ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

UMBERTO MOSCO

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APPROXIMATION OF THE SOLUTIONS OF SOME VARIATIONAL INEQUALITIES (*)

by UMBERTO MOSCO

J. L. LIONS and G. STAMPACCHIA have recently considered a class of variational inequalities and obtained some results on the existence and the approximation of their solutions (1). In particular they have considered the following problem

Given a positive continuous bilinear form a on a real Hilbert space V, a closed convex non empty subset $\mathbb{K}$ of V and a vector v' in the dual V' of V, to determine all vectors u of V such that

$$(\mathrm{p}) \qquad \begin{cases} u \in \mathbb{K} \\ a(u, v-u) \geq \langle v', v-u \rangle \quad \text{for all} \quad v \in \mathbb{K} \end{cases}$$

where $\langle .,. \rangle$ denotes the pairing between V and V'.

In this paper we study the stability and the approximation of solutions of (p) taking into account not only perturbations of a and v', as is done in [3] of Ref (1), but also possible perturbations of the convex set $\mathbb{K}$.

1. Henceforth V will be a given real Hilbert space whose inner product is denoted by $(.,.)$ and the associated norm by $\|\cdot\|$. V' is the strong dual of V, the pairing between V and V' is denoted by $\langle .,. \rangle$ and the dual norm in V' is again denoted by $\|\cdot\|$. We shall also consider V as endowed with its weak topology, thus we shall write as usual s-lim or w-lim to denote the strong or weak convergence in V.

2. We assume that a positive continuous bilinear form a on V is given, together with a vector v' of V and a closed convex non empty

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subset $\mathbb{K}$ of V. We shall always refer to the problem stated at the beginning as to problem (p) and we shall denote by

$$X$$

the set (possibly empty) of all solutions of (p), for the given a, v' and $\mathbb{K}$.

We briefly recall from [3] of Ref (1) some results that we shall use in what follows:

X is a closed convex subset of $\mathbb{K}$, and X is non-empty if $\mathbb{K}$ is bounded.

If the form a is coercive on V, which is to say

$$a(v, v) \geq c \| v \|^2 \quad \text{for some} \quad c > 0 \quad \text{and all} \quad v \in V,$$

then problem (p), even if $\mathbb{K}$ is unbounded, always has one and only one solution, that is X consists of a single vector.

In case a is the inner product in V, i.e. $a(u, v) = (u, v)$ for all $u, v \in V$, then the (unique) solution of (p) is the vector

$$u = P_{\mathbb{K}} \Lambda v'$$

where Λ is the canonical isomorphism of V' onto V (which carries $v' \in V'$ into the vector $\Lambda v' \in V$ such that $(\Lambda v, v') = \langle v', v \rangle$ for all $v \in V$) and

$$P_{\mathbb{K}} : V \rightarrow V$$

is the Riesz projection on V (which carries each vector $v \in V$ into the unique vector $P_{\mathbb{K}} v$ of $\mathbb{K}$ such that

$$\| v - P_{\mathbb{K}} v \| = \inf \{ \| v - w \| : w \in \mathbb{K} \}).$$

3. Let now C_ε, $\varepsilon > 0$, be a family of closed convex subsets of V. We consider the subset of V

$$s\text{-Lim inf } C_\varepsilon$$

of all limit points of C_ε as $\varepsilon \rightarrow 0$ in the strong topology of V, which is to say all $v \in V$ such that for any strong neighbourhood $S(v)$ of v and for some $\varepsilon_0 > 0$ we have

$$C_\varepsilon \cap S(v) \neq \Phi \quad \text{for all} \quad \varepsilon \leq \varepsilon_0 .$$

We also consider the subset of V

$$w\text{-Lim sup } C_\varepsilon$$

of all cluster points of C_ε as $\varepsilon \to 0$ in the weak topology of V, that is of all $v \in V$ such that for any weak neighbourhood $W(v)$ of v and any $\varepsilon > 0$ we have

$$C_\eta \cap W(v) \neq \Phi \quad \text{for some} \quad \eta \leq \varepsilon.$$

These limits are a special case of the notion of Lim inf and Lim sup of a directed family of subsets of a topological space (2).

We give the following

DEFINITION 1. We say that the family C_ε *converges* as $\varepsilon \to 0$ if

$$s\text{-Lim inf } C_\varepsilon = w\text{-Lim sup } C_\varepsilon .$$

If C is a closed convex subset of V, we say that C_ε *converges to* C as $\varepsilon \to 0$, and write

$$C = \text{Lim } C_\varepsilon ,$$

if C_ε converges as $\varepsilon \to 0$ and

$$s\text{-Lim inf } C_\varepsilon = w\text{-Lim sup } C_\varepsilon = C.$$

REMARK 1. Clearly $C = \text{Lim } C_\varepsilon$ if and only if the following conditions are satisfied

(I) $$C \subseteq s\text{-Lim inf } C_\varepsilon$$

(II) $$C \supseteq w\text{-Lim sup } C_\varepsilon .$$

Note that (I) and (II) are equivalent, if C is non-empty, to

(I′) 0 is a strong limit point of $C_\varepsilon - v$ as $\varepsilon \to 0$ for any $v \in C$

(II′) 0 is a weak cluster point of $C - v_\varepsilon$ as $\varepsilon \to 0$ for any bounded set $\{v_\varepsilon\}$, $v_\varepsilon \in C_\varepsilon$ for any ε.

REMARK 2. If C_ε is decreasing as $\varepsilon \to 0$, that is $C_{\varepsilon'} \supseteq C_{\varepsilon''}$ for all $\varepsilon'' \leq \varepsilon'$, then C_ε converges and $\text{Lim } C_\varepsilon = \cap C_\varepsilon$. If C_ε is increasing as $\varepsilon \to 0$, that is $C_{\varepsilon'} \subseteq C_{\varepsilon''}$ for all $\varepsilon'' \leq \varepsilon'$, then C_ε converges and $\text{Lim } C_\varepsilon = \overline{\cup C_\varepsilon}$, closure in V.

4. We now consider perturbations a_ε, v'_ε and $\mathfrak{K}_\varepsilon$, $\varepsilon > 0$, of a, v' and $\mathfrak{K}$ satisfying the assumptions I, II and III listed below. For any continuous

bilinear form α on V we put

$$\| \alpha \| = \sup \{ | \alpha(u, v) | : \| u \| \leq 1, \ \| v \| \leq 1 \}$$

$$| \alpha |_v = \sup \{ | \alpha(u, v) | : \| u \| \leq 1 \} \quad \text{for any} \quad v \in V.$$

I a_ε is a positive (continuous bilinear) form on V which satisfies either (u) or (s) below

(u) $\lim \| a_\varepsilon - a \| = 0$ as $\varepsilon \to 0$

(s) $a_\varepsilon - a$ is positive on V, moreover for any $v \in V$ we have $| a_\varepsilon |_v < c$, for some $c > 0$ — possibly depending on v — and all ε, and $\lim | a_\varepsilon - a |_v = 0$ as $\varepsilon \to 0$.

II v'_ε is a vector of V' such that

$$\lim \| v'_\varepsilon - v' \| = 0 \quad \text{as} \quad \varepsilon \to 0.$$

III $\mathbb{K}_\varepsilon$ is a non empty closed convex subset of V such that

$$\mathbb{K} = \operatorname{Lim} \mathbb{K}_\varepsilon \quad \text{as} \quad \varepsilon \to 0,$$

in the sense of DEFINITION 1.

We also consider for any ε the perturbed problem

$$(\mathrm{p}_\varepsilon) \qquad \begin{cases} u_\varepsilon \in \mathbb{K}_\varepsilon \\ a_\varepsilon(u_\varepsilon, v - u_\varepsilon) \geq \langle v'_\varepsilon, v - u_\varepsilon \rangle \quad \text{for all} \quad v \in \mathbb{K}_\varepsilon . \end{cases}$$

We denote by

$$X_\varepsilon$$

the set (possibly empty) of all solutions u_ε of (p_ε).

Our object in the following sections is to study the convergence properties of the approximate solutions u_ε as $\varepsilon \to 0$. More precisely, we shall study as to whether approximate solutions u_ε exist converging weakly or strongly to a solution u of (p) as $\varepsilon \to 0$.

5. In case the form a is coercive on V we can prove the following theorem

THEOREM 1. *Assume that conditions* I, II *and* III *above are satisfied and suppose further that a and a_ε are coercive on V. Then the (unique) solution u_ε of* (p_ε) *converges strongly in V to the (unique) solution u of* (p) *as* $\varepsilon \to 0$.

When the approximate family $\mathbb{K}_\varepsilon$ of $\mathbb{K}$ is monotone THEOREM 1 admits the following refinement

COROLLARY 1. *Let $\mathbb{K}$ be a non empty closed convex subset of V and let $\mathbb{K}_\varepsilon$, $\varepsilon > 0$, be a monotone family either decreasing or increasing as $\varepsilon \to 0$ of non empty closed convex subsets of V. Then the following conditions are equivalent*

(i) $\mathbb{K} = \text{Lim}\, \mathbb{K}_\varepsilon$, *that is* $\mathbb{K} = \bigcap \mathbb{K}_\varepsilon$ *if* $\mathbb{K}_\varepsilon$ *is decreasing, or* $\mathbb{K} = \overline{\bigcup \mathbb{K}_\varepsilon}$ *if* $\mathbb{K}_\varepsilon$ *is increasing.*

(ii) *If a and a_ε are any coercive forms on V satisfying condition* I *above and v', v'_ε are vectors of V' satisfying condition* II, *then the solution u_ε of* (p_ε) *converges strongly in V to the solution u of* (p) *as $\varepsilon \to 0$.*

(iii) $$P_{\mathbb{K}}\, v = s\text{-lim}\, P_{\mathbb{K}_\varepsilon} v \quad \textit{as } \varepsilon \to 0 \textit{ for any } v \in V.$$

A special case of a monotone approximate family $\mathbb{K}_\varepsilon$ is considered in the following

COROLLARY 2. *Let $\mathbb{K}$ be a closed convex subset of V whose interior is non-empty. Let V_ε, $\varepsilon > 0$, be a family of closed subspaces of V such that $V = \text{Lim}\, V_\varepsilon$ as $\varepsilon \to 0$ and let $\mathbb{K}_\varepsilon = \mathbb{K} \cap V_\varepsilon$ for any ε. Then proposition* (ii) *above holds.*

Thus, for example, if V is separable and V_n is an increasing sequence of finite dimensional subspaces of V such that $V = \overline{\bigcup V_n}$, then the solution u of (p) can be obtained as the strong limit in V of the sequence u_n of solutions of problem (p) in which $\mathbb{K}$ has been replaced by its finite dimensional section $\mathbb{K}_n = \mathbb{K} \cap V_n$.

6. Let us go back to the case of a positive form a. For any $R > 0$ we shall denote by X_ε^R, $\varepsilon > 0$, the bounded section

$$X_\varepsilon^R = \{v \in X_\varepsilon : \| v \| \leq R\}$$

of the set X_ε of all solutions of (p_ε).

THEOREM 2. *Under the assumptions* I, II, III *above, let us suppose that*

$$X_\varepsilon^R \neq \Phi \textit{ for some } R > 0 \textit{ and all } \varepsilon \leq \varepsilon_0 .$$

Then problem (p) *has solutions,* i. e. $X \neq \Phi$, *and any weak cluster point of X_ε as $\varepsilon \to 0$ belongs to X, that is* $w\text{-Lim sup}\, X_\varepsilon \subseteq X$.

For instance, let V, V_n and $\mathbb{K}_n$ be as at the end of the previous section. It follows from THEOREM 2 that if problem (p) with $\mathbb{K}$ replaced by its finite dimensional section $\mathbb{K}_n$ has a solution u_n and such u_n remains in a bounded subset of V as $n \longrightarrow \infty$, then problem (p) for the given $\mathbb{K}$ also has a solution and this solution is the weak limit of a subsequence of u_n (3).

7. The result of THEOREM 2 can be improved in order to obtain strong convergence of perturbed solutions by use of the same device — the « elliptic regularization » — which has been used by J. L. LIONS and G. STAMPACCHIA in [3] of Ref. (1). It consists of adding a coercive perturbation $\varepsilon\beta$ to the given positive form a, then solving problem (p) with a replaced by $a_\varepsilon = a + \varepsilon\beta$ and finally letting $\varepsilon \longrightarrow 0$. However, we must require that the perturbed $\mathbb{K}_\varepsilon$ converges to $\mathbb{K}$ rapidly enough to keep the form a_ε acting coercively on $\mathbb{K}_\varepsilon$ as $\varepsilon \longrightarrow 0$. Therefore we give the following

DEFINITION 2. We say that $\mathbb{K}_\varepsilon$ *converges of order* ε *to* $\mathbb{K}$ as $\varepsilon \longrightarrow 0$ if the following conditions are satisfied

(k) 0 is a strong limit point of $\varepsilon^{-1}(\mathbb{K}_\varepsilon - v)$ as $\varepsilon \longrightarrow 0$ for any $v \in V$

(kk) 0 is a weak cluster point of $\varepsilon^{-1}(\mathbb{K} - v_\varepsilon)$ as $\varepsilon \longrightarrow 0$ for any bounded set $\{v_\varepsilon\}$, $v_\varepsilon \in \mathbb{K}_\varepsilon$ for any ε.

Clearly if $\mathbb{K}_\varepsilon$ converges of order ε to $\mathbb{K}$, then $\mathbb{K} = \operatorname{Lim} \mathbb{K}_\varepsilon$ in the sense of DEFINITION 1, for (k) and (kk) imply (l′) and (ll′) hence also (l) and (ll).

Now we make the following assumptions

I′ β is a fixed coercive form on V and a_ε is given by

$$a_\varepsilon = a + \varepsilon\beta_\varepsilon$$

where β_ε is a coercive (continuous bilinear) form on V, which satisfies either (u) or (s) below

(u) $$\lim \| \beta_\varepsilon - \beta \| = 0 \qquad \text{as} \qquad \varepsilon \longrightarrow 0$$

(s) $\beta_\varepsilon - \beta$ is positive on V, moreover for any $v \in V$ we have $|\beta_\varepsilon|_v < c$, for some $c > 0$ — possibly depending on v — and all ε, and $\lim |\beta_\varepsilon - \beta|_v = 0$ as $\varepsilon \longrightarrow 0$.

II′ φ' is a fixed vector in V' and v'_ε is given by

$$v'_\varepsilon = v' + \varepsilon\varphi'_\varepsilon$$

where $\varphi'_\varepsilon \in V$ satisfies

$$\lim \| \varphi'_\varepsilon - \varphi' \| = 0 \qquad \text{as} \qquad \varepsilon \to 0.$$

III′ $\mathbb{K}_\varepsilon$ is a non-empty closed convex subset of V which converges of order ε to $\mathbb{K}$ as $\varepsilon \to 0$, in the sense of DEFINITION 2.

Then we have

THEOREM 3. *Under the assumptions* I′, II′ *and* III′ *above the solution* u_ε *of* (p_ε) *converges strongly in* V *to a solution of* (p) *as* $\varepsilon \to 0$, *provided* u_ε *remains in a bounded subset of* V, *that is*

$$\| u_\varepsilon \| < R \quad \textit{for some} \quad R > 0 \quad \textit{and all} \quad \varepsilon \leq \varepsilon_0 .$$

Such a solution of (p) *is uniquely determined as the solution* u_0 *of the problem*

$$(\mathrm{p}_0) \qquad \begin{cases} u_0 \in X \\ \beta(u_0, v - u_0) \geq \langle \varphi', v - u_0 \rangle \quad \textit{for all} \quad v \in X. \end{cases}$$

REMARK. It will be clear from the proof that THEOREM 3 is still true if condition (k) of DEFINITION 2 is replaced by condition (l) of REMARK 1 together with the following

(k_0) 0 is a strong limit point of $\varepsilon^{-1}(\mathbb{K}_\varepsilon - v)$ as $\varepsilon \to 0$ for any $v \in X$.

However, condition (k_0) could turn out to be easier to verify in the applications then condition (k) provided that we know some regularity properties of the solutions of (p), in other words, provided that we know that X must belong to a certain « smoother » subspace of V.

8. From THEOREM 3 and a result of [3] we can deduce other sufficient conditions for the existence and the strong approximation of solutions of (p). Indeed we shall consider in the two corollaries below two cases in which by perturbing only a bounded section $\mathbb{K}^R$ of $\mathbb{K}$,

$$\mathbb{K}^R = \{ v \in \mathbb{K} : \| v \| \leq R \}, \quad R > 0,$$

one can obtain a solution of (p) relative to the whole $\mathbb{K}$ as a strong limit of approximate solutions or convex combinations of such.

The following corollary of THEOREM 3 holds

COROLLARY 1. *Assume conditions* I′ *and* II′ *above and suppose that condition* III′ *with* $\mathbb{K}$ *replaced by* $\mathbb{K}^R$ *is satisfied for any* $R \geq R_0$, *with* $\mathbb{K}_\varepsilon$ *possibly depending on* R. *Suppose further that for some* R *there exists a weak cluster point* u *of solutions* u_ε *of* (p_ε) *such that*

$$\| u \| < R.$$

Then u *is a solution of* (p) *and* u_ε *converges strongly to* u *as* $\varepsilon \to 0$. *Moreover* u *coincides with the (unique) solution* u_0 *of* (p_0).

Let's now assume that conditions I′ and II′ are satisfied by $\beta = \beta_i$ and $\varphi' = \varphi'_i$ both for $i = 1$ and $i = 2$. That is, we assume that β_1, β_2 are fixed coercive forms on V and we have

$$a_{i\varepsilon} = a + \varepsilon \beta_{i\varepsilon}, \qquad i = 1, 2,$$

where $\beta_{i\varepsilon}$ satisfies either (u) or (s) of I′ — with β_ε replaced by $\beta_{i\varepsilon}$ and β replaced by β_i; moreover we assume that φ'_1, φ'_2 are fixed vectors in V' and

$$\varphi'_{i\varepsilon} = \varphi' + \varepsilon \varphi'_{i\varepsilon}, \qquad i = 1, 2,$$

where $\varphi'_{i\varepsilon}$ belongs to V' and converges to φ'_i in V' as $\varepsilon \to 0$. Finally we assume that condition III′ is satisfied with $\mathbb{K}$ replaced by $\mathbb{K}^R$, for all $R \geq R_0$, $\mathbb{K}_\varepsilon$ possibly depending on R.

Let $u_{i\varepsilon}$, $i = 1, 2$, be the solution of the problem

$$(\mathrm{p}_{i\varepsilon}) \qquad \begin{cases} u_{i\varepsilon} \in \mathbb{K}_\varepsilon \\ a_{i\varepsilon}(u_{i\varepsilon}, v - u_{i\varepsilon}) \geq \langle v'_{i\varepsilon}, v - u_{i\varepsilon} \rangle \quad \text{for all} \quad v \in \mathbb{K}_\varepsilon \end{cases}$$

We then have the following

COROLLARY 2. *With the hypotheses and notation above, suppose that for some* R *there exists a weak cluster point* u_i, $i = 1, 2$, *of solutions* $u_{i\varepsilon}$ *of* $(\mathrm{p}_{i\varepsilon})$ *and that*

$$u_1 \neq u_2 .$$

Then the vector $u = (1 - \theta) u_1 + \theta u_2$ *is a solution of* (p) *for some* θ, $0 \leq \theta \leq 1$, *and the vector* $u_\varepsilon = (1 - \theta) u_{1\varepsilon} + \theta u_{2\varepsilon}$ *converges strongly to* u *in* V *as* $\varepsilon \to 0$.

8. We state here a partial converse to THEOREM 2 in which we show, under stronger assuptions then those required in THEOREM 2, that the existence

of solutions of problem (p) implies that the approximate solutions u_ε remain in a bounded subset of V as $\varepsilon \to 0$.

We make the following assumptions

I″ a_ε is given by

$$a_\varepsilon = a + \gamma_\varepsilon$$

where γ_ε is a coercive form on V such that

$$\limsup c_\varepsilon^{-1} \| \gamma_\varepsilon \| < + \infty \quad \text{as} \quad \varepsilon \to 0$$

— c_ε being a positive constant such that for any ε, $c_\varepsilon \| v \|^2 \leq \gamma_\varepsilon(v, v)$ for all $v \in V$ and $\lim c_\varepsilon = 0$ as $\varepsilon \to 0$.

II″ v'_ε is given by

$$v'_\varepsilon = v' + \psi'_\varepsilon$$

where ψ'_ε belongs to V' and satisfies

$$\limsup c_\varepsilon^{-1} \| \psi'_\varepsilon \| < + \infty \quad \text{as} \quad \varepsilon \to 0$$

III″ $\mathbb{K}_\varepsilon$ is a closed convex non empty subset of V which satisfies the following conditions

(m) For any $v \in \mathbb{K}$, there exists a strong neighbourhood $S(0)$ of 0 such that

$$c_\varepsilon^{-1} (\mathbb{K}_\varepsilon - v) \cap S(0) \neq \Phi \quad \text{for all} \quad \varepsilon \leq \varepsilon_0$$

(mm) $\limsup c_\varepsilon^{-1} S(\mathbb{K}, \mathbb{K}_\varepsilon) < + \infty$ as $\varepsilon \to 0$ where

$$S(\mathbb{K}, \mathbb{K}_\varepsilon) = \sup \{(1 + \| v \|)^{-1} \| v - P_{\mathbb{K}} v \| : v \in \mathbb{K}_\varepsilon\}.$$

Then we have the following

THEOREM 4. *In the presence of assumptions* I″, II″ *and* III″ *above, if problem* (p) *has solutions then the solution* u_ε *of* (p$_\varepsilon$) *satisfies the following condition*

$$\| u_\varepsilon \| \leq R \quad \textit{for some} \quad R > 0 \quad \textit{and all} \quad \varepsilon \leq \varepsilon_0 .$$

10. We first prove THEOREM 2.

PROOF OF THEOREM 2. Since $X_\varepsilon^R \neq \Phi$ for some $R > 0$ and all $\varepsilon \leq \varepsilon_0$, then w-Lim sup $X_\varepsilon \neq \Phi$. Therefore it suffices to prove that

w-Lim sup $X_\varepsilon \subseteq X$. Let $v_\eta \in X_\eta$ for infinitely many $\eta \to 0$ and let

$$u = w\text{-lim } v_\eta \quad \text{as} \quad \eta \to 0.$$

Since $\mathbb{K} \supseteq w$-Lim sup $\mathbb{K}_\varepsilon$ by assumption III, we know that $u \in \mathbb{K}$. Moreover, it follows from (p_ε) that for any $v \in V$ and all η

$$(1) \qquad a(u_\eta, u_\eta) \leq a(u_\eta, v) + \langle v', u_\eta - v \rangle + M'_\eta(v) + M''_\eta(v)$$

with

$$M'_\eta(v) = (a_\eta - a)(u_\eta, v - u_\eta) + \langle v'_\eta - v', u_\eta - v \rangle$$

$$M''_\eta(v) = a_\eta(u_\eta, P_{\mathbb{K}_\eta} v - v) + \langle v'_\eta, v - P_{\mathbb{K}_\eta} v \rangle .$$

Now we prove that in both cases (u) and (s) of condition I we have

$$(2) \qquad \limsup M'_\eta(v) \leq 0 \quad \text{for any} \quad v \in V$$

$$(3) \qquad \lim M''_\eta(v) = 0 \quad \text{for any} \quad v \in \mathbb{K}.$$

Let us first consider case (u): We then have

$$|M'_\eta(v)| \leq \| a_\eta - a \| \, \| u_\eta \| \, \| v - u_\eta \| + | \langle v'_\eta - v', u_\eta - v \rangle |$$

with $\lim \| a_\eta - a \| = 0$. By II, since $u_\eta - v$ converges weakly in V, we have $\lim \langle v'_\eta - v', u_\eta - u \rangle = 0$. Hence, since u_η remains in a bounded subset of V as $\eta - 0$, we obtain $\lim | M'_\eta(v) | = 0$ and (2) is proved. To prove (3), since

$$| M''_\eta(v) | \leq \| a_n \| \, \| u_\eta \| \, \| P_{\mathbb{K}_\eta} v - v \| + \| v'_\eta \| \, \| v - P_{\mathbb{K}_\eta} v \|,$$

note that $\| a_\eta \|$, $\| u_\eta \|$ and $\| v'_\eta \|$ are bounded as $\eta \to 0$ and that by III we have $\mathbb{K} \subseteq s$-Lim sup $\mathbb{K}_\varepsilon$, which implies $\lim \| P_{\mathbb{K}_\eta} v - v \| = 0$ as $\eta \to 0$ for any $v \in \mathbb{K}$. Let us now consider case (s) of I: Since the form $a_\eta - a$ is positive on V for any η, we have

$$M'_\eta(v) \leq M^0_\eta(v)$$

where

$$M^0_\eta(v) = (a_\eta - a)(u_\eta, v) + \langle v'_\eta - v', u_\eta - v \rangle$$

satisfies the inequality

$$| M^0_\eta(v) | \leq c \, | a_\eta - a |_v + | \langle v'_\eta - v', u_\eta - v \rangle |$$

for some $c > 0$ and all η sufficiently small. Hence (2) holds. To prove (3) in case (s) we first apply Banach-Stheinaus Theorem and obtain that $\| a_\eta \|$ is bounded as $\eta \to 0$. Henceforth we can use the same argument than in case (u).

We now recall that $v \to a(v, v)$ is a lower semicontinuous function in the weak topology of V (see Lemma 3.1 of [3]); therefore we have

$$a(u, u) \leq \liminf a(u_\eta, u_\eta) \quad \text{as } \eta \to 0.$$

From this together with (2) and (3) we obtain by (1) that

$$a(u, u) \leq a(u, v) + \langle v', u - v \rangle \quad \text{for all} \quad v \in \mathbb{K},$$

Therefore u satisfies (p), that is $u \in X$.

To have handy as a ready reference we state below an immediate corollary of THEOREM 2.

COROLLARY. *Under the assumptions* I, II *and* III, *if the forms a and a_ε are coercive on V and the solution u_ε of* (p$_\varepsilon$) *satisfies the condition*

(4) $$\| u_\varepsilon \| \leq R \text{ for some } R > 0 \text{ and all } \varepsilon \leq \varepsilon_0,$$

then the solution u of (p) *is the unique weak cluster point of u_ε as $\varepsilon \to 0$.*

11. Here we prove THEOREM 1 and its corollaries.

PROOF OF THEOREM 1. We first prove that u_ε satisfies condition (4) above. In virtue of (p$_\varepsilon$) we have for any $v \in V$

$$a_\varepsilon(u_\varepsilon, u_\varepsilon) \leq a_\varepsilon(u_\varepsilon, P_{\mathbb{K}_\varepsilon} v) + \langle v'_\varepsilon, u_\varepsilon - P_{\mathbb{K}_\varepsilon} v \rangle.$$

In case (u) of I, since $\lim \| a_\varepsilon - a \| = 0$ as $\varepsilon \to 0$, there exists some $c > 0$ such that

$$c \| u_\varepsilon \|^2 \leq a_\varepsilon(u_\varepsilon, u_\varepsilon) \qquad \text{for all } \varepsilon \text{ sufficiently small.}$$

In case (s) of I, since then $a_\varepsilon - a$ is a positive on V, we still have

$$c \| u_\varepsilon \|^2 \leq a(u_\varepsilon, u_\varepsilon) \leq a_\varepsilon(u_\varepsilon, u_\varepsilon)$$

for some $c > 0$ and all ε. Thus in both cases we obtain for any $v \in V$

(5) $$c \| u_\varepsilon \|^2 \leq a_\varepsilon(u_\varepsilon, P_{\mathbb{K}_\varepsilon} v) + \langle v'_\varepsilon, u_\varepsilon - P_{\mathbb{K}_\varepsilon} v \rangle$$

for some $c > 0$ and all ε small enough. Let now v be a vector of K. By III we have that $\| P_{\mathbb{K}_\varepsilon} v \|$ is bounded as $\varepsilon \to 0$. Furthermore we obtain as in the proof of THEOREM 2 that $\| a_\varepsilon \|$ is bounded as $\varepsilon \to 0$ in both cases (u) or (s) of I. Finally, $\| v'_\varepsilon \|$ is bounded by II. Therefore it follows from (5) that there exists some $c > 0$ such that

$$\| u_\varepsilon \|^2 \leq c (1 + \| u_\varepsilon \|) \quad \text{as} \qquad \varepsilon \to 0$$

and this clearly implies (4). As a consequence of the COROLLARY of THEOREM 1 we conclude that the solution u of (p) is the unique weak cluster point of u_ε as $\varepsilon \to 0$. Since u_ε, $\varepsilon \leq \varepsilon_0$, is by (4) a relatively weakly compact subset of V, it follows that u_ε converges weakly to u in V. Actually we can prove, using a similar argument to one found in [3], that u_ε converges strongly to u in V. In fact we have for some $c > 0$ and all ε

$$c \| u_\varepsilon - u \|^2 \leq a (u_\varepsilon - u, u_\varepsilon - u) = a (u_\varepsilon, u_\varepsilon - u) + a (u, u - u_\varepsilon).$$

By (1) with $\eta = \varepsilon$ and $v = u$ we have

$$a (u_\varepsilon, u_\varepsilon - u) \leq \langle v', u_\varepsilon - u \rangle + M'_\varepsilon (u) + M''_\varepsilon (u)$$

hence by (2) and (3) we find $\limsup a (u_\varepsilon, u_\varepsilon - u) = 0$. Since also $\lim a(u, u - u_\varepsilon) = 0$, we have $u = s\text{-}\lim u_\varepsilon$.

PROOF OF COROLLARY 1 OF THEOREM 1. The implication (i) $\Longrightarrow$ (ii) is a special case of THEOREM 1 (recall REMARK 2). If for any ε we take $a_\varepsilon = a =$ inner product of V and $v'_\varepsilon = v' = \Lambda^{-1} v$, where v is a given vector of V, we find by (ii), in light of the remarks at the end of Sec. 2, that $u_\varepsilon = P_{\mathbb{K}_\varepsilon} v$ converges strongly to $u = P_{\mathbb{K}} v$ in V as $\varepsilon \to 0$. Hence (ii) implies (iii). Finally, (iii) implies $\mathbb{K} \subseteq s\text{-}\mathrm{Lim\,inf}\, \mathbb{K}_\varepsilon$, which is to say $\mathbb{K} \subseteq \mathrm{Lim}\, \mathbb{K}_\varepsilon$ for $\mathbb{K}_\varepsilon$ is monotone. Moreover, we have $v = s\text{-}\lim P_{\mathbb{K}_\varepsilon} v$ for any $v \in \mathrm{Lim}\, \mathbb{K}_\varepsilon$, while (iii) implies $P_{\mathbb{K}} v = s\text{-}\lim P_{\mathbb{K}_\varepsilon} v$. Hence (iii) implies (i) and this concludes the proof.

The proof of COROLLARY 2 of THEOREM 1 is based on the following LEMMA, whose proof we give for the sake of completeness.

LEMMA *Let* $\mathbb{K}$ *be a closed convex subset of* V *whose interior is non-empty. Let* V_ε, $\varepsilon > 0$, *be closed subspaces of* V *such that* $V = \mathrm{Lim}\, V_\varepsilon$ *as* $\varepsilon \to 0$ *and let* $\mathbb{K}_\varepsilon = \mathbb{K} \cap V_\varepsilon$ *for any* ε. *Then* $\mathbb{K} = \mathrm{Lim}\, \mathbb{K}_\varepsilon$ *as* $\varepsilon \to 0$.

PROOF. Let $u_0 \in \mathring{\mathbb{K}}$, $\mathring{\mathbb{K}}$ the interior of $\mathbb{K}$, and let $S(u_0)$ be a strong neighbourhood of u_0 contained in $\mathbb{K}$. Let u be an arbitrary vector of $\mathbb{K}$ and let C be the convex cone generated by u and $S(u_0)$. Clearly $C \subseteq K$. Let $S(u)$ be any strong neighbourhood of u and take $u_1 \in \mathring{C} \cap S(u)$, $\mathring{C}$ the interior of C. Such a u_1 exists, because it can be chosen of the type $u_1 = \theta u_0 + (1 - \theta) u$ for θ small enough. Let $S(u_1)$ be a strong neighbourhood of u_1 contained in $C \cap S(u)$. Since $V = s\text{-}\mathrm{Lim\,inf}\, V_\varepsilon$, we have $V_\varepsilon \cap S(u_1) \neq \Phi$ for all $\varepsilon \leq \varepsilon_0$, for some ε_0. Hence $\mathbb{K}_\varepsilon \cap S(u) \neq \Phi$ for all $\varepsilon \leq \varepsilon_0$, that is $u = s\text{-}\mathrm{Lim\,inf}\, \mathbb{K}_\varepsilon$. Therefore $\mathbb{K} \subseteq s\text{-}\mathrm{Lim\,inf}\, \mathbb{K}_\varepsilon$. Since $\mathbb{K}_\varepsilon \subseteq \mathbb{K}$ for all ε, we then have $\mathbb{K} = \mathrm{Lim}\ \mathbb{K}_\varepsilon$.

PROOF OF COROLLARY 2 OF THEOREM 1. By the LEMMA above $\mathbb{K} = \mathrm{Lim}\, \mathbb{K}_\varepsilon$, hence (ii) of COROLLARY 1 follows from THEOREM 1.

12. In this section we prove THEOREM 3 and its corollaries.

PROOF OF THEOREM 3. Clearly assumption I′, II′ and III′ of the theorem at hand imply conditions I, II and III assumed in THEOREM 2. Therefore, since a_ε is coercive on V and the solution u_ε of (p_ε) is supposed to remain in a bounded subset of V as $\varepsilon \to 0$, we can apply THEOREM 2 and obtain that the set of all weak cluster points of u_ε as $\varepsilon \to 0$ is a non-empty subset of X. Furthermore we prove that any weak cluster point u of u_ε actually coincides with the (unique) solution u_0 of problem (p_0). We have for any ε

$$a_\varepsilon = a + \gamma_\varepsilon$$

with

$$\gamma_\varepsilon = \varepsilon\beta + \varepsilon(\beta_\varepsilon - \beta)$$

and

$$v'_\varepsilon = v' + \psi'_\varepsilon$$

with

$$\psi'_\varepsilon = \varepsilon\varphi' + \varepsilon(\varphi'_\varepsilon - \varphi).$$

Thus the inequality (1) in PROOF of THEOREM 2 becomes

$$a(u_\varepsilon, u_\varepsilon - v) \leq \langle v', u_\varepsilon - v \rangle + M'_\varepsilon(v) + M''_\varepsilon(v) \tag{6}$$

with

$$M'_\varepsilon(v) = \gamma_\varepsilon(u_\varepsilon, v - u_\varepsilon) + \langle \psi'_\varepsilon, u_\varepsilon - v \rangle$$

$$M''_\varepsilon(v) = a_\varepsilon(u_\varepsilon, P_{\mathbb{K}_\varepsilon} v - v) + \langle v'_\varepsilon, v - P_{\mathbb{K}_\varepsilon} v \rangle.$$

On the other hand, it follows from (p) that for any $v \in X$ and all ε

$$a(v, v - u_\varepsilon) \leq \langle v', v - u_\varepsilon \rangle + M'''_\varepsilon(v) \tag{7}$$

with

$$M'''_\varepsilon(v) = a(v, w_\varepsilon - u_\varepsilon) + \langle v', u_\varepsilon - w_\varepsilon \rangle$$

provided $w^\varepsilon \in \mathbb{K}$.

Therefore, adding (6) to (7) we obtain, since a is positive, that

$$\gamma_\varepsilon(u_\varepsilon, u_\varepsilon) \leq \gamma_\varepsilon(u_\varepsilon, v) + \langle \psi'_\varepsilon, u_\varepsilon - v \rangle + M''_\varepsilon(v) + M'''_\varepsilon(v)$$

for any $v \in X$ and all ε. Replacing γ_ε and ψ'_ε by their explicit expressions, we find for any $v \in X$

$$\beta(u_\varepsilon, u_\varepsilon) \leq \beta(u_\varepsilon, v) + \langle \varphi', u_\varepsilon - v \rangle + M_\varepsilon(v) \tag{8}$$

with

$$M_\varepsilon(v) = (\beta_\varepsilon - \beta)(u_\varepsilon, v - u_\varepsilon) + \langle \varphi'_\varepsilon - \varphi', u_\varepsilon - v \rangle +$$
$$+ \varepsilon^{-1} M''_\varepsilon(v) + \varepsilon^{-1} M'''_\varepsilon(v).$$

Let now u be a weak cluster point of u_ε, that is $u = w\text{-}\lim u_\eta$ as $\eta \to 0$. We know that $u \in X$ and we want to prove, as we said at the beginning of this proof, that $u = u_0$. Let us assume for a moment that for some subsequence of u_η, say u_η, we have

$$\limsup M_\eta(v) \leq 0 \qquad \text{as } \eta \to 0 \text{ for any } v \in X. \tag{9}$$

Then we can conclude the proof of the theorem as follows. In consequence of (9) we obtain from (8) by Lemma 3.1 of [3] that

$$\beta(u, u) \leq \beta(u, v) + \langle \varphi', u - v \rangle \qquad \text{for all } v \in X.$$

Therefore $u = u_0$, hence u_0 is the unique weak cluster point of u_ε. Since u_ε is supposed to remain in a bounded subset of V as $\varepsilon \to 0$, we have that u_ε converges weakly to u_0 as $\varepsilon \to 0$. Actually u_ε converges strongly to u_0 as $\varepsilon \to 0$. The proof of this follows from (8) and (9) using a similar argument to the one given in the proof of THEOREM 1, thus we omit here the details.

Therefore we have only to prove (9). By the assumption that $\mathbb{K}_\varepsilon$ converges of order ε to $\mathbb{K}$ as $\varepsilon \to 0$ we know that for some subsequence of u_η, say u_η, we have $w\text{-}\lim \eta^{-1}(u_\eta - w_\eta) = 0$ as $\eta \to 0$ for suitable $w_\eta \in \mathbb{K}$, and also $\lim \eta^{-1} \| P_{\mathbb{K}_\eta} v - v \| = 0$ as $\eta \to 0$. Let us first consider case (u) of I'. Since $\lim \| \beta_\eta - \beta \| = 0$, $\lim \| \varphi'_\eta - \varphi' \| = 0$ and $\| u_\eta \|$ is bounded as $\eta \to 0$, then the first and the second term in the expression of $M_\eta(v)$ converge to zero as $\eta \to 0$. Moreover we have

$$\eta^{-1} | M''_\eta(v) | \leq \| a_\eta \| \, \| u_\eta \| \, \eta^{-1} \| P_{\mathbb{K}_\eta} v - v \| + \| v'_\eta \| \, \eta^{-1} \| v - P_{\mathbb{K}_\eta} v \|$$

$$\eta^{-1} | M'''_\eta(v) | \leq | a(v, \eta^{-1}(w_\eta - u_\eta)) | + | \langle v', \eta^{-1}(u_\eta - w_\eta) \rangle |.$$

Since $\| a_\eta \|$, $\| v'_\eta \|$ and $\| u_\eta \|$ are bounded as $\eta \to 0$ we then find

$$\lim \eta^{-1} M''_\eta (v) = 0 \quad \text{as} \quad \eta \to 0$$
$$\lim \eta^{-1} M'''_\eta (v) = 0 \quad \text{as} \quad \eta \to 0$$

for any $v \in X$. Therefore $\lim M_\eta (v) = 0$ as $\eta \to 0$ for any $v \in X$ and (9) is proved. Let us now consider case (s) of I'. Since $\beta_\eta - \beta$ is positive for any η, we have

$$M_\eta (v) \leq \widetilde{M}_\eta (v)$$

where

$$\widetilde{M}_\eta (v) = (\beta_\eta - \beta)(u_\eta, v) + \langle \varphi'_\eta - \varphi', u_\eta - v \rangle + \eta^{-1} M''_\eta (v) + \eta^{-1} M''' (v).$$

Therefore it suffices to prove that

$$\lim |\widetilde{M}_\eta (v)| = 0 \text{ as } \eta \to 0 \text{ for any } v \in X.$$

In fact by (s) and the fact that u_η is bounded as $\eta \to 0$ we obtain $\lim (\beta_\eta - \beta)(u_\eta, v) = 0$. Moreover $\lim \langle \varphi'_\eta - \varphi', u_\eta - v \rangle = 0$ by II'. As in case (u) above we also prove that $\lim \eta^{-1} M'''_\eta (v) = 0$. Finally, since $\| a_\eta \|$ is bounded as a consequence of Banach-Stheinaus Theorem, the proof of $\lim \eta^{-1} M''_\eta (v) = 0$ as $\eta \to 0$ also is along the line of the proof given in case (u).

Therefore, the proof of (7), and hence of THEOREM 3, is now complete.

PROOF OF COROLLARY 1 OF THEOREM 3. For any $R \geq R_0$, where R_0 is such that $\mathbb{K}^{R_0} \neq \Phi$, we shall denote by

$$X^{(R)}$$

the non-empty closed convex subset of $\mathbb{K}^R$ of all solutions of

$$(\mathrm{p}_R) \qquad \begin{cases} u \in \mathbb{K}^R \\ a(u, v - u) \geq \langle v', v - u \rangle \quad \text{for all} \quad v \in \mathbb{K}^R. \end{cases}$$

Assuming $\mathbb{K} = \mathbb{K}^R$ in THEOREM 3, for any $R \geq R_0$, we find that u_ε converges strongly to u and moreover that u belongs to $X^{(R)}$ and satisfies the inequality

$$\beta(u, v - u) \geq \langle \varphi', v - u \rangle \quad \text{for all} \quad v \in X^{(R)}.$$

Since $\| u \| < R$, Theorem 4.2 of [3] implies that $u \in X$. In particular, if

$$X^R = \{ v \in X : \| v \| \leq R \},$$

we have $u \in X^R$ and

$$\beta(u, v - u) \geq \langle \varphi', v - u \rangle \quad \text{for all} \quad v \in X^R,$$

for clearly $X^R \subseteq X^{(R)}$ for any R. Thus u coincides with the unique solution in X^R of the inequality above. Therefore, applying again Theorem 4.2 of [3], we have $u = u_0$.

PROOF OF COROLLARY OF THEOREM 3. By applying THEOREM 3 again, both for $i = 1$ and $i = 2$, we obtain as above that $u_1, u_2 \in X^{(R)}$ and that $u_{i\varepsilon}$ converges strongly to u_i as $\varepsilon \to 0$, $i = 1, 2$. Therefore there is some θ, $0 \leq \theta \leq 1$, such that the vector $u = (1 - \theta) u_1 + \theta u_2$ — which belongs to $X^{(R)}$ for $X^{(R)}$ is a convex set — satisfies the inequality $\| u \| < R$. Hence, again by Theorem 4.2 of [3], we obtain that $u \in X$.

13. Finally we prove THEOREM 4.

PROOF OF THEOREM 4. Let v be a vector of X. From (p) and (p$_\varepsilon$) it follows, as in the proof of THEOREM 3, that

$$\gamma_\varepsilon(u_\varepsilon, u_\varepsilon) \leq \gamma_\varepsilon(u_\varepsilon, v) + \langle \psi'_\varepsilon, u_\varepsilon - v \rangle + M''_\varepsilon(v) + M'''_\varepsilon(v)$$

with

$$M''_\varepsilon(v) = a_\varepsilon(u_\varepsilon, P_{\mathbb{K}_\varepsilon} v - v) + \langle v'_\varepsilon, v - P_{\mathbb{K}_\varepsilon} v \rangle$$

$$M'''_\varepsilon(v) = a(v, P_{\mathbb{K}} u_\varepsilon - u_\varepsilon) + \langle v', u_\varepsilon - P_{\mathbb{K}} u_\varepsilon \rangle .$$

Therefore we have

$$c_\varepsilon \| u_\varepsilon \|^2 \leq \gamma_\varepsilon(u_\varepsilon, u_\varepsilon) \leq \| \gamma_\varepsilon \| \, \| u_\varepsilon \| \, \| v \| + \| \psi'_\varepsilon \| (\| u_\varepsilon \| + \| v \|) +$$

$$+ | M''_\varepsilon(v) | + | M'''_\varepsilon(v) |$$

with

$$| M''_\varepsilon(v) | \leq (\| a \| + \| \gamma_\varepsilon \|) \, \| u_\varepsilon \| \, \| P_{\mathbb{K}_\varepsilon} v - v \| + (\| v' \| + \| \psi'_\varepsilon \|) \, \| v - P_{\mathbb{K}_\varepsilon} v \|$$

$$| M'''_\varepsilon(v) | \leq \| a \| \, \| v \| (1 + \| u_\varepsilon \|) \, S(\mathbb{K}, \mathbb{K}_\varepsilon) + \| v' \| (1 + \| u_\varepsilon \|) \, S(\mathbb{K}, \mathbb{K}_\varepsilon).$$

Hence by I″, II″ and III″ we find

$$\| u_\varepsilon \|^2 \leq c (1 + \| u_\varepsilon \|) \qquad \text{for some } c > 0 \text{ and all } \varepsilon \leq \varepsilon_0 .$$

Therefore u_ε remains in a bounded subset of V as $\varepsilon \to 0$.

14. Our object in this final section is to clarify the meaning of the convergence notions we have introduced by DEFINITION 1 and DEFINITION 2. To this end we shall consider: *a*) A very simple application of THEOREM 1

to a « perturbed » Dirichlet problem for an elliptic partial differential operator of second order; *b*) A trivial example which shows that the assumption of « order ε » convergence of the $\mathbb{K}_\varepsilon$'s, in the sense of DEFINITION 2, cannot be weakened and replaced by the simple convergence of $\mathbb{K}_\varepsilon$'s, in the sense of DEFINITION 1, without infirming the general validity of THEOREM 3.

a) Let Ω be a bounded open set in the euclidean n-space E^n, $\partial\Omega$ being the boundary of Ω. The space $H_0^1(\Omega)$ is defined as usual, see for istance Ref (4) and $H^{-1}(\Omega)$ is the strong dual of $H_0^1(\Omega)$. We assume that the norm in $H_0^1(\Omega)$ is given by

$$\| v \|_1 = \int_\Omega | v_x |^2 \, dx$$

where $v_x = (v_{x_1}, \dots, v_{x_n})$, $v_{x_i} = \dfrac{\partial v}{\partial x_i}$, and we denote by $\langle .,. \rangle$ the pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. It turns out that

$$\langle v, T \rangle = \sum_i^{1,n} \int_\Omega f_i \, v_{x_i} \, dx$$

for any $v \in H_0^1(\Omega)$ and any $T \in H^{-1}(\Omega)$, $T = \sum_i^{1,n} (f_i)_{x_i}$, $f_i \in L^2(\Omega)$.

Let L be an elliptic partial differential operator of second order of type

$$Lu = - \sum_{i,j}^{1,n} (a_{ij} \, u_{x_i})_{x_j}$$

where a_{ij} are bounded mesurable functions on Ω satisfying the condition $\sum_{i,j}^{1,n} a_{ij} \xi_i \xi_j \geq c \, | \xi |^2$ for some $c > 0$ and all $\xi = (\xi_1, \dots, \xi_n)$, $| \xi |^2 = \sum_i^{1,n} \xi_i^2$.

Let $u \in H_0^1(\Omega)$ be the solution of the Dirichlet problem

$$\text{(d)} \qquad \begin{cases} Lu = T & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where T is a fixed distribution of $H^{-1}(\Omega)$.

Now let E_ε, $\varepsilon > 0$, be a compact subset of Ω and let $u_\varepsilon \in H_0^1(\Omega)$ be the solution of the problem

$$(\mathrm{d}_\varepsilon) \qquad \begin{cases} Lu_\varepsilon = T & \text{in} \quad \Omega \\ u_\varepsilon = 0 & \text{on} \quad \partial(\Omega - E_\varepsilon), \quad u_\varepsilon \equiv 0 \quad \text{in} \quad E_\varepsilon . \end{cases}$$

We recall that for any compact subset E of Ω the capacity of E, cap E, is defined by

$$\operatorname{cap} E = \inf \{ \| \alpha \|_1 : \alpha \in H_0^1(\Omega), \ \alpha \geq 1 \ \text{on} \ E \}$$

where $\alpha \geq 1$ on E is intended in the sense of $H_0^1(\Omega)$, see Ref (4).

By applying THEOREM 1 we can prove the following

THEOREM 5. *If* cap $E_\varepsilon \to 0$ *as* $\varepsilon \to 0$, *then the solution* u_ε *of* (d_ε) *converges strongly to the solution* u *of* (d) *in* $H_0^1(\Omega)$.

PROOF. Let a be the (coercive continuous bilinear) form

$$a(u, v) = \int_\Omega a_{ij} u_{x_i} v_{x_j} \, dx, \qquad u, v \in H_0^1(\Omega).$$

Then the solution u of (d) is also the unique solution of

$$\begin{cases} u \in H_0^1(\Omega) \\ a(u, v - u) \geq \langle T, v - u \rangle \ \text{for all} \ v \in H_0^1(\Omega) \end{cases}$$

see [3]. Since $H_0^1(\Omega - E_\varepsilon)$ is canonically isomorphic to the subspace of all functions of $H_0^1(\Omega)$ which vanish on E_ε, we also have that the solution u_ε of (d_ε) is the unique solution of

$$\begin{cases} u \in H_0^1(\Omega - E_\varepsilon) \\ a(u_\varepsilon, v - u_\varepsilon) \geq \langle T, v - u_\varepsilon \rangle \ \text{for all} \ v \in H_0^1(\Omega - E_\varepsilon) . \end{cases}$$

Therefore the theorem can be proved by applying THEOREM 1, provided we prove the following

LEMMA. *If* cap $E_\varepsilon \to 0$ *as* $\varepsilon \to 0$, *then* $H_0^1(\Omega) = \operatorname{Lim} H_0^1(\Omega - E_\varepsilon)$ *in the sense of* DEFINITION 1.

Proof. We only have to prove that

$$H_0^1(\Omega) \subseteq s\text{-}\lim\inf H_0^1(\Omega - E_\varepsilon) \quad \text{as} \quad \varepsilon \to 0,$$

that is for any $v \in H_0^1(\Omega)$ there exists $v_\varepsilon \in H_0^1(\Omega - E_\varepsilon)$ such that $\|v - v_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$.

Let v_ε be the projection of v on $H_0^1(\Omega - E_\varepsilon)$, which is to say v_ε is the solution of

$$\begin{cases} v_\varepsilon \in H_0^1(\Omega - E_\varepsilon) \\ (v_\varepsilon - v, w - v_\varepsilon)_1 \geq 0 \text{ for all } w \in H_0^1(\Omega - E_\varepsilon) \end{cases}$$

$(.,.)_1$ being the inner product of $H_0^1(\Omega)$, see Sec 2. Then it is easy checked that the function

$$w_\varepsilon = v - v_\varepsilon$$

satisfies

$$\begin{cases} w_\varepsilon = v \text{ on } E_\varepsilon \\ \Delta w_\varepsilon = 0 \text{ on } \Omega - E_\varepsilon . \end{cases}$$

Δ being the Laplace operator.

Since cap $E_\varepsilon \to 0$ as $\varepsilon \to 0$, there exists for any ε a function $\alpha_\varepsilon \in H_0^1(\Omega)$, such that $\|\alpha_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$, $\alpha_\varepsilon \leq 1$ on Ω and $\alpha_\varepsilon \equiv 1$ on E_ε in the sense of $H_0^1(\Omega)$ (see Lemma (1.2) of Ref. (4)). Let us assume for a moment that v is bounded on Ω, i.e. $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Since $w_\varepsilon = \alpha_\varepsilon v$ on E_ε and w_ε is armonic in $\Omega - E_\varepsilon$, we have by the minimum principle

$$\|w_\varepsilon\|_1 \leq \|\alpha_\varepsilon v\|_1 \qquad \text{for any } \varepsilon.$$

Moreover

$$\|\alpha_\varepsilon v\|_1 \leq \sup_\Omega |v| \, \|\alpha_\varepsilon\|_1 + \left(\int_\Omega |\alpha_\varepsilon v_x|^2 dx\right)^{1/2} .$$

Let η be a fixed positive number. For any ε let $I_{\varepsilon\eta}$ be the subset of Ω where $\alpha_\varepsilon \geq \eta$ in the sense of $H_0^1(\Omega)$. Then we have

$$\left(\int_\Omega |\alpha_\varepsilon v_x|^2 dx\right)^{1/2} \leq \eta \|v\|_1 + \left(\int_{I_{\varepsilon\eta}} |\alpha_\varepsilon v_x|^2 dx\right)^{1/2}$$

$$\leq \eta \|v\|_1 + \left(\int_{I_{\varepsilon\eta}} |v_x|^2 dx\right)^{1/2} .$$

We note that cap $I_{\varepsilon\eta} \to 0$ as $\varepsilon \to 0$. In fact, $\frac{1}{\eta} \min\{\alpha_\varepsilon, \eta\} \equiv 1$ on $I_{\varepsilon\eta}$ for any ε, hence cap $I_{\varepsilon\eta} \leq \frac{1}{\eta} \| \alpha^\varepsilon \|_1$. Thus we have

$$\int_{I_{\varepsilon\eta}} | v_x |^2 \, dx \to 0 \text{ as } \varepsilon \to 0$$

therefore

$$\lim_{\varepsilon \to 0} \left(\int_\Omega | \alpha_\varepsilon v_x |^2 \, dx \right)^{1/2} \leq \eta \| v \|_1 .$$

Since $\eta > 0$ is arbitrary we find that

$$\left(\int_\Omega | \alpha_\varepsilon v_x |^2 \, dx \right)^{1/2} \to 0 \text{ as } \varepsilon \to 0 .$$

Therefore we conclude that $\| w_\varepsilon \|_1 \to 0$ as $\varepsilon \to 0$, that is v_ε converges strongly to v in $H_0^1(\Omega)$ as $\varepsilon \to 0$.

Up to this point we have proved that $H_0^1(\Omega) \cap L^\infty(\Omega) \subseteq s\text{-Lim inf}\, H_0^1(\Omega - E_\varepsilon)$ However since $s\text{-lim inf}\, H_0^1(\Omega - E_\varepsilon)$ is closed and $H_0^1(\Omega) \cap L^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, we also have $H_0^1(\Omega) \subseteq s\text{-Lim inf}\, H_0^1(\Omega - E_\varepsilon)$ and the LEMMA is proved.

(*b*) Let us consider the following example :

V is the euclidean space of all vectors $v = \{v_1, v_2\}$, $v_1, v_2 \in \mathbf{R}$: $(u, v) = u_1 v_1 + u_2 v_2$ for $u, v \in V$, a is the form

$$a(u, v) = u_2 v_2 , \qquad u, v \in V,$$

$v' = \{v_1', v_2'\}$ is a fixed vector of $V' \equiv V$, and $\mathbb{K}$ is the straight line

$$\mathbb{K} = \{v \in V : v_2 = 0\}.$$

Problem (p) reduces now to find a vector $u = \{u_1, 0\}$ such that $0 \geq v_1'(v_1 - u_1)$ for all $v_1 \in \mathbf{R}$. Such a solution u exists only if $v_1' = 0$, in which case each vector $u = \{u_1, 0\}$ of $\mathbb{K}$ is a solution.

Let us consider now for any $\varepsilon > 0$ the form

$$a_\varepsilon(u, v) = a(u, v) + \varepsilon(u, v) = (1 + \varepsilon) u_2 v_2 + \varepsilon u_1 v_1 , \qquad u, v \in V$$

and the vector

$$v_\varepsilon' = v' + \varepsilon\varphi',$$

where $\varphi' = \{\varphi_1', \varphi_2'\}$ is a fixed vector of $V' \equiv V$. Moreover, for any fixed $\alpha > 0$, let $\mathbb{K}_\varepsilon$, $\varepsilon > 0$, be the straight line

$$\mathbb{K}_\varepsilon = \{v \in V : v_2 = \eta v_1 \, , \, \eta = \varepsilon^\alpha\}.$$

Then problem (p_ε) has for any $\varepsilon > 0$ a unique solution $u_\varepsilon = \{u_1^\varepsilon, u_2^\varepsilon\}$ and one finds

$$u_1^\varepsilon = \frac{v_1' + \varepsilon\varphi_1' + \eta\,(v_\varepsilon' + \varepsilon\varphi_2')}{\varepsilon + (1+\varepsilon)\,\eta^2} = \frac{\varepsilon^{-1}\, v_1' + \varphi_1' + \varepsilon^{\alpha-1}(v_\varepsilon' + \varepsilon\varphi_2')}{1 + (1+\varepsilon)\,\varepsilon^{2\alpha-1}} \, .$$

Therefore we can draw the following conclusions:

It $v_1' \neq 0$, in which case (p) has no solution, then for any fixed $\alpha > 0$, $\mathbb{K}_\varepsilon$ converges to $\mathbb{K}$ in the sense of DEFINITION 1 but the solution u_ε of (p_ε) diverges as $\varepsilon \to 0$, i.e. $u_1^\varepsilon \to \infty$ as $\varepsilon \to 0$. This agrees with THEOREM 2.

If $v_1' = 0$ then we have:

(i) If $0 < \alpha < 1$: $\mathbb{K}_\varepsilon$ converges to $\mathbb{K}$, but u_ε still diverges as $\varepsilon \to 0$. In particular, this shows that condition III$''$ in THEOREM 4 cannot be replaced by condition III of Sec. 4.

(ii) If $\alpha = 1$: $\mathbb{K}_\varepsilon$ converges to $\mathbb{K}$, moreover condition III$''$ of Sec. 8 is satisfied: u_ε is now bounded as $\varepsilon \to 0$. This agrees with THEOREM 4. Actually u_ε converges to the vector $\{\varphi_1' + v_2', 0\}$, which shows that condition III$'$ in THEOREM 3 cannot be weakened.

(iii) If $\alpha > 1$: $\mathbb{K}_\varepsilon$ converges of order ε to $\mathbb{K}$ in the sense of DEFINITION 2 : u_ε converges to the vector $\{\varphi_1', 0\}$ as $\varepsilon \to 0$. This agrees completely with THEOREM 3. We note finally that any solution $u = \{u_1, 0\}$ of (p) can be obtained as a limit of approximate u_ε, by choosing φ' with $\varphi_2' = u_1$.

* * *

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Università di Pisa

REFERENCES

(1) See [1] G. STAMPACCHIA, *Formes bilinéaires coercitives sur les ensembles convexes,* C. R. Acad. Sc. Paris, t. 258 (1964), p. 4413-4416 ; [2] J. L. LIONS and G. STAMPACCHIA, *Inéquations variationnelles non coercives* C. R. Acad. Sc. Paris, t. 261 (1965), p. 25-27 ; [3] J. L. LIONS and G. STAMPACCHIA, *Variational Inequalities,* to appear. For the « elliptic regularization » see also J. L. LIONS, *Some aspects of operator differential equations,* Lectures at C.I.M.E., Varenna, May 1963.

(2) See for instance G. T. WHYBURN, *Analytic Topology,* Amer. Math. Soc. Colloq. Publs., Vol. 26, 1942, p. 10 or C. BERGE, *Espaces topologiques,* 1959, Dunod, Paris p. 124.

(3) A similar argument has been used by P. HARTMANN and G. STAMPACCHIA to prove the existence of the solution of a non linear variational inequality, see P. H. - G. S., *On some non-linear elliptic differential functional equations,* Acta Mat. Vol. 115, 1966.

(4) W. LITTMAN, G. STAMPACCHIA, H. F. WEINBERGER *Regular points for elliptic equations with discontinuous coefficients,* Ann. Sc. Norm. Sup. Pisa XVII (1963), p. 45-79.