\infty (t - \lambda)^{-1} \, d\sigma(t) - c_0 (-\lambda)^\alpha = O(|\lambda|^\beta) \tag{5.4}$$

as $\lambda \to \infty$ in the complex plane along the curve: $|\operatorname{Im} \lambda| = |\lambda|^\gamma$, $\operatorname{Re} \lambda \geq 0$, where $-1 < \beta < \alpha < 0$, $0 < \gamma < 1$; c_0 some non-negative constant. Then

$$\sigma(t) = \frac{\sin \pi(\alpha + 1)}{\pi(\alpha + 1)} c_0 t^{\alpha+1} + O(t^{\alpha+\gamma}) + O(t^{\beta+1}) \tag{5.5}$$

as $t \to \infty$.

We shall now prove the following result:

THEOREM 5.2: Let all the assumptions of Theorem 5.1 be satisfied, except that we do not require now that $\Sigma_{h=1}^{n} \frac{1}{m_h} < w$. Let $E_t(x, y)$ be the spectral function of $\mathcal{A}$. Then for every $1 \leq r, s \leq p$,

$$(5.6) \qquad E_{t,r,s}(x, x) - D_{r,s}(x)\, t^{\frac{1}{w}\Sigma_{h=1}^{n}\frac{1}{m_h}} = O(t^{\frac{1}{w}\Sigma_{h-1}^{n}\frac{1}{m_h} - \frac{\theta(x) b}{w} + \varepsilon})$$

as $t \to \infty$ for any $\varepsilon > 0$, uniformly in x in any compact subset of Ω. The matrix $D(x)$ is defined by

$$(5.7) \qquad D(x) = (2\pi)^{-n} L^{\varkappa}(\varkappa(x)) \int \mathcal{A}'_{\varkappa}(\varkappa(x), \xi)^{-\Sigma_{h=1}^{n}\frac{1}{m_h}} \omega(\xi).$$

Here $\omega(\xi)$ is the differential form

$$\omega(\xi) = \frac{1}{\Sigma_{h=1}^{n}\frac{1}{m_h}} \left(\frac{\xi_1}{m_1} d\xi_2 \wedge \ldots \wedge d\xi_n + \ldots + (-1)^n \frac{\xi_n}{m_n} d\xi_1 \wedge \ldots \wedge d\xi_{n-1}\right)$$

in R^n, the integration takes place over the sphere $(R^n - \{0\})/R_+$, oriented by $\omega > 0$.

In particular (5.6) holds if we replace $\theta(x)$ by $\min\left(\frac{1}{2}, \frac{a}{b}\right)$. If the semi-principal part of $\mathcal{A}_{\varkappa}$ has constant coefficients and the matrices $\mathcal{A}_{\varkappa, \alpha}$ vanish identically for $w > |\alpha; \mathbf{m}| > w - i_0 a \left(i_0 \leq \frac{b}{a}\right)$, it is possible to replace $\theta(x)$ by $i_0 a/b$. In the elliptic case or any other case where $a = b$ it is possible to replace θ by $\frac{1}{2}$ in general and by 1 if the semi-principal part has constant coefficients.

PROOF: Without loss of generality we may assume that $\tilde{\mathcal{A}}$ is positive. Suppose first that $\Sigma_{h=1}^{n} \frac{1}{m_h} < w$. Set $i_0 = \min\left(\frac{b}{a}, \min_{q_i = 0} i\right)$. Using the representation formula (4.7) and the first term which does not vanish in the asymptotic expansion (5.2), we have

$$(5.8) \qquad \int_0^\infty (t - \lambda)^{-1} dE_{t,r,s}(x, x) - C_{0,r,s}(x)(-\lambda)^{\frac{1}{w}\Sigma_{h=1}^{n}\frac{1}{m_h} - 1} =$$

$$= O(|\lambda|^{\frac{1}{w}\Sigma_{h=1}^{n}\frac{1}{m_j} i_0 a/w - 1}) \qquad 1 \leq r, s \leq p$$

as $\lambda \to \infty$ along the curve $|\operatorname{Im} \lambda| = |\lambda|^{1-\theta(x)\frac{h}{w}+\varepsilon}$, $\operatorname{Re}\lambda \geq 1$, for any $\varepsilon > 0$. Let now, r, s be fixed and let z be any (fixed) complex number. According to (4.6) and using the hermitian character of $C_0(x)$ and $E_t(x, x)$ it follows from (5.8) that we may apply the tauberian theorem to

$$\sigma(t) = E_{t,r,r}(x, x)\,|z|^2 + 2\operatorname{Re}[E_{t,r,s}(x, x)\, z] + E_{t,s,s}(x, x)$$

with

$$\alpha = \frac{1}{w}\Sigma_{h=1}^{n}\frac{1}{m_h} - 1, \quad \beta = \frac{1}{w}\Sigma_{h=1}^{n}\frac{1}{m_h} - \frac{i_0\, a}{w} - 1, \quad \gamma = 1 - \frac{\theta(x)\, b}{w} + \varepsilon.$$

By definition always $\theta(x) \leq i_0\, a/b$, so that $\alpha + \gamma \geq \beta + 1$. Hence it follows from (5.5) that

$$\sigma(t) = \frac{\sin \pi(\alpha + 1)}{\pi(\alpha + 1)}\left(C_{0,r,r}(x)\,|z|^2 + 2\operatorname{Re}[C_{0,r,s}(x)\, z] + C_{0,s,s}(x)\right)\cdot t^{\frac{1}{w}\Sigma_{h=1}^{n}\frac{1}{m_h}} + O\left(t^{\frac{1}{w}\Sigma_{h=1}^{n}\frac{1}{m_h} - \frac{\theta(x)\, b}{w} + \varepsilon}\right) \tag{5.9}$$

as $t \to \infty$. By checking the constants in Pleijel's proof of Malliavin's theorem [23] one also finds that the O estimate in (5.9) is uniform in x in any compact subset of Ω. If we let z vary in the unit disk in the complex plane (compare [8] and [16]) we get (5.6) from (5.9) with

$$D(x) = \frac{\sin \pi(\alpha + 1)}{\pi(\alpha + 1)}\, C_0(x)\,.$$

I learned from L. Hörmander that transforming suitably (5.3) one gets (5.7) which is the analogous form of the well-known formulas of the case of a single operator.

Suppose now that $\Sigma_{h=1}^{n}\frac{1}{m_h} \geq w$. Choose an integer $k > \frac{1}{w}\Sigma_{h=1}^{n}\frac{1}{m_h}$ and consider the spectral function $E_t^{(k)}(x, y)$ of $\tilde{\mathcal{A}}^k$. Clearly $E_t^{(k)}(x, y) = E_{t^{1/k}}(x, y)$. Noreover, $\tilde{\mathcal{A}}^k$ is a self-adjoint realization of $\mathcal{A}^k$, a semi elliptic differential operator of reduced order $kw > \Sigma_{h=1}^{n}\frac{1}{m_h}$. Let now $x \in \Omega$. It is easy to see that the number $\theta(x)$ related to $\mathcal{A}^k$ is not less than the $\theta(x)$ associated with $\mathcal{A}$. Hence this case follows from the special case of the theorem just proved in the same way as in [4].

The remainder of chapter two is devoted to the proof of a general asymptotic formula for resolvent kernels containing Theorem 5.1 as a special case.

6. Fundamental solutions and related kernels.

In this section we consider integral operators acting on functions defined on R^n. We denote by $H_{\infty,p} = H_\infty(R^n)^p$ the class of functions $u \in C^\infty(R^n)^p$ such that $D^\alpha u \in L_2(R^n)^p$ for all α.

From now on through the remainder of this section we shall assume that $\mathcal{A}$ is a semi elliptic (in the restricted sense) differential operator of reduced order w with respect to $\mathbf{m} = (m_1, \dots, m_n)$, that $\mathcal{A} = \mathcal{A}'$, that $\mathcal{A}$ is formally positive and that $\mathcal{A}$ has constant coefficients.

A function $f(x)$ is said to be semi-homogeneous of reduced order w (with respect to $\mathbf{m}$) if for any $d > 0$ we have

$$f(d^{\frac{1}{m_1}} x_1, \dots, d^{\frac{1}{m_n}} x_n) = d^w f(x_1, \dots, x_n).$$

By assumption the matrix $\mathcal{A}(\xi)$ is Hermitian for all $\xi \in R^n$ and is positive definite for $\xi \neq 0$, and its entries are semi-homogeneous of reduced order w. We shall denote by $\{\mu_\varkappa(\xi)\}$, $k = 1, \dots, p$, the eigenvalues of $\mathcal{A}(\xi)$ (with multiplicities) arranged in non-decreasing order: $\mu_1(\xi) \leq \mu_2(\xi) \leq \dots \mu_p(\xi)$. The functions $\mu_i(\xi)$ are positive for $\xi \neq 0$ since $\mathcal{A}(\xi)$ is positive definite and are semi-homogeneous of reduced order w since the elements of $\mathcal{A}(\xi)$ are such. Hence there exists a constant $\gamma \geq 1$ such that

$$\gamma^{-1}(1 + \Sigma_{j=1}^n |\xi_j|^{2m_j})^{w/2} \leq 1 + \mu_i(\xi) \leq \gamma(1 + \Sigma_{j=1}^n |\xi_j|^{2m_j})^{w/2} \tag{6.1}$$

for $1 \leq i \leq p$.

It is clear that there exists a unitary $p \times p$ matrix $(u_{i,j}(\xi))$, $1 \leq i,j \leq p$ for $\xi \neq 0$, such that

$$\mathcal{A}_{i,j}(\xi) = \Sigma_{k=1}^p u_{i,k}(\xi)\, \mu_k(\xi)\, \overline{u}_{j,k}(\xi), \qquad 1 \leq i,j \leq p, \tag{6.2}$$

and the scalar functions $u_{i,j}(\xi)$ are semi-homogeneous of reduced order zero.

It is well known (e. g. [16]) that $\mathcal{A}$ has a unique self-adjoint realization in $L_2(R\,)^p$ which we shall denote by $\widetilde{\mathcal{A}}$. The operator $\widetilde{\mathcal{A}}$ is positive and its domain of definition is $H_{wm}(R^n)^p$. For any $u \in H_{wm}(R^n)^p$ we have:

$$\widehat{\tilde{\mathscr{A}} u} = \mathscr{A}(\xi)\,\widehat{u}(\xi). \tag{6.3}$$

Let $F_\lambda = (\tilde{\mathscr{A}} - \lambda)^{-1}$ be the resolvent of $\tilde{\mathscr{A}}$ which exists for any complex λ not contained in the non-negative axis. We denote by $F_\lambda(j)$ a (scalar) operator of the form

$$F_\lambda(j) = (F_\lambda)_{q_1, r_1}(F_\lambda)_{q_2, r_2} \cdots (F_\lambda)_{q_j, r_j} \tag{6.4}$$

$1 \le q_i,\ r_i \le p,\ 1 \le i \le j$, for a natural number j. We denote the class of operators $F_\lambda(j)$ by $\mathbf{F}_\lambda(j)$. As before we denote by $d(\lambda)$ the distance of λ from the positive axis.

Lemma 6.1: Let $F_\lambda(j) \in \mathbf{F}_\lambda(j)$, j a natural number. The operator $F_\lambda(j)$ defines a one-to-one map of H_∞ onto itself. For any two real numbers s, t with $s \le t \le s + jw$ the following inequality holds:

$$\| F_\lambda(j) f \|_{tm, R^n} \le (3p\gamma)^j \frac{|\lambda|^{(t-s)/w}}{d(\lambda)^j} \| f \|_{sm, R^n} \tag{6.5}$$

for $f \in H_\infty$ and $|\lambda| \ge 1$ where γ is the constant occurring in (6.1). For $t = s$ the constant in (6.5) can be replaced by p^j.

Proof: By Fourier transformation,

$$\widehat{F_\lambda g}(\xi) = [\mathscr{A}(\xi) - \lambda I]^{-1}\,\widehat{g}(\xi) \tag{6.6}$$

for any $g \in H_\infty$. Using (6.2) and unitarity we have that

$$([\mathscr{A}(\xi) - \lambda I]^{-1})_{q, r} = \Sigma_{k=1}^{p} u_{q, k}(\xi)\,(\mu_k(\xi) - \lambda)^{-1}\,\bar{u}_{r, k}(\xi) \tag{6.7}$$

so that by (6.6)

$$\widehat{(F_\lambda)_{q, r} g}(\xi) = \Sigma_{k=1}^{p} u_{q, k}(\xi)\,(\mu_k(\xi) - \lambda)^{-1}\,\bar{u}_{r, k}(\xi)\,\widehat{g}(\xi) \tag{6.8}$$

for $\xi \neq 0$.

Hence $F_\lambda(j)f$ may be expressed by

$$(6.9)\qquad \widehat{F_\lambda(j)f}(\xi) = \Sigma^p_{k_1,\dots,k_j=1} u_{q_1,k_1}(\xi)\dots u_{q_j,k_j}(\xi)\cdot$$

$$\cdot\, \bar{u}_{r_1,k_1}(\xi)\dots \bar{u}_{r_j,k_j}(\xi)\,(\mu_{k_1}(\xi)-\lambda)^{-1}\dots(\mu_{k_j}(\xi)-\lambda)^{-1}\hat{f}(\xi), \qquad \xi \neq 0,$$

for all $f \in H_\infty$. This implies that $F_\lambda(j)$ yields a one-to-one map of H_∞ onto itself. It is clear that $|u_{q,r}(\xi)| \leq 1$ for $\xi \neq 0$, $1 \leq q, r \leq p$. From (6.9) and (1.7) it follows further after a simple calculation that

$$(6.10)\qquad (\| F_\lambda(j) f\|_{tm,R^n})^2 \leq p^j C^2_\lambda (\| f \|_{sm,R^n})^2$$

where

$$C^2_\lambda = \sup_{\xi\in R^n} \frac{[1+\Sigma^n_{h=1}|\xi_h|^{2m_h}]^{t-s}}{|\mu_{k_1}(\xi)-\lambda|^2\dots|\mu_{k_j}(\xi)-\lambda|^2}.$$

Clearly $C_\lambda = d(\lambda)^{-j}$ for $t = s$. Using the estimate $|\mu_k(\xi)-\lambda| \geq d(\lambda)$ and (6.1) we have for $|\lambda| \geq 1$ and $1 \leq k \leq p$:

$$\frac{(1+\Sigma^n_{h=1}|\xi_h|^{2m_h})^{w/2}}{\mu_k(\xi)-\lambda} \leq \frac{\gamma(1+\mu_k(\xi))}{|\mu_k(\xi)-\lambda|} \leq \gamma \left| 1 + \frac{\lambda+1}{\mu_k(\xi)-\lambda}\right| \leq 3\gamma|\lambda|/d(\lambda).$$

Since $t - s \leq jw$ we get

$$C^2_\lambda \leq (3\gamma\,|\lambda|)^{2(t-s)/w}\, d(\lambda)^{-2j}$$

and inserting this in (6.10) we obtain the desired inequality (6.5).

Every element of the matrix F^i_λ is sum of operators in $\mathbf{F}_\lambda(j)$. Therefore it follows from Lemma 6.1 that for s, t, j satisfying $s \leq t \leq s + jw$ there exists a constant C depending on r, j, p but independent of λ such that for $|\lambda| \geq 1$ and $f \in H_{\infty,p}$ we have:

$$(6.11)\qquad \| F^j_\lambda f\|_{tm,R^n} \leq C \frac{|\lambda|^{(t-s)/w}}{d(\lambda)^j} \| f \|_{sm,R^n}.$$

Suppose now that $wj > \Sigma^n_{h=1} \frac{1}{m_h}$. It follows from (6.9) that $\mathbf{F}_\lambda(j)$ is a class of integral (convolution) operators and that any $F_\lambda(j) \in \mathbf{F}_\lambda(j)$ has a continuous and bounded kernel $F_\lambda(j)(x,y) = F_\lambda(j)(x-y, 0)$ given by

$$
\begin{aligned}
(6.12)\qquad F_\lambda(j)(x,y) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi}\cdot\\
\cdot\,\Sigma^{p}_{k_1,\dots,k_j=1} u_{q_1,k_1}(\xi)\dots u_{q_j,k_j}(\xi)\,\overline{u}_{r_1,k_1}(\xi)\dots\overline{u}_{r_j,k_j}(\xi)\,\cdot\\
\cdot\,(\mu_{k_1}(\xi)-\lambda)^{-1}\dots(\mu_{k_j}(\xi)-\lambda)^{-1}\,d\xi.
\end{aligned}
$$

Moreover the kernel (6.12) has continuous bounded derivatives up to the reduced order $wj - \Sigma^n_{h=1}\dfrac{1}{m_h} - a$. In particular it follows from the semi-homogeneity of $\mu_k(\xi)$ and $u_{q,r}(\xi)$ by a straightforward computation that for $|\alpha;\mathrm{m}| \leq wj - \Sigma^n_{h=1}\dfrac{1}{m_h} - a$

$$
(6.13)\qquad [D^\alpha_x F_\lambda(j)](x,x) = (-\lambda)^{\Sigma^n_{h=1}\frac{1}{wm_h} + \frac{|\alpha;\mathrm{m}|}{w} - j}\cdot\Sigma^{p}_{k_1,\dots,k_j=1} C(k_1,\dots,k_j)
$$

where

$$
\begin{aligned}
C(k_1,\dots,k_j) = (2\pi)^{-n}\int_{R^n}\xi^\alpha\cdot u_{q_1,k_1}(\xi)\dots u_{q_j,k_j}(\xi)\,\overline{u}_{r_1,k_1}(\xi)\dots\overline{u}_{r_j,k_j}(\xi)\,\cdot\\
\cdot\,[\mu_{k_1}(\xi)+1]^{-1}\dots[\mu_{k_j}(\xi)+1]^{-1}\,d\xi
\end{aligned}
$$

and $(-\lambda)^{\frac{1}{w}\left[\Sigma\frac{1}{m_h}+|\alpha;\mathrm{m}|\right]-j}$ is the branch of the power which is positive on the negative axis. As a particular (and even simpler) case we get that F^j_λ has a continuous and bounded kernel given by

$$
(6.14)\qquad F^j_\lambda(x,y) = (2\pi)^{-n}\int_{R^n} e^{i(x-y)\cdot\xi}\,[\mathscr{A}(\xi)-\lambda I]^{-j}\,d\xi.
$$

This kernel has continuous bounded derivatives up to the reduced order $wj - \Sigma^n_{h=1}\dfrac{1}{m_h} - a$. It is simple to calculate that for $|\alpha;\mathrm{m}| \leq wj - \Sigma^n_{h=1}\dfrac{1}{m_h} - a$,

$$
(6.15)\qquad (D^\alpha_x F^j_\lambda)(x,x) = (-\lambda)^{\frac{1}{w}\left[\Sigma^n_{h=1}\frac{1}{m_h}+|\alpha;\mathrm{m}|\right]-j}(2\pi)^{-n}\int_{R^n}\xi^\alpha\,[\mathscr{A}(\xi)+I]^{-j}\,d\xi
$$

where the determination for the power in (6.15) is the same as in (6.13).

We consider now (following [4], pp. 11-13) a scalar operator S_λ of the form:

$$S_\lambda = B_{k+1}(x, D)\, F_\lambda(j_k)\, B_k(x, D)\, F_\lambda(j_{k-1}) \ldots F_\lambda(j_1)\, B_1(x, D) \tag{6.16}$$

where the $B_\nu(x, D)$ are (scalar) differential operators of reduced order $l_\nu \geq 0$, $1 \leq \nu \leq k+1$ with coefficients belonging to $C_*^\infty(R^n)$, and $F_\lambda(j_\nu) \in \mathbf{F}_\lambda(j_\nu)$ $1 \leq \nu \leq k$. We set

$$l = \Sigma_{\nu=1}^{n} l_\nu \, , \, j = \Sigma_{\nu=1}^{k} j_\nu \, .$$

Using (4.3) and Lemma 6.1 we find that S_λ which is a well-defined operator: $H_\infty \longrightarrow H_\infty$ is (after completion) a bounded linear operator: $H_{sm} \longrightarrow H_{tm}$ for any, s, t such that $t \leq s + w_j - l$.

By an alternate application of (4.3) and Lemma 6.1 to the factors of S_λ it is easy to see that the following estimates hold for $s - l \leq t \leq s + w_j - l$;

$$\| S_\lambda f \|_{tm, R^n} \leq (3p\gamma)^j \, C \frac{|\lambda|^{\frac{t-s+l}{w}}}{d(\lambda)^j} \| f \|_{sm, R^n} \tag{6.17}$$

for $f \in H_\infty$, $|\lambda| \geq 1$, where γ is the constant occurring in (6.1) and C is a constant depending only on w, m, l, j and on a common bound for the coefficients of $\{B_\nu\}$ and their derivatives up to a certain order. The verification of (6.17) follows immediately from the interpolation inequality (1.8) and the estimates for the norms of the operators $F_\lambda(j_\nu)$ in the appropriate spaces, exactly as in the proof of inequality (4.10) in [4].

Let us assume now that $wj \geq l$. Thus we may regard S_λ as a bounded linear operator: $L_2(R^n) \longrightarrow L_2(R^n)$. It is clear that S_λ^*, the adjoint of S_λ in $L_2(R^n)$, is an operator of the same type:

$$S_\lambda^* = B_1^* \, F_{\bar\lambda}(j_1) \ldots F_{\bar\lambda}(j_k)\, B_{k+1}^*$$

where B_ν^* denotes the formal adjoint of B_ν, and $F_{\bar\lambda}(j_\nu) \in \mathbf{F}_{\bar\lambda}(j_\nu)$ is the $L_2(R^n)$-adjoint of the operator $F_\lambda(j_\nu)$; this adjoint can be represented in the form (6.4) when λ is replaced by $\bar\lambda$ (since the elements of the operator matrix F_λ commute). We have

THEOREM 6.1: Suppose that $wj - l \geq w > \Sigma_{h=1}^{n} \frac{1}{m_h}$. Then S_λ is a (scalar) integral operator with a continuous bounded kernel $S_\lambda(x, y)$ on $R^n \times R^n$, satisfying the following estimate:

(6.18) $$| S_\lambda (x, y) | \leq (3p\gamma)^j C_0 \frac{|\lambda|^{\frac{1}{w}\left(\Sigma_{h=1}^n \frac{1}{m_h} + l\right)}}{d(\lambda)^j} \quad |\lambda| \geq 1$$

where γ is the constant appearing in (6.1), and C_0 is a constant depending only on $\mathfrak{m}, j, l, w$ and on a bound for the coefficient of $\{B_\nu\}$ and their derivatives up to a certain order.

Remark: It is possible, in analogy to Theorem 4.1 in [4], to obtain differentiability properties for the kernel $S_\lambda (x, y)$, and to weaken the assumptions on $wj - l$. However, Theorem 6.1 is sufficient for the sequel.

Proof: According to a remark made after the definition of semi ellipticity we may say that $w(\mathcal{A}) = 1$ with respect to the multi-index $w\mathfrak{m}$. The sum of the reduced orders of the differential operators B_ν with respect to $w\mathfrak{m}$ is l/w. It is clear that the norms of $H_{w\mathfrak{m}}(R^n)$ based on $\mathfrak{m}$ as a multi-index and w as an exponent (1.7) or those which are based on the multi-index $w\mathfrak{m}$, are equivalent. From (6.17) the estimates

(6.19) $$\| S_\lambda \|_{w\mathfrak{m}, R^n} \leq C \frac{|\lambda|^{1+\frac{l}{w}}}{d(\lambda)^j}$$

$$\| S_\lambda^* \|_{w\mathfrak{m}, R^n} \leq C \frac{|\lambda|^{1+\frac{l}{w}}}{d(\lambda)^j}$$

$$\| S_\lambda \|_{0, R^n} \leq C \frac{|\lambda|^{l/w}}{d(\lambda)^j}$$

follow for $|\lambda| \geq 1$, where C is a constant which depends only on $\mathfrak{m}, w, j, l$ and on a bound for the coefficients of B_ν and their derivatives up to a certain order. Theorem 3.1 implies that S_λ is an integral operator with a continuous kernel. Using (6.19) in inequality (3.2) (with $w\mathfrak{m}$ replacing $\mathfrak{m}$) we find that

$$| S_\lambda (x, y) | \leq C (\| S_\lambda \|_{w\mathfrak{m}, R^n} + \| S_\lambda^* \|_{w\mathfrak{m}, R^n})^{\Sigma_{h=1}^n \frac{1}{w m_h}} \cdot$$

$$\cdot (\| S_\lambda \|_{0, R^n})^{1 - \Sigma_{h=1}^n \frac{1}{w m_h}} \leq C_0 (3p\gamma)^j \frac{|\lambda|^{\Sigma_{h=1}^n \frac{1}{w m_h} + \frac{l}{w}}}{d(\lambda)^j}$$

and the theorem is proved.

REMARK: It is possible to get estimates similar to (6.10) and (6.11) for several classes of hypoelliptic differential operators which are more general than the class of semi-elliptic operators. Perhaps even some formulas similar to (but weaker than) (6.13) could be obtained for certain classes of hypoelliptic operators. However, since an analogue to Theorem 3.1 and especially to inequality (3.2) is lacking for operators with ranges in H_{k_s} spaces for more general hypoelliptic weight functions k (see remark at the end of section 3), we have no sufficiently strong analogue for Theorem 6.1. Therefore, the present author does not know whether a complete asymptotic expansion exists for resolvent kernels of more general hypoelliptic operators.

7. Some properties of commutators.

In this section we shall extend the method of commutators, introduced in [4] to deal with scalar operator, so as to be able to treat semi-elliptic systems. The main trouble lies, of course, in the fact that the (reduced) order of $\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}$ is in general no less than the sum of the (reduced) orders of $\mathcal{A}$ and $\mathcal{B}$ — if $\mathcal{A}$ and $\mathcal{B}$ are matrix differential operators. In the following we shall see how to overcome this difficulty.

Although the case of interest to us is that of differential operators, it will be convenient (as in [4]) to start by considering a more general situation.

Let M be a linear space over a field K, and let M^p be the direct sum of p copies of M. Let $\mathcal{A}: M^p \rightarrow M^p$ be a linear operator. We may regard $\mathcal{A}$ as a $p \times p$ matrix of operators $\mathcal{A}_{i,j}: M \rightarrow M$, $1 \leq i, j \leq p$. Denote by $\mathbf{A}$ the collection of the operators $\mathcal{A}_{i,j}$, $1 \leq i, j \leq p$. Let $\mathbf{B}$ be a set of linear operators. If $\mathbf{C}$, $\mathbf{D}$ are sets of linear operators mapping M into itself, we define the following sets by:

$$\mathbf{C} + \mathbf{D} = \{C + D : C \in \mathbf{C}, D \in \mathbf{D}\}$$

$$\mathbf{C} - \mathbf{D} = \{C - D : C \in \mathbf{C}, D \in \mathbf{D}\}$$

$$\mathbf{C}\,\mathbf{D} = \{C \cdot D : C \in \mathbf{C}, D \in \mathbf{D}\}.$$

As in [4], denote by $S(r, t)$ the set of r-vectors $J = (j_1, \dots, j_r)$ with integral components $0 \leq j_i \leq t$, $1 \leq i \leq r$. (The elements of $S(r, t)$ are multi-indices in R^r; to avoid confusion we use here Latin and not Greek letters). Set $|J| = j_1 + \dots + j_r$, $S(r) = \bigcup_{t=1}^{\infty} S(r, t)$ and denote by $J \cup (j_{r+1})$ $(J \in S(r))$

the vector $(j_1, \dots, j_r, j_{r+1}) \in S(r+1)$. Define a zero dimensional vector (belonging to $S(0, t)$) to be the empty vector. For the empty vector $J = \Phi$ set: $|J| = 0$ and $\Phi \cup (j_1) = (j_1)$.

Let us recall the operation of commutation (ad $\mathbf{D}$) $\mathbf{C} = \{CD - DC : C \in \mathbf{C}, D \in \mathbf{D}\}$. We define now classes of multiple commutators: $[\mathbf{C}, \mathbf{D}; J]$, $J \in S(r)$ non-empty, by

$$[\mathbf{C}, \mathbf{D}; J] = (\text{ad } \mathbf{D})^{j_r} \mathbf{C} \dots (\text{ad } \mathbf{D})^{j_1} \mathbf{C}. \tag{7.1}$$

Note that $[\mathbf{C}, \mathbf{D}; (1)] = [\mathbf{C}, \mathbf{D}] = (\text{ad } \mathbf{D})\, \mathbf{C}$ is the class of usual commutators of members of $\mathbf{C}$ and members of $\mathbf{D}$. Note also that if both $\mathbf{C}$ and $\mathbf{D}$ contain just one element (C and D respectively) then $[C, D; J]$ as defined according to (7.1) is the same operator as defined inductively in section 5 of [4].

Let $\lambda \in K$ be such that $\mathcal{A} - \lambda I$ is a one to one mapping of M^p onto itself and let $F_\lambda = (\mathcal{A} - \lambda I)^{-1}$. We denote by $\mathbf{F}_\lambda(j)$ the set of products

$$(F_\lambda)_{r_1, s_1} (F_\lambda)_{r_2, s_2} \dots (F_\lambda)_{r_j, s_j}, \qquad 1 \le r_i, j_i \le p,\ 1 \le i \le j.$$

(In our notation $\mathbf{F}_\lambda(j) = [\mathbf{F}_\lambda(1)]^j$). For any set $\mathbf{E}$ of linear operators $M \to M$ we denote by $\{\mathbf{E}\}$ the additive semi-group generated by elements of $\mathbf{E}$, i.e. the set of finite sums of elements of $\mathbf{E}$.

THEOREM 7.1: Let r and k be positive integers. Then

$$[\mathbf{F}_\lambda(1) B]^r \subset \Sigma_{J \in S(r, k-1)} \{[\mathbf{B}, \mathbf{A}; J]\, \mathbf{F}_\lambda(|J| + r)\} + \tag{7.2}$$

$$+ \Sigma_{s=0}^{r-1} [\mathbf{F}_\lambda(1)\, \mathbf{B}]^s \{\mathbf{F}_\lambda(1)\, \Sigma_{J \in S(r-s-1, k-1)} [\mathbf{B}, \mathbf{A}; J \cup (k)]\, \mathbf{F}_\lambda(|J| + k + r - s - 1)\}.$$

PROOF: The proof proceeds inductively in several steps as the proof of theorem 5.1 in [4]. Consider first the case $r = k = 1$. A typical element of $\mathbf{F}_\lambda(1)\, \mathbf{B}_l$ is $(F_\lambda)_{t,s}\, \mathbf{B}_l$ where $1 \le t, s \le p$ and $\mathbf{B}_l \in B$. One may regard this element as the t, s element of the matrix $F_\lambda B_l I$ (where I is the unit operator in M^p).

The identity

$$F_\lambda B_l I = B_l I F_\lambda + F_\lambda (B_l I \mathcal{A} - \mathcal{A} B_l I) F_\lambda \tag{7.3}$$

is immediately verified by applying $\mathcal{A} - \lambda I$ on both sides of (7.3) from the left. The matrix (operator in M^p) identity (7.3) implies equality of the t, s-matrix elements:

$$(7.4)\qquad (F_\lambda)_{t,s}\, B_l = B_l\,(F_\lambda)_{t,s} + \Sigma^{p}_{h,i=1}\,(F_\lambda)_{t,h}\,[B_l\,\mathcal{A}_{h,i} - \mathcal{A}_{h,i}\,B_l]\,(F_\lambda)_{i,s}\,.$$

But the right-hand side of (7.4) is contained in $\underline{B}\underline{F}_\lambda(1) + \{\underline{F}_\lambda(1)[\underline{B}, \underline{A}\,;(1)]\,\underline{F}_\lambda(1)\}$. Therefore

$$(7.5)\qquad \mathbf{E}_\lambda(1)\,\mathbf{B} \subset \mathbf{B}\mathbf{F}_\lambda(1) + \{\mathbf{F}_\lambda(1)\,[\mathbf{B}, \mathbf{A}\,;(1)]\,\mathbf{F}_\lambda = (1)\}$$

which is (7.2) reduced to the special case $r = k = 1$.

It is in (7.5) that the main difference between the scalar and the matrix cases lies. The rest of the proof is very similar to the corresponding parts of the proof of theorem 5.1 in [4], and will be sketched only.

One may rewrite (7.5) as

$$(7.6)\qquad \mathbf{F}_\lambda(1)\,\mathbf{B} \subset \mathbf{B}\mathbf{F}_\lambda(1) + \{\mathbf{F}_\lambda(1)\,[(\mathrm{ad}\,\mathbf{A})^1\,\mathbf{B}]\,\mathbf{F}_\lambda(1)\}.$$

Suppose that (7.2) has been established already for $r = 1$ and some k, i. e. (according to definition (7.1)) suppose that

$$(7.7)\qquad \mathbf{F}_\lambda(1)\,\mathbf{B} \subset \Sigma^{k-1}_{j=0}\,\{[\mathbf{B}, \mathbf{A}\,;(j)]\,\mathbf{F}_\lambda(j+1)\} + \{\mathbf{F}_\lambda(1)\,[(\mathrm{ad}\,\mathbf{A})^k\,\mathbf{B}]\,\mathbf{F}_\lambda(k)\}.$$

Using (7.6) for $(\mathrm{ad}\ \mathbf{A})^k\,\mathbf{B}$ replacing $\mathbf{B}$ and applying definition (7.1) we get easily (7.7) with $k+1$ replacing k, which proves the theorem for $r = 1$. Assume now that the theorem has been proved for some r. Then

$$(7.8)\qquad [\mathbf{F}_\lambda(1)\mathbf{B}]^{r+1} = \mathbf{F}_\lambda(1)\,\mathbf{B}[\mathbf{F}_\lambda(1)\,\mathbf{B}^r] \subset \mathbf{F}_\lambda(1)\,\mathbf{B}\Sigma_{J\in S(r,k-1)}\{[\mathbf{B}, \mathbf{A}\ ;J]\,\mathbf{F}_\lambda(|J|+r)\} +$$

$$+ \Sigma^{r-1}_{s=0}[\mathbf{F}_\lambda(1)\,\mathbf{B}]^{s+1}\{\mathbf{F}_\lambda(1)\,\Sigma_{J\in S(r-s-1,k-1)}\,[\mathbf{B}, \mathbf{A}\,;J\cup(k)]\,\mathbf{F}_\lambda(|J|+k+r-s-1)\}.$$

According to (7.7) with $\mathbf{B}$ replaced by $\mathbf{B}\,[\mathbf{B}, \mathbf{A}\,;J]$ we may rewrite every set of operators which generates the classes in the first sum in the right-hand side of (7.8) in a form which (after repeated use of definition (7.1)) will lead us to (7.2) with r replaced by $r+1$, in essentially the same way which leads from (5.8) to (5.9) in [4], and this concludes the proof.

REMARK: It would be possible to replace the inclusion (7.2) by an equation generalizing (7.4), but then the quantity of indices would have been discouragingly large. The less explicit form of theorem 7.1 is adequate enough for our purposes.

Assume now that there exists a subring $\mathbf{R}$ of the ring of linear transformations from M to M and a function w from $\mathbf{R}$ to the real line so that the following conditions hold:

$$w(0) = -\infty \tag{7.9}$$

$$w(AB) \leq w(A) + w(B) \tag{7.10}$$

$$w(AB - BA) \leq w(A) + w(B) - b, \ b \text{ a positive constant} \tag{7.11}$$

$$w(I) = 0. \tag{7.12}$$

We denote also (for $\mathbf{C} \subset \mathbf{R}$)

$$w(\mathbf{C}) = \sup_{C \in \mathbf{C}} w(C). \tag{7.13}$$

THEOREM 7.2: Let $J \in S(r)$, $r > 0$. Then

$$w([\mathbf{C}, \mathbf{D}; J]) \leq |J| \, w(\mathbf{D}) + r w(\mathbf{C}) - |J| \, b. \tag{7.14}$$

PROOF: the assumption (7.11) implies that $w((\text{ad } \mathbf{D})\mathbf{C}) \leq w(\mathbf{C}) + w(\mathbf{D}) - b$. The assumption (7.10) implies that $w(\mathbf{CD}) \leq w(\mathbf{C}) + w(\mathbf{D})$. The conclusion

$$w((\text{ad } \mathbf{D})^{j_r} \mathbf{C} \ldots (\text{ad } \mathbf{D})^{j_1} \mathbf{C}) \leq j_r w(\mathbf{D}) - j_r w(\mathbf{D}) + \\ + w(\mathbf{C}) + \ldots + j_1 w(\mathbf{D}) - j_1 b + w(\mathbf{C}) \tag{7.15}$$

follows immediately. By definitions (7.15) is (7.14).

Theorem 7.2 is a (trivial) generalization of lemma 5.1 of [4].

Theorems 7.1 and 7.2 will be applied in the sequel to the case where M is the linear space $H_\infty(R^n)$ and $\mathbf{R}$ is the ring of (scalar) differential operators with C_*^∞ coefficients. For any multi-index with positive components m a function $w(A)$ is defined ($A \in \mathbf{R}$) to be the reduced order of A with respect to m. (The property (7.11) follows immediately from Leibnitz's rule, where $b = b(\mathsf{m})$ is defined as at the beginning of section 5).

It is possible to generalize theorem 5.2 of [4] (which describes zeros of coefficients of commutators of differentiai operators), but this generalization would not be explicit enough to yield the same results as in [4], since (in the semi-elliptic case) it is impossible to have a good estimate for the order of coefficients in a multiple-commutator which do vanish.

8. Several localization and comparison lemmas.

It is well known (e. g. [16], [19]) that the asymptotic behavoir of the resolvent kernel is essentially a local property of the coefficients of the differential operator. We shall need the following lemma which is an extension of lemma 4.2 of [5] to semi-elliptic systems. Note that for results in the interior a weaker localization property (lemma 6.1 of [4]) suffices in the elliptic case. (See semark at the end of the preceding section).

LEMMA 8.1: Let $\mathbf{m}$ be a multi-index with positive components and let w be a rational positive number such that w is a reduced order of a positive semi-elliptic differential operator with respect to $\mathbf{m}$. For every complex λ which is not on the non-negative axis, let T_λ be a bounded linear operator in $L_2(\Omega)^p$ $(\Omega \subset R^n)$ with range contained in $H_{w\mathbf{m}}(\Omega)^p$. Suppose that

$$(8.1) \qquad \| T_\lambda \|_{0,\Omega} \leq \frac{C_1}{d(\lambda)}, \quad \| T_\lambda \|_{w\mathbf{m},\Omega} \leq \frac{C_1 |\lambda|}{d(\lambda)}, \quad |\lambda| \geq 1$$

where C_1 is a constant.

For any point $y \in R^n$ and positive r set

$$(8.2) \qquad \Sigma_{r,y} = \{x \in R^n : |x_i - y_i| < r^{\frac{1}{wm_i}}, \quad 1 \leq i \leq n\}$$

(i) Assume that there exist $x^0 \in \Omega$ and a positive r such that $\Sigma_{r,x^0} \subset \Omega$ and there exists a positive differential operator $\mathcal{A}$ which is semi elliptic in the restricted sense with respect to $\mathbf{m}$ with reduced order w, such that

$$(8.3) \qquad (\mathcal{A}(x, D) - \lambda)(T_\lambda f)(x) = 0$$

for $x \in \Sigma_{r,x^0}$ and $f \in L_2(\Omega)^p$.

Then for every integer $j \geq 0$ and every μ, $0 < \mu < 1$, there exists a constant C (independent of λ and r) such that for all $f \in L_2(\Omega)^p$

$$(8.4) \qquad \| T_\lambda f \|_{0,\Sigma_{\mu r,x^0}} \leq \frac{C}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \|f\|_{0,\Omega}$$

$$(8.5) \qquad \| T_\lambda f \|_{w\mathbf{m},\Sigma_{\mu r,x^0}} \leq \frac{C|\lambda|}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \|f\|_{0,\Omega}$$

for $|\lambda| \geq 1 \left(b = \min\limits_{1 \leq i \leq m} \dfrac{1}{m_i}\right)$.

(ii) Suppose in addition that $\Sigma_{h=1}^{n} \frac{1}{m_h} < w$. If the range of T_λ^* is also contained in $H_{w\mathbf{m}}(\Omega)^p$ then T_λ is an integral operator with a continuous matrix kernel. If the inequality

$$\| T_\lambda^* \|_{w\mathbf{m},\,\Omega} \leq \frac{C_1 |\lambda|}{d(\lambda)} \tag{8.1'}$$

holds for $|\lambda| \geq 1$ and if there exists a positive differential operator $\mathcal{A}_1$ which is semi-elliptic in the restricted sense with respect to $\mathbf{m}$ with reduced order w, such that

$$(\mathcal{A}_1(x, D) - \bar{\lambda})(T_\lambda^* f)(x) = 0 \tag{8.3'}$$

for $x \in \Sigma_{r,x^0}$ and all $f \in L_2(\Omega)^p$, then for every integer $j \geq 0$,

$$|T_{\lambda,s,t}(x,y)| \leq \frac{C |\lambda|^{\Sigma_{h=1}^{n} \frac{1}{wm_h}}}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \tag{8.6}$$

for $1 \leq s,\ t \leq p$, $|\lambda| \geq 1$, and $x, y \in \Sigma_{\mu r, x^0}$. ($C$ depends on μ and j).

PROOF: Without loss of generality assume x^0 is the origin. We prove first (8.4) and (8.5) by induction on j. For $j = 0$ they follow immediately from assumption (8.1). We may assume that $r \geq |\lambda|^{-1}$, since (8.4) and (8.5) are otherwise weaker for $j > 0$ than for $j = 0$. Let $\varphi \in C_0^\infty(R^n)$ be such that $\varphi(x) \equiv 1$ for $\max_{1 \leq h \leq n} |x_h| \leq \mu$, $\varphi(x) \equiv 0$ for $\max_{1 \leq h \leq n} |x_h| \geq \frac{1+\mu}{2} = \mu_1$. Then

$$\begin{aligned}(\mathcal{A} - \lambda)\,\varphi\left(x_1 r^{-\frac{1}{wm_1}}, \ldots, x_n r^{-\frac{1}{wm_n}}\right) T_\lambda f = \\ = \varphi\left(x_1 r^{-\frac{1}{wm_1}}, \ldots, x_n r^{-\frac{1}{wm_n}}\right)(\mathcal{A} - \lambda)\, T_\lambda f + \\ + \sum_{\substack{\alpha > 0 \\ |(\alpha+\beta):\mathbf{m}| \leq w}} C_{\alpha,\beta}\,(D^\alpha \varphi)\left(x_1 r^{-\frac{1}{wm_1}}, \ldots, x_n r^{-\frac{1}{wm_n}}\right) D^\beta\, T_\lambda f.\end{aligned} \tag{8.7}$$

(The matrices $C_{\alpha,\beta}$ depend only on $\mathcal{A}$). The first term on the right-hand side of (8.7) vanishes, and the second one may be estimated by

$$(8.8) \qquad C \sum_{\substack{\alpha>0 \\ |(\alpha+\beta):\mathbf{m}|\leq w}} r^{-|\alpha:\mathbf{m}|/w} \| D^\beta T_\lambda f \|_{0,\Sigma_{\mu_1 r,0}} .$$

The induction assumption is that (8.4) and (8.5) have been proved already (for j) for *all* $0 < \mu < 1$, in particular for $\mu_1 = \dfrac{1+\mu}{2}$. Using the local interpolation inequality (2.8) with $w\mathbf{m}$ replacing $\mathbf{m}$ we get (from the induction assumption) for all β satisfying $|\beta : \mathbf{m}| \leq w$,

$$(8.9) \qquad \| D^\beta T_\lambda f \|_{0,\Sigma_{\mu_1 r,0}} \leq$$

$$\leq \frac{C r^{-|\beta:w\mathbf{m}|}}{d(\lambda)} [1 + |\lambda|^{|\beta:w\mathbf{m}|} r^{|\beta:w\mathbf{m}|}] \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \| f \|_{0,\Omega} \leq$$

$$\leq \frac{C |\lambda|^{|\beta:w\mathbf{m}|}}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \| f \|_{0,\Omega},$$

the last inequality following from the assumption $r \geq |\lambda|^{-1}$. Hence we may estimate (8.8) by

$$C \sum_{\substack{\alpha>0 \\ |(\alpha+\beta):\mathbf{m}|\leq w}} r^{-|\alpha:\mathbf{m}|/w} \frac{|\lambda|^{|\beta:w\mathbf{m}|}}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \| f \|_{0,\Omega} .$$

But $\alpha > 0$ implies $|\alpha : \mathbf{m}| \geq b$ so that for $|\lambda| \geq 1$, $r \geq |\lambda|^{-1}$ we have that

$$r^{-|\alpha:\mathbf{m}|/w} |\lambda|^{|\beta:w\mathbf{m}|} \leq \frac{|\lambda|^{1-b/w}}{r^{b/w}} .$$

Therefore we find, by (8.7), (8.8) and (8.9), that

$$(8.10) \quad \| (\mathscr{A} - \lambda) \varphi (x_1 r^{-\frac{1}{wm_1}}, \dots, x_n r^{-\frac{1}{wm_n}}) T_\lambda f \|_{0,\Omega} \leq C \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^{j+1} \| f \|_{0,\Omega} .$$

It is easily shown that, for $\mathscr{A}$ positive,

$$(8.11) \qquad \| v \|_{0,\Omega} \leq \frac{\| (\mathscr{A} - \lambda) v \|_{0,\Omega}}{d(\lambda)}$$

fro all $v \in H_{w\mathbf{m}}(\Omega)^p$ with compact support contained in Ω. Hence, (8.4) fol lows from (8.10) for $j+1$, since

$$\| T_\lambda f \|_{0, \Sigma_{\mu r, 0}} \leq \| \varphi (x_1 r^{-\frac{1}{wm_1}}, \dots, x_n r^{-\frac{1}{wm_n}}) T_\lambda f \|_{0, \Omega} .$$

Let us note that from (8.11) and well known a priori estimates for semi-elliptic systems ([26]; [1], [9]) that if $v \in H_{w\mathsf{m}}(\Omega)^p$ has a compact support contained in Ω, then

$$(8.12) \qquad \| v \|_{w\mathsf{m}, \Omega} \leq C (\| \mathcal{A} v \|_{0, \Omega} + \| v \|_{0, \Omega}) \leq C (\| (\mathcal{A} - \lambda) v \|_{0, \Omega} + (| \lambda | + 1) \| v \|_{0, \Omega}) \leq \frac{C | \lambda |}{d(\lambda)} \| (\mathcal{A} - \lambda) v \|_{0, \Omega} , \qquad | \lambda | \geq 1.$$

The inequality (8.5) follows (for $j+1$) from (8.10) and (8.12), since

$$\| T_\lambda f \|_{w\mathsf{m}, \Sigma_{\mu r, 0}} \leq \| \varphi (x_1 r^{-\frac{1}{wm_1}}, \dots, x_n r^{-\frac{1}{wm_n}}) T_\lambda f \|_{w\mathsf{m}, \Omega}$$

and (i) is proved.

If the assumptions of (ii) are also satisfied the estimates (8.4) and (8.5) follow with T_λ^* replacing T_λ from (8.1′) and (8.3′) in the same way as above (one has to substitute $\mathcal{A}_1$ for $\mathcal{A}$ and $\bar{\lambda}$ for λ). Using theorem 3.2 in the special case $\varkappa =$ identity we find from inequality (3.14) (with $w\mathsf{m}$ replacing m) and (8.4), (8.5) that

$$| T_{\lambda, s, t}(x, y) | \leq \frac{C}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)} \right]^j \prod_{h=1}^{n} (| \lambda | + r^{-1})^{\frac{1}{wm_h}} \leq \frac{C |\lambda|^{\sum_{h=1}^n \frac{1}{wm_h}}}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)} \right]^j$$

for $1 \leq s,\ t \leq p,\ r \geq |\lambda|^{-1}$. If $r \leq |\lambda|^{-1}$ the desired inequality follows immediately from (8.4) and (8.5) and theorem 3.2.

REMARK: Lemma 8.1 replaces the exponential decrease outside the diagonal of the fundamental solution ([16] in the elliptic case, [19] in the elliptic case with λ in the complex plane, [22] in general hypoelliptic case). In our case it is inconvenient to proceed as in [16], [19] and [22], i. e., to estimate kernels directly, since it is not a simple matter to do this for operators with non-constant coefficients, especially in those regions of the complex plane where we need the estimates.

Let now m, w, $\mathscr{A}$ and F_λ have the same meaning as in section 6. Let G_λ be a linear operator in $L_2(R^n)^p$ defined for complex λ not contained in the non-negative axis, such that the ranges of G_λ and G_λ^* are contained in $H_{w\mathsf{m}}(R^n)^p$ and such that the inequalities

$$\| G_\lambda \|_0 \leq \frac{C}{d(\lambda)}, \quad \| G_\lambda \|_{w\mathsf{m},\, R^n} \leq \frac{C\,|\,\lambda\,|}{d(\lambda)}$$

(8.11)

$$\| G_\lambda^* \|_{w\mathsf{m},\, R^n} \leq \frac{C\,|\,\lambda\,|}{d(\lambda)}$$

hold for $|\,\lambda\,| \geq 1$, with a constant C. Let $\mathscr{B}$ be a differential (matrix) operator with coefficients whose entries are $C_*^\infty(R^n)$ functions, and let $w(\mathscr{B}) = w$. Let k be a positive integer and assume that all the elements of the matrices $\mathscr{B}_\alpha$ with $|\,\alpha : \mathsf{m}\,| = w - ia$ ($a = a(\mathsf{m})$ is defined at the beginning of section 5) have a zero of type q_i (at least) at x^0, $0 \leq i \leq k-1$. Let r_i be positive numbers $1 \leq i \leq k-1$ and let φ be a (fixed) $C_0^\infty(R^n)$ function such that $\varphi(x) \equiv 1$ for $\max\limits_{1 \leq h \leq n} |\,x_h\,| \leq \frac{1}{2}$, $\varphi(x) \equiv 0$ for $\max\limits_{1 \leq h \leq n} |\,x_n\,| \geq 1$. Denote $\mathscr{B}_i = \sum\limits_{|\,\alpha : \mathsf{m}\,| = w - ia} \mathscr{B}_\alpha(x) D^\alpha$ and set

$$\mathscr{B}_{\underline{r}} = \varphi\left((x_1 - x_1^0)\, r_0^{-\frac{1}{m_1}}, \ldots, (x_n - x_n^0)\, r_0^{-\frac{1}{m_n}}\right) \mathscr{B}_0 +$$

(8.12)
$$+ \ldots + \varphi\left((x_1 - x_1^0)\, r_{k-1}^{-\frac{1}{m_1}}, \ldots, (x_n - x_n^0)\, r_{k-1}^{-\frac{1}{m_n}}\right) \mathscr{B}_{k-1} +$$

$$+ \sum\nolimits_{|\,\alpha : \mathsf{m}\,| \leq w - ka} \mathscr{B}_\alpha(x)\, D^\alpha .$$

LEMMA 8.2: Let $w > \sum_{h=1}^n \frac{1}{m_h}$. Then the operator $(F_\lambda \mathscr{B}_{\underline{r}})^j G_\lambda$ (j a positive integer) is an integral operator with a continuous bounded matrix kernel and the following inequality

(8.13)
$$|\,[(F_\lambda \mathscr{B}_{\underline{r}})^j G_\lambda]_{s,t}(x, y)\,| \leq \frac{C\,|\,\lambda\,|^{\sum_{h=1}^n \frac{1}{w m_h}}}{d(\lambda)}\, g(\lambda)^j$$

holds for $1 \leq s, t \leq p$, $|\lambda| \geq 1$ and $1 \geq r_i \geq |\lambda|^{-1}$, where $g(\lambda)$ is defined by

$$(8.14) \qquad g(\lambda) = \max \left(r_0^{q_0} \frac{|\lambda|}{d(\lambda)}, \; r_1^{q_1} \frac{|\lambda|^{1-a/w}}{d(\lambda)}, \ldots, r_{k-1}^{q_{k-1}} \frac{|\lambda|^{1-(k-1)a/w}}{d(\lambda)}, \; \frac{|\lambda|^{1-ka/w}}{d(\lambda)} \right)$$

and C is a constant which depends only on the coefficients of $\mathcal{B}$, $\mathbf{m}$, w, φ, j, the constant appearing in (8.11), and the constant γ which appears in (6.1). Here and in the sequel we agree to replace $r_i^{q_i}$ (where q_i is a zero type) by r_i^N with any positive N, if $q_i = \infty$.

PROOF: Without loss of generality assume that $x^0 = 0$. Note that if $u \in C^\infty$ has a zero of type q at the origin then

$$u(x) = \underset{x \to 0}{O} \left([\Sigma_{i=1}^n |x_i|^{m_i}]^q \right)$$

(and if $q = \infty$ then $u(x) = O([\Sigma |x_i|^{m_i}]^N)$ for every $N > 0$). Using (6.11) and completion it is easy to see that

$$(8.15) \qquad \| \mathcal{B}_{\underline{r}} F_\lambda \|_{0, R^n} \leq C g(\lambda)$$

where here and in the following C denotes a constant depending only on $\mathbf{m}$, w, φ, j, γ, the constant appearing in (8.11) and the coefficients of $\mathcal{B}$. Using (8.11) and interpolation we see in the same way that

$$(8.16) \qquad \| \mathcal{B}_{\underline{r}} G_\lambda \|_{0, R^n} \leq C g(\lambda).$$

Let us point out that

$$\mathcal{B}_{\underline{r}}^* = \mathcal{B}_0^* \varphi\left(x_1 r_0^{-\frac{1}{m_1}}, \ldots, x_n r_0^{-\frac{1}{m_n}}\right) + \ldots +$$
$$+ \mathcal{B}_{k-1}^* \varphi\left(x_1 r_{k-1}^{-\frac{1}{m_1}}, \ldots, x_n r_{k-1}^{-\frac{1}{m_n}}\right) + \left(\Sigma_{|\alpha : \mathbf{m}| \leq w - ka} \mathcal{B}_\alpha D^\alpha\right)^*.$$

From (6.11) and Leibnitz's rule it follows that

$$\| \mathcal{B}_i^* \varphi\left(x_1 r_i^{-\frac{1}{m_1}}, \ldots, x_n r_i^{-\frac{1}{m_n}}\right) F_\lambda \|_{0, R^n} \leq$$

$$\leq \sum_{\substack{|\alpha : \mathbf{m}| = w - ia \\ \beta + \gamma \leq \alpha}} \sup_x \left| (D^\beta \mathcal{B}_\alpha)(x) D^\gamma \varphi\left(x_1 r_i^{-\frac{1}{m_1}}, \ldots, x_n r_i^{-\frac{1}{m_n}}\right) \right| \cdot$$
$$\cdot \frac{|\lambda|^{1 - ia/w - |(\beta+\gamma) : \mathbf{m}|}}{d(\lambda)}.$$

(We recall that $\beta + \gamma \leq \alpha$ if $\beta_h + \gamma_h \leq \alpha_h$ for all $1 \leq h \leq n$).

Since the coefficients of $\mathcal{B}_i$ have a zero of type q_i at the origin we may estimate the right-hand side of the preceding inequality by

$$C \sum_{|(\beta+\gamma):\mathfrak{m}| \leq w-ia} (|\lambda| r_i)^{-|(\beta+\gamma):\mathfrak{m}|} r_i^{q_i} \frac{|\lambda|^{1-ia/w}}{d(\lambda)} \leq C r_i^{q_i} \frac{|\lambda|^{1-ia/w}}{d(\lambda)}$$

where the last inequality follows from the assumption $r_i \geq |\lambda|^{-1}$. Combining this and the obvious estimate for $(\Sigma_{|\alpha:\mathfrak{m}| \leq w-k\alpha} \mathcal{B}_\alpha D^\alpha)^* F_\lambda$ we obtain the estimate

$$\| \mathcal{B}_r^* F_\lambda \|_{0, R^n} \leq C g(\lambda). \tag{8.17}$$

Using (6.11), (8.15) and (8.16) we see that

$$\| (F_\lambda \mathcal{B}_{\underline{r}})^j G_\lambda \|_{0, R^n} \leq \| F_\lambda \|_{0, R^n} (\| \mathcal{B}_{\underline{r}} F_\lambda \|_{0, R^n})^{j-1} \| \mathcal{B}_{\underline{r}} G_\lambda \|_{0, R^n} \leq \frac{C g(\lambda)^j}{d(\lambda)} \tag{8.18}$$

$$\| (F_\lambda \mathcal{B}_{\underline{r}})^j G_\lambda \|_{w\mathfrak{m}, R^n} \leq \| F_\lambda \|_{w\mathfrak{m}, R^n} (\| \mathcal{B}_{\underline{r}} F_\lambda \|_{0, R^n})^{j-1} \| \mathcal{B}_{\underline{r}} G_\lambda \|_{0, R^n} \leq \frac{C|\lambda|}{d(\lambda)} g(\lambda)^j. \tag{8.19}$$

Using (8.11) and (8.17) we find that

$$\begin{aligned} \| [(F_\lambda \mathcal{B}_{\underline{r}})^j G_\lambda]^* \|_{w\mathfrak{m}, R^n} &= \| G_\lambda^* [(F_\lambda \mathcal{B}_{\underline{r}})^*]^j \|_{w\mathfrak{m}, R^n} = \\ &= \| G_\lambda^* (\mathcal{B}_{\underline{r}}^* F_{\bar\lambda})^j \|_{w\mathfrak{m}, R^n} \leq \| G_\lambda^* \|_{w\mathfrak{m}, R^n} (\| \mathcal{B}_{\underline{r}}^* F_{\bar\lambda} \|_{0, R^n})^j \leq \\ &\leq \frac{C|\lambda|}{d(\lambda)} g(\lambda)^j. \end{aligned} \tag{8.20}$$

Applying theorem 3.1 and formula (3.2) (with the multi-index $w\mathfrak{m}$), together with (8.18), (8.19) and (8.20), we obtain the desired inequality (8.13).

In order to prove theorem 5.1 in a general case we need an estimate for the kernel of the operator $(F_\lambda \mathcal{B})^l G_\lambda$, at least near x^0. The difference between this kernel and the kernel estimated in lemma 8.2 is treated in the following:

LEMMA 8.3: Suppose that all the assumptions of lemma 8.2 are fulfilled. Suppose also that there exist positive differential operators $\mathcal{A}_1$ and $\mathcal{A}_2$, semi-elliptic in the restricted sense (in R^n) with respect to $\mathfrak{m}$ and of reduced order w, such that

$$(\mathcal{A}_1 - \lambda I) G_\lambda f = f, \; (\mathcal{A}_2 - \bar\lambda I) G_\lambda^* f = f$$

for all $f \in L_2(R^n)^p$. Set $r = \left[\min_{1\le h\le n}\left(\frac{1}{2}\right)^{wm_h}\right]\left[\min_{0\le i\le k-1} r_i^w\right]$. Then for all positive integers j and l and for all μ, $0 < \mu < 1$, there exists a constant C which depends on μ, m, w, φ, l, j, γ, $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{B}$ and the constant appearing in (8.11), but which does not depend on r and λ (for $|\lambda| \ge 1$), such that the estimate

$$|[F_\lambda \mathcal{B})^l G_\lambda - (F_\lambda \mathcal{B}_{\underline{r}})^l G_\lambda]_{s,t}(x, y)| \le$$

$$\le \frac{C|\lambda|^{\sum_{h=1}^n \frac{1}{wm_h}}}{d(\lambda)} \left(\frac{|\lambda|}{d(\lambda)}\right)^l \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \tag{8.21}$$

holds for $x, y \in \Sigma_{\mu r, x^0}$, $1 \le s, t \le p$.

PROOF: We may assume that $r \ge |\lambda|^{-1}$ since otherwise (8.21) is immediate. From now on let C denote a generic constant having the same dependence as the constant in the statement of our lemma. Set

$$F(\underline{r}, \lambda, q) = (F_\lambda \mathcal{B})^q G_\lambda - (F_\lambda \mathcal{B}_{\underline{r}})^q G_\lambda$$

for q integer. Denote by R_μ the restriction operator, $R_\mu : L_2(R^n)^p \to L_2(\Sigma_{\mu r, x^0})^p$, and let $E_\mu : L_2(\Sigma_{\mu r, x^0})^p \to L_2(R^n)^p$ be an extension operator extending $f \in L_2(\Sigma_{\mu r, x^0})^p$ as zero in $R^n - \Sigma_{\mu r, x^0}$. We have:

$$F(\underline{r}, \lambda, q+1) = F_\lambda E_\mu R_\mu \mathcal{B} F(\underline{r}, \lambda, q) + F_\lambda (I - E_\mu R_\mu) \mathcal{B} F(\underline{r}, \lambda, q) + F_\lambda (\mathcal{B} - \mathcal{B}_{\underline{r}}) (F_\lambda \mathcal{B}_{\underline{r}})^q G_\lambda . \tag{8.22}$$

Using part (i) of lemma 8.1 (the estimates (8.4) and (8.5) for $T_\lambda = F_\lambda(I - E_\mu R_\mu)$) we find that for any $0 < \mu < \mu_1 < 1$ there exists a C such that for all $f \in L_2(R^n)^p$,

$$\| F_\lambda (I - E_{\mu_1} R_{\mu_1} \mathcal{B} F(\underline{r}, \lambda, q) f \|_{0, \Sigma_{\mu r, x^0}} \le$$

$$\le \frac{C}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \| \mathcal{B} F(\underline{r}, \lambda, q) f \|_{0, R^n} \le \tag{8.23}$$

$$\le \frac{C}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w} d(\lambda)}\right]^j \left(\frac{|\lambda|}{d(\lambda)}\right)^{q+1} \| f \|_{0, R^n} .$$

(We use also (8.15) and (8.16)). We find also that

$$\| F_\lambda (I - E_{\mu_1} R_{\mu_1}) \mathcal{B} F(\underline{r}, \lambda, q) f \|_{wm,\, \Sigma_{\mu r,\, x^0}} \leq C \left(\frac{|\lambda|}{d(\lambda)}\right)^{q+2} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)}\right]^j \|f\|_{0, R^n}. \tag{8.24}$$

By assumptions the differential operators $\mathcal{B}$ and $\mathcal{B}_{\underline{r}}$ coincide in $\Sigma_{r,\, x^0}$. Applying lemma 8.1 once again we see that

$$\begin{aligned} &\| F_\lambda (\mathcal{B} - \mathcal{B}_{\underline{r}}) (F_\lambda \mathcal{B}_{\underline{r}})^q G_\lambda f \|_{0,\, \Sigma_{\mu r,\, x^0}} = \\ &= \| F_\lambda (I - E_{\mu_1} R_{\mu_1}) (\mathcal{B} - \mathcal{B}_{\underline{r}}) (F_\lambda \mathcal{B}_{\underline{r}})^q G_\lambda f \|_{0,\, \Sigma_{\mu r,\, x^0}} \leq \\ &\leq \frac{C}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)}\right]^j \| (F_\lambda \mathcal{B}_{\underline{r}})^q G_\lambda f \|_{wm,\, R^n} \leq \\ &\leq \frac{C}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)}\right]^j \left(\frac{|\lambda|}{d(\lambda)}\right)^{q+1} \| f \|_{0,\, R^n} \end{aligned} \tag{8.25}$$

and

$$\| F_\lambda (\mathcal{B} - \mathcal{B}_{\underline{r}}) (F_\lambda \mathcal{B}_{\underline{r}})^q G_\lambda f \|_{wm,\, \Sigma_{\mu r,\, x^0}} \leq C \left(\frac{|\lambda|}{d(\lambda)}\right)^{q+2} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)}\right]^j \|f\|_{0,\, R^n}. \tag{8.26}$$

Furthermore, we know that

$$\| F_\lambda E_\mu R_\mu \mathcal{B} F(\underline{r}, \lambda, q) f \|_{0,\, R^n} \leq \frac{C}{d(\lambda)} \| F(\underline{r}, \lambda, q) \|_{wm,\, \Sigma_{\mu r,\, x^0}} \tag{8.27}$$

$$\| F_\lambda E_\mu R_\mu \mathcal{B} F(\underline{r}, \lambda, q) f \|_{wm,\, R^n} \leq \frac{C|\lambda|}{d(\lambda)} \| F(\underline{r}, \lambda, q)) \|_{wm,\, \Sigma_{\mu r,\, x^0}}. \tag{8.28}$$

Since the right-hand sides of (8.27) and (8.28) vanish for $q = 0$, we get estimating the terms in (8.22) recursively using the inequalities (8.23) through (8.28) and choosing a sequence $0 < \mu < \mu_1 < \ldots < \mu_q < 1$, that

$$\| F(\underline{r}, \lambda, q) f \|_{0,\, \Sigma_{\mu r,\, x^0}} \leq \frac{C}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)}\right]^j \left(\frac{|\lambda|}{d(\lambda)}\right)^q \|f\|_{0,\, R^n} \tag{8.29}$$

$$\| F(\underline{r}, \lambda, q) f \|_{wm,\, \Sigma_{\mu r,\, x^0}} \leq C \left(\frac{|\lambda|}{d(\lambda)}\right)^{q+1} \left[\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)}\right]^j \|f\|_{0,\, R^n}. \tag{8.30}$$

We can treat the adjoint operator in the same way, since the differential operator $\mathcal{B}^* - (\mathcal{B}_{\underline{r}})^*$ also vanishes identically on $\Sigma_{r,\, x^0}$ and the inequality (8.16) (and an analogous inequality for $\| (\mathcal{B}_{\underline{r}})^* G_\lambda \|_{0,\, R^n}$) are known to hold.

(To apply lemma 8.1 we use of course also the additional assumptions on G_λ made in lemma 8.3).

The conclusion (8.21) follows now by application of theorem 3.2 and inequality (3.14) in the same way as at the end of the proof of lemma 8.1, using the condition $|\lambda| \geq 1,\ 1 \geq r \geq |\lambda|^{-1}$.

9. The asymptotic expansion of resolvent kernels.

We shall discuss first a class of operators on R^n. Since we wish to treat operators connected to operators which are semi-elliptic in the extended sense, we use L_2 spaces with a weight function. These spaces are discussed in [4]. We shall repeat here briefly their definition and some of their properties.

Let $\varrho(x)$ be a function in $C_*^\infty(R^n)$ such that $\varrho(x) \geq d > 0$, d some constant. We denote by $L_{2,\varrho}(R^n)^p$ the Hilbert space which is the completion of $C_0^\infty(R^n)^p$ under the norm $\left[\Sigma_{i=0}^p \int_{R^n} |u_i(x)|^2 \varrho(x)\, dx\right]^{1/2}$. The differential operator $\mathscr{A}$ is said to be ϱ-formally self-adjoint if

$$\Sigma_{i=1}^p \int (\mathscr{A} u)_i(x) \overline{v_i(x)}\, \varrho(x)\, dx = \Sigma_{i=1}^p \int u_i(x) \overline{(\mathscr{A} v)_i(x)}\, \varrho(x)\, dx$$

for all $u, v \in C_0^\infty(R^n)^p$.

Let $\mathscr{A}(x, D)$ a ϱ-formally self-adjoint defferential operator, semi-elliptic in the restricted sense with respect to $\mathbf{m}$ with reduced order $w > \Sigma_{h=1}^n \frac{1}{m_h}$. We assume that the coefficients of $\mathscr{A}$ are matrices whose entries belong to $C_*^\infty(R^n)$, and that at every point $x \in R^n$, the matrix $\mathscr{A}'(x, \xi)$ is positive definite for all $\xi \in R^n$, $\xi \neq 0$. We also assume that $\mathscr{A}'$ is « uniformly » semi-elliptic. That is, we assume that the inequalities (6.1) hold for the eigenvalues $\mu_i(x, \xi)$ of $\mathscr{A}'$ with a constant γ which is *independent* of x.

Considering $\mathscr{A}$ as a symmetric operator in the Hilbert space $L_{2,\varrho}(R^n)$ with a domain $C_0^\infty(R^n)^p$ we denote its closure by $\widetilde{\mathscr{A}}$. The a priori estimate

$$\|u\|_{w\mathbf{m}, R^n} \leq \text{Const}\,(\|\mathscr{A} u\|_{0,R^n} + \|u\|_{0\, R^n}) \tag{9.1}$$

which holds for $u \in H_{w\mathbf{m}}(R^n)^p$ is essentially well known ([26]; compare also (8.12)). The L_2 regularity theory of weak solutions of semi-elliptic equations is parallel to the regularity theory for elliptic equations (e. g. [9], [17]; it is clear that the proofs of the usual elliptic theory [1] carry over easily with some modifications to the present situation). From this it follows easily (and this is also essentially well-known) that $\tilde{\mathcal{A}}$ is a self-adjoint operator in $L_{2,\varrho}(R^n)^p$ with domain of definition $\overset{\circ}{H}_{wm}(R^n)^p$, and that $\tilde{\mathcal{A}}$ is the unique self-adjoint realization of $\mathcal{A}$ in $L_{2,\varrho}(R^n)^p$. The Garding inequality for semi-elliptic (scalar) operators is demonstrated in [17]; it is not difficult to prove that it holds for $\mathcal{A}$. Hence $\tilde{\mathcal{A}}$ is bounded from below, and we shall assume in the following without loss of generality that $\tilde{\mathcal{A}}$ is positive.

Consider now the resolvent operator $R_\lambda = (\tilde{\mathcal{A}} - \lambda)^{-1}$, $R_\lambda : L_{2,\varrho}(R^n)^p \longrightarrow L_{2,\varrho}(R^n)^p$. The range of R_λ is contained in $\mathrm{H}_{wm}(R^n)^p$. Since $L_{2,\varrho}(R^n)^p$ and $L_2(R^n)^p$ are the same function spaces on which two equivalent Hilberte norms are defined, we may regard $\tilde{\mathcal{A}}$ and R_λ as operators in $L_2(R^n)^p$. We shall denote by G_λ the resolvent operator R_λ when considered as an operator in $L_2(R^n)^p$. Note that G_λ^*, the $L_2(R^n)^p$-adjoint of G_λ, is given by

$$G_\lambda^* = \varrho G_{\bar{\lambda}} \varrho^{-1}. \tag{9.2}$$

The norm of G_λ is contained in $H_{wm}(R^n)^p$ and the following estimates hold

$$\| G_\lambda \|_{0,R^n} \leq \frac{C}{d(\lambda)}, \ \| G_\lambda \|_{wm,R^n} \leq \frac{C\,|\lambda|}{d(\lambda)}, \ \| G_\lambda^* \|_{wm,R^n} \leq \frac{C\,|\lambda|}{d(\lambda)} \qquad (|\lambda| \geq 1): \tag{9.3}$$

where C is a constant. The estimates (9.3) are completely analogous to the estimates (6.1) of [4] and may be proved in the same way (or in another fashion).

It follows from the properties of G_λ described above and from theorem 3.1 that G_λ is an integral operator with a continuous matrix kernel. We are interested in the asymptotic expansion of the kernel $G_\lambda(x, y)$. In order to derive this expansion we fix an arbitrary point x^0 in R^n and set:

$$\mathcal{A}_0(D) = \mathcal{A}'(x^0, D)$$

$$\mathcal{B}(x, D) = \mathcal{A}'(x^0, D) - \mathcal{A}(x, D).$$

As in section 6, we denote by $\tilde{\mathcal{A}}_0$ the (unique) self-adjoint realization of $\mathcal{A}_0$ in $L_2(R^n)^p$ and by F_λ the resolvent of $\tilde{\mathcal{A}}_0$, $F_\lambda = (\tilde{\mathcal{A}}_0 - \lambda I)^{-1}$. It is easily seen (as in [4]) that for every integer l,

$$(9.4)\qquad G_\lambda = F_\lambda + F_\lambda \mathcal{B} G_\lambda = F_\lambda + F_\lambda \mathcal{B} F_\lambda + (F_\lambda \mathcal{B})^2 G_\lambda =$$

$$= \dots = \Sigma_{r=0}^{l-1} (F_\lambda \mathcal{B})^r F_\lambda + (F_\lambda \mathcal{B})^l G_\lambda .$$

The formula (9.4) (which is really the Neumann series expansion of G_λ in terms of F_λ) is a matrix equation. Written out explicitly for the matrices elements, it reads:

$$G_{\lambda, s, t} = F_{\lambda, s, t} +$$

$$(9.5)\qquad + \Sigma_{r=1}^{l-1} \Sigma_{\substack{1 \le s_i,\, t_i \le p \\ 1 \le i \le r}} F_{\lambda, s, t_1} \mathcal{B}_{t_1, s_1} F_{\lambda, s_1, t_2} \cdots \mathcal{B}_{t_r, s_r} F_{\lambda, s_r, t} +$$

$$[(F_\lambda \mathcal{B})^l G_\lambda]_{s, t'}, \quad \text{for } 1 \le s, t \le p.$$

Denote by $\mathbf{B}$ the set $\{\mathcal{B}_{s, t}\}_{1 \le s, t \le p}$ and by $\mathbf{A}_0$ the set $\{\mathcal{A}_{0, s, t}\}_{1 \le s, t \le p}$. (The elements of $\mathbf{A}_0$ and $\mathbf{B}$ are scalar operators). We can consider $\mathcal{A}_0$, $\mathcal{B}$ and F_λ as linear operators in $H_\infty(R^n)^p$. Using the notation of section 7 we rewrite (9.5) as

$$(9.6)\qquad G_{\lambda, s, t} - F_{\lambda, s, t} - [(F_\lambda \mathcal{B})^l G_\lambda]_{s, t} \in \Sigma_{r=1}^{l-1} \{\mathbf{F}_\lambda (1) \mathbf{B}]^r \mathbf{F}_\lambda (1)\}.$$

Noting that $\mathcal{A}_0 - \lambda I$ is one-to-one from $H_\infty(R^n)^p$ onto itself, we apply theorem 7.1 for $[\mathbf{F}_\lambda (1) \mathbf{B}^r]$. After completion in $L_2(R^n)^p$ is follows from (9.6) and (7.2) that

$$(9.7)\qquad G_{\lambda, s, t} - F_{\lambda, s, t} - [(F_\lambda \mathcal{B})^l G_\lambda]_{s, t} \in$$

$$\in \Sigma_{r=1}^{l-1} \Sigma_{J \in S(r, k, 1)} \{[\mathbf{B}, \mathbf{A}_0 ; J] \mathbf{F}_\lambda (|J| + r + 1)\} +$$

$$+ \Sigma_{r=1}^{l-1} \Sigma_{i=0}^{r-1} \{[\mathbf{F}_\lambda (1) \mathbf{B}]^i \mathbf{F}_\lambda (1) \Sigma_{J \in S(r-i-1, -k-1)} [\mathbf{B}, \mathbf{A}_0, J \cup (k)] \mathbf{F}_\lambda (|J| + k + r - i)\}$$

where k is an arbitrary positive integer, $1 \le s, t \le p$.

We proceed now (in analogy to [4]) to use (9.7) in order to get the desired asymptotic expansion. According to theorem 7.2 the reduced order of a differential operator in $[\mathbf{B}, \mathbf{A}_0 ; J]$ for $J \in S(r)$ is at most $|J|(w - b) +$

$+ rw(\mathbf{B})$. We have $w(\mathbf{B}) \leq w$ hence

$$w(\mathbf{B}, \mathbf{A}_0 ; J) \leq (|J| + r)(w - b) + rb. \tag{9.8}$$

It follows from section 6 that F_λ is an integral operator with a continuous bounded matrix kernel. From theorem 6.1 and (9.8) it follows that every operator in $[\mathbf{B}, \mathbf{A}_0 ; J] F_\lambda (|J| + r + 1)$ which appears in the sums which generate the first sum in (9.7) is a (scalar) integral operator with a continuous bounded kernel. It follows from the results of section 8 that $(F_\lambda \mathcal{B})^l G_\lambda$ is an integral operator with a continuous bounded matrix kernel. Hence the terms in the last sum of the ight-hand side of (9.7) also describe an integral operator with a continuous bounded kernel.

Applying theorem 6.1 to an operator S_λ which is in

$$(\mathbf{F}_\lambda (1)\, \mathbf{B})^i\, \mathbf{F}_\lambda (1) [\mathbf{B}, \mathbf{A}_0 ; J \cup (k)]\, \mathbf{F}_\lambda (|J| + k + r - i)$$

where $1 \leq r \leq l - 1, 0 \leq i \leq r - 1, J \in S(r - i - 1, k - 1)$ we find that (since $w([\mathbf{B}, \mathbf{A}_0 ; J \cup (k)]) \leq (|J| + k)(w - b) + (r - i) w(\mathbf{B})$) S_λ is an integral operator with a continuous kernel such that

$$| S_\lambda (x, y) | \leq \frac{C |\lambda|^{\frac{1}{w}\left(\Sigma_{h=1}^n \frac{1}{m_h} + (|J| + k)(w - b) + rw(\mathbf{B})\right)}}{d(\lambda)^{|J| + k + r + 1}} \tag{9.9}$$

for $|\lambda| \geq 1$. Here and in the following C denotes a generic constant which is independent of λ, x, y and x^0 (but depends on k and l).

If, in particular, the coefficients of $\mathcal{A}'$ are constant and the matrices $\mathcal{A}_\alpha$ vanish identically for $w > |\alpha : \mathrm{m}| > w - i_0 a$, we may rewrite (9.9) as

$$| S_\lambda (x, y) | \leq \frac{C |\lambda|^{\Sigma_{h=1}^n \frac{1}{w m_h}}}{d(\lambda)} \left[\frac{|\lambda|^{1 - b/w}}{d(\lambda)}\right]^{|J| + k} \left[\frac{|\lambda|^{1 - i_0 a/w}}{d(\lambda)}\right]^r \tag{9.10}$$

for $|\lambda| \geq 1$, since in this case $w(B) \leq w - i_0 a$. In the general case $i_0 = 0$ and $w(\mathbf{B}) = w$. If we choose $k > l + \frac{(l - 1) b}{\varepsilon w}$ for some fixed $\varepsilon > 0$ it follows from (9.9) (since $r \leq l - 1$) that in all cases

$$| S_\lambda (x, y) | \leq \frac{C |\lambda|^{\Sigma_{h=1}^n \frac{1}{w m_h}}}{d(\lambda)} \left[\frac{|\lambda|^{1 - \varepsilon/w}}{d(\lambda)}\right]^l \tag{9.11}$$

for λ which is in the domain of the complex plane given by $d(\lambda) \geq |\lambda|^{1-b/w+\varepsilon}$, $|\lambda| \geq 1$. (Note that if $i_0 = \frac{b}{a}$ then (9.11) holds in the larger domain $d(\lambda) \geq \geq |\lambda|^{1-b/w}$ for $k \geq l-1$). We get from (9.11) that under the same restrictions (on k, $|\lambda|$, $d(\lambda)$) it is possible to estimate the kernel S_λ of an operator belonging to

$$\Sigma_{r=1}^{l-1} \Sigma_{i=0}^{r-1} \{[\mathbf{F}_\lambda(1)\,\mathbf{B}]^i\, \mathbf{F}_\lambda(1)\, \Sigma_{J \in S(r-i-1,\, k-1)} [\mathbf{B}, \mathbf{A}_0; J \cup (k)]\, \mathbf{F}_\lambda(|J| + k + r - i)\}$$

by the right-hand of (9.11). We remark that the number of the operators which generate the terms in the curled brackets in (9.7) depends only on k, l and p.

Consider now the last member of the left-hand side of (9.7). Here also matters simplify if $\mathcal{A}'$ has constant coefficients and the matrices $\mathcal{A}_\alpha$ vanish identically for $w > |\alpha : \mathbf{m}| > w - i_0 a$, since then $w(\mathcal{B}) \leq w - i_0 a$, so that (by interpolation) the $L_2(R^n)^p$ norms of the operators $\mathcal{B}F_\lambda$, $\mathcal{B}G_\lambda$, $\mathcal{B}^* F_\lambda$ $G_\lambda^* \mathcal{B}^*$ are all bounded by $\frac{C|\lambda|^{1-i_0 a/w}}{d(\lambda)}$. Hence it follows immediately from theorem 3.1 that for all $x, y \in R^n$,

$$|[(F_\lambda \mathcal{B})^l G_\lambda]_{s,t}(x, y)| \leq \frac{C|\lambda|^{\Sigma_{h=1}^n \frac{1}{w m_h}}}{d(\lambda)} \left[\frac{|\lambda|^{1-i_0 a/w}}{d(\lambda)}\right]^l. \tag{9.12}$$

(Compare (6.8) in [4]).

For the general case (where $w(\mathcal{B}) = w$) we shall use lemmas 8.2 and 8 3. By construction the leading terms of $\mathcal{B}$ have a zero of type b (at least) at the point x^0. Let $\{q_i\}$ be the (finite) sequence of type of zeros (at x^0) associated with $\mathcal{A}$, $q_i \in Q \cup \{+\infty\}$ (see the beginning of section 5) and let i_0 be $\min(b/a, \min_{q_i = 0}\{i\})$. Then the operator $\mathcal{B}$, defined above by $\mathcal{B} = \mathcal{A}_0 - \mathcal{A}$, the operators F_λ, G_λ, and the sequence $\{q_i\}$, $0 \leq i \leq i_0 - 1$ satisfy all the conditions of lemma 8.2 (with i_0 replacing k). Moreover the assumptions of lemma 8.3 are also satisfied by definition of G_λ. Hence, combining the inequalities (8.13) and (8.21) we find that, given a sequence $\{r_i\}$ of positive numbers, $1 \geq r_i \geq |\lambda|^{-1}$, and with r defined as in lemma 8.3 (by

$$r = \left[\min_{1 \leq h \leq n} \left(\frac{1}{2}\right)^{w m_h}\right]_{0 \leq i \leq i_0 - 1} [\min\, r^w]\Big),$$

$$|[(F_\lambda \mathcal{B})^l G_\lambda]_{s,t}(x, y)| \leq \frac{C|\lambda|^{\Sigma_{h=1}^n \frac{1}{w m_h}}}{d(\lambda)} \left\{g(\lambda)^l + \left(\frac{|\lambda|}{d(\lambda)}\right)^l \left[\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)}\right]^j\right\} \tag{9.13}$$

for $x, y \in \Sigma_{\mu r, x^0}$, $|\lambda| \geq 1, j \geq 0$ arbitrary, and $g(\lambda)$ is defined in (8.14), with C depending also on j and μ, $0 < \mu < 1$.

In order to utilize (9.13) set $r_i = |\lambda|^{-s_i}$. Then it follows from (8.14) that

$$g(\lambda) = \frac{|\lambda|}{d(\lambda)} \max(|\lambda|^{-s_0 q_0}, |\lambda|^{-a/w - s_1 q_1}, \ldots, |\lambda|^{-(i_0-1)a/w - s_{i_0-1} q_{i_0-1}}, \lambda^{-i_0 a/w})$$

whereas

$$\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)} = [\max_{1 \leq h \leq n} 2^{bm_h}] \frac{|\lambda|}{d(\lambda)} [\max_{0 \leq i \leq i_0 - 1} |\lambda|^{-b/w + s_i b}].$$

Hence the optimal s_i satisfies $-ia/w - s_i q_i = -b/w + s_i b$, so that $s_i = \frac{1}{w}\frac{b - ia}{q_i + b}$. If $q_i = \infty$ we choose $r_i = |\lambda|^{-s}$ with an arbitrary positive s (but independent of x^0), and we use the convention $\frac{q_i + ia}{q_i + b} = 1$. With these optimal s_i - s,

$$g(\lambda) = \frac{|\lambda|}{d(\lambda)} \max_{0 \leq i \leq i_0} (|\lambda|^{-\frac{b}{w}\frac{q_i + ia}{q_i + q}}) = \frac{|\lambda|^{1-\theta(x^0)b/w}}{d(\lambda)}$$

(we use the notation of the beginning of section 5) and

$$\frac{|\lambda|^{1-b/w}}{r^{b/w}\, d(\lambda)} \leq (\max_{1 \leq h \leq n} 2^{bm_h}) \frac{|\lambda|^{1-\theta(x^0)b/w}}{d(\lambda)}.$$

(Note that the numbers r_i defined by $r_i = |\lambda|^{-s_i}$ for optimal s_i satisfy $1 \geq r_i \geq |\lambda|^{-1}$). Hence, we get from (9.13) that

$$(9.14) \qquad |[(F_\lambda \mathcal{B})^l G_\lambda]_{s,t}(x, y)| \leq$$

$$\leq \frac{C|\lambda|^{\sum_{h=1}^{n} \frac{1}{wm_h}}}{d(\lambda)} \left[\left(\frac{|\lambda|}{d(\lambda)}\right)^l \left(\frac{|\lambda|^{1-\theta(x^0)b/w}}{d(\lambda)}\right)^j + \left(\frac{|\lambda|^{1-\theta(x^0)b/w}}{d(\lambda)}\right)^l\right]$$

for $x, y \in \Sigma_{\mu r, x^0}$ where

$$r = \left[\min_{1 \leq h \leq n} \left(\frac{1}{2}\right)^{wm_h}\right] |\lambda|^{-\min_{0 \leq i \leq i_0} \frac{b - ia}{b + q_i}}.$$

For any given $\varepsilon > 0$ we choose $j = l + \frac{l\,b}{\varepsilon\,w}\,\theta(x^0)$ and find from (9.14) that

$$(9.15)\qquad |[(F_\lambda \mathcal{B})^l G_\lambda]_{s,t}(x,y)| \leq \frac{C\,|\lambda|^{\Sigma_{h=1}^n \frac{1}{wm_h}}}{d(\lambda)} \left[\frac{|\lambda|^{1-\theta(x^0)b/w}}{d(\lambda)}\right]^l$$

for $x, y \in \Sigma_{\mu r, x^0}$ and $|\lambda| \geq 1$, $d(\lambda) \geq |\lambda|^{1-\theta(x^0, b/w+\varepsilon}$. Combining the estimates (9.10), (9.11), (9.15), and applying them to the representation (9.7), we get in conclusion the following result:

THEOREM 9.1: The kernel $G_\lambda(x,y)$ of the resolvent $R_\lambda = (\tilde{\mathcal{A}} - \lambda I)^{-1}$ (considered as an operator in $L_2(R^n)^p$) has an asymptotic representation of the form:

$$(9.16)\qquad G_\lambda(x,y) = F_\lambda(x,y) + H_\lambda(x,y\,;\,l) + O\left(\frac{|\lambda|^{\Sigma_{h=1}^n \frac{1}{wm_h}}}{d(\lambda)}\left[\frac{|\lambda|^{1-\theta b/w}}{d(\lambda)}\right]^{l+1}\right)$$

where $F_\lambda = (\tilde{\mathcal{A}}_0 - \lambda I)^{-1}$ with $\mathcal{A}_0 = \mathcal{A}'(x^0, D)$ (x^0 a fixed point),

$$H_{\lambda,\,i,\,j}(x,y\,;\,l) \in \Sigma_{r=1}^{l-1} \Sigma_{J \in S(r,\,k-1)} \{[\mathbf{B}, \mathbf{A}_0\,;\,J]\,\mathbf{F}_\lambda(|J| + r + 1)\}$$

$1 \leq i, j \leq p$ ($\mathbf{B}$ is the set of the entries of the operator matrix $\mathcal{A}_0 - \mathcal{A}$) such that:

(i) If $\mathcal{A}'$ has constant coefficients and the matrices $\mathcal{A}_\alpha$ vanish identically for $w > |\alpha:\mathbf{m}| > w - i_0$ a where $0 \leq i_0 \leq b/a$, then $\theta = i_0\,a/b$ and the O estimates hold for any positive integers k, l satisfyng $k \geq l$ and for $\lambda \rightarrow \infty$ in the region $d(\lambda) \geq |\lambda|^{1-\theta b/w}$, $|\lambda| \geq 1$, uniformly in x, y and x^0.

(ii) If $\mathcal{A}'$ has variable coefficients then θ is the number $\theta(x^0)$ given by (5.1), k and l any positive integers with $k \geq l + \frac{lb}{\varepsilon\,w}$ for any given $\varepsilon > 0$; the O estimates hold for $\lambda \rightarrow \infty$ in the region $|\lambda| \geq 1$, $d(\lambda) \geq |\lambda|^{1-\theta(x^0)b/w+\varepsilon}$, for any x, y restricted to the neighborhood of x^0 where (9.15) holds. Under these restrictions the O estimate is uniform in x, y and x^0.

On the diagonal of $R^n \times R^n$ the formula (9.16) takes a much more explicit form, since in this case $(-\lambda)^{1-\Sigma_{h=1}^n \frac{1}{wm_h}} H_\lambda(x^0, x^0\,;\,l)$ is really a polynomial in $(-\lambda)^{-a/w}$. It follows from (6.13) and (9.8) that every (matrix) function $(-\lambda)^{1-\Sigma_{h=1}^n \frac{1}{wm_h}} S_\lambda(x,x)$ where $S_\lambda(x,y)$ is the kernel of a matrix operator S_λ whose entries belong to the set $[\mathbf{B}, \mathbf{A}_0\,;\,J]\,\mathbf{F}_\lambda(|J| + r + 1)$ is a polynomial in $(-\lambda)^{-a/w}$ with coefficients which are matrices whose entries are C_*^∞ functions of x^0. Moreover, by (6.13) the coefficients of $(-\lambda)^{-ia/w}$ are the sums of coefficients of differentiations D_β which occur in differential operators contained in $[\mathbf{B}, \mathbf{A}_0; J]$ with $\frac{|\beta : m|}{w} - |J| - r = ia$. Using (9.8) it is easily seen that all such coefficients vanish for $i < i_0$. Also, by (6.15),

$$F_\lambda(x^0, x^0) = C_0(x^0)(-\lambda)^{-1+\Sigma_{h=1}^n \frac{1}{wm_h}} \tag{9.17}$$

where

$$C_0(x^0) = (2\pi)^{-n} \int_{R^n} [\mathscr{A}'(x^0, \xi) + I]^{-1} \, d\xi.$$

These observations and theorem 9.1 show that on the diagonal the kernel G_λ has an asymptotic expansion

$$G_\lambda(x^0, x^0) \backsim (-\lambda)^{\Sigma_{h=1}^n \frac{1}{wm_h} - 1} \Sigma_{i=0}^\infty (-\lambda)^{-ia/w} C_i(x^0) \tag{9.18}$$

where the coefficients $C_i(x^0)$ are matrices whose elements are C_*^∞ functions of x^0. Moreover $C_i(x^0) = 0$ for $0 < i < i_0(x^0)$. In general, $C_i(x^0)$ $(i > 0)$ is the coefficient of $(-\lambda)^{-ia/w}$ in the polynomial $H_\lambda(x^0, x^0; l)(-\lambda)^{1-\Sigma_{h=1}^n \frac{1}{wm_h}}$, with l satisfying $(l+1)\theta b > ia$ (for such l-s no additional terms of the i-th power can enter). The asymptotic expansion (9.18) holds in the complex plane regions of λ described in parts (i) and (ii) of theorem 9.1 and in those regions the expansion is uniform in $x^0 \in R^n$. (Note that in order to demonstrate that the remainder decreases at least as $|\lambda|^{-Na/w}$ one has to apply (9.16) with l satisfying $l + 1 > \frac{Na}{\varepsilon w}$.

The asymptotic expansion (9.18) is the asymptotic expansion (5.2) of theorem 5.1 for the special case of differential operators which are semi-elliptic in the restricted sense in R^n (for this special case it is sufficient to prove theorem 9.1 with $\varrho \equiv 1$). We now extend theorem 9.1 to the case of a self-adjoint realization of a differential operator which is semi-elliptic in the extended sense on a manifold Ω.

Let Ω be a manifold, let $\tilde{\mathscr{A}}$ be a positive self-adjoint operator in $L_2(\Omega)^p$ which is a realization of a formally self-adjoint differential operator $\mathscr{A}$. Assume that $\mathscr{A}$ is semi-elliptic in the extended sense in Ω (with respect to $\mathbf{m}$), $w(\mathscr{A}) = w$, and $\Sigma_{h=1}^n \frac{1}{m_h} < w$. Let $\mathscr{F}$ be a complete family of coordinate systems belonging to $\mathscr{A}$ such that for each $\varkappa \in \mathscr{F}$ the matrix $\mathscr{A}_\varkappa(\varkappa(x), \xi)$ is positive definite (for real $\xi \neq 0$) and $L^\varkappa$ is positive in $\tilde{\Omega}_\varkappa$. Let $R_\lambda(x, y)$ be the resolvent kernel of $\tilde{\mathscr{A}}$. Let x^0 be a (fixed) point in Ω, and let $x^0 \in \Omega_\varkappa$

with $\varkappa \in \mathcal{F}$. It is obvious that

$$\int_{\tilde{\Omega}_\varkappa} L^\varkappa (\mathcal{A}_\varkappa u) \bar{v}\, dx = \int_{\tilde{\Omega}_\varkappa} L^\varkappa u \,(\overline{\mathcal{A}_\varkappa v})\, dx \tag{9.19}$$

for $u, v \in C_0^\infty (\tilde{\Omega}_\varkappa)$. Let $\Omega_\varkappa^0 \subset \Omega_\varkappa$ be an open neighborhood of x^0 which has a compact closure contained in $\Omega_\varkappa$. Let us choose a real positive function $\varrho(x) \in C_*^\infty(R^n)$ such that $\varrho(x) \equiv L^\varkappa(x)$ on $\tilde{\Omega}_\varkappa^0$ (where $\tilde{\Omega}_\varkappa^0 = \varkappa(\Omega_\varkappa^0)$, $\varrho(x) \geq d > 0$ on R^n, and then choose a ϱ-formally self-adjoint differential operator $\mathcal{A}_\varkappa^0$ defined on R^n with coefficients whose entries are $C_*^\infty(R^n)$ function such that $\mathcal{A}_\varkappa^0$ coincides with $\mathcal{A}_\varkappa$ on $\tilde{\Omega}_\varkappa^0$ and $\mathcal{A}_\varkappa^0$ is uniformly semi-elliptic in the restricted sense on R^n (compare the proof of theorem 6.2 in [4]). Thus $\mathcal{A}_\varkappa^0$ satisfies all the assumptions listed at the beginning of this section. Let $\tilde{\mathcal{A}}_\varkappa^0$ be the unique self-adjoint realization of $\mathcal{A}^0$ in $L_{2,\varrho}(R^n)^p$. Then $\tilde{\mathcal{A}}_\varkappa^0$ is semi bounded and without loss of generality $\tilde{\mathcal{A}}_\varkappa$ is positive. Let $R_{\varkappa,\lambda}^0$ be the resolvent of $\tilde{\mathcal{A}}_\varkappa^0$ in $L_{2,\varrho}(R^n)^p$ and let $G_{\varkappa,\lambda}^0$ denote $R_{\varkappa,\lambda}^0$ when considered as an operator in $L_2(R^n)^p$. We apply theorem 9.1 to $G_{\varkappa,\lambda}^0$ and find that the kernel of $G_{\varkappa,\lambda}^0$ may be represented in the form (9.16) and more explicitly on the diagonal of $R^n \times R^n$ it has the asymptotic expansion (9.18). On $\tilde{\Omega}_\varkappa^0$ the kernels $H_\lambda(x, y; l)$ and the matrices $C_j(x)$ are determined by $\mathcal{A}_\varkappa$, since $\mathcal{A}_\varkappa$ coincided with $\mathcal{A}_\varkappa^0$ there. Using the comparison lemma (8.1) it is easy to see (as in the proof of theorem 6.2 in [4]) that

$$| G_{\varkappa,\lambda}^0(x, y) - L^\varkappa(y) R_\lambda(\varkappa^{-1}(x), \varkappa^{-1}(y)) | \leq C \frac{|\lambda|^{\sum_{h=1}^n \frac{1}{wm_h}}}{d(\lambda)} \left[\frac{|\lambda|^{1-b/w}}{d(\lambda)} \right]^j \tag{9.20}$$

for any positive integer j, $|\lambda| \geq 1$, and x, y restricted to a compact subset of $\tilde{\Omega}_\varkappa^0$. Hence we have an asymptotic expansion of R_λ which, restricted to the diagonal of $\Omega \times \Omega$, have all the properties asserted in theorem 5.1.

Chapter three

ASYMPTOTIC DISTRIBUTION OF EIGENVALUES

0. Obtaining « rough estimates up to the boundary ».

We now turn our attention to the problem of the asymptotic distribution of eigenvalues. It is well known that one way of obtaining asymptotic formulas for eigenvalues is to integrate the formulas for the eigenfunctions (spectral function) over Ω, (using (4.9)). However, the formulas for the spectral functions (such as (5.6), or even weaker formulas such as (0.1)) are obtained for compact subsets contained in the interior of Ω, so that it is not obvious why (if Ω is not a compact manifold without boundary) it is permissible to pass to the limit (as $t \longrightarrow \infty$) under the integral sign.

The Agmon kernel method has been particularly successful in overcoming this difficulty. In [2] and [3] it was shown (among other things) that one has only to assume that the domain of definition of a self-adjoint realization $\widetilde{A}$ of an elliptic operator A is contained in $H_m(\Omega)$, where Ω satisfies the (very mild) cone condition, in order to be able to integrate (0.1), so as to obtain (0.2). In [5] it was shown that if Ω satisfies a not too restrictive additional condition (condition (3.4) of [5]) then it is possible to integrate the formulas of [4] for the spectral function of an elliptic operator and to obtain remainder estimates for the eigenvalues. That those integrations are justified follows from the relatively easy rough estimate (which follows easily from the Agmon kernel theorem) for the spectral function:

$$(10.1) \qquad | E_{i,j}(x, y ; t) | \leq C t^{\frac{n}{m}} \qquad 1 \leq i,j \leq p$$

with a C which is independent of $x, y \in \Omega$, or from the similar estimate for the resolvent kernel

$$(10.2) \qquad | R_{\lambda,i,j} (x, y) | \leq C \frac{|\lambda|^{\frac{n}{m}}}{d(\lambda)} \qquad (|\lambda| \leq 1).$$

By contrast, the asymptotic distribution of eigenvalues has been unknown in the semi-elliptic case ([9], [10]). The kernel theorems of section 3 yield estimates only for very special kind of domains: n-dimensional boxes whose edges are parallel to the coordinate axes. Hence one cannot expect to have an estimate

$$(10.3) \qquad | E_{t,i,j}(x, y) | \leq Ct^{\Sigma_{j=1}^{n} \frac{1}{m_j}}$$

with C independent of $x, y \in \Omega$, even if we know that $D_{\tilde{\mathcal{A}}} \subset H_{\mathrm{m}}(\Omega)^p$. It is possible to have C in (10.3) replaced by an integrable function if Ω satisfies a certain geometric condition (see below). But it is not obvious at all that there exist realizations of $\mathcal{A}$ in $L_2(\Omega)^p$ such that $D_{\tilde{\mathcal{A}}} \subset H_{\mathrm{m}}(\Omega)^p$, since regularity theory for semi-elliptic boundary value problems (for a curved boundary) is not a simple matter. Since the differential operator is not elliptic, the boundary of a bounded smooth domain in R^n must contain points where this boundary is characteristic. These points are bound to cause trouble in trying to prove « coerciveness », see [20] for a treatment of elliptic-parabolic second order case.

Let us note, however, that several of the older methods may be applied to various special cases also in the semi-elliptic situation.

(i) Consider first the Dirichlet realization of a semi-elliptic positive operator with constant coefficients, where $w > \Sigma_{h=1}^{n} \frac{1}{m_h}$. We consider for simplicity a scalar operator A. For $u \in H_{(w/2)\mathrm{m}}(R^n)$, $(Au + \lambda u, u)$ is well defined. Following Garding [14] we note that for such u,

$$u(x) = (2\pi)^{-\frac{n}{2}} \int e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi =$$

$$= (2\pi)^{-\frac{n}{2}} \int e^{ix\cdot\xi} \hat{u}(\xi) [A(\xi) + t]^{\frac{1}{2}} [A(\xi) + t]^{-\frac{1}{2}} d\xi, \; t > 0.$$

So that by the Cauchy-Schwartz inequality and Parseval's theorem,

$$(10.4) \qquad |u(x)| \leq (2\pi)^{-\frac{n}{2}} (Au + tu, u) \int [A(\xi) + t]^{-1} d\xi \leq$$

$$\leq Ct^{\Sigma_{h=1}^{n} \frac{1}{wm_h}} (Au + tu, u)$$

for $t > 0$ sufficiently large. Since $D_{\tilde{A}}$ is the intersection of $H_{w\mathrm{m}}(\Omega)$ with the closure in $H_{(w/2)\mathrm{m}}(R^n)$ of $C_0^\infty(\Omega)$ it follows easily from (10.4) (compare

[14]) that

$$| G_t(x, x) | \leq C t^{\Sigma_{h=1}^{n} \frac{1}{w m_h} - 1}$$

for $t > 0$ sufficiently large, where $G_t(x, x)$ is Green's functions (the resolvent kernel) of $\widetilde{A}$, *uniformly in* $x \in \Omega$. Hence it follows in a standard way (using Lebesgue convergence theorem and the Hardy-Littlewood tauberian theorem) that

$$(10.5) \qquad \sum_{\lambda_j \leq t} 1 = d t^{\Sigma_{h=1}^{n} \frac{1}{w m_h}} + o\left(t^{\Sigma_{h=1}^{n} \frac{1}{w m_h}}\right)$$

$$t \to \infty$$

where $\{\lambda_j\}$ is the sequence of eigenvalues of $\widetilde{A}$ and

$$d = \int_{\Omega} d(x)\, dx$$

where $d(x)$ is defined as in (5.6) and (5.7).

(ii) In [15], p. 251, Gårding describes a method which works for Dirichlet realizations of general elliptic operators, removing the assumptions that $w > \Sigma_{h=1}^{n} \frac{1}{m_h}$ and that A has constant coefficients. The essence of the method lies in comparing $\int_{\Omega} G_t^k(x, x)\, dx$ with the integral (over Ω) of $(E \dot{G}_t E)^k$ (x, x), where $\dot{G}_t$ is the Green function of an operator A_1 which is defined on a set Ω_1 containing $\overline{\Omega}$ in its interior and such that $A = A_1$ on Ω. This method can be extended without change to the semi-elliptic case.

(iii) A variant of a method due to Ehrling [13] may be used for a certain type of problems. Assume that A is of the form

$$A = \Sigma_{\substack{|\alpha : \mathrm{m}| = w/2 \\ |\beta : \mathrm{m}| = w/2}} D^\beta A_{\alpha, \beta}(x) D^\alpha$$

and that there exists a positive constant γ such that for all complex numbers ξ_α, $|\alpha : \mathrm{m}| = w/2$ and all $x \in \Omega$,

$$(10.6) \qquad \gamma \sum_{|\alpha : \mathrm{m}| = w/2} |\xi_\alpha|^2 \geq \Sigma_{\substack{|\alpha : \mathrm{m}| = w/2 \\ |\beta : \mathrm{m}| = w/2}} A_{\alpha, \beta}(x)\, \xi_\alpha \bar{\xi}_\beta \geq$$

$$\geq \gamma^{-1} \Sigma_{|\alpha : \mathrm{m}| = w/2} |\xi_\alpha|^2 .$$

Assume also that the domain of definition of the self-adjoint realization $\widetilde{A}$ is such that for all $u \in D_{\widetilde{A}}$, one has (e. g. by integration by parts)

$$(10.7) \qquad (\widetilde{A}u, u) = \int_{\Omega} \Sigma_{\substack{|\alpha : m| = w/2 \\ |\beta : m| = w/2}} A_{\alpha,\beta}(x)\, D^{\alpha} u\, D^{\beta} \bar{u}\, dx .$$

It follows from (10.6) and (10.7) that for $t > 0$ sufficiently large there exists a positive constant C such that

$$((\widetilde{A} + t)\, u, u)^{\frac{1}{2}} \geq C[\| u \|_{(w/2)\, m, \Omega} + t \| u \|_{0, \Omega}] .$$

Let now $\Sigma \subset \Omega$ be an n-dimensional box, the edges of which are parallel to the coordinate axes and their lengths given by $r_1, \dots, r_n$. Using a local form of (2.11) it follows easily that if $w > \Sigma_{h=1}^{n} \frac{1}{m_h}$, then $u \in D_{\widetilde{A}}$ is a continuons function and

$$(10.8) \qquad | u(x) |^2 \leq C t^{\Sigma_{h=1}^{n} \frac{1}{wm_h} - 1} (r_1 \dots r_n)^{-1} ((\widetilde{A}u + tu, u)$$

for $x \in \Sigma$.

The estimate (10.8) is the analogue of the basic inequality (4) in [13]. It follows easily from (10.8) (compare (33), (35) in [13] and the method of [14]) that the resolvent kernel R_t of $\widetilde{A}$ satisfies, for t positive and sufficiently large and $x \in \Sigma$, that

$$(10.9) \qquad | R_t(x, x) | \leq C t^{\Sigma_{h=1}^{n} \frac{1}{wm_h} - 1} (r_1 \dots r_n)^{-1} .$$

We see from (10.9) that the following condition is of importance:

DEFINITION: Let $K \subset R^n$ be a bounded measurable set. For each $x \in K$ denote by $h(x)$ the supremum of the (n-dimensional) volumes of boxes $\Sigma \subset R^n$ whose edges are parallel to the coordinate axes, which contain x. If the function $\frac{1}{h(x)}$ is integrable over K, then the set K is said to satisfy the condition (I).

If the set Ω satisfies the condition (I) then, using (10.9) and Lebesgue dominated integration theorem, we may arrive at (10.5).

Let us note that condition (I) is not invariant under rotation, so that this condition is not strictly « geometric ». (The condition that K satisfies condition (I) for *every* coordinate system *is* geometric. Is it equivalent to a known geometric condition?). It is very easy to see that a plane convex domain with a smooth boundary satisfies condition (I). The non-convex plane set bounded by the parabolar $x_2 = x_1^2$, $x_2 = -x_1^2$ and the line $x_1 = 1$ has the property (I). A triangle with at least one angle which does not contain a ray which is parallel to one of the coordinate axes, does not satisfy condition (I). I did not find any condition which is equivalent to (I) nor any simple sufficient condition. It is clear, however, that there exist families of domains in R^n which satisfy the condition (I).

Let us turn our attention now to the problem, whether it is possible to have a theory of eigenvalue distribution for semi-elliptic operators which generalizes the more modern theories of Agmon [3], [5] (or of Browder [10]; compare also the elliptic part of [9]). The answer is, that it is possible to have such a theory for certain manifolds, but not (in general) for subsets of R^n, if the operator is semi-elliptic in the restricted sense.
We recall that the definition of a manifold with boundary ([18], p. 32) coincides with the definition of manifold except for the fact that the sets $\widetilde{\Omega}_\varkappa$ are required to be open subsets of the closed half space $\overline{R}_n^+ = \{x : x \in R^n, x_n \geq 0\}$. The set of all $x \in \Omega$ such that $\varkappa(x)$ belongs to the boundary of $\overline{R}_n^+$ for some (and hence for every) $\varkappa$ with $x \in \Omega_\varkappa$ forms a manifold of dimension $n-1$, called the boundary of Ω, and is denoted by $\partial\Omega$. The interior of Ω is $\Omega - \partial\Omega$.

A regularity theory for certain hypoelliptic (or even more general) boundary value problems does exist for the case of a plane boundary [23], [25]. A corresponding theory for curved boundaries is lacking. However, for a special type of surfaces, called « normal » by Cavalluci [12], it is possible to have a regularity theory for semi elliptic operators. Let $A(\xi)$ be a semi elliptic polynomial with respect to the multi-index $\mathbf{m} = (m_1, \ldots, m_n)$. A major cause of trouble lies in the fact that semi-ellipticity is not invariant under general coordinate transformations. Consider the real invertible linear transformation

$$X_j^1 = \Sigma_{k=1}^n a_{jk}^1 x_k \quad 1 \leq j \leq n.$$

Denote by $A^1(D^1)$ the operator $A(D)$ in the transformed coordinates. The transformation is called *stable* with respect to $A(\xi)$ if (i) $A(\xi)$ and $A^1(\xi)$ have the same order with respect to each variable. (ii) The equations (in ξ_j) $A(\xi_1, \ldots, \xi_j, \ldots, \xi_n) = 0$, $A^1(\xi_1, \ldots, \xi_j, \ldots, \xi_n) = 0$ have the *same* number m_j^+ of solutions with positive imaginary part for real $(\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_n) \to \infty$. Cavallucci proved that if the transformation is stable then $A^1(\xi)$ is also semi elliptic with respecto m. Moreover, a necessary and sufficient condition for the transformation to be stable with respect to the positive semi-elliptic polynomial $A(\xi)$ is the vanishing of $a_{j,k}$ for $m_j < m_k$. The transformation $x_j^1 = f_j(x)$, $1 \leq j \leq n$ is called stable if at every point x the linear transformation $x_j^1 = \Sigma_{k=1}^n \frac{\partial f_j}{\partial x_k}(x)\, x_k$, $1 \leq j \leq n$ is stable. A surface S in R^n is called *normal* if there exists (locally) a stable transformation mapping S into a coordinate hyperplane. Cavallucci proved also that a surface $S \subset R^n$ is *normal* with respect to $A(\xi)$ if and only if S can be represented (locally) by means of the equation $x_i = g(x)$ with g independente of x_i such that $\frac{\partial g}{\partial x_k} \equiv 0$ if $m_k > m_i$. (Note that a non-characteristic surface is normal.)

Hence it is obvious from [23], [25] that if the positive semi-elliptic operator A and the boundary operators $B_1, \dots, B_{m_j^+}$ satisfy the regularity (analogs of the «complementing» or Lopatinsky conditions) hypotheses 1, 2, 3, 4, 5 of [25] ((7) of [23]) for the hyperplane $x_j = 0$ then they are satisfied also for a normal surface $x_j = g(x)$, so that the a-priori estimates are valid also for functions u with $B_i u = 0$, $1 \leq i \leq m_j^+$.

If Ω is a bounded open subset of R^n with a smooth boundary and the m_i are not all equal, it is impossible for $\partial\Omega$ to be normal everywhere. Consider however the following example. Let $n = 2$ and $m_1 > m_2$. Let g_1 and g_2 be C^∞ functions of a single variable such that the functions g_1 and $g_2 - 1$ have zeros of infinite order at $x = 0$ and $x = 1$ and $g_2 > g_1$. Let each one of the systems $\{B_1^{(1)}, \dots, B_{m_1}^{(1)}\}$, $\{B_1^{(2)}, \dots, B_{m_1}^{(2)}\}$ of ordinary differential operators with constant coefficients «cover» the operator D^{2m_1}. Assume moreover that if $(B_i^{(1)}u)(a) = (B_i^{(1)}v)(a) = (B_i^{(2)}u)(b) = (B_i^{(2)}v)(b) = 0$ then

$$\int_a^b u(x)\, D^{2m_1} \overline{v}(x)\, dx = \int_a^b (D^{2m_1} u)(x)\, \overline{v}(x)\, dx.$$

Define $D_{\tilde{A}}$ to be the $H_{2m_1, 2m_2}$ closure of C^∞ functions u which are periodic with period, in the variable x_2 and which satisfy $(B_j^{(1)}(D_1)\, u)(x_1, x_2) = 0$ on $x_1 = g_1(x_2)$, $(B_j^{(2)}(D_1)\, u, (x_1, x_2) = 0$, on $x_1 = g_2(x_2)$, $1 \leq j \leq m_1$, and let A be the semi-elliptic operator $D_1^{2m_1} + D_2^{2m_2}$.

Then

$$\|u\|_{(2m_1, 2m_2), \Omega} \leq C\,[\|\tilde{A}u\|_{0,\Omega} + \|u\|_{0,\Omega}]$$

for $u \in D_{\tilde{A}}$, if Ω is defined to be the strip obtained from the subset of R^2: $\{(x_1, x_2);\, g_1(x_2) \leq x_1 \leq g_2(x_2),\ 0 \leq x_2 \leq 1\}$ by identifying $[0, 1] \times \{0\}$

with $[0, 1] \times \{1\}$. We note that the boundaries of Ω are normal everywhere. Note also that if we required that $u(x_1, 0) = u(1 - x_1, 1)$ instead of requiring periodicity in x_2, we should obtain a Möbius strip.

Let us point out that using the transformation $r = x_1 + 1$, $\theta = 2\pi x_2$, we map Ω into a deformed annulus in the plane, and A is transformed into $\left(\frac{\partial}{\partial r}\right)^{2m_1} + \frac{1}{(2\pi)^{2m_2}} \left(\frac{\partial}{\partial \theta}\right)^{2m_2}$, which is semi-elliptic in an extended sense. It is easy to see by means of the well-known Hopf's theorem about the sum of indices of singularities of vector fields that no differential operator A which is semi-elliptic in the extended sense in Ω and such that the boundary is normal with respect to A exists, if Ω is a bounded plane domain which have more than one « hole » or which has no holes at all. This fact shows that topological obstructions exist for constructing semi-elliptic differential operators on manifolds with boundary such that the boundary will be normal. It might be interesting to find out the nature of these obstructions.

11. Asymptotic formulas for eigenvalues of semi-elliptic operators.

We shall prove now a theorem about asymptotic distribution of eigenvalues. The discussion of the preceding section explains, first of all, why some of the assumptions of the theorem are necessary, and secondly, that there exist non-trivial cases to which the theorem applies.

THEOREM 11.1 : Let Ω be a compact manifold with a non-empty boundary $\partial\Omega$. Let $\tilde{\mathscr{A}}$ be a self-adjoint semi-bounded operator in $L_2(\Omega)^p$ which is a realization of a formally self-adjoint differential operator $\mathscr{A}$. We assume that $\mathscr{A}$ is semi-elliptic in Ω with respect to the multi index m, where $m_i > 0$, $1 \leq i \leq n$, and that $\mathscr{A}$ has a complete family $\mathcal{F}$. Let $w(\mathscr{A}) = w$. Assume that if $\Omega_\varkappa \cap \partial\Omega$ is non-empty for $\varkappa \in \mathcal{F}$ then the set $\tilde{\Omega}_\varkappa$ satisfies the condition (I). Assume also that there exists a constant C, independent of $\varkappa$, such that for all $u \in D_{\tilde{\mathscr{A}}}$

$$(11.1) \qquad \| u \circ \varkappa^{-1} \|_{w\mathsf{m}, \tilde{\Omega}_\varkappa} \leq C [\| \tilde{\mathscr{A}} u \|_{0, \Omega} + \| u \|_{0, \Omega}].$$

If $w < \Sigma_{h=1}^n \frac{1}{m_h}$ assume that if $u \in D_{\tilde{\mathscr{A}}^k}$ then

$$(11.1') \qquad \| u \circ \varkappa^{-1} \|_{kw\mathsf{m}, \tilde{\Omega}_\varkappa} \leq C [\| \tilde{\mathscr{A}}^k u \|_{0, \Omega} + \| u \|_{0, \Omega}]$$

for some positive integer k such that $kw > \Sigma_{h=1}^{n} \frac{1}{m_h}$. Then $\tilde{\mathscr{A}}$ has a discrete spectrum. Let $\{\lambda_j\}$ be the sequence of eigenvalues, each repeated according to its multiplicity. Then

$$\sum_{\lambda_j \leq t} 1 = d t^{\Sigma_{h=1}^{n} \frac{1}{wm_h}} + o\left(t^{\Sigma_{h=1}^{n} \frac{1}{wm_h}}\right) \tag{11.2}$$

at $t \to \infty$, where

$$d = \Sigma_{i=1}^{p} \int_{\Omega} D_{i,i}(x)\, dx \tag{11.3}$$

and $D(x)$ is the matrix defined in (5.7).

PROOF: According to the formula (4.9)

$$\sum_{\lambda_j \leq t} 1 = \int_{\Omega} \Sigma_{i=1}^{p} E_{t,i,i}(x, x)\, dx$$

where $E_t(x, y)$ is the spectral function of $\tilde{\mathscr{A}}$. By theorem 5.2,

$$E_{t,i,i}(x, x) = D_{i,i}(x)\, t^{\Sigma_{h=1}^{n} \frac{1}{wm_h}} + o\left(t^{\Sigma_{h-1}^{n} \frac{1}{wm_h}}\right), \tag{11.4}$$

as $t \to \infty$, for x in the *interior* of Ω, and the o estimate is not uniform.

Assume without loss of generality that $\tilde{\mathscr{A}}$ is positive, and assume at first that $w > \Sigma_{h=1}^{n} \frac{1}{m_h}$. The operator E_t is a self-adjoint bounded projection by definition. Hence

$$\| E_t \|_{0, \Omega} \leq 1 \tag{11.5}$$

$$\| \tilde{\mathscr{A}} E_t \|_{0, \Omega} \leq t \tag{11.6}$$

since $\tilde{\mathscr{A}}$ is positive. It follows by (11.1), (11.5) and (11.6) that theorem 3.2 may be applied to the self-adjoint operator E_t. Let Σ be an n-dimensional box whose edges are parallel to the coordinate axes, $\Sigma \subset \tilde{\Omega}_\varkappa$ for $\varkappa \in \mathcal{F}$. From the inequality (3.14) it follows that

$$| E_{t,j,j}(x, x) | \leq C \prod_{h=1}^{n} \left(t + r_h^{-wm_h}\right)^{\frac{1}{wm_h}} \tag{11.7}$$

for $1 \leq j \leq p$, $x \in \varkappa^{-1}(\Sigma)$, where $r_1, \dots, r_n$ are the lengths of the edges of Σ and $t \geq 1$. (Recall that the numbers wm_h are integers). Since $t \geq 1$ and the numbers r_i are bounded, it follows that there exists a constant C_1 with $t + r_i^{-wm_i} \leq C_1 r_i^{-wm_i} t$. Hence we may rewrite (11.7) as

$$(11.8) \qquad | E_{t,j,j}(x, x) | \leq C_2 (r_1 \dots r_n)^{-1} t^{\Sigma_{h=1}^n \frac{1}{wm_h}}.$$

We choose a finite covering of $\partial\Omega : \partial\Omega \subset U_{i=1}^N \Omega_{\varkappa_i}$. For each $x \in \Omega_{\varkappa_i}$ choose an n-dimensional box $\Sigma \subset \widetilde{\Omega}_{\varkappa_i}$ whose edges are parallel to the coordinate axes such that the volume of Σ is greater than $h_i(\varkappa(x))/2$ and such that $\varkappa(x) \in \Sigma$. Here $h_i(x)$ is the function defined in condition (I), which is associated with $\widetilde{\Omega}_{\varkappa_i}$. It follows from (11.8) that

$$(11.9) \qquad | E_{t,j,j}(x, x) | \leq \frac{2C_2}{h_i(\varkappa(x))} t^{\Sigma_{h=1}^n \frac{1}{wm_h}}.$$

On $\Omega - \cup_{i=1}^N \Omega_{\varkappa_i}$ the o estimate in (11.4) is uniform. From condition (I) and the strict positivity of the density on Ω it follows that the function $\frac{1}{h_i(\varkappa(x))}$ is integrable over $\Omega_{\varkappa_i}$. Hence we may use the Lebesgue dominated convergence theorem to obtain (11.2) from (11.4).

In the general case we consider (as usual) the operator $\widetilde{\mathscr{A}}^k$ which is a self-adjoint realization of $\mathscr{A}^k$ in $L_2(\Omega)^p$. The eigenvalues of $\widetilde{\mathscr{A}}^k$ are $\{\lambda_j^k\}$. We have proved already that

$$\Sigma_{\lambda_j^k \leq t} 1 = dt^{\Sigma_{h=1}^n \frac{1}{kwm_h}} + o\,(t^{\Sigma_{h=1}^n \frac{1}{kwm_h}})$$

as $t \longrightarrow \infty$. Replacing t by t^k we obtain (11.2) in the general case.

We now indicate briefly how to obtain extensions of the remainder estimates for the eigenvalues, due to Agmon [5]. Essentially, no new idea (beyond those of Agmon) is required in order to make the extension for the semi-elliptic case. First we state a « geometric » condition which stands for condition (3.4) of [5].

DEFINITION: The bounded measurable set $K \subset R^n$ is said to posses the strong (I) property if for every $0 < \sigma < 1$ the function $h(x)^{-1} \delta(x)^{-\sigma}$ is integrable over K. Here $h(x)$ denotes the same function (supremum of certain volumes) as in the definition of the (I) condition (in section 10), whereas $\delta(x) = \min\{1, \operatorname{dist}(x, \partial K)\}$.

It is clear that smooth convex planar sets have the strong (I) property. We now state the following:

THEOREM 11.2: Suppose that all the conditions of theorem 11.1 hold. Assume moreover that each set $\tilde{\Omega}_\varkappa$ for which $\varkappa \in \mathcal{F}$ and $\Omega_\varkappa \cap \partial\Omega$ is not empty possesses the strong (I) property. Set

$$\bar{\theta} = \min_{x \in \Omega} \theta(x)$$

where $\theta(x)$ is the number associated with $\mathcal{A}$ (defined in (5.1)). Then for every $\varepsilon > 0$,

$$\sum_{\lambda_j \leq t} 1 = d t^{\sum_{h=1}^{n} \frac{1}{w m_h}} + O\left(t^{\sum_{h=1}^{n} \frac{1}{w m_h} - \frac{\bar{\theta} b}{w} + \varepsilon}\right)$$

as $t \to \infty$.

To prove this theorem one first has to prove an extension of theorem 3.2 of [5]. This is done by means of theorem 5.1 (which plays the role of theorem 4.1 of [5]) and the localization lemma 8.1 (which corresponds to lemma 4.2 of [5]). Thus, it is easy to see from lemma 8.1, that if $|\lambda| \geq 1$, $d(\lambda) \geq |\lambda|^{1 - \bar{\theta} b / w + \varepsilon}$ (i. e., in the same region of the complex plane where the interior asymptotic formula (5.2) holds) we have, with a constant which depends on N, ε and p, that

$$(11.10) \quad \left|(-\lambda)^{1 - \sum_{h=1}^{n} \frac{1}{w m_h}} R_{\lambda, s, t}(x, x) - \sum_{j=0}^{N-1} C_{j, s, t}(x, x)(-\lambda)^{-ja/w}\right| \leq$$

$$\leq \text{Const}\left[|\lambda|^{-Na/w} + \frac{1}{d(\lambda)}\left(\frac{|\lambda|^{1-b/w}}{\delta(\varkappa(x))\, d(\lambda)}\right)^p\right]$$

for any $p > 0$, if $x \in \Omega_\varkappa$ $(\varkappa \in \mathcal{F})$ such that $\Omega_\varkappa \cap \partial\Omega$ is not empty, and $\delta(\varkappa(x))$ is defined with respect to the set $\tilde{\Omega}_\varkappa$. Note that one applies lemma 8.1 with $r = \delta^{w/b}$. Theorem 11.2 follows from (11.10) in exactly the same way as theorem 3.1 of [5] follows from theorem 3.2 of [5].

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