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A GENERALIZATION OF A THEOREM OF J. STEINBERG

by M. DAVID

1. Introduction. We shall deal here with integral operators

$$Kf(t) = \int_a^b k(s, t) f(t)\, dt$$

for which the kernel $k(s, t)$ is non-degenerate (that is, not a finite sum of products of a function of s and a function of t), is analytic in its two variables, and satisfies a differential equation of the form

$$A_s k = B_t k \tag{1}$$

where A_s and B_t are ordinary differential operators relative to the variables s and t respectively. Moreover, we shall assume that their coefficients have an infinite number of continuous derivatives. If the order of A (B) is other than zero, we shall assume that the coefficient of the highest derivative in A (B) is other than zero in each point of $[a, b]$. The interval $[a, b]$ may be considered finite or infinite. It can be shown that the relation (1) implies the formula

$$A_s \int_a^b k(s, t) f(t)\, dt = \int_a^b k(s, t)\, B_t^* f(t)\, dt + \{M_B [k(s, t) f(t)]\}_a^b$$

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where B^* is the adjoint of B and M_B the differential bilinear form of Lagrange corresponding to B. (see [1] p. 186). The nature of the last expression of the second member will not be of interest in the sequel, and we shall follow Steinberg ([see 2]) in writting down the relation between the operators AK and KB^* defined by this equality, in short by

$$AK \longrightarrow KB^*$$

and in calling it a commutation law of K.

We shall prove in this paper a theorem concerning commutation laws which is a generalization of the theorem proved by J. Steinberg in [2]. The method of proof is essentially that used in [2]. We shall resort to the following lemma:

LEMMA (Steinberg). If the integral operator K satisfies the commutation law

$$EK \longrightarrow KF$$

then F is uniquely defined by E and at least one of the operators E and F is of order ≥ 1. This was proved in [2] p. 27-28.

2. We shall now state our result.

THEOREM. Let $\Phi(s)$ be a polynomial of degree at least 1, $\psi(s)$ a continuous function other than zero in each point of $[a, b]$ and α and β two real numbers such that

$$(2) \qquad -\psi \frac{d\Phi}{ds} = \alpha\Phi + \beta.$$

Let χ be another continuous function, and D the differentiation operator.

A necessary and sufficient condition for the existence of an integral operator K having a non-degenerate kernel and satisfying the commutation relations.

$$(4) \qquad \begin{aligned} &\Phi K \longrightarrow KG \\ &(\psi D + \chi) K \longrightarrow KH \end{aligned}$$

is that

(*a*) G be of order at least 1, and that

(*b*) $GH - HG \equiv \alpha G + \beta$.

REMARK 1. It is easy to show that relation (*b*) holds also for operators Φ and $\psi D + \chi$ which appear in the left members of the commutation relations (4).

REMARK 2. Steinberg's theorem, proved in (2) is obtained from the above theorem taking $\Phi(s) = s$ $\psi = 1$ $\chi = 0$ $\alpha = 0$, $\beta = -1$.

REMARK 3. We shall give here an example of an integral operator K which satisfies the conditions of the theorem. Let K be defined by the equality

$$Kf = \int_a^b e^{st} f(t)\, dt$$

where $0 < a < b$. Also, let $\Phi = s^2$ $\psi = s$ $\chi = s$. It is clear that (2) is satisfied in this case for $\alpha = -2$ $\beta = 0$. Moreover, the kernel $k(s, t) = e^{st}$ satisfies the relations

$$\Phi_s k = D_t^2 k$$

$$(\psi D + \chi)_s k = (tD_t + D_t) k.$$

Hence the commutation relations (4) hold for

$$G = (D_t^2)^* = (D_t^*)^2 = D_t^2$$

$$H = (tD_t + D_t)^* = D_t^* t + D_t^* = - D_t t - D_t$$

and it is easily seen that conditions (a) and (b) are satisfied.

Proof of the theorem.

Necessity. Condition (a) follows immediately from the lemma. It remains to prove condition (b). The relations (4) show that the kernel $k(s, t)$ of the operator K satisfies the two equations

$$\Phi_s k = G_t^* k \tag{5}$$

$$(\psi_s D_s + \chi_s) k = H_t^* k. \tag{6}$$

Hence

$$(\psi_s D_s + \chi_s)\, \Phi_s k = (\psi_s D_s + \chi_s)\, G_t^* k = G_t^* (\psi_s D_s + \chi_s) k = G_t^* H_t^* k$$

$$\Phi_s (\psi_s D_s + \chi_s) k = \Phi_s H_t^* k = H_t^* \Phi_s k = H_t^* G_t^* k.$$

Subtraction yields

$$(G_t^* H_t^* - H_t^* G_t^*) k = (\psi_s D_s + \chi_s)\, \Phi_s k - \Phi_s (\psi_s D_s + \chi_s) k =$$

$$= (\psi_s D_s \Phi_s + \chi_s \Phi_s - \Phi_s \psi_s D_s - \Phi_s \chi_s) k = (\psi_s D_s \Phi_s - \Phi_s \psi_s D_s) k =$$

$$= \psi(s)\, \Phi(s) \frac{\partial k}{\partial s} + \psi_{(s)} k(s, t) \frac{d\Phi}{ds} - \Phi_{(s)} \psi_{(s)} \frac{\partial k}{\partial s} = \psi(s) \frac{d\Phi}{ds} k(s, t).$$

Thus by (2) we obtain

$$(G_t^* H_t^* - H_t^* G_t^*) k = -(\alpha \Phi + \beta)_s k.$$

Hence, by (5) we have

$$(G_t^* H_t^* - H_t^* G_t^* + \alpha G_t^* + \beta) k(s, t) = 0. \tag{7}$$

For convenience we denote

$$G_t^* H_t^* - H_t^* G_t^* + \alpha G_t^* + \beta = R_t.$$

Let n be the order of R. We shall show that $n = 0$. Suppose that $n \geq 1$. Let $y_1(t), y_2(t), \dots, y_n(t)$ be a fundamental system of solutions of the equation

$$Ry = 0.$$

By (7), k is a solution of this equation. k is therefore of the form

$$c_1(s) y_1(t) + c_2(s) y_2(t) + \dots + c_n(s) y_n(t).$$

This contradicts the assumption that k is non-degenerate. Hence R_t is of order zero, i. e. a function. Denoting this function by $r(t)$, we have

$$r(t) k(s, t) \equiv 0.$$

Since k is considered analytic, if follows that $R_t = r(t) \equiv 0$. Hence

$$G_t^* H_t^* - H_t^* G_t^* = -\alpha G_t^* - \beta$$

and condition (b) is obtained by passing to the adjoints.

Sufficiency. We shall show that eqs. (5) and (6) admit a common solution if conditions (a) and (b) are satisfied. By (a), eq. (5) is an ordinary differential equation of order $n \geq 1$, with respect to t, s being a parameter. It has therefore a fundamental system of n solutions.

$$g_1(s, t), g_2(s, t), \dots, g_n(s, t)$$

which are continuous and have an infinite number of continuous derivatives with respect to t. Moreover, the functions and their derivatives with respect to t are continuos-differentiable with respect to s, (see [3], chapters 1,2). The general solution is therefore of the form

$$k(s, t) = h_1(s) g_1(s, t) + h_2(s) g_2(s, t) + \dots + h_n(s) g_n(s, t)$$

Substituting this expression in (6), we obtain

$$\sum_{r=1}^{n} \psi(s)\, h_r'(s)\, g_r + \sum_{r=1}^{n} h_r(s)\left(\chi(s)\, g_r + \psi(s)\frac{\partial g_r}{\partial s} - H_t^* g_r\right) = 0. \tag{7}$$

For convenience, we introduce the notation $l_r(s, t) = \chi(s)\, g_r(s, t) + \psi(s)\frac{\partial g_r}{\partial s} - H_t^* g_r$ for $r = 1, 2, \dots, n$. We shall show that l_r is a solution of (5). The function $\chi(s)\, g_r(s, t)$ for $r = 1, 2, \dots, n$ satisfies eq. (5), since g_r satisfies it. Hence, it suffices to show that

$$\Phi(s)\left(\psi(s)\frac{\partial g_r}{\partial s} - H_t^* g_r\right) = G_t^*\left(\psi(s)\frac{\partial g_r}{\partial s} - H_t^* g_r\right). \tag{8}$$

Since $b)$ holds and g_r is a solution of (5), we have

$$G_t^*\left(\psi(s)\frac{\partial g_r}{\partial s} - H_t^* g_r\right) = \psi(s)\frac{\partial(G_t^* g_r)}{\partial s} - G_t^* H_t^* g_r = \psi(s)\frac{\partial(G_t^* g_r)}{\partial s} -$$

$$- (H_t^* G_t^* - \alpha G_t^* - \beta)\, g_r = \psi\frac{\partial \Phi}{\partial s} g_r + \psi\Phi\frac{\partial g_r}{\partial s} - (H_t^* G_t^* - \alpha G_t^* - \beta)\, g_r.$$

Hence, by (2) we obtain

$$G_t^*\left(\psi(s)\frac{\partial g_r}{\partial s} - H_t^* g_r\right) = -(\alpha\Phi + \beta)\, g_r + \psi\Phi\frac{\partial g_r}{\partial s} - H_t^*(\Phi g_r) + \alpha\Phi g_r + \beta g_r =$$

$$= \Phi(s)\,(\psi(s)\frac{\partial g_r}{\partial s} - H_t^* g_r).$$

Equality (8) is proved. Hence, l_r satisfies (5). There exist therefore, for each $r = 1, 2, \dots, n$, n continuous functions $f_{rq}(s)$ satisfying

$$l_r(s, t) = \sum_{q=1}^{n} f_{rq}(s)\, g_q(s, t).$$

Substituting this expression in (7), we obtain

$$\sum_{r=1}^{n} \psi(s)\, h_r'(s)\, g_r(s, t) + \sum_{r=1}^{n} h_r(s)\left(\sum_{q=1}^{n} f_{r,q}(s)\, g_q(s, t)\right) = 0.$$

Hence

$$\sum_{r=1}^{n} \psi(s)\, h_r'(s)\, g_r(s, t) + \sum_{q=1}^{n}\left(\sum_{r=1}^{n} h_r(s)\, f_{r,q}(s))\, g_q(s, t)\right) = 0.$$

Replacing the summation index r by q in the first sum, we obtain

$$\sum_{q=1}^{n}\left(\psi(s)\, h_q'(s) + \sum_{r=1}^{n} h_r(s)\, f_{r,q}(s)\right) g_q(s, t) = 0.$$

The functions g_q being linear-independent, it follows that

$$\psi(s)\, h'_q(s) + \sum_{r=1}^{n} h_r(s)\, f_{r,q}(s) = 0 \qquad q = 1, 2, \ldots, n.$$

Since $f_{r,q}$ and ψ are continuous and $\psi(s)$ is non-zero in $[a, b]$, this system has a solution $h_1, h_2, \ldots, h_n$. The corresponding function $k(s, t)$ is therefore a solution of both eqs. (5) and (6).

It remains to show that this solution is non-degenerate Steinberg proved [[2], p. 30] that for $\Phi(s) = s$ every solution of (5) is non-degenerate. We conclude by noting that simple examination of this proof shows that the proposition is true in the more general case, when $\Phi(s)$ is a polynomial of degree $n \geq 1$.

REFERENCES

[1] E. L. INCE, « *Ordinary differential equations* », 1956.

[2] J. STEINBERG, « *Sur les lois de commutation de certaines transformations integrales* », Annali della Scuola Normale Superiore di Pisa Serie III, Vol. X. Fasc. I-II, 1956 p. 25-33.

[3] CODDINGTON and LEVINSON, « *Theory of ordinary differential equations* », 1955.