

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

VIOREL BARBU

Differentiable distribution semi-groups

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 23,
n° 3 (1969), p. 413-429

http://www.numdam.org/item?id=ASNSP_1969_3_23_3_413_0

© Scuola Normale Superiore, Pisa, 1969, tous droits réservés.

L'accès aux archives de la revue « *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

DIFFERENTIABLE DISTRIBUTION SEMI-GROUPS

by VIOREL BARBU

Introduction.

Distribution semi-groups of operators in a Banach space were introduced and studied by Lions [1] (cf. also Foiaş [2], Yoshinaga [10], [11], Peetre [3]). J. L. Lions has obtained the characterization of the infinitesimal generator of an exponential distribution semi-group and recently his result has been generalized by Chazarain [4], [5] (cf. also Foiaş [2], Larsson [9]), for regular distribution and hyper-distribution semi-groups. In their works, Da Prato-Mosco [6], [7] and Fujiwara [8] have generalized the notion of holomorphic semi-group (cf. Yosida [14]) to that of holomorphic distribution semi-groups and have given a characterization of the infinitesimal generator of such a distribution semi-group.

In this paper we extend some of their results for differentiable distribution semi-groups.

§. 1. General results on distribution semi-groups.

We use the notations and the terminologies of L. Schwartz [12], [13] for infinitely differentiable functions and for distributions. We set $R = \{t; -\infty < t < \infty\}$ and denote: \mathcal{D} the space of all infinitely differentiable functions with compact support in R , \mathcal{C} the space of infinitely differentiable function on R ; \mathcal{D}^+ the space of all $\varphi \in \mathcal{D}$ such that $\text{supp } \varphi \subset [0, \infty)$ topologized as in Schwartz [12]; \mathcal{S} the space of rapidly decreasing \mathcal{C} functions and \mathcal{C}' the space of scalar distributions with compact support. We denote also by \mathcal{D}_- the strict inductive limit of the spaces $\mathcal{C}_a = \{\varphi \in \mathcal{C}; \text{supp } \varphi \subset (-\infty, a]\}$. Let X be a Banach space and $L(X, X)$ the space of all continuous linear

Pervenuto alla Redazione il 20 Gennaio 1969 ed in forma definitiva il 27 Febbraio 1969.

operator on X topologized with the operator norm. We denote also by $\mathcal{D}'(L(X, X))$, $\mathcal{D}'_+(L(X, X))$ and $\mathcal{S}'(L(X, X))$ the vector-valued distribution spaces: $L(\mathcal{D}, L(X, X))$, $L(\mathcal{D}_-, L(X, X))$ and $L(\mathcal{S}, L(X, X))$ respectively.

A vector-valued distribution $T \in \mathcal{D}'_+(L(X, X))$ is called a distribution semi-group (D. S. G. in short) if it satisfies the following conditions:

- i) $T(\varphi^* \psi) = T(\varphi) T(\psi)$ for any $\varphi, \psi \in \mathcal{D}^+$.
- ii) The support of T is contained in $[0, \infty)$.
- iii) The linear subspace $[T(\mathcal{D}^+)X]$ generated by $T(\mathcal{D}^+)X$ is dense in X .
- iv) If $x \in X$ and $T(\varphi)x = 0$ for any $\varphi \in \mathcal{D}^+$, then $x = 0$.

Let $R_+ \{t; t > 0\}$ and $\bar{R}_+ \{t; t \geq 0\}$. If $\mu \in \mathcal{C}'(\bar{R}_+)$ then we define a closable and densely defined operator $T(\mu)$ on $[T(\mathcal{D}^+)X]$, by the formula

$$(1.1) \quad T(\mu)x = \sum_{i=1}^n T(\varphi_i^* \mu) x_i, \quad \text{for} \quad x = \sum_{i=1}^n T(\varphi_i) x_i; \quad x_i \in X, \varphi_i \in \mathcal{D}^+.$$

Let us denote the closure of $T(\mu)$ again by $T(\mu)$. The linear operator $A = T(-D \delta_0)$ is called the infinitesimal generator of T . Here δ_t is the Dirac measure concentrated at $\mathcal{C} = t$ and D is the derivation symbol.

For any $\varphi(t)$ defined on R we denote by $\varphi_+(t)$ the function

$$\varphi_+(t) = \begin{cases} \varphi(t) & \text{for } t \geq 0, \\ 0 & \text{for } t < 0. \end{cases}$$

We say that a D. S. G. T is regular if $T(\varphi_+) = T(\varphi)$ for any $\varphi \in \mathcal{D}_-$. A regular D. S. G., T is called of exponential growth (E. D. S. G. in short) if there exists a number α such that $e^{-\alpha t} T \in \mathcal{S}'(L(X, X))$.

THEOREM 1 (Lions). A closed linear operator A in X with domain $D(A)$ dense in X , generates an E. D. S. G. if and only if there is a number $\alpha \geq 0$ such that

i) for any λ with $\text{Re } \lambda > \alpha$, $\lambda I - A$ defines an isomorphism of $D(A)$ onto X .

ii) $\|(\lambda I - A)^{-1}\| \leq \text{pol}(|\lambda|)$ for $\text{Re } \lambda > \alpha$ where $\text{pol}(|\lambda|)$ denote a polynomial with non-negative coefficients.

For the proof see [1] and [10]. The following theorem is due to Chazarain [4] (cf. also Foiaş [2]).

THEOREM 2. Let A be a closed and dense operator on X . Then A is the infinitesimal generator of a regular S. G. D. if and only if the following conditions hold:

i) There exist the constants α, β, γ ; $\alpha, \gamma \geq 0$ such that $(\lambda I - A)^{-1} \in L(X, X)$ for any λ in the domain

$$(1.2) \quad A = \{\lambda \in U; \operatorname{Re} \lambda \geq \alpha \log |\operatorname{Im} \lambda| + \beta; \operatorname{Re} \lambda \geq \gamma\}$$

$$\text{ii) } \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq \operatorname{pol}(|\lambda|), \text{ for } \lambda \in A.$$

We shall give a sketch of the proof for this theorem.

Necessity. Since $T \in \mathcal{D}'_+(L(X, X))$ is regular it follows (cf. Yoshinaga [10]) that $T \in \mathcal{D}'_+(L(X, D_A))$ and

$$(1.3) \quad \left(\frac{d}{dt} - A\right) * T = \delta_0 \otimes I_X; \quad T * \left(\frac{d}{dt} - A\right) = \delta_0 \otimes I_{D_A}$$

where D_A is the domain of A topologized by the norm $\|x\| = \|x\| + \|Ax\|$ and I_X (resp. I_{D_A}) is the identical application on X (resp. D_A). Let $\varrho(t)$ be a \mathcal{D} -function such that $\varrho(t) = 1$ on $\{t; |t| < 1\}$ and $\varrho(t) = 0$ for $|t| > 2$. We denote by E (resp. Φ) the distribution ϱT (resp. $\varrho' T$) and set $\widehat{E}(\varrho) = E(e^{-\lambda t}); \widehat{\Phi}(\lambda) = \Phi(e^{-\lambda t})$ for any complex λ . From (1.3) we have

$$(1.4) \quad (\lambda I - A) \widehat{E}(\lambda) = I_X - \widehat{\Phi}(\lambda); \quad \widehat{E}(\lambda)(\lambda I - A) = I_{D_A} - \widehat{\Phi}(\lambda).$$

Since $\Phi \in \mathcal{C}'(L(X, X))$ and $\operatorname{supp} \Phi \geq 1$, by a well known argument it follows

$$(1.5) \quad \|\widehat{\Phi}(\lambda)\|_{L(X, X)} \leq C(1 + |\lambda|)^N \exp(-\operatorname{Re} \lambda), \text{ for any } \lambda \in C.$$

From (1.4) this implies that $(\lambda I - A)^{-1} \in L(X, X)$ for $\lambda \in A = \{\lambda; \operatorname{Re} \lambda \geq \alpha \log |\operatorname{Im} \lambda| + \beta; \operatorname{Re} \lambda \geq \gamma\}$, with $\alpha, \gamma > 0$ convenient chosen. Moreover we get

$$(1.6) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq c \|\widehat{E}(\lambda)\|_{L(X, X)}, \text{ for } \lambda \in A.$$

But $\operatorname{supp} E \subset [0, 1]$ and by a Paley-Wiener theorem argument it follows

$$\|\widehat{E}(\lambda)\|_{L(X, X)} \leq \operatorname{pol}(|\lambda|) \text{ for any } \lambda \in A.$$

This inequality together (1.6) proves (ii).

Sufficiency. Define $T \in \mathcal{D}'(L(X, X))$ by the formula

$$(1.7) \quad T(\varphi) = (2\pi i)^{-1} \int_{\Gamma} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda) d\lambda,$$

where Γ is the frontier of Λ and

$$\widehat{\varphi}(\lambda) = \int e^{-\lambda t} \varphi(t) dt.$$

From (i), (ii) it follows that T is a regular D. S. G.

§. 2. Differentiable distribution semi-groups.

DEFINITION. A regular D. S. G. T is called differentiable if for every $t > 0$, $T(\delta_t) \in L(X, X)$ and the application $t \rightarrow T(\delta_t)$ from \mathbb{R}^+ in $L(X, X)$ is differentiable.

REMARKS. 1° If T is differentiable, then the distribution $T \in \mathcal{D}'(L(X, X))$ is given on \mathbb{R}^+ by a differentiable $L(X, X)$ -valued function. In fact for any $x \in [T(\mathcal{D}^+)X]$, we have

$$(2.1) \quad T(\varphi)x = \int_0^{\infty} T(\delta_t)x \varphi(t) dt \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^+).$$

Since the space $[T(\mathcal{D}^+)X]$ is dense in X , this implies that $T = T(\delta_t)$ on \mathbb{R}^+ .

2° Let T be a differentiable D. S. G. and $A = T(-\delta'_0)$ its infinitesimal generator. Then for every $t > 0$, $T(\delta_t)X \subset D_A$ and

$$(2.2) \quad \frac{d}{dt} T(\delta_t)x = AT(\delta_t)x, \quad \text{for any } x \in X \text{ and } t > 0.$$

To prove this, we consider x an arbitrary element of X and set $y(t) = T(\delta_t)x$ for $t > 0$. Let x_n be a sequence of $[T(\mathcal{D}^+)X]$ such that $x_n \rightarrow x$. It is obvious that $AT(\delta_t)x_n = d/dt T(\delta_t)x_n \rightarrow d/dt T(\delta_t)x$ for $n \rightarrow \infty$. Since A is closed, this implies that $y(t) \in D(A)$ and $y'(t) = Ay(t)$ for any $t > 0$.

The following theorem gives a characterization for the generator of a differentiable D. S. G.

THEOREM 3. Let A be a closed operator on X with domain D_A dense in X . A necessary and sufficient condition for A generate a differentiable regular D. S. G. is : for every $\delta > 0$ there exist positive constants C_δ and M_δ such that $(\lambda I - A)^{-1} \in L(X, X)$ for any complex λ in the domain

$$A_\delta = \{\lambda; \operatorname{Re} \lambda \geq -\delta \log |\operatorname{Im} \lambda| + C_\delta\} \cup \{\lambda; \operatorname{Re} \lambda \geq \gamma\}$$

and for $\lambda \in A_\delta$,

$$(2.3) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq M_\delta \text{pol}(|\lambda|),$$

where γ is a non-negative constant independent of δ .

PROOF. Necessity. Let $\varphi(t)$ be a \mathcal{D} -function so that $\text{supp } \varphi \subset \{t; |t| \leq 1\}$ and $\varphi(t) = 1$ in $|t| < 2^{-1}$. Denote by $\varphi_\varepsilon(t)$, $\varepsilon > 0$, the function $\varphi(t/\varepsilon)$ and by $E_\varepsilon, \widehat{\Phi}_\varepsilon$ the vector-valued distribution $\varphi_\varepsilon T$ and $\varphi'_\varepsilon T$ respectively. It is obvious that $\widehat{\Phi}_\varepsilon$ is differentiable and $\text{supp } \widehat{\Phi}_\varepsilon \subset \{t; 2^{-1}\varepsilon \leq t \leq \varepsilon\}$. Put

$$M_\varepsilon = \sup_{2^{-1}\varepsilon < t < \varepsilon} \|D^1(\varphi'_\varepsilon(t) T(t))\|.$$

It is easy to see that

$$\|\widehat{\Phi}_\varepsilon(\lambda)\| \leq 2^{-1}\varepsilon M_\varepsilon |\text{Im } \lambda|^{-1} \sup_{\varepsilon/2 \leq t \leq \varepsilon} \exp(-t \text{Re } \lambda), \quad \lambda \in \mathbb{C}.$$

Hence $\|\widehat{\Phi}_\varepsilon(\lambda)\| \leq 2^{-1}$ for any complex λ in the domain

$$\Sigma_\varepsilon = \{\lambda; \log |\text{Im } \lambda| \geq -\varepsilon \text{Re } \lambda + \log M_\varepsilon; \text{Re } \lambda \leq 0\} \cup \\ \cup \{\lambda; \log |\text{Im } \lambda| \geq -\varepsilon/2 \text{Re } \lambda + \varepsilon/2 \log M_\varepsilon; \text{Re } \lambda \geq 0\}.$$

Remembering (1.4) this implies that

$$(2.4) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq 2^{-1} \|\widehat{E}_\varepsilon(\lambda)\|_{L(X, X)}, \quad \text{for } \lambda \in \Sigma_\varepsilon.$$

On the other hand, since $\text{supp } E \subset [0, \varepsilon]$ we have (see L. Schwartz 12, Th.)

$$\|\widehat{E}_\varepsilon(\lambda)\|_{L(X, X)} \leq \sup_{t \in [0, \varepsilon]} \sum_{j=0}^m |D^j(e^{-\lambda t} \varphi_\varepsilon(t))|, \quad \lambda \in \mathbb{C}.$$

Hence

$$(2.5) \quad \|\widehat{E}_\varepsilon(\lambda)\|_{L(X, X)} \leq M_\varepsilon \text{pol}(|\lambda|) |\text{Im } \lambda|, \quad \text{for } \lambda \in \Sigma_\varepsilon,$$

where the degree of the polynomial $\text{pol}(|\lambda|)$ is equal to the order of the distribution T in a neighbourhood of the origin. Therefore $(\lambda I - A)^{-1} \in L(X, X)$ and satisfies (2.3) for any $\lambda \in \Sigma_\varepsilon$. From theorem 2 it follows then, that there exists a non-negative constant γ such that $\|(\lambda I - A)^{-1}\| \leq \text{pol}(|\lambda|)$ for $\text{Re } \lambda \geq \gamma$. If we choose N_ε so that $\log N_\varepsilon/M_\varepsilon \geq \gamma$, we deduce that the

estimate (2.3) is verified for any λ in the domain

$$\{\lambda; \operatorname{Re} \lambda \geq -\varepsilon^{-1} \log |\operatorname{Im} \lambda| + \varepsilon^{-1} \log N_\varepsilon\} \cup \{\lambda; \operatorname{Re} \lambda \geq \gamma\}.$$

Choosing $\delta = \varepsilon^{-1}$ this implies that $(\lambda I - A)^{-1}$ satisfies (2.3) in any domain A_δ .

Sufficiency. It is obvious that T is an E. D. S. G. Hence we may write

$$(2.6) \quad T(\varphi) = (2\pi i)^{-1} \int_{\operatorname{Re} \lambda = \gamma} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda) d\lambda, \quad \varphi \in \mathcal{D}.$$

As $(\lambda I - A)^{-1}$ is holomorphic in every A_δ and $\|(\lambda I - A)^{-1} \widehat{\varphi}(-\lambda)\|$ rapidly tends to zero at infinity, we can change the path of integration and obtain

$$T(\varphi) = (2\pi i)^{-1} \int_{\Gamma_\delta} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda) d\lambda,$$

where Γ_δ is the boundary of the domain $\{\lambda; \operatorname{Re} \lambda \geq -\delta \log |\operatorname{Im} \lambda| + C_\delta; \operatorname{Re} \lambda \leq \gamma\}$. Let $\{\varrho_k\}_{k=0}^\infty \subset \mathcal{D}^+$ be a sequence of regularization for Dirac distribution, i. e. $\varrho_n(t) \geq 0$, $\int \varrho_n(t) dt = 1$ and $\operatorname{supp} \varrho_n \rightarrow 0$. We have

$$(2.7) \quad D_t^k T(\delta_t * \varrho_n) = (2\pi i)^{-1} \int_{\Gamma_\delta} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho}_n(-\lambda) d\lambda$$

for any non-negative integer k . We set $\Gamma_\delta = \Gamma_\delta^1 \cup \Gamma_\delta^2$, where Γ_δ^1 is given by $\{\operatorname{Re} \lambda = -\delta \log |\operatorname{Im} \lambda| + C_\delta; |\operatorname{Im} \lambda| \geq A_\delta = \exp(\delta^{-1}(C_\delta - \gamma))\}$ and Γ_δ^2 by $\{\operatorname{Re} \lambda = \gamma; |\operatorname{Im} \lambda| \leq \exp(\delta^{-1}(C_\delta - \gamma))\}$. We write

$$T_j^{(k)}(t) = (2\pi i)^{-1} \int_{\Gamma_\delta^j} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho}_n(-\lambda) d\lambda; \quad j = 1, 2, \dots$$

It is obvious that $T_2^{(k)}(t)$ is defined for every $t \geq 0$ and

$$(2.8) \quad \|T_2^{(k)}(t)\|_{L(X, X)} \leq M_\delta^{m+k+1} \exp(\gamma t) \quad \text{for } t \geq 0.$$

where M_δ is another non-negative constant. Let $\lambda = \sigma + i\eta$; then since on Γ_δ^1 , $\sigma = -\delta \log |\eta| + C_\delta$ we have

$$\|T_1^{(k)}(t)\|_{L(X, X)} \leq M_\delta \exp(C_\delta t) \int |\eta|^{m+k-t\delta} d\eta.$$

Hence

$$(2.9) \quad \|T_1^{(k)}(t)\| \leq M_\delta \exp(C_\delta t), \quad \text{for } t > (m+k+1)\delta^{-1}.$$

Therefore we find a constant $M_{k, \delta}$ such that

$$(2.10) \quad \|D_t^k T(\delta_t * \varrho_n)\|_{L(X, X)} \leq M_{k, \delta} \exp(C_\delta t)$$

for $t > (m+k+1)\delta^{-1}$. But for any $x \in [T(\mathcal{D}^+)X]$ we have

$$D_t^k T(\delta_t * \varrho_n) x \rightarrow D_t^k T(\delta_t) x, \quad t > 0, \quad k = 0, 1, \dots$$

uniformly on every compact. Since the space $[T(\mathcal{D}')X]$ is dense in X this implies that $D_t^k T(\delta_t) \in L(X, X)$ for $t > (m+k+1)\delta^{-1}$. Since δ is arbitrary this proves the differentiability of T . Moreover we have proved that $D_t^k T(\delta_t) \in L(X, X)$ for any $t > 0$ and $k = 0, 1, \dots$. Combining with the first part of the proof it follows that if a regular D. S. G., T is differentiable then the application $t \rightarrow T(\delta_t)$ from R^+ in $L(X, X)$ is infinitely differentiable.

COROLLARY. Let A be a closed and densely defined operator on the Banach space X . If the conditions of Theorem 3 are satisfied, then the abstract Cauchy problem $(ACP)_0$:

$$(2.11) \quad \begin{aligned} \frac{du(t)}{dt} - Au(t) &= 0, & \text{for } t > 0, \\ u(0) &= 0, \end{aligned}$$

has a solution $u \in C^\infty(R^+, X)$ for every $x \in X$.

PROOF. Let T be the D. S. G. generated by A . Then from remark 2 it follows that $T(\delta_t)x$ solves $(ACP)_0$ for any $x \in X$.

REMARKS 1° If T is a differentiable regular D. S. G., then

$$(2.1') \quad \|D^k T(\delta_t)\|_{L(X, X)} = 0(\exp(\gamma_0 t)), \quad \text{for } t \rightarrow \infty$$

and $k = 0, 1, \dots$, where γ_0 is a non-negative constant.

2° In particular if T is a strongly continuous semi-group of bounded linear operators on X , then according to formula (2.5) it follows that T is differentiable if and only if for $\lambda \in A_\delta$

$$(2.12) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq M_\delta |\operatorname{Im} \lambda|.$$

Thus we find a result proved by Pazy [16].

§ 3. Analytic and non-quasianalytic D. S. G.

Let Y be a Banach space and $L = \{L_k\}_{k=0}^\infty$ and increasing sequence of non-negative numbers such that

$$(3.1) \quad L_k^{2k} \leq L_{k-1}^{k-1} L_{k+1}^{k+1}; \quad L_{mk+n} \leq q(m) L_k, \quad k = 0, 1, \dots$$

where m and n are non-negative integers and $r \rightarrow q(r)$ is a positive and monotone increasing function. If Ω is an open set of \mathbb{R} we denote by $C^L(\Omega, Y)$ the space of infinitely differentiable Y -valued functions $u(t)$ in Ω , such that for any compact subset K there exists $M > 0$ such that

$$(3.2) \quad \sup_{t \in K} \|D^j u(t)\| \leq M^{j+1} L_j^j; \quad j = 0, 1, \dots$$

The space $C^L(\Omega, Y)$ is topologized as projective limit of all $\{C^L(K, Y); K \subset \Omega\}$. The function class $C^L(\Omega, Y)$ is called non-quasi-analytic if it contains a non-trivial regular function with compact support contained in Ω . The Carleman-Denjoy criterion states that C^L is on-quasianalytic if and only if

$$\sum L_j^{-1} < \infty.$$

If $Y = \mathbb{R}$ we often omit \mathbb{R} and write $C^L(Y)$. In particular, if $L_j = (j!)^{\rho j}$, C^L is the classical Gevrey class \mathcal{G} which is non-quasianalytic for $1 < \rho < \infty$. For $\rho = 1$ we obtain the class of real analytic functions. If L is a non-quasianalytic sequence we denote by $C_0^L(\Omega, Y)$ the space $C^L(\Omega, Y) \cap C_0^\infty(\Omega, Y)$.

DEFINITION. A D. S. G., $T \in \mathcal{D}'_+(L(X, X))$ is said to be of class C^L if the mapping $t \rightarrow T(\delta_t)$ is of class C^L on \mathbb{R}^+ .

In particular, for $L_j = (j!)^{\rho j}$ the semi-group T is called ρ -hypoanalytic; $1 \leq \rho < \infty$. As above we remark that if the semi-group T is of class C^L then the distribution $T \in \mathcal{D}'(L(X, X))$ is defined on $\mathbb{R}^+ = \{t; t > 0\}$ by a $L(X, X)$ -valued function of class C^L .

Let $L = \{L_j\}_{j=0}^\infty$ be a non-quasianalytic sequence and $\omega_L(t)$ be a scalar function defined by

$$\omega_L(t) = \sum_{j=0}^\infty t^j/L_j^j; \quad t \geq 0.$$

Then we have

THEOREM 4. Let T be a regular D.S.G. and $A = T(-\delta_0)$ be its infinitesimal generator. T is of class C^L if and only if for every $0 < \varepsilon < 1$ there exist C_ε and $M_\varepsilon > 0$ such that

i) $(\lambda I - A)^{-1} \in L(X, X)$ for any λ in the domain

$$(3.3) \quad \Sigma_\varepsilon = \{\lambda; \operatorname{Re} \lambda \geq -\log \omega_L(C_\varepsilon |\operatorname{Im} \lambda|) + \gamma\}$$

and

ii) $\|(\lambda I - A)^{-1}\|_{L(X, X)} \leq M_\varepsilon \operatorname{pol}(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|)$, for $\lambda \in \Sigma_\varepsilon$ where γ is a positive constant independent of ε .

PROOF. Necessity. Assume that for every $0 < \varepsilon < 1$.

$$(3.4) \quad \|D^k T(t)\|_{L(X, X)} \leq M_\varepsilon^{k+1} L_k^k, \quad t \in [\varepsilon/2, \varepsilon], \quad k = 0, 1, \dots$$

We choose $\varphi \in C_0^L$ such that $\operatorname{supp} \varphi \subset \{t; |t| \in 1\}$, $\varphi(t) = 1$ in $|t| \leq 2$, and denote: $\Phi_\varepsilon = \varphi'_\varepsilon T$; $E_\varepsilon = \varphi_\varepsilon T$ where $\varphi_\varepsilon(t) = \varphi(t/\varepsilon)$. From (3.4) we obtain

$$(3.5) \quad \|\Phi_\varepsilon^{(k)}(t)\| \leq M M_\varepsilon \varepsilon^{-1} (2 N_\varepsilon)^{-k} L_k^k, \quad k = 0, 1, \dots$$

where $M > 0$ and $N_\varepsilon^{-1} = 2 \max(M\varepsilon^{-1}, M\varepsilon)$. Or,

$$\|\widehat{\Phi}_\varepsilon(\lambda)\| = M(L_k/2N_\varepsilon |\operatorname{Im} \lambda|)^k \int_{\varepsilon/2}^\varepsilon \exp(-t \operatorname{Re} \lambda) dt, \quad k = 0, 1, \dots$$

Thus for any λ complex in the domain

$$A_\varepsilon = \{\lambda; \operatorname{Re} \lambda \geq \varepsilon^{-1} \log \omega_L(N_\varepsilon |\operatorname{Im} \lambda|) + M_\varepsilon^{-1}; \operatorname{Re} \lambda \leq \gamma_\varepsilon\}$$

we have $\|\widehat{\Phi}_\varepsilon(\lambda)\| \leq 2^{-1}$. Here M_ε^{-1} and γ_ε are another non-negative constants. Since the semi group T is regular we may assume that $\gamma_\varepsilon = \infty$. As in the proof of theorem 3, this implies that $(\lambda I - A)^{-1} \in L(X, X)$, and

$$(3.6) \quad \|(\lambda I - A)^{-1}\| \leq p_\varepsilon(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|), \quad \text{for } \lambda \in A_\varepsilon.$$

where p_ε is a polynomial with non-negative coefficients. Since the sequence $\{L_k\}$ satisfies (3.1), the function $r \rightarrow \log \omega_L(r)$ is sub-additive. Hence we may find another constant $C_\varepsilon > 0$ such that (3.6) to be satisfied for any λ in the domain

$$\{\lambda; \operatorname{Re} \lambda \geq -2 \log \omega_L(C_\varepsilon |\operatorname{Im} \lambda|) + M_\varepsilon^1\}.$$

Without loss of the generality we may assume that $\varepsilon \rightarrow C_\varepsilon$ is bounded and $\lim_{\varepsilon \rightarrow 0} M_\varepsilon^1 = \infty$. Let a be a non-negative constant such that $C_\varepsilon \leq a$ for $0 < \varepsilon < 1$. Using the above argument it follows that there exist $b > 0$ such that

$$(3.8) \quad \|(\lambda I - A)^{-1}\| \leq \operatorname{pol}(|\lambda|) \exp(N |\operatorname{Re} \lambda|)$$

for $\operatorname{Re} \lambda \geq -\log \omega_L(a |\operatorname{Im} \lambda|) + b$. For ε enough small we may suppose $b < M_\varepsilon^1$. Hence

$$(3.9) \quad \|(\lambda I - A)^{-1}\| \leq p_\varepsilon(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|)$$

for $\operatorname{Re} \lambda \geq -\log \omega_L(C_\varepsilon |\operatorname{Im} \lambda|) + b$; and, $|\operatorname{Im} \lambda| \geq C_\varepsilon^{-1} \omega_L^{-1}(\exp(M_\varepsilon^{-1} - b))$. Using (3.8) we get that the estimate (3.3) satisfied in the whole domain Σ_ε with $\gamma = b$.

Sufficiency. From (3.3) it follows that $\|(\lambda I - A)^{-1}\| = 0$ ($\operatorname{pol}(|\lambda|)$) for $\operatorname{Re} \lambda > \gamma$. Hence, as in the proof of theorem 3, we get

$$(3.10) \quad D_t^k T(\delta_t * \varrho_n) = (2\pi i)^{-1} \int_{\Gamma_\varepsilon} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho}_n(-\lambda) d\lambda$$

where $\{\varrho_n\}$ is a sequence of regularization of \mathcal{D}^+ and Γ_ε is the frontier of the domain

$$\{\operatorname{Re} \lambda \geq -\log \omega_L(C_\varepsilon |\operatorname{Im} \lambda|) + \gamma; \operatorname{Re} \lambda \leq \gamma\}.$$

It is easily verified that

$$(3.11) \quad \|D_t^k T(\delta_t * \varrho_n)\|_{L(X, X)} \leq M_\varepsilon^{m+k+1} \exp(\gamma t) \int_{\bar{K}_+} \eta^{m+k} \omega_L^{\varepsilon-t}(\eta) d\eta$$

where m is the degree of the polynomial $\operatorname{pol}(|\lambda|)$.

Since, for any non-negative integer k , $\omega_L(\eta) \geq \eta^j L_j^{-j}$, it follows that the right side of (3.11) is bounded by $M^{m+k+1} \exp(\gamma t) L_{(m+k+1)p+1}^{m+k+2}$

where p is the largest integer smaller than $(t - \varepsilon)^{-1}$. Because of the properties of the sequence $\{L_k\}$ we find another constant M_ε we such that

$$\| D_t^k T(\delta_t * \varrho_n) \|_{L(X, X)} \leq M_\varepsilon^{k+1} L_k^k(\exp(\gamma t))$$

for $t > \varepsilon$ and $k = 0, 1, \dots$. According to an argument used in the proof of theorem 3, this implies that

$$(3.13) \quad \| D^k T(\delta_t) \|_{L(X, X)} \leq M_\varepsilon^{k+1} + L_k^k \exp(\gamma t), \quad k = 0, 1, \dots; \quad t > \varepsilon > 0.$$

Since ε is arbitrary, the proof is complete.

COROLLARY. Let T be a regular D.S.G. and A its infinitesimal generator. The semi-group T is ϱ -hyppoanalytic; $1 \leq \varrho < \infty$, if and only if for every $\varepsilon > 0$ there exist constants C_ε and M_ε such that $(\lambda I - A)^{-1} \in L(X, X)$ and satisfies

$$(3.14) \quad \| (\lambda I - A)^{-1} \| \leq M_\varepsilon \operatorname{pol}(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|)$$

for

$$(3.15) \quad \operatorname{Re} \lambda \geq - C_\varepsilon |\operatorname{Im} \lambda|^{1/\varepsilon} + \gamma$$

where γ is a non-negative constant independent of ε .

PROOF. The non-quasianalytic case $\varrho > 1$ is a consequence of theorem 4. We assume that $\varrho = 1$. It is easily proved that there exists a sequence $\varphi_k \in \mathcal{D}$, $k = 0, 1, \dots$ such that

$$\operatorname{supp} \varphi_k \subset \{t; |t| \leq 1\}; \quad \varphi_k(t) \equiv 1 \quad \text{for } |t| \leq 2^{-1}$$

and

$$(3.16) \quad |\varphi_k^{(j)}(t)| \leq M^{j+1} k^j, \quad \text{for } j \leq k.$$

Put

$$\varphi_{\varepsilon, k}(t) = \varphi_k(t/\varepsilon); \quad E_{\varepsilon, k} = \varphi_{\varepsilon, k} T; \quad \Phi_{\varepsilon, k} = \varphi'_{\varepsilon, k} T.$$

If $T(\delta_t) \in G^1(\mathbb{R}^+, L(X, X))$, then as in the proof of theorem 4 we find a constant $M_\varepsilon > 0$ such that

$$(3.17) \quad \| \widehat{\Phi}_{\varepsilon, k}(\lambda) \|_{L(X, X)} \leq M_\varepsilon (k/M_\varepsilon |\operatorname{Im} \lambda|)^k \int_{\varepsilon/2}^\varepsilon \exp(-t \operatorname{Re} \lambda) dt$$

for any non-negative integer k . Take k equal to the largest integer smaller than $M_\varepsilon |\operatorname{Im} \lambda| e^{-1}$. Thus from (3.17) we obtain

$$\|\widehat{\Phi}_{\varepsilon, k}(\lambda)\|_{L(X, X)} \leq M_\varepsilon^1 \exp(-M_\varepsilon e^{-1} |\operatorname{Im} \lambda|) \int_{\varepsilon/2}^{\varepsilon} \exp(-\operatorname{Re} \lambda t) dt.$$

As above this implies that, $(\lambda I - A)^{-1} \in L(X, X)$ and satisfies

$$\|(\lambda I - A)^{-1}\| \leq \operatorname{pol}(|\lambda|) \exp(\varepsilon |\operatorname{Re} \lambda|)$$

for

$$\operatorname{Re} \lambda \geq 2\varepsilon^{-1} \log(2M_\varepsilon) - M_\varepsilon(e\varepsilon)^{-1} |\operatorname{Im} \lambda|; \quad \text{and } |\operatorname{Im} \lambda| < M_\varepsilon^{-1} e(k+1).$$

Since k is arbitrary, this implies that $(\lambda I - A)^{-1}$ satisfies the estimate (3.14) in a domain of the form

$$\operatorname{Re} \lambda \geq -C_\varepsilon |\operatorname{Im} \lambda| + D_\varepsilon; \quad \operatorname{Re} \lambda \geq \gamma$$

Sufficiency of (3.14) follows just in the proof of theorem 4.

§ 4. Distribution semi-groups of class A^e .

If a D.S.G. T is differentiable, then for any integer $k \geq 0$, $\|D_t^k T(\delta_t)\|_{L(X, X)}$ is of exponential growth for $t \rightarrow \infty$. In this section we also impose a restriction of the origin for $D_t^k T(\delta_t)$.

DEFINITION. Let $1 \leq \varrho < \infty$. A regular D.S.G., T is said to be of class A^e , if for $t > 0$,

$$(4.1) \quad \|D_t^k T(\delta_t)\|_{L(X, X)} \leq p(t^{-\varrho})(Mt)^{-ek}(k!)^e \exp(\gamma t); \quad k = 0, 1, \dots$$

where M, γ are non-negative constants and $p(r)$ is a polynomial with non negative coefficients.

The semi-groups of class A^e can be characterized in the following way (see theorem 4).

THEOREM 5. Let A be a closed operator on X with the domain D_A dense in X . Then A is the infinitesimal generator for a D.S.G. of class A^e if and only if there exist positive constants α and β such that

$$(4.2) \quad \begin{aligned} (i) \quad & (\lambda I - A)^{-1} \in L(X, X) \text{ for} \\ & \lambda \in A = \{\lambda \mid \operatorname{Re} \lambda > -\alpha |\operatorname{Im} \lambda|^{1/e} + \beta\}. \\ (ii) \quad & \|(\lambda I - A)^{-1}\| \leq \operatorname{pol}(|\lambda|) \text{ for any } \lambda \in A. \end{aligned}$$

PROOF. Necessity. From (4.1) it is obvious that $e^{-\gamma t} T \in \mathcal{D}'(L(X, X))$ where γ is a non-negative constant. Therefore the semigroup T is of exponential growth and from Lions's theorem it follows that $(\lambda I - A)^{-1}$ exists and satisfies the estimate (ii) for $\text{Re } \lambda > \gamma$. Moreover by a well known result (cf. Schwartz [12], Yoshinaga [11]) there exists a function $f \in \mathcal{C}^0(L(X, X))$ and an integer $m \geq 0$ such that

$$(4.3) \quad \|f(t)\|_{L(X, X)} = O((1 + t^2)^m) \quad \text{for } t \rightarrow \infty$$

and $e^{-\gamma t} T$ may be expressed as

$$(4.4) \quad e^{-\gamma t} T = D^m f.$$

The function $f(t)$ is regular on R^+ and from (4.1) we have

$$(4.5) \quad \|D^k f(t)\|_{L(X, X)} \leq M_p (t-e)^k (k!)^e, \quad k \geq m, t > 0$$

and

$$(4.6) \quad \|D^k f(t)\|_{L(X, X)} \leq M_1 p (t-e) t^{m-k} \quad \text{for } 0 \leq k \leq m$$

Let j and k be two non-negative integers such that $\varrho(k+p) < j \leq \varrho(k+p) + 1$, where p is the degree of $p(v)$. Since $\text{supp } f \subset [0, \infty)$, for j and k taken as above we have

$$\lambda^k D^j \widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t} D^k (t^j f(t)) dt, \quad \text{Re } \lambda > \varepsilon$$

where ε is an arbitrary positive number. Then (4.5) and (4.6) imply that

$$\|D^j \widehat{f}(\lambda)\|_{L(X, X)} \leq M^{j+1} j! |\lambda|^{-k}$$

for $\text{Re } \lambda > \varepsilon$ and $\varrho(k+p) < j \leq \varrho(k+p) + 1$. Hence

$$(4.7) \quad \|D^j \widehat{f}(\lambda)\|_{L(X, X)} \leq M^{j+1} j! |\lambda|^{-(j-1)e^{-1} + p}, \quad \text{for } \text{Re } \lambda > \varepsilon > 0.$$

Then the analyticity of $\widehat{f}(\lambda)$ in the domain $\{\lambda : \text{Re } \lambda > 0\}$ and the estimate (4.7) imply that $\widehat{f}(\lambda)$ can be extended holomorphically in a domain of the form

$$\Sigma = \{\lambda \in C; |\text{Re } \lambda - \varepsilon| < M^{-1} |\text{Im } \lambda|^{1/e}\}$$

and $\|\widehat{f}(\lambda)\|_{L(X, X)} \leq M |\lambda|^{p+e-1}$ for $\lambda \in \Sigma$. We observe that

$$T(e^{-\lambda t}) = \widehat{T}(\lambda) = (\lambda - \gamma)^m \widehat{f}(\lambda - \gamma), \quad \text{for } \operatorname{Re} \lambda > \gamma.$$

Hence we have proved that $\widehat{T}(\lambda)$ exists and satisfies the estimate

$$(4.8) \quad \|\widehat{T}(\lambda)\|_{L(X, X)} \leq \operatorname{pol}(|\lambda|)$$

for $|\operatorname{Re} \lambda - \gamma - \varepsilon| < M^{-1} |\operatorname{Im} \lambda|^{1/e}$. Because $\widehat{T}(\lambda) = (\lambda I - A)^{-1} \in L(X, X)$ for $\operatorname{Re} \lambda > \gamma$, the analyticity of $(\lambda I - A)^{-1}$ implies that it satisfies the estimate (ii) for

$$\operatorname{Re} \lambda > -M^{-1} |\operatorname{Im} \lambda|^{1/e} + \gamma.$$

Sufficiency. From (i) and (ii) it follows that the operator A generates an E. D. S. G. and

$$T(\varphi) = (2\pi i)^{-1} \int_{\operatorname{Re} = \beta + \varepsilon} (\lambda I - A)^{-1} \widehat{\varphi}(-\lambda) d\lambda, \quad \varphi \in \mathcal{D}$$

where ε is an arbitrary positive number. As in the proof of theorem 5 we have

$$(4.9) \quad D_t^k T(\delta_t * \varrho_n) = (2\pi i)^{-1} \int_{\Gamma_\varepsilon} e^{\lambda t} (\lambda I - A)^{-1} \lambda^k \widehat{\varrho}_n(-\lambda) d\lambda$$

where Γ_ε is the curve given by

$$\Gamma_\varepsilon = \{\lambda = \sigma + i\eta; \sigma = -\alpha |\eta|^{1/e} + \beta + \varepsilon; -\infty < \sigma \leq \beta + \varepsilon\}.$$

Then our estimates of $(\lambda I - A)^{-1}$ and $\widehat{\varrho}_n(-\lambda)$ imply that

$$\|D_t^k T(\delta_t * \varrho_n)\|_{L(X, X)} \leq C^{k+1} \exp(\beta + \varepsilon) t \int_0^\infty \eta^{p+k} \exp(-t\alpha\eta^{1/e}) d\eta$$

for any $t > 0$ and $k = 0, 1, \dots$. Hence

$$(4.10) \quad \|D_t^k T(\delta_t * \varrho_n)\|_{L(X, X)} \leq \operatorname{pol}(t^{-e}) (Ct^{-e})^k \Gamma(\varrho k) \exp(\beta + \varepsilon) t.$$

Here $\Gamma(r)$ is Euler's function and C is a positive constant independent of ε . Consequently the semi-group T is of class A^e and the proof is complete.

For $\varrho \geq 1$ and $\gamma \geq 0$ we denote by A_γ^ϱ the class of regular D. S. G., T such that for any $\varepsilon > 0$

$$(4.11) \quad \|D_t^k T(\delta_t)\|_{L(X, X)} \leq \underset{\varepsilon}{\text{pol}}(t^{-\varepsilon})(Mt)^{-\varepsilon k} (k!)^\varepsilon \exp(\gamma + \varepsilon)t.$$

As a consequence of theorem 5 and its proof we obtain (see also Da Prato-Mosco [7]).

COROLLARY. A closed and densely defined operator A on X generates a D. S. G. of class A_γ^ϱ if and only if there exists $\alpha > 0$ such that $(\lambda I - A)^{-1} \in L(X, X)$ and satisfies

$$\|(\lambda I - A)^{-1}\| \leq \underset{\varepsilon}{\text{pol}}(|\lambda|)$$

for $\text{Re } \lambda > -\alpha$ $|\text{Im } \lambda|^{1/\varepsilon} + \gamma + \varepsilon$, where ε is an arbitrary non-negative number.

Let $T \in \mathcal{D}'_+(L(X, X))$ be a regular D. S. G. T is said holomorphic (cf. Fujiwara [8], Da Prato-Mosco [6]) in the sector $\Sigma = \{\mu; |\arg \mu| < \alpha; 0 < \alpha < \pi/2\}$ if $t \rightarrow T(\delta_t)$ can be extended at an holomorphic function T_μ in this sector. It is obvious that a D. S. G. of class A^ϱ with $\varrho = 1$ is holomorphic in a sector of the complex plane. Conversely from Cauchy's formula it follows that any holomorphic D. S. G. in a sector Σ is of class A^1 . We can now formulate the following result (cf [8]).

THEOREM 6. A closed and dense linear operator A generates a D. S. G. which is holomorphic in the sector $\Sigma = \{\mu; |\arg \mu| < \alpha < \pi/2\}$ if and only if there exists a real γ such that for any $\varepsilon > 0$ and any λ in the sector

$$A = \{\lambda \mid |\arg(\lambda - \gamma)| < \pi/2 + \alpha - \varepsilon\}$$

we have $(\lambda I - A)^{-1} \in L(X, X)$ with the estimate

$$(4.12) \quad \|(\lambda I - A)^{-1}\|_{L(X, X)} \leq \text{pol}(|\lambda|).$$

PROOF. The sufficiency of condition (4.12) is a consequence of theorem 5. Also the necessity can be obtained by an adaptation of the proof of theorem 5, but we shall give a direct proof. If the semi-group T is holomorphic in the sector $\Sigma = \{\mu; |\arg \mu| < \alpha < \pi/2\}$, then according to theorem 3, there exists a real γ such that $e^{-\gamma t} T = D^m f$ where $f(t)$ is a \mathcal{C}^0

$(L(X, X))$ -function satisfying (4.3). Put

$$\widehat{f}(\lambda) = \int_0^{\infty} e^{-\lambda t} f(t) dt, \quad \text{for } \operatorname{Re} \lambda > 0.$$

Because $f(t)$ is analytic in Σ and $\|e^{-\lambda t} f(t)\|$ rapidly tends to zero at infinity, we may write

$$(4.13) \quad \widehat{f}(\lambda) = \int_{\Gamma} e^{-\lambda \mu} f(\mu) d\mu, \quad \text{for } \operatorname{Re} \lambda > 0.$$

where $\Gamma = \{\mu; \mu = te^{-i(\alpha-\varepsilon)}; t > 0\}$ for $\operatorname{Im} \lambda \geq 0$

and $\Gamma = \{\mu; \mu = te^{i(\alpha-\varepsilon)}; t > 0\}$ for $\operatorname{Im} \lambda < 0$.

This implies that $\widehat{f}(\lambda)$ can be extended at an holomorphic function $\widehat{f}(\lambda)$ in the domain

$$\{\lambda; \operatorname{Re} \lambda > -(\alpha - \varepsilon) |\operatorname{Im} \lambda|\}.$$

Again following the proof of theorem 5 we obtain that $(\lambda I - A)^{-1} \in (L(X, X))$ and satisfies (4.12) for

$$\operatorname{Re} \lambda > -(\alpha - \varepsilon) |\operatorname{Im} \lambda| + \gamma.$$

Thus theorem 6 is proved.

*Faculty of Mathematics
University of Jussy, Romania*

BIBLIOGRAPHY

- [1] J. L. LIONS, *Les semi-groupes distributions*. Portugaliae Math., 19 (1960), 141-164.
- [2] C. FOIAS, *Remarques sur le semi-groupes distributions d'opérateurs*. Portugaliae Math. 19 (1960), 227-243.
- [3] J. PEETRE, *Sur la théorie des semi-groupes distributions. Sémin. sur les équations aux dérivées partielles*. Coll. France, 1963-1964, 76-98.
- [4] J. CHAZARAIN, *Problèmes de Cauchy au sens des distributions vectorielles et applications*. C. R. Acad. Sc. Paris, t. 226 (1968), 10-13.
- [5] J. CHAZARAIN, *Problèmes dans les espaces d'ultra-distributions*. C. R. Acad. Sc. Paris, t. 266 (1968), 564-566.
- [6] G. DA PRATO - U. MOSCO, *Semigrupperi Distribuzioni Analitici*. Annali Scuola Normale Superiore di Pisa, XIX (1965), 367-396.
- [7] G. DA PRATO - U. MOSCO, *Regolarizzazione dei semigrupperi Distribuzioni Analitici*. Annali Scuola Normale Superiore di Pisa, XIX (1965), 563-570.
- [8] D. FUJIWARA, *A characterization of exponential distribution semi-groups*. J. Math. Soc. Japan, 18, 3 (1966), 265-274.
- [9] E. LARSSON, *Generalized distributions semi-groups of bounded linear operators*. Ann. Scuola Norm. Sup. Pisa, 19 (1967), 137-140.
- [10] K. YOSHINAGA, *Ultradistributions and semi-groups distributions*. Bull. Kyushu Inst. Techn. 10 (1963), 1-24.
- [11] K. YOSHINAGA, *Values of vector-valued distributions and smoothness of semi-groups distributions*. Bul. Kyushu Inst. Techn. 12 (1965), 1-27.
- [12] L. SCHWARTZ, *Théorie des distributions*. Hermann, Paris, 1966.
- [13] L. SCHWARTZ, *Théorie des distributions à valeurs vectorielles*. Ann. Inst. Fourier, 7 (1957), 1-141.
- [14] K. YOSIDA, *Functional Analysis*. Springer, Verlag (1965).
- [15] G. DA PRATO, *Semigrupperi di crescita n*. Ann. Scuola Norm. Sup. Pisa, XX (1966), 753-782.
- [16] A. PAZZY, *On the differentiability and compactness of semigroups of linear operators*. J. of Math. and Mech., 17, 12 (1968), 1131-1141.
- [17] V. BARBU, *Les semi-groupes distributions différentiables*. C. R. Acad. Sc. Paris, t. 268 (1968).