ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

JEAN PIERRE AUBIN

# Behavior of the error of the approximate solutions of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods

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# BEHAVIOR OF THE ERROR OF THE APPROXIMATE SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR LINEAR ELLIPTIC OPERATORS BY GALERKIN'S AND FINITE DIFFERENCE METHODS

JEAN PIERRE AUBIN (*)

CONTENTS.



## § 1. Introduction.

Let $\Omega$ be a sufficiently smooth open subset of $R^n$ with boundary $\Gamma$, and let us consider a Neumann problem for an elliptic operator of order $2m$

$$(1.1) \qquad Au(x) = \sum_{|p|,|q| \leq m} (-1)^{|q|} D^q (a_{pq}(x) D^p u(x)) = f(x)$$

under suitable assumptions (cf. § 2) the operator $A$ is an isomorphism between the Sobolev space $V = H^m(\Omega)$ and its (anti-)dual $V'$.

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Our goal is to construct approximate problems. For this, we associate with a parameter $h$

(1.2) $$\begin{cases} \text{a finite dimensional space } V_h \\ \text{an isomorphism } A_h \text{ mapping } V_h \text{ onto } V_h \\ \text{an element } f_h \text{ of } V_h \end{cases}$$

and we consider the problem :

Find $u_h$ in $V_h$ such that

(1.3) $$A_h u_h = f_h \text{ in } V_h .$$

We will first give a suitable definition of « convergence », and then we will give a process for constructing $A_h$ and $f_h$ if $V_h$ is given. This will be done by *constructing an operator $p_h$ from $V_h$ into $V$*, and then we will say that « *$u_h$ converges to $u$* » if

(1.4) $$\lim_{h \to 0} \| u - p_h u_h \|_V = 0.$$

We can give here the simplest process for the construction of $A_h$ and $f_h$. Let $r_h^*$ denote the transposed operator of $p_h$, so that

$$(p_h u_h , v) = (u_h , r_h^* v)_h .$$

We then obtain the following scheme

$$\begin{array}{ccc} u \in V & \xrightarrow{A} & V' \ni f \\ \uparrow p_h & & \downarrow r_h^* \\ u_h \in V_h & \xrightarrow{A_h} & V_h \ni f_h \end{array}$$

and we can take

(1.5) $$\begin{cases} \text{(i)} \quad A_h = r_h^* A p_h \\ \text{(ii)} \quad f_h = r_h^* f. \end{cases}$$

Then if $V_h$ and $p_h \in \mathcal{L}(V_h , V)$ are given, formula (1.5) permits us to construct approximate problems for any choice of operators $A$ and elements $f$ of $V'$.

For a given class of operators $A$ (for example, linear coercive operators) we have to look for suitable assumptions about the spaces $V_h$ and the operators $p_h$ to obtain the following results:

$$(1.6) \qquad \begin{cases} \text{(i)} & \text{there exists a solution } u_h \text{ of } (1.3) \\ \text{(ii)} & u_h \text{ converges to } u \end{cases}$$

and to study the behavior of the error

$$(1.7) \qquad \| u - p_h u_h \|_V .$$

An example of the operators $p_h$ is well known. Let $(\omega_n)_n$ be a « basis » for the (real separable) Hilbert space $V$. If $h = \frac{1}{n}$ we take

$$(1.8) \qquad \begin{cases} V_h = R^n \\ u_h = (u_h^i)_{1 \leq i \leq n} \\ p_h u_h = \sum_{i=1}^{n} u_h^i \omega_i . \end{cases}$$

Then the process of construction (1.5) is called Galerkin's Method. (See § 4).

We now return to our original problem. The space $V$ is then a Sobolev space. We shall construct a class of operators $p_h$ such that the approximate operators associated with $A$ are « finite-difference » operators. We will meet during this construction all the « technical aspects » of numerical analysis: but, this being done once and for all, we will be able to use these operators for the construction of « finite-difference » schemes for other classes of differential operators defined on Sobolev spaces.

We will associate with $\Omega$ a parameter $h = (h_1, \dots, h_n)$, a suitable mesh $R_h(\Omega)$ of multi-integers $\alpha = (\alpha_1, \dots, \alpha_n)$, and the space $V_h$ of sequences $u_h = (u_h^\alpha)$ defined on $R_h(\Omega)$. We consider a function $\sigma(x)$ which is an $m$-th fold convolution of the characteristic functions of the cube $[-1, 1]^n$, and we then define

$$(1.9) \qquad \sigma_h^\alpha(x) = \frac{1}{h} \sigma\left(\frac{1}{h} x - \alpha\right) = \frac{1}{h_1 h_2 \dots h_n} \sigma(h_1^{-1} x_1 - \alpha_1, \dots, h_n^{-1} x_n - \alpha_n).$$

The operators $p_h$ will have the form

$$(1.10) \qquad p_h u_h(x) = \sum_{\alpha \in R_h(\Omega)} u_h^\alpha \sigma_h^\alpha(x).$$

We shall construct operators $r_h$ from $V$ into $V_h$ so that we have

$$\| u - p_h r_h u \|_{H^m(\Omega)} \leq C h^{q-m} \| u \|_{H^q(\Omega)} \qquad (q \geq m). \tag{1.11}$$

We will be able to deduce from this inequality a similar inequality for the errors $u - p_h u_h$, so that if the solution $u$ of (1.1) belongs to $H^q(\Omega)$ we have

$$\| u - p_h u_h \|_{H^m(\Omega)} \leq C h^{q-m}. \tag{1.12}$$

## § 2. Coercive Boundary Value Problems.

We summarize here some known results concerning variational methods in the study of boundary value problems.

Let $\Omega$ be a sufficiently smooth bounded open subset of $R^n$ with boundary $\Gamma$. We denote by $H^m(\Omega)$ the Sobolev space of functions $u$ in $L^2(\Omega)$ that have all derivatives $D^k u$ (in the sense of distributions) of order $|k| \leq m$ also in $L^2(\Omega)$. This is a Hilbert space with the norm

$$\| u \|_m = \Big( \sum_{|k| \leq m} | D^k u |^2 \Big)^{1/2} \tag{2.1}$$

where

$$| v | = \left( \int_\Omega | v(x) |^2 \, dx \right)^{1/2}$$

and

$$D^k u = \frac{\partial^{|k|} u}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}, \quad | k | = k_1 + \ldots + k_n .$$

We will consider a Neumann problem for the differential operator

$$A u = \sum_{|p|, |q| \leq m} (-1)^{|q|} D^q (a_{pq} D^p u). \tag{2.2}$$

Introduce the sesquilinear form

$$a(u, v) = \sum_{|p|, |q| \leq m} \int_\Omega a_{pq}(x) \, D^p u(x) \, D^q \bar{v}(x) \, dx. \tag{2.3}$$

If we assume that the coefficients $a_{pq}(x)$ belong to $L^\infty(\Omega)$, then this form is continuous on $H^m(\Omega)$.

Assume that $\Omega$ and the coefficients $a_{pq}(x)$ satisfy conditions sufficient to obtain the *coercivness inequality*

$$\text{Re } a(v, v) \geq C \| v \|_m^2 \quad (C > 0, v \in H^m(\Omega)). \tag{2.4}$$

Write the formal Green's formula in the form

$$\int_\Omega Au(x)\, \overline{v}(x)\, dx - a(u, v) = \sum_{j=0}^{m-1} \int_\Gamma S_j u\, \gamma_j \overline{v}\, d\sigma \tag{2.5}$$

where $\gamma_j v$ is the trace operator of order $j$ on $\Gamma$, $d\sigma$ is the surface measure on $\Gamma$, and $S_j u$ is a differential operator of order $2m - j - 1$. Then we can prove that the following two problems are equivalent:

PROBLEM $P$. *Let $f$ be a given function in $L^2(\Omega)$. Find a function $u$ in $H^m(\Omega)$ that satisfies*

$$\begin{cases} \text{(i)} & Au = f \text{ in } L^2(\Omega) \\ \text{(ii)} & S_j u|_\Gamma = 0 \text{ for } 0 \leq j \leq m - 1 \text{ (in a suitable sense).} \end{cases} \tag{2.6}$$

PROBLEM $P'$. *Let $f$ be a given function in $L^2(\Omega)$. Find a function $u$ in $H^m(\Omega)$ satisfying the variational equation*

$$a(u, v) = (f, v) \text{ for all } v \text{ in } H^m(\Omega). \tag{2.7}$$

The problem $P'$ is a particular example of the following abstract situation: Denote by $V$ the Hilbert space $H^m(\Omega)$, and by $H$ the Hilbert space $L^2(\Omega)$. On $H$ we use the scalar product

$$(u, v) = \int_\Omega u(x)\, \overline{v}(x)\, dx, \quad |u| = \sqrt{(u, u)} \tag{2.8}$$

and we identify $H$ with its (anti-)dual $H'$. Then we have:

$$V \text{ is a dense subspace of } H, \text{ and } |u| \leq k \| u \|. \tag{2.9}$$

Thus the space $H$ is identified as a dense subspace of the (anti-) dual $V'$ of $V$, and $V'$ is a Hilbert space with the dual norm

$$\| f \|_* = \sup_{u \neq 0} \| u \|^{-1} | (f, u) |. \tag{2.10}$$

We then have the following situation:

$$(2.11)\qquad \begin{cases} V \subset H \subset V' \\ \text{each space is dense in the one that contains it} \\ k^{-1} \| u \|_* \leq | u | \leq k \| u \|. \end{cases}$$

If we are given the continuous sesquilinear form $a(u, v)$, define a continuous linear operator $A$ from $V$ into $V'$ by the variational equality

$$(2.12)\qquad (Au, v) = a(u, v) \quad \text{for all } v \in V.$$

However, this also defines an unbounded linear operator $A$ on the Hilbert space $H$.

Denote by $D(A)$ the space (possibly null)

$$(2.13)\qquad D(A) = \left\{ \begin{array}{l} u \in V \text{ such that there exists } k_u \text{ with} \\ | a(u, v) | \leq k_u | v | \text{ for all } v \in V \end{array} \right\}.$$

Then if $u \in D(A)$ the map $Au : u \longrightarrow a(u, v)$ is continuous on the space $V$ with the $H$-norm. Thus by an extension argument this map belongs to $H' = H$. The subspace $D(A)$ is a pre-Hilbert space for the graph norm

$$[u]_{D(A)} = (| u |^2 + | Au |^2)^{1/2}$$

and $A$ is a continuous linear operator from $D(A)$ onto $H$. Define $a^*(u, v) = \overline{a(v, u)}$ and $(A^* u, v) = a^*(u, v)$, so that we have the following:

THEOREM 2.1. *Assume that the form $a(u, v)$ is coercive, so that*

$$(2.14)\qquad \operatorname{Re}\, a(v, v) \geq C \| v \|^2 \quad \textit{for all } v \in V.$$

*Then $D(A)$ is a Hilbert space with the graph norm, $D(A)$ is dense in $V$ and $H$, and $A$ is an isomorphism*

$$(2.15)\qquad \begin{cases} \text{(i)} & \textit{from } D(A) \textit{ onto } H \\ \text{(ii)} & \textit{from } V \textit{ onto } V' \\ \text{(iii)} & \textit{from } H \textit{ onto } D(A^*)'. \end{cases}$$

Since $A^*$ has the same properties as $A$, we see immediately that $D(A^*)$ is dense in $V$, and that $V'$ is dense in $D(A^*)'$. But since $V$ is dense in $H$ the transpose $A^{*\prime}$ of $A^*$ maps $H$ into $D(A^*)'$ and is an extension of $A$. This is the reason for putting $A^{*\prime} = A$ in (2.15 iii).

We conclude from this theorem that there exist unique solutions to the problems $P$ and $P'$.

For the study of Sobolev spaces and boundary value problems, we can refer to lectures of J. L. Lions at the University of Montreal (« Problemes aux limites pour les equations aux derivees partielles ». Editions de l'universite de Montreal (2nd edition, 1965)) where supplementary bibliographical notes can be obtained. We can find more precise results in a book of J. L. Lions and H. Magenes (to appear).

We now look for suitable approximations for solutions to the problem $P$.

## § 3. **Abstract theorems about approximation.**

We want to prove here some « abstract results » which are due to J. Cea, J. L. Lions, and J. P. Aubin.

3.1 *Approximation by restriction schemes.*

Let $V$ be a Hilbert space, and let $a(u, v)$ be a continuous coercive sesquilinear form on $V$. We then look for approximations to the solution $u$ of the

PROBLEM $P$. *Let $f$ be an element of $V'$. Find an element $u$ of $V$ such that*

$$a(u, v) = (f, v) \quad \textit{for all } v \in V. \tag{3.1}$$

Let $h$ be a « parameter », which will eventually converge to 0. Associate with $h$ the following:

$$\begin{cases} \text{(i)} & \text{a finite dimensional space } V_h \\ \text{(ii)} & \text{an injective linear operator } p_h \\ & \text{from } V_h \text{ into } V. \end{cases} \tag{3.2}$$

Let $r_h$ be the map from $V$ onto $V_h$ such that $p_h r_h$ is the orthogonal projection onto $p_h V_h$ (1).

(1) Since $V$ is a Hilbert space, this map exists and satisfies

$$\| u - p_h r_h u \| = \min_{v_h \in V_h} \| u - p_h v_h \| .$$

If $A$ is the canonical isomorphism between $V$ and $V'$ defined by

$$(Au, v) = ((u, v)), \text{ then } r_h = (r_h^* A p_h)^{-1} r_h^* A.$$

In other words, $p_h r_h u$ is the best approximation of $u$ by elements of $p_h V_h$.

Assume that we have

$$\lim_{h \to 0} \| u - p_h r_h u \| = 0 \quad \text{for all fixed } u \text{ in } V. \tag{3.3}$$

We will construct such operators when $V = H^m(\Omega)$ in § 7. We then define a norm on $V_h$ by

$$\| u_h \|_h = \| p_h u_h \| \quad \text{for all } u_h \in V_h . \tag{3.4}$$

Now consider the following approximate problem:

PROBLEM $P_h^1$. *Let $f$ be an element of $V'$. Find an element $u_h$ in $V_h$ that satisfies*

$$a(p_h u_h, p_h v_h) = (f, p_h v_h) \quad \text{for all } v_h \in V_h . \tag{3.5}$$

THEOREM 3.1. *The solutions $u_h$ of (3.5) converge to the solution $u$ of (3.1) in the sense that*

$$\lim_{h \to 0} \| u - p_h u_h \| = 0. \tag{3.6}$$

*More precisely, the errors $u - p_h u_h$ satisfy the inequality*

$$\| u - p_h u_h \| \leq c^{-1} M \| u - p_h r_h u \| \tag{3.7}$$

*and the asymptotic behavior of $\| u - p_h u_h \|$ is the same as the best approximation to the solution $u$ by elements of $p_h V_h$. If we define*

$$\gamma(h) = \sup_{u \in D(A^*)} [u]_{D(A^*)}^{-1} \| u - p_h r_h u \| \tag{3.8}$$

*then the errors $u - p_h u_h$ satisfy the inequality in $H$:*

$$| u - p_h u_h | \leq M_1 \gamma(h) \| u - p_h u_h \| \leq M_2 \gamma(h) \| u - p_h r_h u \| . \tag{3.9}$$

Since $a(u, v) = (f, v)$ and $a(p_h u_h, p_h v_h) = (f, p_h v_h)$ we can set $v = p_h v_h$ to obtain the equation $a(u - p_h u_h, p_h v_h) = 0$. Putting $v_h = r_h u$, we deduce:

$$a(u - p_h u_h, u - p_h u_h) = a(u - p_h u_h, u - p_h r_h u)$$

and this implies that

$$c \| u - p_h u_h \|^2 \leq M \| u - p_h u_h \| \| u - p_h r_h u \| .$$

Notice that this inequality is independent of the choice of the operator $r_h$ from $V$ onto $V_h$. In particular, we can take for $r_h$ the orthogonal projection in $V$ of $u$ onto the Hilbert space $p_h V_h$. Now define $\varepsilon_h = u - p_h u_h$ and $\tau_h = 1 - p_h r_h$ so that we have

$$a(\varepsilon_h, v) = a(\varepsilon_h, \tau_h v) \quad \text{for all } v \in D(A^*) \subset V. \tag{3.10}$$

Denote by $\tau_h^*$ the transposed operator of $\tau_h$, so that equation (3.10) is equivalent to

$$\varepsilon_h = A^{-1} \tau_h^* A \varepsilon_h . \tag{3.11}$$

But $\tau_h$ is an operator from $D(A^*)$ into $V$ with norm $\gamma(h)$, so that $\tau_h$ is an operator from $V'$ into $D(A^*)'$ with the same norm $\gamma(h)$. Then we have, by Theorem 2.1, the scheme

$$V \underset{A}{\longrightarrow} V' \underset{\tau_h^*}{\longrightarrow} D(A^*)' \underset{A^{-1}}{\longrightarrow} H$$

and this implies that

$$|\varepsilon_h| \leq M\gamma(h) \| \varepsilon_h \| .$$

NOTE 3.1.

In the following examples we will compute the function $\gamma(h)$. If the injection from $V$ into $H$ is compact, then the injection from $D(A^*)$ into $V$ is also compact. Since

$$\begin{cases} \| p_h r_h u \| \leq \| u \| \\ \lim\limits_{h \to 0} \| u - p_h r_h u \| = 0 \quad \text{for all } u \in V \end{cases} \tag{3.12}$$

then the function $\gamma(h)$ converges to 0 with $h$ by the Banach-Steinhaus Theorem.

NOTE 3.2.

Suppose that $A$ is a continuous operator from $V$ into a Banach space $E$ with the property that [2]

$$\begin{cases} \text{there exists a continuous linear operator } L \text{ from} \\ V \text{ into } V' \text{ such that} \\ \operatorname{Re}(Au, Lu) \geq c \| u \|^2 \text{ for } c > 0 \text{ and all } u \in V. \end{cases} \tag{3.13}$$

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[2] This class of operators, called « $L$-positive definite » operators, was first introduced by Martyniuk (See [9], [10]).

Denote by $L'$ the transpose of $L$. Then if $A$ is an isomorphism the equations

$$Au = f \tag{3.14}$$

and

$$L' Au = L' f \tag{3.15}$$

are equivalent. We can approximate the solution $u$ to (3.15) by the solutions $u_h$ to

$$(Ap_h u_h , Lp_h v_h) = (f, Lp_h v_h) \qquad \text{for all} \quad v_h \in V_h . \tag{3.16}$$

Then if $c_1$ is the norm of $L$ and $M$ the norm of $A$, we can prove in the same way that

$$\| u - p_h u_h \| \leq Mc^{-1} c_1 \| u - p_h r_h u \| . \tag{3.17}$$

For example, if $E$ is a Hilbert space and $A$ is an isomorphism we may take $L = A$. If $E = V'$ and $A$ is coercive, we may take $L = I$.

NOTE 3.3. (regularised restriction schemes).

Let $W$ be a Hilbert space contained in the space $V$, and assume that the solution $u$ to $Au = f$ belongs to the subspace $W$. If we do not make suitable assnmptions about the operators $p_h$ and $r_h$, then we cannot deduce the convergence of approximate solutions $u_h$ to $u$ in $W$. Nevertheless, we can construct a perturbed scheme which gives approximations in the space $W$.

Let $b(u, v)$ be a coercive sesquilinear continuous from on $W$ (for example, take $b(u, v)$ to be the scalar product of $W$). Let $\varepsilon(h)$ be a positive numerical function, and define $\gamma(h)$ by

$$\gamma(h) = \sup_{u \in W} |||u|||^{-1} \| u - p_h r_h u \| \tag{3.18}$$

where $|||\cdot|||$ is the norm of $W$. Assume also that $p_h V_h$ is contained in $W$. We then propose the following approximate problem :

PROBLEM $P_h^2$ . *Find an element $u_h$ in $V_h$ that satisfies*

$$\varepsilon(h) b(p_h u_h , p_h v_h) + a(p_h u_h , p_h v_h) = (f, p_h v_h) \tag{3.19}$$

*for all* $v_h \in V_h$ .

THEOREM 3.2. *Assume that*

$$|||p_h r_h u||| \leq m |||u||| \qquad \textit{for all} \quad u \in W \tag{3.20}$$

*and that there exists a constant $m_1$ such that*

$$\gamma(h) \leq m_1 \sqrt{\varepsilon(h)} \tag{3.21}$$

*where*

$$\lim_{h \to 0} \varepsilon(h) = 0. \tag{3.22}$$

*Then if $u_h$ is a solution to $P_h^2$ we have*

$$\begin{cases} \text{(i)} \quad \| u - p_h u_h \| \leq c \sqrt{\varepsilon(h)} \\ \text{(ii)} \quad p_h u_h \textit{ converges weakly to } u \textit{ in } W. \end{cases} \tag{3.23}$$

*If we assume in addition that*

$$\begin{cases} \lim\limits_{h \to 0} \gamma(h)(\varepsilon(h))^{-1/2} = 0, \quad \textit{and} \\ \lim\limits_{h \to 0} ||| u - p_h r_h u ||| = 0 \quad \textit{for all} \quad u \in W \end{cases} \tag{3.24}$$

*then we have*

$$\lim_{h \to 0} ||| u - p_h u_h ||| = 0. \tag{3.25}$$

Let $v = p_h v_h$ in (3.1) and put $\delta_h = p_h u_h - p_h r_h u$ to obtain

$$\varepsilon(h) b(p_h u_h, \delta_h) + a(\delta_h + p_h r_h u - u, \delta_h) = 0. \tag{3.26}$$

Thus we have

$$\varepsilon(h) b(p_h u_h, p_h u_h) + a(\delta_h, \delta_h) = \tag{3.27}$$

$$= a(u - p_h r_h u, \delta_h) + \varepsilon(h) b(p_h u_h, p_h r_h u).$$

But we also have

$$| a(u - p_h r_h u, \delta_h) | \leq M \| u - p_h r_h u \| \, \| \delta_h \| \leq c \gamma(h) ||| u ||| \, \| \delta_h \|$$

and

$$| b(p_h u_h, p_h r_h u) | \leq c ||| p_h u_h ||| \, ||| p_h r_h u ||| \leq mc ||| u ||| \, ||| p_h u_h ||| .$$

Use these equations to obtain the inequality

$$\varepsilon(h) ||| p_h u_h |||^2 + \| \delta_h \|^2 \leq c(\varepsilon(h) + \gamma(h)^2) ||| u ||| . \tag{3.28}$$

This inequality implies (3.23 i), and :

$$|||\, p_h u_h \,||| \leq \text{constant.} \tag{3.29}$$

Then $p_h u_h$ converges weakly in $W$ to an element which is necessarily equal to $u$ (since $p_h u_h$ converges to $u$ in $V$).

We then have

$$b(\delta_h, \delta_h) = b(p_h u_h, \delta_h) - b(p_h r_h u, \delta_h)$$

and so $b(\delta_h, \delta_h)$ converges to 0, because from (3.26) we have

$$|\, b(p_h u_h, \delta_h \,| \leq c \frac{\gamma(h)}{\sqrt{\varepsilon(h)}} |||\, u \,|||$$

and the right hand side converges to 0. In addition, $p_h r_h u$ converges strongly to $u$ in $W$, and $\delta_h$ converges weakly to 0 in $W$.

Under the same hypotheses of Theorem 3.2 we can also approximate the transposed problems and obtain, in this fashion, approximations of non-homogeneous problems for elliptic boundary value problems. See a paper of J. L. Lions and J. P. Aubin (to appear).

3.2. *Approximation by partial restriction schemes.*

We now consider a particular case of the problem. Assume that $V$ is a closed subspace of a finite intersection of Hilbert spaces $V_q$, which are all subsets of the same space $H$. We define a norm on $V$ by

$$\begin{cases} V \subset \bigcap\limits_q V_q \subset H \\ \|\, v \,\| = (\sum\limits_q \|\, v \,\|_q^2)^{1/2}. \end{cases} \tag{3.30}$$

Assume that we also have

$$\begin{cases} a(u, v) = \sum\limits_{p,q} a_{pq}(u, v) \\ |\, a_{pq}(u, v) \,| \leq M_{pq} \,\|\, u \,\|_p \,\|\, v \,\|_q \end{cases} \tag{3.31}$$

and that $a(u, v)$ is *strongly coercive* in the sense that

$$\operatorname{Re} \sum_{p,q} a_{pq}(u_p, u_q) \geq c \sum_q \|\, u_q \,\|_q^2 \qquad \text{for} \quad u_q \in V_q. \tag{3.32}$$

We shall then construct approximation schemes under these conditions.

Assume that we have the following:

$$\text{(3.33)} \qquad \begin{cases} \text{(i)} & \text{a finite dimensional space } V_h \\ \text{(ii)} & \text{operators } p_h^q \text{ from } V_h \text{ into } V_q \\ \text{(iii)} & \text{an operator } r_h \text{ from } H \text{ into } V_h \end{cases}$$

and that these satisfy:

$$\text{(3.34)} \qquad \begin{cases} \text{(i)} & \| u_h \|_h = (\sum_q \| p_h^q u_h \|_q^2)^{1/2} \text{ is a norm for } V_h \\ \text{(ii)} & \lim_{h \to 0} \| u - p_h^q r_h u \|_q = 0 \text{ for all } q \text{ and } u \in V_q \\ \text{(iii)} & \text{If } p_h^q u_h \text{ converges weakly to } u_q \text{ in } V_q \text{ for all } q, \\ & \text{then there exists } u \in V \text{ such that } u = u_q \text{ for all } q. \end{cases}$$

We can then write an element $f \in V'$ in the form

$$\text{(3.35)} \qquad f = \sum_q f_q \qquad \text{for} \qquad f_q \in V_q'.$$

We consider now the problem:

PROBLEM $P_h^3$. *Find a solution* $u_h \in V_h$ *of*

$$\text{(3.36)} \qquad \sum_{p,q} a_{pq}(p_h^p u_h, p_h^q v_h) = \sum_q (f_q, p_h^q v_h) \qquad (\textit{for all } v_h \in V_h).$$

It is clear from (3.32) that there is a unique solution to the problem $P_h^3$. In face, we can prove the following:

THEOREM 3.3. *Assume that conditions* (3.30.34) *are satisfied. Then the solutions* $u_h$ *of Problem* $P_h^3$ *converge to* $u$ *in the sense that*

$$\text{(3.37)} \qquad \lim_{h \to 0} \sum_q \| u - p_h^q u_h \|_q^2 = 0.$$

*If we also assume that there is a mapping* $p_h$ *from* $V_h$ *into* $V$ *that satisfies*

$$\text{(3.38)} \qquad \begin{cases} \text{(i)} & \| p_h u_h \| \leq M \| u_h \|_h \\ \text{(ii)} & \lim_{h \to 0} \| u_h \|_h^{-1} \| p_h u_h - p_h^q u_h \|_q = 0 \quad \textit{for all } q \end{cases}$$

*then we have*

$$\lim_{h \to 0} \| u - p_h u_h \| = 0. \tag{3.39}$$

We will first prove that

$$\| u_h \|_h \leq M.$$

By taking $v_h = u_h$ in (3.36), it follows from (3.22) that

$$c \| u_h \|_h^2 \leq \operatorname{Re} \sum_q (f_q, p_h^q u_h) \leq M \| u_h \|_h (\sum_q \| f_q \|_q^2)^{1/2}.$$

Then, for each $q, p_h^q u_h$ is bounded in $V_q$, and a suitable subsequence of $p_h^q u_h$ converges weakly to $u_q$. Thus by (3.34iii) $u_q = u \in V$, and this $u$ is a solution to the Problem $P$. To see this, it is sufficient to take $v_h = r_h v$ in (3.36) and, by (3.33ii), to take the limit, which converges to 0. We will now prove that $p_h^q u_h$ converges strongly to $u$ in $V_q$. Notice that

$$\sum_{p,q} a_{pq} (u - p_h^p u_h, u - p_h^q u_h) = \sum_{p,q} a_{pq} (u, u - p_h^q u_h) + \sum_q (f_q, p_h^q (u_h - r_h u))_q$$
$$+ \sum_{p,q} a_{pq} (p_h^p u_h, p_h^q r_h u - u)$$

and the right hand side of this equation converges to 0.

Methods for obtaining the solution of (3.36) are avalaible in [13].

3.3. *General approximation criteria.*

We have constructed approximate problems $P_h$. We shall now consider an equation

$$A_h u_h = f_h \quad \text{in} \quad V_h$$

and we will give sufficient conditions to ensure the strong convergence of the solutions $u_h$.

Let $A_h$ be an operator from $V_h$ into $V_h$, and define

$$a_h (u_h, v_h) = (A_h u_h, v_h)_h.$$

We also define the functions

$$\text{(3.40)} \quad \begin{cases} \text{(i)} & \varepsilon (f, f_h) = \sup_{v_h \in V_h} \| v_h \|_h^{-1} \, | (f, p_h v_h) - (f_h, v_h)_h | \\ \text{(ii)} & \psi_u (h) = \sup_{v_h \in V_h} \| v_h \|_h^{-1} \, | a (u, p_h v_h) - a_h (r_h u, v_h) |. \end{cases}$$

We consider the problem:

PROBLEM $P_h^4$. *Find a solution* $u_h \in V_h$ *of*

$$(3.41) \qquad a_h(u_h, v_h) = (f_h, v_h)_h \qquad \textit{for all} \quad v_h \in V_h.$$

If we assume that

$$\operatorname{Re} a_h(v_h, v_h) \geq c \| v_h \|_h^2 \qquad \text{for all} \quad v_h \in V_h$$

then there is a unique solution to (3.41). In fact, we can prove the following:

THEOREM 3.4. *The solution $u_h$ of* (3.41) *satisfies the inequality*

$$(3.42) \qquad \| u_h - r_h u \|_h \leq c^{-1} (\varepsilon(f, f_h) + \psi_u(h)).$$

*If we also have*

$$(3.43) \quad \begin{cases} \text{(i)} & \| p_h u_h \| \leq M \| u_h \|_h \quad \textit{and} \quad \lim\limits_{h \to 0} \| u - p_h r_h u \| = 0 \\ \text{(ii)} & \lim\limits_{h \to 0} \varepsilon(f, f_h) = \lim\limits_{h \to 0} \psi_u(h) = 0 \end{cases}$$

*then*

$$(3.44) \qquad \lim_{h \to 0} \| u - p_h u_h \| = 0.$$

We compute $a_h(u_h - r_h u, u_h - r_h u)$:

$$a_h(u_h - r_h u, u_h - r_h u) = (f_h, u_h - r_h u)_h - (f, p_h(u_h - r_h u)))$$

$$+ a(u, p_h(u_h - r_h u)) - a_h(r_h u, u_h - r_h u).$$

From this we conclude that

$$c \| u_h - r_h u \|_h^2 \leq (\varepsilon(f, f_h) + \psi_u(h)) \| u_h - r_h u \|_h$$

and $\| u_h - r_h u \|_h$ converges to 0 if (3.43ii) is satisfied. This then implies (3.44) by equation (3.43i).

## § 4. Study of the error for a self adjoint Galerkin's method: optimal approximation.

Let $V$ and $H$ be Hilbert spaces, with $V$ dense in $H$. Assume that

(4.1) the injection from $V$ into $H$ is compact.

Let $\Lambda$ be the self-adjoint positive operator defined in terms of the scalar product $((u, v))$ of $V$ by

$$(4.2) \qquad (\Lambda u, v) = ((u, v)).$$

By (4.1) $\Lambda^{-1}$ is compact, and so there exists an orthonormal basis $\omega_n$ for $H$ such that

$$(4.3) \qquad \begin{cases} \Lambda\omega_n = \lambda_n \, \omega_n \\ (\omega_n \, , \, \omega_p) = \delta_{np} \\ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots . \end{cases}$$

We now consider the operators $\Lambda^\theta$ (for $\theta > 0$) and their domains

$$(4.4) \qquad D(\Lambda^\theta) = V_\theta$$

supplied with the inner product $((u, v))_\theta = (\Lambda^\theta u, \Lambda^\theta v)$, so that $\| v \|_\theta = | \Lambda^\theta v |$. (We will take $V_0 = H$, and note that $V = V_{1/2}$). Since $\Lambda^\theta$ is an isomorphism from $V_\theta$ onto $H$, by transposition $\Lambda^\theta$ is an isomorphism from $H$ onto $V_{-\theta} = V'_\theta$ supplied with the norm $\| u \|_{-\theta} = | \Lambda^{-\theta} u |$, where $\Lambda^{-\theta} = (\Lambda^\theta)^{-1}$. We want to construct approximations which hold for all the spaces $V_\theta$, and we will call these « self-adjoint Galerkin's approximations ».

Set $h = \dfrac{1}{n}$, and consider

$$(4.5) \qquad V_h = R^n \quad \text{or} \quad C^n .$$

The self-adjoint Galerkin's approximation will be defined by giving the operators $p_h$ and $r_h^*$, where

$$(4.6) \qquad \begin{cases} p_h \, u_h = \sum\limits_{i=1}^{n} u_h^i \, \omega_i \\ r_h^* \, u = ((u, \omega_i))_{1 \leq i \leq n} \end{cases}$$

and $\{\omega_i\}$ is the basis consisting of the eigenvectors of $\Lambda$.

Then we obtain the following commutation property :

$$(4.7) \qquad \Lambda^\theta \, p_h \, r_h^* \, u = p_h \, r_h^* \, \Lambda^\theta u.$$

On the other hand, since $\| \omega_i \|_\theta = | \lambda_i^\theta \, \omega_i | = \lambda_i^\theta$ the basis $(\lambda_n^{-\theta} \, \omega_n)_n$ is orthonormal in $V_\theta$, and

$$(4.8)\qquad \begin{cases} p_h r_h^* u = \sum_{i=1}^{n} ((u, \lambda_i^{-\theta} \omega_i))_\theta \lambda_i^{-\theta} \omega_i \\ \text{is the continuous orthogonal projection} \\ \text{from } V_\theta \text{ onto } V_h^*. \end{cases}$$

More precisely, we have:

PROPOSITION 4.1. *Assume that* $\alpha \geq \beta$ *(so that* $V_\alpha \subset V_\beta$*) and* $h = \dfrac{1}{n}$. *Then :*

$$(4.9)\qquad \gamma_\alpha^\beta (p_h) = \sup_{\| u \|_\alpha \leq 1} \| u - p_h r_h^* u \|_\beta = \sup_{\| u \|_\alpha \leq 1} \inf_{v_h} \| u - p_h v_h \|_\beta = \frac{1}{\lambda_{n+1}^{\alpha-\beta}}$$

$$(4.10)\qquad \| p_h u_h \|_\alpha \leq \lambda_n^{\alpha-\beta} \| p_h u_h \|_\beta$$

$$(4.11)\qquad \| p_h r_h^* u \|_\alpha \leq \| u \|_\alpha \quad \text{and} \quad \lim_{n \to 0} \| u - p_h r_h^* u \|_\alpha = 0.$$

We have:

$$\begin{aligned} \| u - p_h r_h^* u \|_\beta^2 &= \sum_{i=n+1}^{\infty} \lambda_i^{2\beta} \, | (u, \omega_i) |^2 \\ &= \sum_{i=n+1}^{\infty} \lambda_i^{2(\beta-\alpha)} \, | ((u, \lambda_i^{-\alpha} \omega_i))_\alpha |^2 \\ &\leq \lambda_{n+1}^{2(\beta-\alpha)} \sum_{i=0}^{\infty} | ((u, \lambda_i^{-\alpha} \omega_i))_\alpha |^2 \\ &= \lambda_{n+1}^{2(\beta-\alpha)} \| u \|_\alpha^2 . \end{aligned}$$

Thus

$$\sup_{\| u \|_\alpha \leq 1} \inf_{v_h} \| u - p_h v_h \|_\beta \leq \sup_{\| u \|_\alpha \leq 1} \| u - p_h r_h^* u \|_\beta \leq \frac{1}{\lambda_{n+1}^{\alpha-\beta}} .$$

If we take $u_0 = \dfrac{\omega_{n+1}}{\lambda_{n+1}^{\alpha}}$ then $\| u_0 \|_\alpha = 1$, and

$$\inf_{v_h} \| u_0 - p_h v_h \|_\beta = \| u_0 \|_\beta = \frac{1}{\lambda_{n+1}^{\alpha-\beta}}$$

so that

$$\frac{1}{\lambda_{n+1}^{\alpha-\beta}} \leq \sup_{\| u \|_\alpha \leq 1} \inf_{v_h} \| u - p_h v_h \|_\beta .$$

This proves equation (4.9). To prove (4.10), let $u = p_h u_h$, so that

$$\| u \|_\alpha^2 = \sum_{i=1}^{n} \lambda_i^{2\alpha} \, | (u, \omega_i) |^2$$

$$= \sum_{i=1}^{n} \lambda_i^{2(\alpha-\beta)} \, | ((u, \lambda_i^{-\beta} \omega_i))_\beta |^2$$

$$\leq \lambda_n^{2(\alpha-\beta)} \sum_{i=1}^{n} | ((u, \lambda_i^{-\beta} \omega_i))_\beta |^2$$

$$\leq \lambda_n^{2(\alpha-\beta)} \| u \|_\beta^2 .$$

Finally, (4.11) follows immediately from (4.8).

Moreover, we will show that the self-adjoint Galerkin's method is « optimal ». For that, we will use the notion of $n$-width of Kolmogorov. (See [5], [6]).

We now introduce the set $Q_n$ of all injective operators $q_n$ from $R^n$ or $C^n$ into $V_\beta$. Define

$$\gamma_\alpha^\beta (q_n) = \sup_{\| u \|_\alpha \leq 1} \inf_{v_n \in R^n} \| u - q_n v_n \|_\beta . \tag{4.12}$$

(Thus $\gamma_\alpha^\beta (q_n)$ is the « distance » in the $\beta$-norm between the unit ball of $V_\alpha$ and $q_n R^n$). We now let $p_n = p_h$ with $h = \dfrac{1}{n}$, and we have the following theorem :

THEOREM 4.1. *If* $q_n \in Q_n$ *and* $\alpha \geq \beta$ *then*

$$\gamma_\alpha^\beta (p_n) = \frac{1}{\lambda_{n+1}^{\alpha-\beta}} \leq \gamma_\alpha^\beta (q_n). \tag{4.13}$$

Since $n = \dim q_n R^n < \dim p_{n+1} R^{n+1} = n + 1$ we have :

$$(q_n V_n)^\perp \cap V_{n+1}^* \neq 0 \tag{4.14}$$

where $V_{n+1}^* = p_{n+1} R^{n+1}$ and $W^\perp$ is the orthogonal subspace of $W$ in $V_\beta$. Choose $u_0 \in (q_n V_n)^\perp \cap V_{n+1}^*$, so then $\inf_{v_n \in R^n} \| u_0 - q_n v_n \|_\beta = \| u_0 \|_\beta$ because $u_0 \in (q_n V_n)^\perp$ and since $u_0 \in V_{n+1}^*$ by (4.10)

$$\| u_0 \|_\beta \geq \frac{1}{\lambda_{n+1}^{\alpha-\beta}} \| u_0 \|_\alpha .$$

Thus we have:

$$\gamma_\alpha^\beta(p_n) = \frac{1}{\lambda_{n+1}^{\alpha-\beta}} \leq \frac{\| u_0 \|_\beta}{\| u_0 \|_\alpha} = \frac{1}{\| u_0 \|_\alpha} \inf_{v_n \in R^n} \| u_0 - q_n v_n \|_\beta \leq \gamma_\alpha^\beta(q_n).$$

We now apply this theorem to the study of the error behavior. From Theorems 3.1, 3.2, and 4.1 we have the following:

COROLLARY 4.1. *Let $a(u, v)$ be a continuous coerciv sesquilinear form on $V \times V$. Assume that the solution $u$ of*

$$a(u, v) = (f, v) \qquad \text{for all } v \in V \tag{4.15}$$

*actually belongs to $D(A^\alpha)$ with $\alpha \geq 1/2$. Let $\widehat{u}_n \in R^n$ or $C^n$ be the solution of*

$$\sum_{j=1}^{n} a(\omega_j, \omega_i) \widehat{u}_n^j = (f, \omega_i) \qquad (1 \leq i \leq n). \tag{4.16}$$

*Then $\widehat{u}_n$ converges to $u$, and satisfies.*

$$\left\| u - \sum_{i=1}^{n} \widehat{u}_n^i \omega_i \right\| \leq Mc^{-1} \| u \|_\alpha \frac{1}{\sqrt{\lambda_{n+1}^{2\alpha-1}}}. \tag{4.17}$$

*If $\theta$ is a parameter with $0 < \theta \leq 1$, then the solution $\bar{u}_n$* of

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} a(\omega_j, \omega_i) \bar{u}_n^j + (a(\omega_i, \omega_i) + \lambda_{n+1}^{\theta(1-2\alpha)} \lambda_i^{2\alpha}) \bar{u}_u^i = (f, \omega_i) \qquad (1 \leq i \leq n) \tag{4.18}$$

*converges to $u$, and satisfies*

$$\left\| u - \sum_{i=1}^{n} \bar{u}_n^i \omega_i \right\| \leq c \| u \|_\alpha \lambda_{n+1}^{\theta\left(\frac{1}{2} - \alpha\right)}. \tag{4.19}$$

*Thus $\bar{u}_n$ converges to $u$ in $V_\alpha$ strongly if $\theta < 1$ and weakly if $\theta = 1$.*

Moreover, we know by Theorem 4.1 that the solutions of (4.16) are optimal in the following sense: for any $q_n \in Q_n$ the solutions $u_n$ of

$$a(q_n u_n, q_n v_n) = (f, q_n v_n) \qquad \text{for all } v_n \in R^n \tag{4.20}$$

satisfy the inequality

$$\| u - q_n v_n \| \leq Mc^{-1} \| u \|_\alpha \gamma_\alpha^\beta(q_n) \tag{4.21}$$

where

$$\gamma_\alpha^\beta(q_n) \geq \frac{1}{\sqrt{\lambda_{n+1}^{2\alpha-1}}}.$$

In the case where $V$ is a Sobolev space and $a(u, v)$ is an integrodifferential form, we will construct in the next section examples of other $p_h$ and $r_h$ operators. If they are not optimal, they yield matrices $A_h$ which contain a small number of non-zero elements, while the elements $a(\omega_i, \omega_j)$ of the matrices obtained by the self-adjoint Galerkin's method are à priori nonzero. On the other hand except in the case of the simplest examples we don't know explicitly the eigenfunctions $\omega_j$, and we can encounter difficulties in the computation of the elements $a(\omega_i, \omega_j)$. (3)

## § 5. Construction of the $p_h$ and $r_h$ operators in Sobolev space.

We first study the construction of the $p_h$ and $r_h$ operators in the Sobolev space $H^m(R)$. For that, we will use the convolution powers of characteristic functions « which map derivative operators into finite difference operators ».

5.1. *Convolution powers of characteristic functions.*

Let $\chi_h(x) = \frac{1}{h}\chi\left(\frac{x}{h}\right)$ be the homothetic of the characteristic function of $(0, 1)$ defined by

$$\text{(5.1)} \quad \begin{cases} \chi_h(x) = \begin{cases} \dfrac{1}{h} & \text{if } 0 \leq \dfrac{x}{h} \leq 1 \\ 0 & \text{otherwise} \end{cases} \\ \chi_h^\alpha(x) = \begin{cases} \dfrac{1}{h} & \text{if } \alpha h \leq x \leq (\alpha+1)h \\ 0 & \text{otherwise.} \end{cases} \end{cases}$$

---

(3) We will only give a simple example.

Let

$$\Omega = (0, \pi), \qquad H = L^2(\Omega), \qquad \Lambda = -D^2 + 1.$$

Then

$$V_{s/2} = H^s(\Omega) \text{ and } V_{m+1/2} = H^{m+1}(\Omega),$$

$$\omega_{2j-1} = \omega_{2j} = \cos jx \text{ and } \lambda_j = 1 + j^2 - (1 \leq j).$$

If $n = 2j - 1$ or $n = 2j$, then $\gamma_{m+1/2}^{s/2}(q_n) = (\sqrt{1 + j^2})^{s-m-1}$.

Letting $*$ denote the convolution product and $\delta(x)$ the Dirac measure at $x$, we note that the derivative of $\chi_h$ is

$$
(5.2)\qquad \begin{cases} D\chi_h = \dfrac{\delta - \delta(h)}{h} \\[2ex] V_h \Phi = D\chi_h * \Phi = \dfrac{\Phi(x) - \Phi(x-h)}{h}. \end{cases}
$$

Then we introduce the following notations:

$$
(5.3)\qquad \begin{cases} \text{(i)}\quad \sigma_{m,h}(x) = \sqrt{h}\,\underbrace{\chi_h * \chi_h * \dots * \chi_h}_{(m+1)\text{ times}}(x) \\[2ex] \qquad = \left(\dfrac{1}{\sqrt{h}}\right)\chi_{m+1}\left(\dfrac{x}{h}\right) \\[2ex] \qquad = \sqrt{h}\,\chi_{m+1,h}(x) \\[2ex] \text{(ii)}\quad \sigma^{\alpha}_{m,h}(x) = \sigma_{m,h}(x - \alpha h) = \sqrt{h}\,\chi_{m,h} * \chi^{\alpha}_{h}(x). \end{cases}
$$

We then see that

$$
(5.4)\qquad \begin{cases} D^m \chi_{m,h} = h^{-m} \sum\limits_{k \le m} (-1)^k c^k_m\, \delta(kh) \\[2ex] V^m_h = D^m \chi_{m,h} * . \end{cases}
$$

After computation we obtain the following results (for $h = 1$, $\alpha = 0$):

$$
(5.5)\qquad \begin{cases} \text{(i)}\quad \sigma_m(x) = \sum\limits_{k \le m} \delta(k) * (\psi^k_m(x)\,\chi(x)) \quad \text{where} \\[2ex] \text{(ii)}\quad \psi^k_m(x) = \sum\limits_{q \le k} a_m(k,q)\dfrac{x^q}{q!} \quad \text{is a polynomial of degree } m \text{ with} \\[2ex] \text{(iii)}\quad a_m(k,q) = \sum\limits_{i \le k} (-1)^i c^i_{m+1} \dfrac{(k-i)^{m-q}}{(m-q)!}. \end{cases}
$$

5.2. *The spaces $H^m(R, h)$.*

We consider the space $l^2(Z)$ of sequences $u_h = (u^{\alpha}_h)_{\alpha}$ that satisfy:

$$
(5.6)\qquad |u_h| = \Big(\sum_{\alpha} |u^{\alpha}_h|^2\Big)^{1/2} < +\infty.
$$

We define finite difference operators on sequences by

(5.7) $$(\nabla_h^q u_h)^\alpha = h^{-q} \sum_{k \leq q} (-1)^q c_q^k u_h^{\alpha-k} .$$

We will denote by $H^m(R, h)$ the space $l^2(Z)$ with the (Sobolev discrete) norm :

(5.8) $$\begin{cases} \| u_h \|_{m,h} = \Big( \sum_{|q| \leq m} | \nabla_h^q u_h |^2 \Big)^{1/2} \\ \| u_h \|_{0,h} = | u_h | . \end{cases}$$

We then have :

(5.9) $$| u_h | \leq \| u_h \|_{m,h} \leq c h^{-m} | u_h | .$$

5.3. *The operators* $p_h^m$.

We now define an operator $p_h^m$ from $l^2(Z)$ into the Sobolev space $H^q(R)$ for $0 \leq q \leq m$ :

(5.10) $$\begin{cases} p_h^m u_h(x) = \sum_\alpha u_h^\alpha \sigma_{m,h}^\alpha(x) \\ \qquad = \sqrt{h} \sum_\alpha \chi_h^\alpha(x) \sum_{k \leq m} u_h^{\alpha-k} \psi_m^k \left( \frac{x}{h} - \alpha \right) \\ \qquad = \sqrt{h} \sum_\alpha \chi_h^\alpha(x) \sum_{q \leq m} (Q_m^q u_h)^\alpha \dfrac{\left( \frac{x}{h} - \alpha \right)^q}{q!} \end{cases}$$

where we have (4)

(5.11) $$\begin{cases} (Q_m^q u_h)^\alpha = \sum_{k \leq m} a_m(k, q) u_h^{\alpha-k} \\ Q_m^m = h^m \nabla_h^m . \end{cases}$$

5.4. *Construction of the* $r_h$ *operators.*

Let $\varrho(x)$ be a function with compact support such that

(5.12) $$\begin{cases} \varrho(x) \in L^\infty(R) \\ \int_R \varrho(x)\, dx = 1. \end{cases}$$

---

(4) On each $(\alpha h, (\alpha + 1) h)$, the function $p_h^m u_h$ is a polynomial of degree $m$. In other words, the $p_h^m$ are « Spline-functions » of degree $m$, introduced by Schoënberg (See for instance [12]).

We then set

$$(5.13) \qquad \varrho_h^\alpha(x) = \frac{1}{h}\,\varrho\left(\frac{x}{h} - \alpha\right)$$

and we define the $r_h$ operators on the spaces $H^m(R)$ for $m \geq 0$ by:

$$(5.14) \qquad \begin{cases} (r_h\, \Phi)^\alpha = \sqrt{h} \displaystyle\int_R \Phi(x)\, \varrho_h^\alpha(x)\, dx \\[2ex] \qquad\quad = \sqrt{h} \displaystyle\int_R \Phi(xh + \alpha h)\, \varrho(x)\, dx. \end{cases}$$

(More generally, we can assume that $\varrho$ is a distribution on $H^m(R)$ with compact support such that $(\varrho, 1) = 1$; for example, if $m \geq 1$ we can take $\varrho$ to be the Dirac measure.)

In the next section we will give examples of functions $\varrho$ to obtain a precise estimate on the truncation error $u - p_h^m r_h u$. In particular, if we take $\varrho = \sigma_m = \chi_{m+1}$ we will set $r_h = r_h^{*m}$ since in this case we have:

$$(5.15) \qquad (p_h^m u_h, \Phi) = (u_h, r_h^{*m}\, \Phi)_h$$

where

$$(u_h, v_h)_h = \sum_\alpha u_h^\alpha\, \overline{v}_h^\alpha$$

and

$$(u, \Phi) = \int_R u(x)\, \overline{\Phi}(x)\, dx.$$

THEOREM 5.1. *The following relations between the operators $D^q$, $\nabla_h^q$, $p_h^m$, and $r_h$ are valid for $q \leq m$:*

$$(5.16) \qquad \begin{cases} \text{(i)} \quad D^q\, p_h^m u_h = p_h^{m-q}\, \nabla_h^q u_h \\[1ex] \text{(ii)} \quad \nabla_h^q\, r_h\, \Phi = r_h\, \nabla_h^q\, \Phi = r_h(\chi_{q,h} * D^q\, \Phi). \end{cases}$$

*These then imply that:*

$$(5.17) \qquad \begin{cases} \text{(i)} \quad \| p_h^m u_h \|_q \leq \| u_h \|_{q,h} \leq c\, \| p_h^m u_h \|_q \\[1ex] \text{(ii)} \quad \| r_h\, \Phi \|_{q,h} \leq c\, \| \Phi \|_q \end{cases}$$

*and also that*

$$(5.18) \qquad \lim_{h \to 0} \| \Phi - p_h^m r_h\, \Phi \|_q = 0.$$

*Finally, there is a restriction* $r_h^{-m}$ *such that*

$$(5.19)\qquad \begin{cases} r_h^{-m}\, p_h^m\, u_h = u_h \quad \text{and} \\ p_h^m\, r_h^{-m} \text{ is a continuous projection on } H^q(R). \end{cases}$$

We first note that the translation operators commute with the $p_h$ and $r_h$ operators :

$$\tau_{\beta h}\, p_h^m\, u_h(x) = \sum_\alpha u_h^\alpha\, \sigma_{m,h}^{\alpha+\beta}(x)$$

$$= \sum_\alpha u_h^{\alpha-\beta}\, \sigma_{m,h}^\alpha(x)$$

$$= p_h^m\, \tau_\beta\, u_h(x)$$

$$(\tau_\beta\, r_h\, \Phi)^\alpha = \sqrt{h}\int_R \Phi(x)\, \varrho_h^{\alpha-\beta}(x)\, dx$$

$$= \sqrt{h}\int_R \tau_{\beta h}\, \Phi(x)\, \varrho_h^\alpha(x)\, dx$$

$$= (r_h\, \tau_{\beta h}\, \Phi)^\alpha .$$

Thus the finite difference operators commute with the $p_h$ and $r_h$ operators :

$$\nabla_h^q\, p_h^k\, u_h = p_h^k\, \nabla_h^q\, u_h$$

and

$$\nabla_h^q\, r_h\, \Phi = r_h\, \nabla_h^q\, \Phi.$$

Then since $D^q\, \chi_{m,h} = \nabla_h^q\, \chi_{m-q,h}$ we set $k = m - q$ to obtain

$$D^q\, p_h^m\, u_h = \nabla_h^q\, p_h^{m-q}\, u_h = p_h^{m-q}\, \nabla_h^q\, u_h$$

and this proves (5.16).

Now notice that

$$p_h^m\, u_h = \chi_{m,h} * p_h^0\, u_h$$

and

$$p_h^0\, u_h(x) = \sqrt{h}\sum_\alpha u_h^\alpha\, \chi_h^\alpha(x).$$

But it is clear that

$$(5.20)\qquad | p_h^0 u_h | = | u_h |$$

since

$$\int_R | p_h^0 u_h(x) |^2 \, dx = h \int_R \sum_\alpha | u_h^\alpha |^2 \, | \chi_h^\alpha(x) |^2 \, dx = \sum_\alpha | u_h^\alpha |^2 .$$

Then since $\int_R \chi_{q,h}(x)\, dx = 1$ we obtain the inequality

$$(5.21)\qquad \begin{aligned} | D^q p_h^m u_h | &= | \chi_{m-q,h} * p_h^0 \nabla_h^q u_h | \\ &\leq | p_h^0 \nabla_h^q u_h | \\ &= | \nabla_h^q u_h | . \end{aligned}$$

This establishes the first inequality of (5.17i).

Similarly, we see that there exists a constant $c$ such that

$$(5.22)\qquad | r_h \Phi | \leq c \, | \Phi | .$$

Indeed, using the Cauchy-Schwarz inequality we have:

$$\sum_\alpha \left| \int_R \Phi(x) \, \varrho_h^\alpha(x) \, dx \right|^2 \leq \sum_\alpha \int_R | \Phi(x) |^2 \, | \varrho_h^\alpha(x) | \, dx \left( \int_R | \varrho_h^\alpha(x) | \, dx \right)$$

$$\leq c_1 \, c_2 \, | \Phi |^2$$

where $c_1 = \int_R | \varrho_h^\alpha(x) | \, dx$ and $c_2 = \sup_x \sum_\alpha \varrho(x-\alpha)$. Then (5.22) implies (5.17ii) with the use of (5.16ii).

We have to now prove (5.18). Since by (5.17) we have

$$\| p_h^m r_h \Phi \|_q \leq c \, \| \Phi \|_q$$

it suffices to prove (5.18) for infinitely differentiable functions with compact support (which are dense in Sobolev spaces).

On the other hand, by (5.16) we have:

$$(5.23)\qquad \begin{aligned} D^q p_h^m r_h \Phi - D^q \Phi &= p_h^{m-q} r_h (\chi_{q,h} * D^q \Phi) - D^q \Phi = \\ &= p_h^{m-q} s_h (D^q \Phi) - D^q \Phi \end{aligned}$$

where $s_h = \varrho_h * \check{\chi}_{q,h}$. It thus suffices to prove that $p_h^k r_h \Phi$ converges to $\Phi$ in $L^2(R)$. But

$$p_h^k r_h \Phi - \Phi = \chi_{k,h} * (p_h^0 r_h \Phi - \Phi) + (\chi_{k,h} * \Phi - \Phi)$$

and it is well known that $\chi_{k,h} * \Phi$ converges to $\Phi$ in $L^2(R)$, and that $p_h^0 r_h \Phi$ converges to $\Phi$ in $L^2(R)$.

Finally, there exists a function $\varrho_m(x)$ such that

$$(5.24) \qquad (\sigma_m^{\alpha-\beta}, \varrho_m) = (\sigma_{m,h}^{\alpha}, \sqrt{h}\, \varrho_{m,h}^{\beta}) = \begin{cases} 0 & \text{if} \quad \alpha \neq \beta \\ 1 & \text{if} \quad \alpha = \beta. \end{cases}$$

For instance, we can try to find $\varrho_m(x)$ in the form

$$(5.25) \qquad \varrho_m(x) = \chi(x) \left( \sum_{q=0}^{m} \varrho_m^q \frac{x^q}{q!} \right).$$

Noting that the polynomials $\dfrac{x^q}{q!}$ and $\psi_m^k$ are linearly independent for $0 \leq$ $\leq q \leq m$ and $0 \leq k \leq m$, (5.24) leads to a Carmer's system for determining the coefficients $\varrho_m^q$.

Then, since $\sum_{\alpha} \sigma_m^{\alpha} = 1$, we have

$$(5.26) \qquad \int_R \varrho_m(x)\, dx = \sum_{\alpha} \int_R \varrho_m(x)\, \sigma_m^{\alpha}(x)\, dx$$

$$= \int_R \varrho_m(x)\, \sigma_m^0(x)\, dx$$

$$= 1.$$

If we denote by $r_h^{-m}$ the operator defined by $\varrho_m$ in (5.14), equation (5.24) implies that :

$$(5.27) \qquad r_h^{-m} p_h^m u_h = u_h.$$

It follows from (5.27) that there is a constant $c$, independent of $h$, such that:

$$\| u_h \|_{q,h} \leq c \, \| p_h^m u_h \|_q \qquad \text{for all } q \leq m.$$

Thus the norms $\| u_h \|_{q,h}$ and $\| p_h^m u_h \|_q$ are equivalent « globally in $h$ ».

## § 6. Behavior of a truncation error $u - p_h^m r_h^m u$ in Sobolev spaces.

In this section we will construct operators $r_h^m$ for which we can obtain estimates of the truncation error.

THEOREM 6.1. *There exists a function $\varrho_m(x)$ which compact support and $\int_R \varrho_m(x)\,dx = 1$ such that if $r_h^m$ is defined by*

$$(6.1) \qquad (r_h^m \Phi)^\alpha = h^{+1/2} \int_R \Phi(x)\, \varrho_{m,h}^\alpha(x)\,dx$$

*then if $\Phi \in H^{m+1}(R)$ whe have*

$$(6.2) \qquad \| D^j (p_h^m r_h^m \Phi - \Phi) \| \leq c h^{s-j} | D^s \Phi | \quad \text{if } j \leq s \leq m+1.$$

Let us denote by $\varrho_{m,j}$ the $j$th moment of the function $\varrho_m$:

$$(6.3) \qquad \varrho_{m,j} = \int_R \varrho_m(x) \frac{x^j}{j!}\,dx \,;\ \varrho_m^0 = \int_R \varrho_m(x)\,dx = 1.$$

Then the truncation error $p_h^m r_h^m \Phi - \Phi$ can be written in the form

$$(6.4) \qquad p_h^m r_h^m \Phi - \Phi = \sum_\alpha \theta_h^\alpha \tau_{m,h}^\alpha(\Phi)$$

where:

$$(6.5) \qquad \tau_{m,h}^\alpha(\Phi) = \sum_{k=0}^{m} (\Phi, \varrho_{m,h}^{\alpha-k}) \psi_m^k \left(\frac{x}{h} - \alpha\right) - \Phi(x).$$

Let us assume that $\Phi$ is infinitely differentiable with compact support. Using the Taylor expansion formula

$$(6.6) \qquad \Phi(x) = \sum_{q=0}^{s} h^q \frac{D^q \Phi(\alpha h)}{q!} \left(\frac{x}{h} - \alpha\right)^q + h^{s+1} \omega_{\alpha h}^{s+1}(D^{s+1}\Phi)$$

where

$$(6.7) \qquad \omega_{\alpha h}^{s+1}(\Phi) = \frac{1}{s!\,h^{s+1}} \int_{\alpha h}^{x} (x-t)^s \Phi(t)\,dt.$$

We obtain the formula

$$(6.8) \qquad \tau_{m,h}^\alpha(\Phi) = \sum_{q=1}^{s} h^q D^q \Phi(\alpha h) p_m^q \left(\frac{x}{h} - \alpha\right) + h^{s+1} \tau_{m,h}^\alpha \omega_{\alpha h}^{s+1}(D^{s+1}\Phi)$$

where

$$
(6.9)\qquad \begin{cases} p^q_m(x) = \dfrac{x^q}{q!} - L^q_m(x) \\[2mm] L^q_m(x) = \sum\limits_{j=0}^{q} \varrho_{m,\,q-j} \sum\limits_{k=0}^{m} \dfrac{(-k)^j}{j\,!}\, \psi^k_m(x). \end{cases}
$$

LEMMA 6.1. *For* $0 \leq q \leq m$ *the polynomials* $B^q_m(x) = \sum\limits_{k=0}^{q} \dfrac{(-k)^q}{q\,!} \psi^k_m(x)$ *have the following properties*:

$$
(6.10)\qquad \begin{cases} B^q_m(x) = \sum\limits_{j=0}^{q} b_m(q-j,0)\dfrac{x^j}{j\,!} \text{ where} \\[2mm] b_m(q,j) = \sum\limits_{k=0}^{m} \dfrac{(-k)^q}{q\,!}\, a_m(k,j) \\[2mm] \qquad\quad = D^j B^q_m(0) \\[2mm] \qquad\quad = b_m(q-j,0) \ \ \textit{if} \ \ 0 \leq j \leq q \leq m \\[2mm] b_m(0,0) = 1 \\[2mm] b_m(j,0) = 0 \ \ \textit{if} \ \ j < 0. \end{cases}
$$

This lemma is true for $m = 0$. Let us assume that it is true for $m-1$, and the proof for $m$ will follow from the recursion formula

$$
(6.11)\qquad \psi^k_m(x) = \int_0^x (\psi^k_{m-1}(t) - \psi^{k-1}_{m-1}(t))\, dt + \int_0^1 \psi^{k-1}_{m-1}(t)\, dt.
$$

This formula is obtained by noting that

$$
(6.12)\qquad \begin{cases} \chi_{m+1} = \chi * \chi_m \\ \psi^k_m = 0 \ \text{ if } \ k < 0 \ \text{ or } \ k > m. \end{cases}
$$

Since $a_m(k,0) = \psi^k_m(0) = \int_0^1 \psi^{k-1}_{m-1}(t)\, dt$ we have

$$
(6.13)\qquad a_m(k,0) = \sum_{j=0}^{m-1} \frac{a_{m-1}(k-1,j)}{(j+1)\,!}
$$

Then, since $b_m(q,j) = \sum\limits_{k=0}^{m} \dfrac{(-k)^q}{q!} a_m(k,j)$ we obtain

$$(6.14) \qquad b_m(q,0) = \sum_{j=0}^{m-1} \frac{1}{(j+1)!} \sum_{p=0}^{q} \frac{(-1)^{q-p}}{(q-p)!} b_{m-1}(p,j).$$

By the inductive assumption

$$b_{m-1}(q,j) = b_{m-1}(q-j,0) \text{ if } 0 \leq q \leq j \leq m-1.$$

In this case, it follows from (6.14) that

$$(6.15) \qquad \begin{cases} b_m(q,0) = -\sum\limits_{k=0}^{q} b_{m-1}(k,0) \dfrac{(-1)^{q-k+1}}{(q-k+1)!} \left( \sum\limits_{j=0}^{m-1} c_{q-k+1}^{j+1} \right) \\[2ex] \quad = \sum\limits_{k=0}^{q} \dfrac{(-1)^{q-k+1}}{(q-k+1)!} b_{m-1}(k,0) \text{ if } 0 \leq q \leq m-1. \end{cases}$$

On the other hand, if $1 \leq j \leq m$ we have from (6.11) the equation:

$$a_m(k,j) = D^j \psi_m^k(0) = D^{j-1} \psi_{m-1}^k(0) - D^{j-1} \psi_{m-1}^{k-1}(0)$$

$$= a_{m-1}(k,j-1) - a_{m-1}(k-1,j-1).$$

Then, since $b_{m-1}(p,j-1) = b_{m-1}(p-j+1,0)$ if $0 \leq p \leq m-1$ and $0 \leq j-1 \leq m-1$, we have:

$$(1.16) \qquad \begin{cases} b_m(r,j) = -\sum\limits_{p=0}^{r-1} \dfrac{(-1)^{r-p}}{(r-p)!} b_{m-1}(p,j-1) = \\[2ex] = \sum\limits_{k=0}^{r-j} \dfrac{(-1)^{r-j-k+1}}{(r-j-k+1)!} b_{m-1}(k,0) \text{ if } 1 \leq j \leq m. \end{cases}$$

Taking $0 \leq q = r-j \leq m-1$ if $0 \leq r \leq m$ and $1 \leq j \leq m$, it follows from (6.15) and (6.16) that

$$(6.17) \qquad b_m(r,j) = b_m(r-j,0) \text{ for } 0 \leq r \leq m, 0 \leq j \leq m.$$

Moreover, (6.15) provides a recursive formula between the $b_m(q,0)$ and $b_{m-1}(q,0)$ for $0 \leq q \leq m-1$.

Now defime the numbers $\varrho_{m,j}\,(0 \leq j \leq m)$ by the following recursive formula :

$$
(6.18) \qquad \begin{cases} \varrho_{m,0} = 1 \\ \sum\limits_{j=0}^{q} \varrho_{m,q-j}\, b_m(j,0) = 0 \ \text{ for } 1 \leq q \leq m. \end{cases}
$$

Then we obtain from Lemma 6.1 :

$$
(6.19) \qquad L_m^q(x) = \sum_{j=0}^{q} \varrho_{m,q-1}\, B_m^j(x) = \frac{x^q}{q\,!} \ \text{ for } 0 \leq q \leq m
$$

and the polinomials $p_m^q(x)$ of (6.9) are identically zero for $0 \leq q \leq m$. Notice also that there exists the following recursive relation between the moments $\varrho_{m,j}$ and $\varrho_{m-1,j}$ ;

$$
(6.20) \qquad \varrho_{m-1,q} = \sum_{k=0}^{q} \varrho_{m,q-1} \frac{(-1)^{j+1}}{(j+1)\,!} \,.
$$

Let $\varrho_m(x)$ be a function with compact support such that

$$
(6.21) \qquad \begin{cases} \int\limits_{R} \varrho_m(x) \frac{x^j}{j\,!}\, dx = \varrho_{m,j} \quad (0 \leq j \leq m) \\ \text{where } \varrho_{m,j} \text{ is defined by (6.18)} \end{cases}
$$

and denote by $r_h^m$ the operator defined by $\varrho_m(x)$ in (6.1). Then, by (6.8) and (6.19), if $s \leq m$ we have

$$
(6.22) \qquad \tau_{m,h}^{a}(\Phi) = h^{s+1}\, \tau_{m,h}^{a}\, \omega_{ah}^{s+1}(D^{s+1}\Phi).
$$

Let $\varrho_{m-r}(x) = -\varrho_m(x) * (-1)^r \chi_r(-x) = \varrho_m(x) * \check{\chi}_{r}(x)$. Then, by (6.20), the moments of $\varrho_{m-r}$ are the numbers $\varrho_{m-r,q}$ , since the $j$th moment of $\check{\chi}(x) = -\chi(-x)$ is equal to $\dfrac{(-1)^{j+1}}{(j+1)\,!}$ .

Then we have the following relation :

$$
(6.23) \qquad D^r \tau_{m,h}^{a}(\Phi) = h^{s+1-r}\, \tau_{m-r,h}^{a}\, \omega_{ah}^{s+1-r}(D^{s+1}\Phi).
$$

Indeed, it is clear from (6.7) that if $r \leq s$ we have

$$
(6.24) \qquad D^r \omega_{ah}^{s+1}(\Phi) = h^{-r}\, \omega_{ah}^{s+1-r}(\Phi).
$$

On the other hand, it follows from (6.11) that:

$$(6.25)\quad \left\{\begin{aligned} & D^r\left(\sum_{k=0}^{m}(\varrho_m^{\alpha-k},\Phi)\,\psi_m^k\left(\frac{x}{h}-\alpha\right)\right)= \\ & = h^{-r}\sum_{k=0}^{m}(\varrho_m^{\alpha-k},\Phi)\left(\sum_{p=0}^{r}(-1)^{\nu}\,C_r^p\,\psi_{m-r}^{k-p}\left(\frac{x}{h}-\alpha\right)\right) \\ & = \sum_{k=0}^{m-r}(\chi_{r,h}*\check{\varrho}_m^{\alpha-k},\,D^r\,\Phi)\,\psi_{m-r}^k\left(\frac{x}{h}-\alpha\right) \\ & = \sum_{k=0}^{m-r}(\varrho_{m-r,h}^{\alpha-k},\,D^r\,\Phi)\,\psi_{m-r}^k\left(\frac{x}{h}-\alpha\right). \end{aligned}\right.$$

Then (6.23) follows from (6.24) and (6.25).

Theorem 6.1 is now a consequence of the following:

LEMMA 6.2.

$$(6.26)\qquad \left|\sum_{\alpha}\theta_h^{\alpha}\,\tau_{m,h}^{\alpha}\,\omega_{\alpha h}^{s}(\Phi)\right|_{L^2}\leq c\,|\,\Phi\,|_{L^2}.$$

We note first that

$$(6.27)\qquad |\,\omega_{\alpha h}^{s+1}(\Phi)(x)\,|^2\leq\frac{c}{h}\left(\frac{x}{h}-\alpha\right)^{2s+1}\int_{\alpha h}^{x}|\,\Phi(x)\,|^2\,dx.$$

On the other hand, since $\int_R \varrho_{m,h}(x)\,dx=1$, we deduce from

$$(6.28)\quad \tau_{m,h}^{\alpha}\,\omega_{\alpha h}^{s+1}(\Phi)(x)=\sum_{k=0}^{m}\psi_m^k\left(\frac{x}{h}-\alpha\right)\left[\int_R\varrho_{m,h}^{\alpha-k}(y)\,(\omega_{\alpha h}^{s+1}(\Phi)(x)-\omega_{\alpha h}^{s}(\Phi)(y))\,dy\right]$$

the following inequality:

$$(6.29)\qquad |\,\theta_h^{\alpha}\,\tau_{m,h}^{\alpha}\,\omega_{\alpha h}^{s+1}(\Phi)(x)\,|^2\leq C\sup_{(a'+\alpha)h\leq x\leq(b'+\alpha)h}|\,\omega_{\alpha h}^{s+1}(\Phi)(x)\,|^2$$

where $(a,b)$ is the support of $\varrho_m(x)$, $a'=\min(1,a-m)$, and $b'=\max(2,b)$. Then integrate (6.29) with respect to $x$ using (6.27), and sum over $\alpha\in Z$ to obtain (6.26).

## § 7. Approximations of the Sobolev spaces $H^m(\Omega)$.

In this section we will extend the results of Sections 5 and 6 to general Sobolev spaces on a bounded open subset $\Omega$ of $R^n$. We first consider the case where $\Omega = R^n$, and we replace $Z$ by $Z^n$. We will introduce the following notations:

$$x = (x_1, \dots, x_n), \quad h = (h_1, \dots, h_n) \text{ or } h = h_1 h_2 \dots h_n,$$

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \sum_{i=1}^{n} |\alpha_i|, \quad m = (m_1, \dots, m_n),$$

$$(m) = (m, \dots, m), \quad \alpha! = \alpha_1! \dots \alpha_n!,$$

$$h^\alpha = h_1^{\alpha_1} h_2^{\alpha_2} \dots h_n^{\alpha_n},$$

$$\alpha \le \beta \text{ if } \alpha_i \le \beta_i \text{ for } 1 \le i \le n, \text{etc.}$$

Then we will take

$$(7.1) \quad \begin{cases} \varrho_m(x) = \varrho_m(x_1, \dots, x_n) = \varrho_{m_1}(x_1)\, \varrho_{m_2}(x_2) \dots \varrho_{m_n}(x_n) \\ \psi_m^k(x) = \psi_{m_1}^{k_1}(x_1) \dots \psi_{m_n}^{k_n}(x_n) \\ \chi_m(x) = \chi_{m_1}(x_1) \dots \chi_{m_n}(x_n) \\ \varrho_h^\alpha(x) = \dfrac{1}{h}\, \varrho\left(\dfrac{x}{h} - \alpha\right) = \dfrac{1}{h_1 \dots h_n}\, \varrho\left(\dfrac{x_1}{h_1} - \alpha_1, \dots, \dfrac{x_n}{h_n} - \alpha_n\right). \end{cases}$$

Let $m$ be a multi-integer, and define:

$$(7.2) \quad \begin{cases} \text{(i) } P_h^m u_h = \sqrt{h}\, \chi_{m,h} \sum\limits_\alpha u_h^\alpha\, \theta_h^\alpha \\ \text{(ii) } (r_h^m \Phi)^\alpha = \sqrt{h}\, (\varrho_{m,h}^\alpha, \Phi) \text{ where} \\ \qquad (\Phi, \psi) = \displaystyle\int \Phi(x)\, \overline{\psi}(x)\, dx. \end{cases}$$

Then Theorems 5.1 and 6.1 remain valid, since the space of functions $\Phi(x) = \Phi_1(x_1) \dots \Phi_n(x_n)$ where each $\Phi_j$ is infinitely differentiable with compact support is dense in the Sobolev space $H^m(R^n)$.

Consider now a bounded open subset $\Omega$ of $R^n$ that is smooth enough to satisfy the following property:

$$(7.3)\qquad \begin{cases} \text{there exists a continuous linear operator } \pi \text{ mapping} \\ H^s(\Omega) \text{ into } H^s(R^n) \text{ for all } s \leq m \text{ such that} \\ \qquad\qquad \pi\Phi|_\Omega = \Phi \text{ in } H^m(\Omega). \end{cases}$$

Let $h = (h_1, .., h_n)$, and define

$$(7.4)\qquad R_h^m(\Omega) = \{\alpha \in Z^n \mid (\text{support } \sigma_{m,h}^\alpha) \cap \Omega \neq \Phi\}.$$

Let $N_\Omega^m(h)$ be the number of elements in $R_h^m(\Omega)$. Since $\Omega$ is bounded, the $N_\Omega^m(h)$ functions $\sigma_{m,h}^\alpha(x)$ are linearly independent on $\Omega$. Thus the restriction to $\Omega$ of

$$(7.5)\qquad \begin{cases} p_h^m u_h = \sqrt{h} \displaystyle\sum_{\alpha \in R_h^m(\Omega)} u_h^\alpha \chi_{m,h} * \theta_h^\alpha \\ \qquad\;\; = \sqrt{h} \displaystyle\sum_\alpha u_h^\alpha \sigma_{m,h}^\alpha \end{cases}$$

defines an injective operator mapping the space $R^{N_\Omega^m(h)}$ of sequences $u_h = (u_h^\alpha)_{\alpha \in R_h^m(\Omega)}$ on $R_h^m(\Omega)$ into the space $L^2(\Omega)$. More precisely, if $s$ is an integer such that $s \leq m$ we will put

$$(7.6)\qquad \begin{cases} H^s(R_h^m(\Omega)) = \text{the space of sequences } u_h \text{ on } R_h^m(\Omega) \\ \text{with the norm } \| p_h^m u_h \|_{H^s(\Omega)}. \end{cases}$$

Then $p_h^m$ is an isometry from $H^s(R_h^m(\Omega))$ into $H^s(\Omega)$. If $\varrho_m(x)$ is a function provided by Theorem 6.1, then the restriction to $R_h^m(\Omega)$ of the operator $r_h^m\pi$ is a continuous operator mapping $H^s(\Omega)$ into $H^s(R_h^m(\Omega))$. We deduce from Theorems 5.1 and 6.1 the following:

THEOREM 7.1. *If $(s) \leq m$, the operators $p_h^m$ are isometries from $H^s(R_h^m(\Omega))$ into $H^s(\Omega)$ that satisfy:*

$$(7.7)\qquad D^q p_h^m u_h = p_h^{m-q} \nabla_h^q u_h \quad \textit{for} \quad q \leq m$$

and

$$(7.8)\quad \begin{cases} \text{(i)} \lim\limits_{h\to 0} \| p_h^m r_h^m \pi\Phi - \Phi \|_{H^s(\Omega)} = 0 \\ \text{(ii) } if\ m = (m) = (m, m, \dots, m)\ then \\ \qquad \| p_h^m r_h^m \pi\Phi - \Phi \|_{H^s(\Omega)} \le c h^{m-s+1} \| \Phi \|_{H^{m+1}(\Omega)} \cdot (5), (6) \end{cases}$$

The assumptions of Theorems 3.1 and 3.2 are then satisfied, so now we have to fulfill the hypoteses (3.33) end (3.34) of Theorem 3.3.

Let $V = H^s(\Omega)$, and if $q = (q_1, \dots, q_n)$ is a multi-integer such that $|q| \le s$, let $V_q$ be the partial Sobolev space $H^q(\Omega)$ consiting of the functions $u \in L^2(\Omega)$ such that $D^q u \in L^2(\Omega)$. Then it is clear that:

$$(7.9)\qquad H^s(\Omega) = \bigcap_{|q| \le s} H^q(\Omega)$$

if $H^q(\Omega)$ is supplied with the norm

$$(7.10)\qquad \| u \|_{H^q(\Omega)} = (|u|^2 + |D^q u|^2)^{1/2}.$$

---

(5) Consider the example where $\Omega = (0, \pi)$. Let $n$ be an integer and let us put $h = \dfrac{\pi}{n - m - 1}$. Then $V_h = H^s(R_h^m(\Omega))$ is a space of dimension $n$. By (7.8), we have

$$\gamma_{m+1/2}^{s/2}(p_h^m) = \sup_{\|u\|_{m+1}} \| u - p_h^m \pi_h^m \|_s \le c \left( \frac{\pi}{n - m - 1} \right)^{m+1-s}.$$

By the foot-note (3), if $q_n$ denotes the optimal approximation, we have

$$\frac{\sqrt{2}}{\sqrt{(n+3)^2 + 2}} \le \gamma_{m+1/2}^{s/2}(q_n) \le \frac{\sqrt{2}}{\sqrt{(n+1)^2 + 2}}.$$

Then, comparing this two estimates, we see that there exists a constant $c$ such that:

$$\lim_{h \to 0} \gamma_{m+1/2}^{s/2}(p_h^m) / \gamma_{m+1/2}^{1/2}(q_n) \le c$$

In other words, we will say that the $p_h^m$ are «*almost optimal*» *in* $H^m(\Omega)$.

(6) If we assume moreover that the derivative $D_1^{m_1+1} D_2^{m_2+1} \dots D_n^{m_n+1} u$ of $u$ belongs to $L^2(\Omega)$, we have the stronger estimate

$$| D^k (u - p_h^m r_h^m u) | \le c h^{n + |m| - |k|} | D_1^{m_1+1} \dots D_n^{m_n+1} u |.$$

Let $\vec{m} = (m_q)_{|q| \leq s}$ be a sequence of multi-integers $m_q \geq q$. We define

$$R_h^{\vec{m}}(\Omega) = \bigcup_{|q| \leq s} R_h^{m_q}(\Omega). \tag{7.11}$$

Then the restriction to $\Omega$ of $p_h^{m_q} u_h$ defines a (non-injective) operator mapping the space of sequences on $R_h^{\vec{m}}(\Omega)$ into $L^2(\Omega)$. It is clear that

$$\| u_h \|_{s, \vec{m}, h} = \Big( \sum_{|q| \leq s} \| p_h^{m_q} u_h \|^2_{H^q(\Omega)} \Big)^{1/2} \tag{7.12}$$

is a norm for the space of sequences on $R_h^{\vec{m}}(\Omega)$. This space with this norm will be denoted by $H^s(R_h^{\vec{m}}(\Omega))$.

The restriction to $R_h^{\vec{m}}(\Omega)$ of the sequences $r_h^s \pi \Phi$ defines an operator mapping $H^s(\Omega)$ into $H^s(R_h^{\vec{m}}(\Omega))$, and it is clear that:

$$\lim_{h \to 0} \| u - p_h^{m_q} r_h^s \pi u \|_q = 0 \qquad \text{for all} \quad |q| \leq s. \tag{7.13}$$

We can now prove the following:

THEOREM 7.2. *If $p_h^{m_q} u_h$ converges weakly to $u_q$ in $H^q(\Omega)$ for all $|q| \leq s$, then there exists $u \in H^s(\Omega)$ such that $u = u_q$ for all $|q| \leq s$.*

Choose an infinitely differentiable function $\Phi$ of compact support. Let $m = \max\limits_{|q| \leq s} m_q$, and $k_q = m - m_q$. Then, for $h$ sufficiently small, the support of $\check{X}_{k_q, h} * \Phi$ is contained in $\Omega$ for all $|q| \leq m$. Thus we have:

$$\begin{aligned}
\int_\Omega (p_h^{m_q} u_h)(\chi_{k_q, h} * \Phi)\, dx &= \int_{R^n} (p_h^{m_q} u_h)(\chi_{k_q, h} * \Phi)\, dx \\
&= \int_{R^n} (p_h^{m} u_h)(\Phi)\, dx \\
&= \int_\Omega p_h^m u_h \Phi\, dx.
\end{aligned}$$

Since $\chi_{q,h} * \Phi$ converges strongly to $\Phi$ in $L^2(\Omega)$, we can take the limit and obtain $\int u_q \Phi\, dx = \int u \Phi\, dx$ for all $\Phi$ with compact support. This proves that all the $u_q$ are equal to some element $u$. Then since $D^q p_h^{m_q} u_h$ converges weakly to $D^q u_q = D^q u$ in $L^2(\Omega)$, $u$ belongs to $L^2(\Omega)$.

## § 8. Construction of finite difference schemes. Behavior of the error.

In this section we will apply the results of Sections 3 and 7 to the problem $P$ of Section 2. Let $V$ be the space $H^m(\Omega)$ and consider the following problem:

$$\sum_{|p|,|q|\leq m} \int_\Omega a_{pq}(x)\, D^p u\, D^q v\, dx = \int_\Omega fv\, dx + \sum_{j=0}^{m-1} \int_\Gamma g_j(x)\, \gamma_j\, v\, d\sigma \tag{8.1}$$

where $a_{pq}(x) \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and $g_j(x)$ is a function belonging to a suitable Sobolev space on the boundary $\Gamma$ of $\Omega$. We will assume that $\Omega$, $a_{pq}$, $f$, and $g_j$ are smooth enough so that the following conditions are satisfied:

$$\sum_{|p|,|q|\leq m} \int_\Omega a_{pq}(x)\, D^p u\, D^q u\, dx \geq c\, \| u \|_m^2 \; (c > 0) \tag{8.2}$$

$$\sum_{0\neq|p|,|q|\leq m} \int_\Omega a_{pq}(x)\, D^p u_p\, D^q u_q\, dx + \frac{1}{N} \sum_{|p|\leq m} \int_\Omega a_0(x)\, | u_p |^2\, dx \geq c \big( \sum_{|p|\leq m} \| u_p \|_p^2 \tag{8.3}$$

where $N$ is the number of multi-integers $p$ such that $|p| \leq m$.

(8.4) The solution $u$ of (8.1) belongs in fact to $H^s(\Omega) = W$.

We associate with $h$ and a sequence $\vec{k} = (k_q)_{|q|\leq m}$ the space $H^m(R_h^{\vec{k}}(\Omega)) = V_h$. We then define $(u_h, v_h)_h = \sum_{\alpha \in R_h^{\vec{k}}(\Omega)} u_h^\alpha v_h^\alpha$. Let $A_h^{\vec{k}} = \sum_{|p|,|q|\leq m} A_{pq,h}^{\vec{k}}$ be the matrix defined by:

$$(8.5)\quad \begin{cases} (A_{pq,h}^{\vec{k}} u_h, v_h)_h = \displaystyle\int_\Omega a_{pq}(x)\, D^p p_h^{k_p} u_h\, D^q p_h^{k_q} v_h\, dx \quad \text{if} \quad p \neq q \\[2ex] (A_{pp,h}^{\vec{k}} u_h, v_h)_h = \\ \qquad = \displaystyle\int_\Omega a_{pp}(x)\, D^p p_h^{k_p} u_h\, D^p p_h^{k_p} v_h\, dx + \frac{1}{N} \int_\Omega a_0(x)\, p_h^{k_p} u_h\, p_h^{k_p} v_h\, dx \\[2ex] (A_{00,h}^{\vec{k}} u_h, v_h)_h = \dfrac{1}{N} \displaystyle\int_\Omega a_0(x)\, p_h^{k_0} u_h\, p_h^{k_0} v_h\, dx. \end{cases}$$

Assume that for each $j$ $(0 \leq j \leq m-1)$ there exists a $q_j$ such that the following expression is well defined :

$$(8.6) \qquad (f_h, v_h)_h = \int_{\Omega} f(x)\, p_h^{k_0}\, v_h\, dx + \sum_{j=0}^{m-1} \int_{\Gamma} g_j(x)\, \gamma_j\, p_h^{k_{q_j}}\, v_h\, d\sigma.$$

THEOREM 8.1. *Under the assumptions* (8.2) *and* (8.4), *if* $k_q = k$ *for all* $|q| \leq m$ *for* $m \leq k \leq s-1$ *the solutions* $u_h$ *of*

$$(8.7) \qquad A_h^k u_h = f_h \quad in \quad V_h$$

*exist, are unique, and the errors* $u - p_h^k u_h$ *satisfy*

$$(8.8) \qquad \| u - p_h^k u_h \|_m \leq c h^{k-m}.$$

*Under the assumption* (8.3), *the solutions* $u_h$ *of* (8.7) *exist and satisfy*

$$(8.9) \qquad \lim_{h \to 0} \sum_q \| u - p_h^{k_q} u_h \|_{H^q(\Omega)} = 0.$$

*If* $k = s$, *and if* $B_h^\theta$ $(0 \leq \theta \leq 1)$ *is defined by*

$$(8.10) \qquad (B_h^\theta u_h, v_h)_h = \sum_{|p| \leq s} \int_{\Omega} h^{2\theta(s+m)} D^p p_h^s u_h D^p p_h^s v_h\, dx$$

*then the solutions* $u_h$ *of*

$$(8.11) \qquad (B_h^\theta + A_h^s)\, u_h = f_h$$

*exist, are unique under the assumptions* (8.2) *and* (8.4), *and satisfy*

$$(8.12) \qquad \| u - p_h^s u_h \|_m \leq c h^{\theta(s-m)}$$

(8.13) $\quad p_h^s u_h$ *converges to* $u$ *in* $H^s(\Omega)$ *strongly if* $\theta < 1$ *and weakly if* $\theta = 1$.

The first part of Theorem 8.1 follows from Theorems 3.1 and 7.1, the second part from Theorems 3.3 and 7.2, and the last part from Theorems 3.2 and 7.1. By ([6]), if we assume that $D_1^{s_1+1} \dots D_n^{s_n+1} u$ belongs to $L^2(\Omega)$, we have stronger estimates of error in (8.8) and (8.12), replacing $h^{k-m}$ by $h^{1|k|-m}$ and $h^{\theta(s-m)}$ by $h^{\theta(ns-m)}$ respectively.

Consider the following example:

$$a(u, v) = \sum_{i=1} \int_\Omega D_i u \, D_i v \, dx + \int_\Omega uv \, dx = \int_\Omega fv \, dx. \tag{8.14}$$

We then obtain the Neumann problem for the operator $-\Delta + 1$, and if $\Omega$ is smooth, conditions (8.2), (8.3), and (8.4) are satisfied. In this example, we obtain the classical finite difference schemes. For example, if $n = 2$ we obtain the particular schemes:

```
  x
x x x      if  k = ((0, 0), (0, 1), (1, 0))
  x

x x x
x x x      if  k = (1, 1)
x x x

    x
    x
x x x x x  if  k = ((0, 0), (0, 2), (2, 0))
    x
    x
```

and so on.

We can also study the special structure of the matrices $A^{\vec{k}}_{pq, h}$. For example, the general coefficient $a^{\vec{k}}_{pq, h}(\alpha, \beta)$ of this matrix is

$$a^{\vec{k}}_{pq, h}(\alpha, \beta) = \int_\Omega a_{pq}(x) \, D^p \sigma^\alpha_{k_p, h}(x) \, D^q \sigma^\beta_{k_q, h}(x) \, dx \qquad (\text{if } p \neq q \neq 0) \tag{8.15}$$

and can be computed easily as a function of the terms

$$\int_\Omega \theta^\alpha_h \left( a_{pq}(x) \frac{\left(\frac{x}{h} - \alpha\right)^{j+r}}{j!\, r!} \right) dx. \tag{8.16}$$

If (support $\theta^\alpha_h) \cap \Gamma \neq \Phi$, this term (8.16) includes the boundary conditions.

The coefficient $a^{\vec{k}}_{pq, h}(\alpha, \beta)$ is zero if the intersection of the supports of the functions $\sigma^\alpha_{k_p, h}(x)$ and $\sigma^\beta_{k_q, h}(x)$ is empty. This class of matrix possesses a small number of non-zero elements. Nevertheless, by (5), *the speed of convergence is almost optimal* (of the same order then the speed of convergence obtained by Galerkin's method for a same order of regularity of the solution).

We can easily see, for example, that if $q \neq 0$, then:

$$\sum_{\beta} a^{\vec{k}}_{pq,\,h}(\alpha, \beta) = 0. \tag{8.17}$$

This is true since

$$\sum_{\beta} a^{\vec{k}}_{pq,h}(\alpha, \beta) = \int_{\Omega} a_{pq}(x)\, D^p \sigma^{\alpha}_{k_p,h}\, D^q \big(\sum_{\beta} \sigma^{\beta}_{k_q,h}(x)\big)\, dx$$

and the restriction to $\Omega$ of the function $\sum_{\beta} \sigma^{\beta}_{k_q,h}(x)$ is equal to the constant $\sqrt{h}$.

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