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GLOBAL LINEAR GOURSAT PROBLEMS FOR FUNCTIONS OF GEVREY-LEDNEV TYPE

JAN PERSSON

1. Introduction.

The purpose of this paper is to prove two theorems for global Goursat problems. Theorem 1 in section 4 covers equations with variable coefficients. It generalizes the theorem in [7]. Theorem 2 in section 5 is restricted to equations with constant coefficients. It covers both the first part of theorem 1 in [4] and theorem 1 in [6]. The proofs depend on a unification of the exponential majorization, used in [6], with a generalization to the classes $\gamma(\beta, \delta, d)$ in [6] of the characterization of entire functions in [7]. The reader is referred to [7] for further details of the background of the problem. We shall also give some comments in section 6 below.

In section 2 we introduce the necessary notation and give some definitions. The lemmas used in the proofs of the theorems in section 4 and 5 are stated and proved in section 3.

2. Preliminaries.

Let $x = (x_1, \dots, x_{n'}, x_{n'+1}, \dots, x_n) = (x', x'') \in R^{n'} \times R^{n-n'}$, where $1 \leq n' \leq n$. A multi-index with non-negative components is denoted by a Greek letter $\alpha = (\alpha_1, \dots, \alpha_n)$. We define $D_x = (D_{x_1}, \dots, D_{x_n}) = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ and we write $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$. We also write $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $|x| = |x_1| + \dots + |x_n|$, $\alpha! = \alpha_1! \dots \alpha_n!$. We define $\alpha \leq \delta \Longleftrightarrow \alpha_j \leq \delta_j$, $1 \leq j \leq n$, and $\binom{\delta}{\alpha} = \delta!\,(\alpha!\,(\delta - \alpha)!)^{-1}$ when $\alpha \leq \delta$. Let $d \in R^n$. We define $\alpha d = \alpha_1 d_1 + \dots + \alpha_n d_n$. We shall also use variables

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$(y, x) \in R^s \times R^n$, $y \in R^s$, $x \in R^n$. For these variables we use the natural extension of the definititions above.

DEFINITION 1. *The real-valued function $p(t)$, $t \geq 0$, is continuously differentiable. If it satisfies the following conditions*

(2.1) $\quad 0 < p(t) < \log(t+4)$,

(2.2) $\quad 0 < p'(t) < (t+4)^{-1}$,

(2.3) $\quad$ *$p(t)$ tends monotonically to $+\infty$ when $t \to +\infty$,*

(2.4) $\quad$ *$p'(t)$ tends monotonically to zero when $t \to +\infty$,*

and

(2.5) $\quad$ *$p(t)/t$ and $p(p(t))/t$ are decreasing,*

then p is said to belong to the class P.

We note that $(p(t)/t)' = p'(t)/t - p(t)/t^2$. The mean value theorem says that

$$p(t) = p(0) + tp'(\theta t), \quad \text{some } \theta,\ 0 < \theta < 1.$$

It now follows from (2.1) and (2.4) that

$$(p(t)/t)' = p'(t)/t - p'(\theta t)/t - p(0)/t^2 < 0.$$

Thus $p(t)/t$ is decreasing. Then it follows from this, (2.1) and (2.3) that

$$p(p(t))/t = p(p(t))/p(t) \times p(t)/t,$$

is decreasing. Therefore (2.5) is a consequence of (2.1), (2.3) and (2.4). It has been incorporated in the definition of P because it will be convenient in the following. It also follows from (2.1)-(2.4) that $p \in P \Longrightarrow kp(p(t)) \in P$, for some $k > 0$.

DEFINITION 2. *The function $g(y, x)$ is complex-valued and defined in R^{s+n}, $\delta \in R^n$ and $\beta \in R^s$ are multi-indices, and $d \in R^n$, $d_j \geq 0$, $1 \leq j \leq n$. The derivatives $D_y^\gamma D_x^\xi g$, $\gamma \leq \beta$, all ξ, exist and are continuous together with g itself. If to every compact set $K \subset R^{s+n}$ there exists a constant $C > 0$ and a function $p \in P$ such that*

$$|D_x^\xi D_y^\gamma D_x^\alpha g(y, x)| \leq C(\xi d/p(\xi d))^{\xi d - 1}, \ (y, x) \in K, \tag{2.6}$$

$$\alpha d + |\gamma| \leq |\beta| + \delta d, \quad \gamma \leq \beta, \quad \text{all } \xi,$$

then g is said to belong to the function class $\gamma(\beta, \delta, d)$. Here $0^{-1} = 1$ in (2.6).

Lemma 3 in [7] and the computing after definition 1 above shows that definition 2 is identical with the definition of $\gamma(\beta, \delta, d)$ in [6], since we may as well use the factor $(\xi d/p(\xi d))^{\xi d}$ in (2.6).

The function $g \in \gamma(\beta, \delta, d)$. We write

$$g = O(y^\beta x^\delta).$$

if

$$D^k_{y_j} g(y, x) = 0, \quad y_j = 0, \quad 0 \leq k < \beta_j, \quad j = 1, \dots, s,$$

and

$$D^k_{x_j} g(y, x) = 0, \quad x_j = 0, \quad 0 \leq k < \delta_j, \quad j = 1, \dots, n.$$

3. The fundamental lemmas.

We are now going to prove the lemmas that will be used in the proofs in sections 4 and 5.

LEMMA 1. *The vector $d \in R^n$ and $d_j \geq 1$, $1 \leq j \leq n$. Then it follows that there exists a constant $c > 0$ independent of ξ and d such that*

$$\sum_{\nu \leq \xi} \binom{\xi}{\nu} \xi d^{-\xi d + 1} ((\xi - \nu) d)^{(\xi-\nu)d - 1} \nu d^{\nu d - 1} \leq c, \quad \text{all } \xi. \tag{3.0}$$

PROOF. The left member in (3.0) is equal to 1 when $\xi = 0$. Since $\xi d \geq |\xi|$, $\xi \neq 0$, we see that the left member of (3.0) is always majorized by

$$A = \sum_{\nu \leq \xi} \binom{\xi}{\nu} |\xi|^{-|\xi|+1} |\xi - \nu|^{|\xi-\nu|-1} |\nu|^{|\nu|-1}.$$

But according to the proof of lemma 1 in [5] there exists a constant $c > 0$ such that

$$A \leq c, \quad \text{all } \xi.$$

Thus lemma 1 is proved.

LEMMA 2. *The function $u(y,x)$ is defined in all $R^s \times R^n$, $\beta \in R^s$, and $\delta = (\delta_1, \dots, \delta_n, 0, \dots, 0) \in R^n$ are multi-indices, and $d \in R^n$, $d_j = 1$, $1 \leq j \leq n'$, $d_j \geq 1$, $1 \leq j \leq n$. The function u and all its derivatives mentioned below are continuous and $u = O(y^\beta x^\delta)$. To every $r > 0$ there exists a $p \in P$. The constant*

$k \geq 2$ is chosen so great that

$$g(t) = te^k/p(p(t)) \geq 1, \quad t \geq 1,$$

and $(g(t))^{t-1}$ is increasing $t \geq 1$.

We define

$$\bar{p}(t) = e^{-k} p(p(t)).$$

Let

$$M = \{t;\ ar/\bar{p}(t) \geq 1,\ a = e^2\}.$$

The constant $K > 1$ is chosen such that

$$(3.1)\quad (t/\bar{p}(t))^{t-1}(K + t/p(t))^{-s} \exp(art/\bar{p}(t)) \leq 1, \quad 1 \leq t \leq 1+s, \quad 1 \leq s \leq |\beta| + |\delta|.$$

and

$$(3.2)\quad [K + (t+s)/\bar{p}(t+s)]^{-1}((t+s/\bar{p}(t+s)) \exp(ar/\bar{p}(t)) \leq 1, \quad t \in M,$$
$$0 \leq s \leq |\beta| + |\delta|.$$

The following inequality is satisfied

$$(3.3)\quad |D_x^\xi D_y^\beta D_x^\delta u(y,x)| \leq (\xi d/\bar{p}(\xi d))^{\xi d-1} \exp((K + \xi d/\bar{p}(\xi d))\, a(|y| + |x|)).$$
$$|y| + |x| \leq r, \quad \text{all } \xi.$$

The multi-indices $\gamma \in R^s$, and $\alpha \in R^n$ satisfy

$$\gamma \leq \beta, \qquad |\beta - \gamma| + |\delta| - \alpha d = \varepsilon' \geq 0.$$

Define t by

$$t = 0, \qquad \xi d - \varepsilon' < 1, \quad \text{and } t = \xi d - \varepsilon', \qquad \xi d - \varepsilon' \geq 1.$$

Then it follows that

$$(3.4)\quad |D_x^\xi D_y^\gamma D_x^\alpha u(y,x)| \leq (t/\bar{p}(t))^{t-1} \exp((K + t/\bar{p}(t))\, a|y| + |x|)),$$
$$|y| + |x| \leq r, \quad \text{all } \xi.$$

REMARK. *It follows from* (2.5) *and the choice of k that the right member of* (3.4) *is an increasing function of ξd. This turns out to be an essential point in the proof in section* 4.

If (3.3) *is true with some K, it is always possible to choose a greater K that satisfies* (3.1) *and* (3.2) *without violating* (3.3). *This we shall do several times in the following when we adjust k upwards, although the adjustment of K is not always stated explicitly.*

PROOF. The proof is a modification of the proof of the lemma in [6]. It is based on the simple principle of exponential majorization. For a given $k > 0$, we get

$$\left| \int_0^t e^{k|s|} ds \right| = k^{-1} (e^{k|t|} - e^0) \leq k^{-1} e^{k|t|}, \quad -\infty < t < +\infty. \tag{3.5}$$

It is obvious that there exist multi indices μ, $\mu \leq \delta$, and η such that

$$\xi + \alpha = \eta + \delta - \mu.$$

Note that $\mu \leq \delta$ implies $\mu_j = 0$, $n' < j \leq n$. It also follows that $\mu d = |\mu|$. We denote integration from the origin by negative powers of D_y and D_x. Remembering that $u = O(y^\beta x^\delta)$ we get

$$A = | D_x^\xi D_y^\gamma D_x^\alpha u | = | D_y^{-(\beta-\gamma)} D_x^{-\mu} D_x^\eta D_x^\beta D_x^\delta u |.$$

It follows from (3.3) and repeated application of (3.5) that

$$A \leq (\eta d/\bar{p}(\eta d))^{\eta d+1} \times a^{-|\beta-\gamma|-|\mu|} (K + \eta d/\bar{p}(\eta d))^{-|\beta-\gamma|-|\mu|} \times$$

$$\times \exp([K + \eta d/\bar{p}(\eta d)]\, a(|y| + |x|)).$$

Now we have with $|\delta| + |\beta| - \alpha d - |\gamma| = \varepsilon' > 0$ and $|\beta - \gamma| + |\mu| = s$,

$$\eta d = \xi d - \varepsilon' + s.$$

If $s \neq 0$ and $\xi d - \varepsilon' < 1$ then $\eta d < 1 + s$ and

$$A \leq \exp(Ka(|y| + |x|))$$

because of (3.1).

If $s = 0$ then

$$\eta d = \xi d - \varepsilon'.$$

If $\eta d = 0$ then $A \leq \exp(Ka(|y| + |x|))$.

If $\eta \neq 0$ then $\eta d = \xi d - \varepsilon' > 1$ because of $d_j \geq 1, 1 \leq j \leq n$. So (3.4) is proved in the case $\xi d - \varepsilon' < 1$.

Let $t = \xi d - \varepsilon' \geq 1$. Then $\eta d = t + s$. So we get

$$A \leq K'[t/\bar{p}(t)]^{t-1} \exp((K + t/\bar{p}(t))\, a(|y| + |x|)).$$

Here we have defined K' by

$$K' = (t/\bar{p}(t))^{-t+1}((t+s)/\bar{p}(t+s))^{t+s-1}(K+(t+s)/\bar{p}(t+s))^{-s} \times$$

$$\times a^{-s} \exp(((t+s)/\bar{p}(t+s) - t/\bar{p}(t))\, ar).$$

If we can prove that $K' \leq 1$, then we are through. Now (2.1), (2.3) and $t \geq 1$ implies

$$(t/\bar{p}(t))^{-t+1}((t+s)/\bar{p}(t+s))^{t-1} \leq (1+s/t)^{t-1} \leq e^s.$$

It follows from (2.1) and (2.3) that

$$(t+s)/\bar{p}(t+s) - t/\bar{p}(t) \leq s/\bar{p}(t).$$

Because of (3.2) we now conclude that

$$B = (K+(t+s)/\bar{p}(t+s))^{-s}((t+s)/\bar{p}(t+s))^s e^{sar/\bar{p}(t)} \leq 1, \quad t \in M,$$

and

$$B \leq e^s, \qquad t \notin M.$$

It is now clear that

$$K' \leq e^{2s} a^{-s} = 1.$$

The lemma is proved.

Next we prove a modification of lemma 2 in [7].

LEMMA 3. *The function p belong to P. The number ε is restricted by $0 < \varepsilon < 4^{-1}$. The number k is the same constant as the k in the hypothesis of lemma 2, with the additional condition that*

$$k > 1 + [p(p(0))]^{-1}.$$

The function $g_n(t)$, $0 \leq t \leq n - 2^{-1}$, is defined by

$$g_n(t) = [p(p(n))]^{n-1}[p(p(n-t-\varepsilon))]^{n-t-\varepsilon-1} \times \tag{3.6}$$

$$\times [e^k p(t)]^{-t+1}(n-t-\varepsilon)^{-\varepsilon}, \; n \geq 0, \; n \in R.$$

It follows that

$$\lim_{n\to\infty} g_n(0) = 0, \quad \text{and} \quad \lim_{n\to\infty} \sup_{1 \leq t \leq n-2^{-1}} g_n(t) = 0. \tag{3.7}$$

It follows that there exist a constant $C > 1$ such that

(3.8) $$e^{-k(1+\varepsilon)} g_n(t) \leq C, \quad t = 0, 1 \leq t \leq n - 2^{-1}, \ n \geq 0, \ n \in R$$

REMARK. Note that $e^{-k(1+\varepsilon)} g_n(t)$ in (3.8) is a decreasing function of k for every n and t.

PROOF. We look at

$$g_n(0) = [p(p(n))]^{n-1} [p(p(n-\varepsilon))]^{n-\varepsilon-1} e^k p(0) (n-\varepsilon)^{-\varepsilon}.$$

The mean value theorem says that for some θ, $0 < \theta < 1$,

$$p(p(n)) = p(p(n-\varepsilon)) + \varepsilon p'(p(n-\theta\varepsilon)) p'(n-\theta\varepsilon).$$

So we now see that

$$p(p(n))/p(p(n-\varepsilon)) \leq 1 + \varepsilon p'(p(n-\theta\varepsilon)) p'(n-\theta\varepsilon) [p(p(n-\varepsilon))]^{-1}.$$

It follows from (2.1) and (2.2) that for n great

$$p(p(n))/p(p(n-\varepsilon)) \leq 1 + \varepsilon n^{-1} \leq 1 + n^{-1}.$$

Now we have

$$[p(p(n))/p(p(n-\varepsilon))]^{n-1-\varepsilon} \leq e,$$

and

$$g_n(0) \leq e [p(p(n))]^{\varepsilon} e^k p(0) (n-\varepsilon)^{-\varepsilon}.$$

It follows from (2.1) that $g_n(0)$ tends to zero when n tends to infinity.

Now t is restricted to $1 \leq t \leq n - 2^{-1}$. The function $f_n(t)$ is defined as

$$f_n(t) = \log g_n(t) = (n-1) \log p(p(n)) -$$

$$- (n-t-\varepsilon-1) \log p(p(n-t-\varepsilon)) - k(t-1) -$$

$$- (t-1) \log p(t) - \varepsilon \log(n-t-\varepsilon).$$

We differentiate f_n.

(3.9) $$f_n'(t) = \log p(p(n-t-\varepsilon)) +$$

$$+ (n-t-\varepsilon-1)[p(p(n-t-\varepsilon))]^{-1} p'(p(n-t-\varepsilon)) p'(n-t-\varepsilon) -$$

$$- k - \log p(t) - (t-1) p'(t)/p(t) + \varepsilon(n-t-\varepsilon)^{-1}.$$

The goal is to determine the maximun point of f_n and then estimate the value of $g_n(t)$ in this point. All the reasoning in the following is based on the assumption that n is great.

If $t \geq 2^{-1} n$, it follows from (2.2) and (2.3) that

$$f_n'(t) \leq \log p(p(n)) - k + [p(p(0))]^{-1} - \log p(2^{-1} n) + 1.$$

It follows from (2.1), (2.3) and the choice of k that

$$f_n'(t) < 0, \qquad t \geq 2^{-1} n.$$

It is also noted that

$$f_n'(1) \geq \log p(p(n - 1 - \varepsilon)) - k - \log p(1).$$

But that means that

$$f_n'(1) > 0 .$$

The maximum point of f_n will be found in

$$1 < t < 2^{-1} n. \tag{3.10}$$

From now on we denote the maximum point of f_n by t. We define

$$a_1 = (n - t - \varepsilon - 1)[p(p(n - t - \varepsilon))]^{-1} \times p'(p(n - t - \varepsilon))\, p'(n - t - \varepsilon),$$

$$a_2 = (t - 1)\, p'(t)/p(t),$$

$$a_3 = \varepsilon (n - t - \varepsilon)^{-\varepsilon} .$$

It follows from $f_n'(t) = 0$ and (3.9) that

$$\log p(p(n - t - \varepsilon)) = \log p(t) + a_2 - a_1 - a_3 + k. \tag{3.11}$$

The mean value theorem says that for some θ, $0 < \theta < 1$,

$$\begin{aligned} \log p(p(n)) &= \log p(p(n - t - \varepsilon)) + \\ &+ (t + \varepsilon)[p(p(n - \theta(t + \varepsilon)))]^{-1} \times p'(p(n - \theta(t + \varepsilon)))\, p'(n - \theta(t + \varepsilon)) = \\ &= \log p(p(n - t - \varepsilon)) + b. \end{aligned} \tag{3.12}$$

Here b is defined as

$$b = (t + \varepsilon)\,[p\,(p\,(n - \theta\,(t + \varepsilon)))]^{-1}\, p'(p\,(n - \theta\,(t + \varepsilon)))\, p'(n - \theta\,(t + \varepsilon)).$$

A look at (2.1)-(2.4) shows that

$$b < (t + \varepsilon)\,[p\,(p\,(n - t - \varepsilon))]^{-1}\, p'(p\,(n - t - \varepsilon))\, p'(n - t - \varepsilon). \tag{3.13}$$

From (3.11) and (3.12) it follows that

$$\log p\,(p\,(n)) = \log p\,(t) + a_2 - a_1 - a_3 + k + b.$$

It follows from the properties of p, and from $1 < t < 2^{-1} n$ that

$$0 < a_1 < 1, \quad 0 < a_3 < 1, \quad a_2 \geq 0, \quad 0 < b \leq 1. \tag{3.14}$$

Thus we estimate $\log p\,(t)$ by

$$\log p\,(t) \leq \log\,(p\,(p\,(n))) + 1 + 1 - k \leq \log p\,(p\,(n))\,.$$

But $p\,(t)$ and $\log t$ are increasing and we get

$$t < p\,(n). \tag{3.15}$$

Nov (2.1) gives

$$t < \log\,(4 + n) \tag{3.16}$$

We may also estimate $\log p\,(t)$ by

$$\log p\,(t) \geq \log p\,(p\,(n)) - k - [p\,(0)]^{-1} - 1.$$

From this we conclude that

$$n \to \infty \Longrightarrow p\,(t) \to \infty\,. \tag{3.17}$$

The maximum point is inserted in g_n. Since

$$p\,(p\,(n - t - \varepsilon)) = p\,(t)\, \exp\,(a_2 - a_1 - a_3 + k),$$

and since

$$p\,(p\,(n)) = p\,(t)\, \exp\,(a_2 - a_1 - a_3 + b + k),$$

it now follows that

$$g_n(t) \leq [p(t) \exp(a_2 - a_1 - a_3 + k + b)]^{n-1} [p(t) \exp(a_2 - a_1 - a_3 + k)]^{-(n-t-\varepsilon-1)} \times$$
$$\times [p(t)]^{-t+1} e^{-k(t-1)} (n - t - \varepsilon)^{-\varepsilon}.$$

We rewrite this as

$$g_n(t) \leq [e^k p(t)]^{1+\varepsilon} (n - t - \varepsilon)^{-\varepsilon} \exp((n-1) b + (t + \varepsilon)(a_2 - a_1 - a_3)).$$

We now use (2.1), (2.2), (2.3), and (3.13) and get

$$B = b(n-1) + (t+\varepsilon)(a_2 - a_1) \leq [(n-1)(t+\varepsilon) - (n-t-\varepsilon-1)(t+\varepsilon)] a_1 (n-t-\varepsilon-1)^{-1} +$$
$$+ (t+\varepsilon)(t-1) p'(t)/p(t) \leq (t+\varepsilon)^2 a_1 (n-t-\varepsilon-1)^{-1} + t/p(t).$$

It follows from (3.15) and $t \leq 2^{-1} n$ that

$$(t+\varepsilon)^2 [p(p(n-t-\varepsilon)]^{-1} p'(p(n-t-\varepsilon)) p'(n-t-\varepsilon) \leq [p(p(2^{-1}n-\varepsilon))]^{-1} \leq 1.$$

Thus we have proved that

$$B \leq 1 + t/p(t) \leq 1 + [p(t)]^{-1} \log(n+4).$$

We now see that

$$g_n(t) \leq e [e^k \log(n+4)]^{1+\varepsilon} n^{-\varepsilon} 2^{-\varepsilon} \times (n+4)^{\frac{1}{p(t)}}.$$

It follows from (4.17) that

$$\lim_{n \to \infty} g_n(t) = 0.$$

So (3.7) is proved. It follows from (3.7) and the continuity that (3.8) is true too. The lemma is proved.

Lemma 3 in [7] is restated as lemma 4 in a somewhat modified form.

LEMMA 4. *Let* $f(\xi, d) > 0$ *be a continuous function in* $\xi \in R^n$, $\xi_j \geq 0$, $1 \leq j \leq n$, *with a fixed* $d \in R^n$, $d_j > 0$, $1, \ldots, n$. *If*

$$\xi d \to +\infty \Longrightarrow f(\xi, d) \to +\infty,$$

then it follows that there exists a $p \in P$ *such that*

$$p(\xi d) \leq f(\xi, d), \qquad \text{all } \xi.$$

PROOF. Let

$$g(s) = \inf f(\xi, d), \quad \xi d \geq s.$$

It follows that $g(s) \to +\infty$ when $s \to +\infty$.

Since $d_j > 0$ it also follows that $\xi d > c|\xi|$, for some $c > 0$.

So if $f(\xi, d) > 1$, $\xi d \geq L$, then we have $f(\xi, d) \geq 1$, $|\xi| \geq c^{-1} L$. Since f is continuous it follows from $f(\xi, d) > 0$ that for some $\varepsilon > 0$

$$f(\xi, d) \geq \varepsilon, \qquad |\xi| \leq c^{-1} L.$$

Thus we have proved that $g(0) > 0$.

It is then easy to construct a piecewise linear strictly increasing continuous function $g_0(s)$ such that $g_0(0) > 0$, $g_0(s) \leq g(s)$, $g_0(s) \to +\infty$ when $s \to +\infty$, and such that the derivative exists except in at most integer points. Then we use the proof of lemma 3 in [7]. Lemma 4 is proved.

LEMMA 5. *The function $u(y, x)$ is defined in all $R^s \times R^n$, $\beta \in R^s$ and $\delta = (\delta_1, \dots, \delta_{n'}, 0, \dots, 0) \in R^n$ are multi-indices, $d \in R^n$, $0 \leq d_j \leq 1$, $1 \leq j \leq n'$, $d_j \geq 0$, $1 \leq j \leq n$. The function u and all its derivatives mentioned below are continuous and*

$$u = O(y^\beta x^\delta).$$

In connection with $r \geq 1$ there exist a $p \in P$ and constants k, and K. Let $\bar{p} = e^{-k} p(p(t))$.

Let

$$M = \{t\,;\, ar/\bar{p}(t) \geq 1,\ a = e^2\}.$$

The following conditions are satisfied

$$\bar{p}(t) \leq p(t),\ t \geq 0. \tag{3.18}$$

$$(1 + (t+s)/\bar{p}(t+s))\,(K + (t+s)/\bar{p}(t+s))^{-1} \exp(ar/\bar{p}(t)) \leq 1, \tag{3.19}$$

$$t \in M,\ 0 \leq s \leq |\beta| + |\delta|.$$

$$(1 + t/\bar{p}(t))^t\,(K + t/\bar{p}(t))^{-s} \exp(art/\bar{p}(t)) \leq 1,\ 0 \leq t \leq s, \tag{3.20}$$

$$1 \leq s \leq |\beta| + |\delta|.$$

$$|D_x^\xi D_y^\beta D_x^\delta u(y, x)| \leq (1 + \xi d/\bar{p}(\xi d))^{\xi d} \exp[(K + \xi d/\bar{p}(\xi d))\, a(|y| + |x|)], \tag{3.21}$$

$$|y| + |x| \leq r,\ all\ \xi.$$

The multi-indices $\gamma \in R^s$, *and* $\alpha \in R^n$ *satisfy*

$$\gamma \leq \beta, \, |\beta - \gamma| + (\delta - \alpha)\, d = \varepsilon' \geq 0. \tag{3.22}$$

The variable t *is now defined by*

$$t = 0, \, \xi d - \varepsilon' < 0, \, t = \xi d - \varepsilon', \, \xi d - \varepsilon' \geq 0. \tag{3.23}$$

Then it follows that

$$| D_x^\xi D_y^\gamma D_x^\alpha u(y, x) | \leq (1 + t/\bar{p}(t))^t \exp((K + t/\bar{p}(t))\, a(|y| + |x|)), \tag{3.24}$$

$$|y| + |x| \leq r, \text{ all } \xi, \text{ with } t \text{ defined in (3.23).}$$

REMARK. *It follows from* (2.1) *that for some* t_0

$$p(t) \leq t, \, t \geq t_0 \, .$$

If we choose $k \geq 1$ *such that*

$$e^{-k} p(p(t_0)) \leq p(0),$$

then it follows that

$$e^{-k} p(p(t)) \leq p(t), \quad t \geq 0.$$

Therefore it is always possible to choose k *such that* (3.18) *is true. Since* M *is a bounded set it is also possible to choose* K *such that* (3.19) *and* (3.20) *are true.*

PROOF OF LEMMA 5. Choose η and μ such that $\mu \leq \delta$, $\xi + \alpha = \eta - \mu + \delta$. We note that $0 \leq d_j \leq 1$, $1 \leq j \leq n'$, and $\mu_j = 0$, $n' \leq j \leq 1$, $\mu d \leq |\mu|$. Just as in the proof of lemma 2 we get

$$A = | D_x^\xi D_y^\gamma D_x^\alpha u | = | D_y^{-(\beta-\gamma)} D_x^{-\mu} D_y^\beta D_x^\delta u |,$$

and

$$A \leq (1 + \eta d/\bar{p}(\eta d))^{\eta d} (a(K + \eta d/\bar{p}(\eta d))^{-|\beta-\gamma|-|\mu|} \times$$
$$\times \exp((K + \eta d/\bar{p}(\eta d))\, a(|y| + |x|)).$$

Let $|\beta - \gamma| + |\mu| = s$. It follows from (3.22) that

$$\eta d = \xi d - \varepsilon' + \mu d - |\mu| + s \leq \xi d - \varepsilon' + s.$$

So

$$\xi d - \varepsilon' < 0 \Longrightarrow s \neq 0 \Longrightarrow s \geq 1,$$

and

$$\xi d - \varepsilon' < 0 \Longrightarrow \eta d < s.$$

It follows from (3.20) that (3.24) is true when $\xi d - \varepsilon' < 0$. Now we look at the case $t = \xi d - \varepsilon' \geq 0$. Then we have $\eta d \leq t + s$, and

$$A \leq K' (1 + t/\bar{p}(t))^t \exp((K + t/\bar{p}(t)\, a(|y| + |x|)).$$

Here K' is defined by

$$K' = (1 + (t + s)/\bar{p}(t + s))^{t+s} (1 + t/\bar{p}(t)^{-t} ((K + (t + s)/\bar{p}(t + s))\, a)^{-s} \times$$

$$\times \exp(((t + s)/\bar{p}(t + s) - t/\bar{p}(t))\, a(|y| + |x|)).$$

Here we have used (2.5) and $\eta d \leq t + s$ in the form $\eta d/\bar{p}(\eta d) \leq (t + s)/\bar{p}(t + s)$, and also the fact that the left member of (3.19) is increasing in s for $a = 0$. We look at

$$g(t) = [(1 + (t + s)/\bar{p}(t + s))/(1 + t/\bar{p}(t))]^t.$$

We get

$$g(t) \leq [(1 + (t + s)/\bar{p}(t))/(1 + t/\bar{p}(t))]^t = (1 + s/(1 + t/\bar{p}(t)))^t.$$

If $r > 1$ and $t \notin M$ then $\bar{p}(t) \geq 1$. Then it follows that

$$g(t) \leq (1 + s/t)^t \leq e^s.$$

Thus we get just as in the proof of lemma 2 that

$$K' \leq e^{2s} a^{-s} = 1.$$

Let $t \in M$. We look at

$$(1 + (t + s)/\bar{p}(t + s))^t (1 + t/\bar{p}(t))^{-t} \leq$$

$$\leq (\bar{p}(t) + t + s)^t (\bar{p}(t) + t)^{-t} = (1 + s/(t + \bar{p}(t)))^t \leq (1 + s/t)^t \leq e^s.$$

It follows from (3.19) that in this case

$$K' \leq e^s a^{-s} = e^{-s} \leq 1.$$

The lemma is proved.

4. Operators with variable coefficients.

We start by stating the theorem.

THEOREM 1. *The integer n' is restricted by $1 \leq n' \leq n$. Let $d \in R^n$, $d_j = 1,\ 1 \leq j \leq n',\ d_j \geq 1,\ 1 \leq j \leq n$.*

The multi-indices $\beta \in R^s$, $\delta \in R^n$, $\gamma^k \in R^s$, $\alpha^k \in R^n$, $1 \leq k \leq N$. They are restricted by

$$\delta_j = 0,\ n' < j \leq n,\ \gamma^k \leq \beta,\ \alpha^k d + |\gamma^k| \leq |\delta| + |\beta|,\ 1 \leq k \leq N. \tag{4.1}$$

The functions $f(y, x)$, $a_k(y, x)$, $1 \leq k \leq N$, belong to $\gamma(0, 0, d)$. To every $r > 0$, there exist a $p \in P$ and a $C_1 \geq 0$ such that

$$|D_x^\xi f(y, x)| \leq C_1 (\xi d / p(\xi d))^{\xi d - 1}, \qquad |y| + |x| \leq r,\ all\ \xi, \tag{4.2}$$

and

$$|D_x^\xi a_k(y, x)| \leq C_1 (\xi d / p(\xi d))^{\xi d - 1},\ |y| + |x| \leq r,\ all\ \xi, 1 \leq k \leq N. \tag{4.3}$$

The functions a_k are further restricted by

$$\alpha^k d + |\gamma^k| = |\beta| + |\delta| \Longrightarrow a_k \text{ depends on } y \text{ only}, \tag{4.4}$$

and

$$\Sigma\, |a_k(y)| < 1,\ y \in R^s,\ \text{the sum taken over } k \text{ with} \tag{4.5}$$

$$\alpha^k d + |\gamma^k| = |\beta| + |\delta|.$$

It follows that the Goursat problem

$$D_y^\beta D_x^\delta u = \Sigma a_k D_y^{\gamma^k} D_x^{\alpha^k} u + f,\ u = O(y^\beta x^\delta), \tag{4.6}$$

has one and only one solution u in $\gamma(\beta, \delta, d)$.

PROOF. We shall prove the theorem by successive approximations. We define the sequence $(u^q)_{q=0}^\infty$ by letting $u^0 = 0$ and

$$D_y^\beta D_x^\delta u^{q+1} = \Sigma a_k D_y^{\gamma^k} D_x^{\alpha^k} u^q + f,\ u^{q+1} = O(y^\beta x^\delta), \qquad q = 0, 1, \ldots. \tag{4.7}$$

It follows from (4.5) and the continuity that for a fixed r there exists a λ, $\lambda < 1$, such that

$$\Sigma\, |a_k(y)| \leq \lambda, \qquad |y| \leq r,$$

The sum is taken over k with $\alpha^k d + |\gamma^k| = |\beta| + |\delta|$. Let C be the constant in (3.8) and c the constant in lemma 1. If in (4.2) and (4.3)

$$C_1 > (1 - \lambda)/8\,NCc,$$

then we make a coordinate transformation

$$y_j' = ty_j,\ 1 \leq j \leq s,\ x_j' = t^{d_j} x_j,\ 1 \leq j \leq n,\ t \geq 1.$$

The new coefficients are of the form

$$a_k'(y', x') = t^{|\gamma^k| - |\beta| - d(\delta - \alpha^k)}\, a_k(y, x), \qquad 1 \leq k \leq N.$$

It is obvious that (4.5) is true for the new coefficients too in the compact set D in the primed space that corresponds to $|y| + |x| \leq r$ in the original coordinates.

We also get

$$f'(y', x') = t^{-|\beta| - |\delta|} f(y, x).$$

We differentiate and get

$$D_{x'}^{\xi}\, a_k'(y', x') = t^{|\gamma^k| - |\beta| - d(\delta - \alpha^k) - d\xi}\, D_x^{\xi}\, a^k(y, x),$$

$$(x', y') \in D, \text{ all } \xi,\ 1 \leq k \leq N.$$

It follws from (4.3) that with $s = |\beta - \gamma| + |\delta| - \alpha^k d$

$$|D_{x'}^{\xi}\, a_k'(y', x')| \leq C_1\, t^{-s - \xi d}\, (\xi d / p\,(\xi d))^{\xi d - 1}, \quad (y', x') \in D,\ 1 \leq k \leq N.$$

With $s = |\beta| + |\delta|$ we get

$$|D_{x'}^{\xi} f'(y', x')| \leq C_1\, t^{-s - \xi d}\, (\xi d / p\,(\xi d))^{\xi d - 1}, \quad (y', x') \in D.$$

There exists an ε, $0 < \varepsilon < 4^{-1}$ such that

$$\alpha^k d + |\gamma^k| < |\beta| + |\delta| \Longrightarrow |\beta - \gamma| + |\delta| - \alpha^k d \geq \varepsilon, \quad 1 \leq k \leq N.$$

It follows from this that we may choose t so great that with $C_1' = (1 - \lambda)/8cNC$

$$\text{(4.2)}' \qquad |D_{x'}^{\xi} f'(y', x')| \leq C_1'\, t^{-\xi d}\, (\xi d / p\,(\xi d))^{\xi d - 1}, \quad (y', x') \in D,$$

and

$$(4.3)' \qquad |D_{x'}^{\xi} a_k(y', x')| \leq C_1' t^{-\xi d} (\xi d/p(\xi d))^{\xi d-1}, (y', x') \in D,$$

$$\alpha^k d + |\gamma^k| < |\beta| + |\delta|, 1 \leq k \leq N.$$

Since D is compact there is always a new r such that

$$(y', x') \in D \Longrightarrow |y'| + |x'| \leq r.$$

We shall prove that $(u^q)_0^\infty$ converges in D. It will follow from the proof that we may assume that (4.2)′ and (4.3)′ are true when $(y', x') \in D$ is substituted by $|y'| + |x'| \leq r$, with the new r. Maybe the inequalities are false outside D, but we only use the result inside D, which corresponds to $|y| + |x| \leq r$, with the original r. Since we do not use (4.3) for those k with $\alpha^k d + |\gamma^k| = |\beta| + |\delta|$, we may delete the primes in (4.2)′ and (4.3)′ and assume that (4.2)′ and (4.3)′ are true with $t = 1$, D given by the new r, and $C_1 = (1 - \lambda)/8NcC$. This we will do in the following.

We now assert that with a, K, and $\bar{p}$ chosen as in the hypothesis of lemma 2

$$(4.8) \quad |D_x^{\xi} D_y^{\beta} D_x^{\delta} u^q(y, x)| \leq (\xi d/\bar{p}(\xi d))^{\xi d-1} \exp(K + \xi d/\bar{p}(\xi d))\, a(|y| + |x|)),$$

$$|y| + |x| \leq r, \text{ all } \xi, \quad q = 0, 1, \ldots .$$

We see that (4.8) is true for $q = 0$. We shall prove that if (4.8) is true, then it is also true when q is substituted by $q + 1$.

Let

$$t = 0, \xi d - |\beta - \gamma^k| - |\delta| + \alpha^k d < 1, \text{and } t = \xi d - |\beta - \gamma^k| - |\delta| + \alpha^k d,$$

$$\xi d - |\beta - \gamma^k| - |\delta| + \alpha^k d \geq 1.$$

It follows from (4.8) and lemma 2 that

$$(4.9) \qquad |D_x^{\xi} D_y^{\gamma^k} D_x^{\alpha^k} u^q| \leq (t/\bar{p}(t))^{t-1} e^{(K + t/\bar{p}(t))a(|y|+|x|)},$$

$$|y| + |x| \leq r, \quad 1 \leq k \leq N.$$

If $t = \xi d$ then a_k depends on y only and a_k is a term in the sum in (4.5). We get

$$A = |D_y^{\xi} D_y^{\beta} D_x^{\delta} u^{q+1}| \leq \sum_{|\beta-\gamma^k|+|\delta|-\alpha^k d=0} |a_k| |D_x^{\xi} D_y^{\gamma^k} D_x^{\alpha^k} u^q| +$$

$$+ \sum_{|\beta-\gamma^k|+|\delta|-\alpha^k d>0} |D_x^{\xi}(a_k D_y^{\gamma^k} D_x^{\alpha^k} u^q)| + |D_x^{\xi} f|.$$

Let

$$\varepsilon = \min\,(|\,\beta - \gamma^k\,| + \alpha^k\, d - |\,\delta\,|,\ 4^{-1}),\ |\,\beta - \gamma^k\,| + \alpha^k\, d - |\,\delta\,| > 0,\quad 1 \leq k \leq N.$$

It is obvious that $\varepsilon > 0$.

If for those k with $|\,\beta - \gamma^k\,| + \alpha^k\, d - |\,\delta\,| > 0$ we redefine t in (4.9) by

$$t = 0, \qquad \xi d - \varepsilon < 1, \quad \text{and} \quad t = \xi d - \varepsilon, \qquad \xi d - \varepsilon \geq 1,$$

then (4.9) is still true because of our choice of $\bar{p}$.

Now we shall use this, (4.5) with $\Sigma\,|\,\alpha_k\,| \leq \lambda < 1$, (4.2) and (4.3). We remember that $\exp\,((K + t/p\,(t))\,a\,(|\,y\,| + |\,x\,|))$ is an increasing function in t. Further it is not smaller that 1. We get

$$A \leq (\lambda\,(\xi d/\bar{p}\,(\xi d))^{\xi d-1} +$$

$$+ C_1\, N \sum_{\substack{\nu \leq \xi \\ \nu d - \varepsilon \geq 1}} \binom{\xi}{\nu} ((\xi - \nu)\, d/p\,((\xi - \nu)\, d))^{(\xi-\nu)d-1}((\nu d - \varepsilon)/\bar{p}\,(\nu d - \varepsilon))^{\nu d - \varepsilon - 1} +$$

$$+ N\, C_1 \sum_{\nu d - \varepsilon < 1} \binom{\xi}{\nu} ((\xi - \nu)\, d/p\,((\xi - \nu)\, d))^{(\xi-\nu)d-1} +$$

$$+ C_1\,(\xi d/p\,(\xi d))^{\xi d-1})\, \exp\,((K + \xi d/\bar{p}\,(\xi d))\, a\,(|\,y\,| + |\,x\,|)).$$

Now we may as well assume that k is chosen so great that

$$e^{-k}\, p\,(\,p\,(t)) \leq p\,(t), \qquad t \geq 0. \tag{4.10}$$

If $\xi = 0$ then we have

$$A \leq (\lambda + C_1\, N + C_1)\exp\,(Ka\,(|\,x\,| + |\,y\,|)) \leq \frac{2}{\lambda + 1} \exp\,(Ka\,(|\,x\,| + |\,y\,|)).$$

If $\xi \neq 0$ we rewrite the inequality

$$A \leq (\xi d/\bar{p}\,(\xi d))^{\xi d-1}\,(\lambda + C_1\, N \sum_{\substack{\nu \leq \xi \\ \nu d - \varepsilon \geq 1}} \binom{\xi}{\nu} ((\xi - \nu)\, d)^{(\xi-\nu)d-1}\,(\nu d - \varepsilon)^{\nu d - 1} \times \tag{4.11}$$

$$\times\,(\xi d^{-\xi d+1}\,(p\,((\xi - \nu)\, d))^{-(\xi-\nu)d+1}\,(\,p\,(\nu d - \varepsilon))^{-\nu d + \varepsilon + 1} \times$$

$$\times\,(\bar{p}\,((\xi d))^{\xi d-1}\,(\nu d - \varepsilon)^{-\varepsilon} + C_1\, N \sum_{\substack{|\,\nu\,| = 1 \\ \nu d - \varepsilon < 1}} \binom{\xi}{\nu} (\xi - \nu)^{(\xi-\nu)d-1} \times$$

$$\times (\xi d)^{\xi d+1} [p((\xi-\nu)d)]^{-(\xi-\nu)d+1} (\bar{p}(\xi d))^{\xi d-1} +$$
$$+ C_1 (N+1) (\bar{p}(\xi d)/p(\xi d)^{\xi d-1}) \exp((K + \xi d/\bar{p}(\xi d))\, a(|y|+|x|)).$$

We look at

$$B = \sum_{\substack{|\nu|=1 \\ \nu d - \varepsilon < 1}} \binom{\xi}{\nu} ((\xi-\nu)d)^{(\xi-\nu)d-1} (\xi d)^{-\xi d+1} p((\xi-\nu)d)^{-(\xi-\nu)d+1} \times \bar{p}(\xi d)^{\xi d-1}.$$

If we let $(\xi-\nu)d = t$, and $\nu d = s$, then $\xi d = t + s$. Let

$$g(t) = [p(t)]^{-t+1} [e^{-k} p(p(t+s))]^{t+s-1}.$$

If we can prove that $g(t) \leq C$ with C from lemma 3, then we use lemma 3, lemma 1, (4.10), and $C_1 \leq (1-\lambda)/8NCc$ on (4.11). We get

$$A \leq 2^{-1}(\lambda+1)[\xi d/p(\xi d)]^{\xi d-1} \exp((K + \xi d/\bar{p}(\xi d))\, a(|y|+|x|)),$$
$$|y|+|x| \leq r, \quad \text{all } \xi.$$

So we look at $g(t)$. Let $f(t) = \log g(t)$.

$$f(t) = -(t-1)\log p(t) - k(t+s-1) + (t+s-1)\log p(p(t+s)).$$

Then we obtain

$$f'(t) = -\log p(t) - (t-1)\cdot p'(t)[p(t)]^{-1} - k + \log p(p(t+s)) +$$
$$+ (t+s-1)[p(p(t+s))]^{-1} p'(p(t+s))\, p'(t+s).$$

We choose k great. Then it follows from (2.1)-(2.4) that

$$f'(t) \leq -\log p(t)/\log p(p(t+s)).$$

Since $t \geq \log(4+t+s) \geq p(t+s)$, t great, it follows that $f'(t) < 0$, t great, $0 \leq s \leq |d|$, uniformly in s. So $g(t)$ is bounded. We may as well assume that $g(t) \leq C$, C from lemma 3. By that we have proved (4.8).

Now we assert that

$$(4.12) \qquad |D_x^\xi D_y^\beta D_x^\delta (u^q - u^{q-1})| \leq$$
$$\leq \left(\frac{1+\lambda}{2}\right)^q (\xi d/\bar{p}(\xi d))^{\xi d-1} \exp(K + \xi d/\bar{p}(\xi d))\, a(|y|+|x|)),$$
$$|y|+|x| \leq r, \quad \text{all } q, q = 1, 2, \ldots .$$

We have just proved that (4.12) is true for $q = 1$. We shall prove that if (4.12) is true for q, then it is also true if q is substituted by $q + 1$. We substitute u^q by $u^q - u^{q-1}$ in the computation above letting $f = 0$ and using (4.12). Then it is immediately seen that (4.12) is true for all q. Since $2^{-1}(\lambda + 1) < 1$, it follows from (4.12) and lemma 2 that all derivatives $D_x^\xi D_y^\gamma D_x^\alpha u^q$, $\alpha d + |\gamma| \leq |\beta| + |\delta|$, $\gamma \leq \beta$, converges uniformly when $|y| + |x| \leq r$. It also follows from (4.8) and lemma 2 that the limit function u with a new C satisfies

$$|D_x^\xi D_y^\gamma D_x^\alpha u| \leq C e^{Kar}(C\xi d/\bar{p}(\xi d))^{\xi d-1}, \ |y| + |x| \leq r, \qquad \gamma \leq \beta,$$
$$\alpha d + |\gamma| \leq |\beta| + |\delta|, \quad \text{all } \xi.$$

This happens in the new coordinate system. We go back to our original space and our original r.

With the same $t > 1$ as was used in the definition of the transformation we now get

$$|D_x^\xi D_y^\gamma D_x^\alpha u| \leq C e^{Kar}\, t(tC\xi d/\bar{p}\,(\xi d))^{\xi d-1}, \ |y| + |x| \leq r, \ \gamma \leq \beta, \ \alpha d + |\gamma| \leq |\beta| + |\delta|.$$

It follows from above, from $\bar{p}(\xi d)/tC \to +\infty$, when $d\xi \to +\infty$, and from lemma 4 that $u \in \gamma(\beta, \delta, d)$. The existence of a solution is proved.

Let v be the difference between two solutions of (4.6) in $\gamma(\beta, \delta, d)$. For the r we have just considered there are always a $p_1 \in P$ and a $C_2 > 0$ such that

$$(4.13) \qquad |D_x^\xi D_y^\beta D_x^\delta v| \leq C_2(\xi d/p_1(\xi d))^{\xi d-1} \times$$
$$\times \exp((K + \xi d/p_1(\xi d))\, a(|y| + |x|)), \quad |y| + |x| \leq r, \quad \text{all } \xi.$$

But then (4.13) is true in the transformed system with new K, p_1, C_2, and r. This is said with the same reservation as before. Maybe the inequality is false outside the compact set D, that corresponds to $|y| + |x| \leq r$ in the original system. If the new p_1 happens to violate (2.1) or (2.2), then it is only divided by a suitable constant. So we assume that (4.13) is true with $p_1 \in P$. If now

$$(4.14) \qquad p_1(t) \geq e^{-k}p(p(t)), \qquad t \geq 0$$

with the p used in (4.2) and (4.3), then we choose C_2 minimal in (4.13), where p_1 has been substituted by $e^{-k}p(p(t))$. With K adjusted to $e^{-k}p(p(t))$ a computution as that which lead to the proof of (4.12), leads to a contradiction if $C_2 \neq 0$. Then the uniqueness is proved in that case.

If (4.14) is not true, then we proceed in the following way. It follows from (2.1), (2.3), and (2.5) that there exists a t_0 such that

$$p_1(p_1(t)) \leq p_1(t), \qquad t \geq t_0.$$

If $k_0 > 1$ is great then

$$k_0^{-1} p_1(k_0^{-1} p_1(t_0)) \leq p_1(0).$$

We then see that

$$k_0^{-1} p_1(k_0^{-1} p_1(t)) \leq p_1(t), \qquad t \geq 0. \tag{4.15}$$

We now choose k_0 such that (4.15) is true. Then we choose $p' \in P$ such that

$$p'(t) \leq \min(k_0^{-1} p_1(t),\ p(t)).$$

Then we use p' as our new p. The uniqueness proof can now be completed just as in the first case. The proof of the theorem is finished.

5. Operators with constant coefficients.

We now treat the special case of constant coefficients. We have abstained from formulating the theorem for the more general case when the coefficients depend on y only.

THEOREM 2. *The integer n' is restricted by $1 \leq n' \leq n$. Let $d \in R^n$. It has the following property*

$$0 \leq d_j \leq 1, \qquad 1 \leq j \leq n', \qquad d_j \geq 0, \qquad 1 \leq j \leq n. \tag{5.1}$$

The multi-indices $\beta \in R^s$, $\delta \in R^n$, $\gamma^k \in R^s$, $\alpha^k \in R^n$, $1 \leq k \leq N$, are restricted by

$$\delta_j = 0, \qquad n' < j \leq n, \qquad \gamma^k \leq \beta, \qquad 1 \leq k \leq N, \tag{5.2}$$

$$|\beta - \gamma^k| + (\delta - \alpha^k)d \geq 0, \qquad 1 \leq k \leq N.$$

The function $f(y, x)$ belongs to $\gamma(0, 0, d)$. So to every $r > 0$ there exists a $p \in P$ and a constant $C_1 \geq 0$ such that

$$|D_x^\xi f(y, x)| \leq C_1 (\xi d / p(\xi d))^{\xi d}, \qquad |y| + |x| \leq r. \tag{5.3}$$

The constants $a_k \in C$, $1 \leq k \leq N$. They are restricted by

(5.4) $\Sigma \mid a_k \mid = \lambda < 1$, *the sum taken over k with* $\mid \beta - \gamma^k \mid + (\delta + \alpha^k) d = 0$.

It follows that the Goursat problem

$$D_y^\beta D_x^\delta u = \Sigma \alpha_k D_y^{\gamma^k} D_x^{\alpha^k} u + f, \qquad u = O(y^\beta x^\delta), \tag{5.5}$$

has one and only one solution in $\gamma(\beta, \delta, d)$.

Proof. We choose new coordinates by

$$y'_j = ty_j, \qquad 1 \leq j \leq s, \qquad x'_j = t^{d_j} x_j, \qquad 1 \leq j \leq n, \qquad t > 1.$$

By analogy from the proof of theorem 4 we choose t so big that (5.3) in the new coordinate system has $C_1 < (\lambda - 1)/8$ and $\Sigma \mid \alpha_k \mid \leq (1 - \lambda)/8$, the sum taken over k with $\mid \beta - \gamma^k \mid + (\delta - \alpha^k) d > 0$. We define the sequence $(u^q)_0^\infty$ as in (4.7). We assert that with $\bar{p}(t) = e^{-k} p(p(t))$

$$\mid D_x^\xi D_y^\beta D_x^\delta u^q(x, y) \mid \leq (1 + \xi d / \bar{p}(\xi d))^{\xi d} \exp((K + \xi d / \bar{p}(\xi d)) a (\mid y \mid + \mid x \mid)), \tag{5.6}$$

$$\mid y \mid + \mid x \mid \leq r, \quad \text{all } \xi, \qquad q = 0, 1, 2, \ldots,$$

in the new system. Since $u^0 = 0$, (5.6) is true for $q = 0$. If it is true for a certain q, we get from above and from lemma 5, noting especially (3.18), that

$$\mid D_x^\xi D_y^\beta D_x^\delta u^{q+1}(x, y) \mid \leq$$

$$\leq 2^{-1}(1 + \lambda)(1 + \xi d / \bar{p}(\xi d))^{\xi d} \exp((K + \xi d / \bar{p}(\xi d)) a (\mid y \mid + \mid x \mid)),$$

$$\mid y \mid + \mid x \mid \leq r, \quad \text{all } \xi.$$

So (5.6) is true for all q. Then we prove the analogue of (4.12) just as in section 4. By that we have proved that a solution u exists, and that in the transformed system u satisfies (5.6).

Since $p(\xi d)/(1 + p(\xi d)/\xi d) \to +\infty$ when $\xi d \to +\infty$ it follows from lemma 4 and lemma 5 that there exist a $p_1 \in P$ and constants C_1 and C_2 such that

$$\mid D_x^\xi D_y^\gamma D_x^\alpha u \mid \leq C_1 (\xi d / p_1(\xi d))^{\xi d} \times C_2^{\xi d}, \qquad \mid y \mid + \mid x \mid \leq r \qquad \gamma \leq \beta,$$

$$\mid \beta - \gamma \mid + (\delta - \alpha) d \geq 0, \quad \text{all } \xi.$$

We may adjust p_1 such that

$$| D_x^{\xi} D_y^{\gamma} D_x^{\dot{\alpha}} u | \leq C_1 (\xi d/p_1 (\xi d))^{\xi d}, \quad \text{all } \xi.$$

In the original system this corresponds to

$$| D_x^{\xi} D_y^{\gamma} D_x^{\alpha} u (y, x) | \leq C_1 t^{(\xi+\alpha) d + |\gamma|} (\xi d/p_1 (\xi d))^{\xi d} \qquad \text{all } \xi.$$

From lemma 4 it then follows that $u \in \gamma (\beta, \delta, d)$, since only those coordinates with $d_j > 0$ are involved in the rigth member.

Let v be the difference between two solutions in $\gamma (\beta, \delta, d)$. If v in the transformed system satisfies

$$(5.7) \qquad | D_x^{\xi} D_y^{\beta} D_x^{\delta} v | \leq C (1 + \xi d/\bar{p} (\xi d))^{\xi d} \exp (K + \xi d/\bar{p} (\xi d)) \, a (| y | + | x |),$$

$$| y | + | x | \leq r, \quad \text{all } \xi,$$

with C chosen minimal, then we get a contradiction if $C \neq 0$, just as in section 4. If we must choose some $p_1 \in P$, $p_1 (t) \leq \bar{p} (t)$, and insert p_1 instead of $\bar{p}$ in (5.7) for this inequality to be true, then we operate just as in the corresponding case in section 4. Thus the solution is unique in $\gamma (\beta, \delta, d)$. Theorem 2 is proved.

6. Comments.

The proof of the theorem shows that (4.5) in the hypothesis of the theorem can be weakened. We shall prove below that it is sufficient to require that to every given compact set $D \subset R^{s+n}$ there exists a linear coordinate transformation of the type

$$x'_j = t^{b_j} x_j, \qquad 1 \leq j \leq n, \qquad b = (b_1, \dots, b_n) \in R^n,$$

$$y'_j = t^{c_j} y_j, \qquad 1 \leq j \leq s, \qquad c = (c_1, \dots, c_s) \in R^s, \quad t > 0,$$

such that the inequality in (4.5) is true in the compact set D' in the primed space that corresponds to D. Indeed, if

$$| D_x^{\xi} f (y, x) | \leq C_1 (\xi d/p (\xi d))^{\xi d - 1}, \qquad (y, x) \in D, \quad \text{all } \xi,$$

then

$$(6.1) \qquad | D_{x'}^{\xi} f' (y', x') | \leq C_1 t^{-\beta c - \delta b} t^{-\xi b} (\xi d/p (\xi d))^{\xi d - 1}, \quad (y', x') \in D', \quad \text{all } \xi.$$

Since $d_j \geq 1$, $1 \leq j \leq n$, it is clear that for fixed t and b

$$t^{\xi b/(\xi d-1)}\, p\,(\xi d) \to +\infty, \quad \text{when } \xi d \to +\infty.$$

According to lemma 4 there must exist a $p' \in P$ such that

$$p'\,(\xi d) < t^{\xi b/(\xi d-1)}\, p\,(\xi d), \quad \text{all } \xi.$$

From this and from (6.1) it follows that for some $C' > 0$

$$|\, D_{x'}^{\xi} f'\,(y', x')\,| \leq C'\,(\xi d/p'\,(\xi d))^{\xi d-1}, \qquad (y', x') \in D', \quad \text{all } \xi.$$

The same applies to the coefficients a_k. The proof then shows that a solution u exists in D'. There is a $p' \in P$ and a $C' > 0$ such that

$$|\, D_{x'}^{\xi}\, D_{y'}^{\gamma}\, D_{x'}^{\alpha}\, u\,(y', x')\,| \leq C'\,(\xi d/p'\,(\xi d))^{\xi d-1}, \qquad (y', x') \in D',$$

$$\gamma \leq \beta, \quad |\gamma| + \alpha d \leq |\beta| + |\delta|, \quad \text{all } \xi.$$

It follows from (6.1) that

$$|\, D_x^{\xi}\, D_y^{\gamma}\, D_x^{\alpha}\, u\,(y, x)\,| \leq C_1\, t^{\gamma c+\alpha b}\, t^{\xi d}\;\; t^{\xi b}\,(\xi d/p'\,(\xi d))^{\xi d-1}, \qquad (y, x) \in D.$$

Since

$$\xi d \to +\infty \Longrightarrow \inf_{\gamma \leq \beta,\ |\gamma| + \alpha d \leq |\beta| + \delta} t^{(\gamma c+\alpha b+\xi b)/(\xi d-1)}\, p'\,(\xi d) \to +\infty,$$

lemma 4 says that there exist a $p \in P$ and a constant C such that

$$|\, D_x^{\xi}\, D_y^{\gamma}\, D_x^{\alpha}\, u\,(y, x)\,| \leq C\,(\xi d/p\,(\xi d))^{\xi d-1}, \qquad (y, x) \in D.$$

The solution u then belongs to $\gamma\,(\beta, \delta, d)$ since D is arbitrary. We have here tacitly used compact sets defined by

$$e_1\,|\,x_1\,| + \dots + e_n\,|\,x_n\,| + e_1'\,|\,y_1\,| + \dots + e_s'\,|\,y_s\,| \leq r, \quad e_j > 0, \quad 1 \leq j \leq n,$$

$$e_j' > 0, \qquad 1 \leq j \leq n.$$

This is obviously enough. See the definition of $\gamma\,(\beta, \delta, d)$.

The uniqueness goes along the same lines. We start with the difference v between two solutions in $\gamma\,(\beta, \delta, d)$. The estimates of v in D gives estimates of v in D'. These estimates in their turn give estimates in the transform of D' used in the proof in section 4. It follows that v must be zero.

Even this weakened from of (4.5) is not a necessary condition although it is essential for our method of proof. The counterexample in Gårding [1], p. 152, note 1, due to Gyunther [2], also applies to entire functions.

One could also give some conditions equivalent to (4.5) that appears to be weaker. Let the inequality in (4.5) be true when the sum is taken only over those k, $\gamma^k = \beta$, $\alpha^k d = |\delta|$, $\alpha^k{}_j = 0$, $n' < j \leq n$. Then we use the transformation

$$y_j' = t y_j, \qquad 1 \leq j \leq s,$$

$$x_j' = t x_j, \qquad 1 \leq j \leq n',$$

$$x_j' = x_j, \qquad 1' \leq j \leq n.$$

For t great enough (4.5) is true in the new coordinates.

For a still apparently weaker condition see the remark in Gårding [1], p. 152 below, due to J. Leray.

The functions in $\gamma(\beta, \delta, d)$, used in theorem 1 can be extended to functions in $R^s \times C^{n'} \times R^{n-n'}$, which are entire functions of the variables $(x_1, \ldots, x_{n'}) \in C^{n'}$, since we have $d_j = 1, 1 \leq j \leq n'$. So letting $s = 0$ and $n = n'$ we see that the hypothesis of the theorem in [7] is that of the theorem in section 4, except that (4.5) is not explicitly satisfied. The proof in [7] shows however that this is the case. Thus the result of this paper generalizes the theorem in [7].

Theorem 2 is apparently stronger than theorem 1 in [6]. The argument with transformations in the proof in section 5 shows howewer that $\lambda < 1$ used in theorem 1 in [6] is equivalent to (4.5). In the same way we may allow $0 \leq d_j \leq 1$ for some j, $1 \leq j \leq n'$, and $d_j = 0$ for some j, $n' < j \leq n$. This is not included in theorem 2 in [6], but the proof in [6] covers this case too with proper notation. The transformations described above for theorem 1 do not apply to theorem 2 when some $d_j = 0$. So the reader is warned on this point. We abstain from going into details how things could be adjusted.

In the following we let $s = 0$ in theorem 2. Then we see that part of theorem 1 in [4] is included in theorem 2. Now we let $d = 0$ in theorem 2. We see that if (5.4) is true then (5.5) has a unique solution in $\gamma(0, \delta, 0)$, without any restrictions on α^k.

With our definition of $\gamma(0, \delta, 0)$ this class is defined as those functions the derivatives of which are uniformly bounded on every compact set. So for $n = 1$, $e^x \in \gamma(0, 0, 0)$ but $e^{2x} \notin \gamma(0, 0, 0)$. We define

$$\gamma(\bar{0}) = \bigcap_d \gamma(0, \delta, d), \qquad d_j > 0, \qquad 1 \leq j \leq n.$$

If to (5.5) there is a vector $b, b_j > 0, 1 \leqq j \leq n, b\alpha^k < b\beta, 1 \leq k \leq N$, then it is proved in [4] that (5.5) has a unique solution $u \in \gamma(\bar{0})$ if $f \in \gamma(\bar{0})$. This is corollary 1 of the first part of theorem 1 in [4]. So it also follows from theorem 2. The class $\gamma(\bar{0})$ is the class of entire functions of order one.

We now discuss the necessity of some conditions in theorem 1 and theorem 2. Let $s = 0$ and let $\delta_1 > \alpha_1^k, 1 \leq k \leq N$, in (5.5). If (5.5) has a solution $u \in \gamma(0, \delta, d)$, $0 < d_j \leq l$, $l \leq j \leq n$, for every $f \in \gamma(0, 0, \delta)$, then it follows from the second part of theorem 1 in [4] that

$$(6.2) \qquad \alpha^k d \leq \delta d, \qquad 1 \leq k \leq N.$$

It is proved by a counterexample. If (6.2) is not true, then there exists an $f \in \gamma(0, 0, d)$, such that the formal solution does not correspond to a function in $\gamma(0, \delta, d)$. So the last part of (5.2) cannot be deleted without violating the conclusion of theorem 2. For more general theorems of this kind, when $s = 0$, $n' = 1$, and $d = (1, \dots, 1)$, see [3]. The result in [3] is concerned with local Cauchy problems. It can however be easily extended to cover global Cauchy problems for entire functions.

The example

$$Du = f(x)(u + 1), \; u(0) = 0, \; u + 1 = \exp \int_0^x f(t)\, dt),$$

where f is an arbitrary entire function, shows that we cannot let $d_j < 1$, some j, $1 \leq j \leq n'$, in theorem 1.

It is shown by a counterexample in [7] that theorem 1 is false if (4.4 is excluded from the hypothesis of the theorem.

We now restate a part of theorem 1 as a theorem for a Cauchy problem for entire functions, since it brings out the essence of theorem 1 in a simple form.

THEOREM 3. *The multi-indices* $\delta = (\delta_1, 0, \dots, 0) \in R^n$, *and* $\alpha^k \in R^n$, $1 \leq k \leq N$, *are restricted by*

$$\alpha^k \neq \delta, \; |\alpha^k| \leq \delta, \qquad 1 \leq k \leq N.$$

The entire functions f, a_k, $1 \leq k \leq N$, *are restricted by*

$$|\alpha^k| = \delta \Longrightarrow a_k \text{ is a constant.}$$

It follows that the Cauchy problem

$$D^\delta u = \Sigma a_k D^{\alpha^k} u + f, \qquad u = 0 \; (x^\delta).$$

has one and only one entire solution.

Here (4.5) in theorem 1 is easily achieved by the transformation

$$x_j' = tx_1 ,$$

$$x_j' = x_j , \qquad 2 \leq j \leq n,$$

if t is chosen great enough.

It seems likely that the results of this paper could be extended to systems using a spectral matrix defined by analogy from this concept defined in [1] and [5]. We have, however, not done this.

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