

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

JÖRAN FRIBERG

Multi-quasielliptic polynomials

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3^e série, tome 21,
n° 2 (1967), p. 239-260

http://www.numdam.org/item?id=ASNSP_1967_3_21_2_239_0

© Scuola Normale Superiore, Pisa, 1967, tous droits réservés.

L'accès aux archives de la revue « *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze* » (<http://www.sns.it/it/edizioni/riviste/annaliscienze/>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
<http://www.numdam.org/>

MULTI-QUASIELLIPTIC POLYNOMIALS

JÖRAN FRIBERG

0. Introduction.

The aim of this paper is to introduce a new relatively simple class of hypoelliptic polynomials, including the quasi-elliptic polynomials studied before by a number of authors (Hörmander [10], Friberg [3], Grušin [8], Volevič [14], ...), but also more complicated examples like polynomials $P(\xi_1, \xi_2)$ of the form

$$(0.1) \quad A \xi_1^{2p} + B \xi_2^{2q} + C \xi_1^{2r} \xi_2^{2s}, \quad A, B, C > 0,$$
$$p > r, \quad q > s, \quad r/p + s/q > 1,$$

considered by Gorenko [6] and Pini [13].

In section 1 are recalled a couple of well-known algebraic necessary and sufficient conditions for hypoellipticity, given by Hörmander [9] and Gorin [7], and involving parameters $b = (b_1, \dots, b_n)$, or $a = (a_{jk}), j, k = 1, \dots, n$, respectively. It is shown how the parameters can be explicitly determined for quasielliptic polynomials, i. e. for polynomials $Q(\xi), \xi \in R^n$, which are, in the sense of Hörmander, of the same strength as a function $\varrho_m(\xi) = \sum_1^n |\xi_k|^{m_k}$, m_k positive integers. The results are then carried over to products $\prod_1^l Q^j$ of such polynomials, which are, consequently, of the same strength as some function

$$(0.2) \quad \prod_{j=1}^l \varrho_{m^j}(\xi) = \sum_{i=1}^N |\xi^{\alpha^i}|, \quad \alpha^i = (\alpha_1^i, \dots, \alpha_n^i), \quad \alpha_j^i \text{ integers } \geq 0.$$

Pervenuto alla Redazione il 3 Febbraio 1966 ed in forma definitiva il 15 Giugno 1966.

In section 2 we consider the convex «Newton polyhedron» $F(P)$ of a polynomial $P(\xi) = \sum c_\alpha \xi^\alpha$. If $n = 2$, $F(P)$ reduces to the Newton polygon for $P(\xi)$, commonly used to determine the Puiseux-expansions of the zeros of $P(\xi)$. (Cf. Friberg [5]). Generalizing what is a well-known fact for $n = 2$, we show that all flat faces of $F(P)$ have strictly positive normals when $P(\xi)$ is hypoelliptic. Then we determine the general structure of the Newton polyhedron for a product of polynomials with simple Newton polyhedrons (as e. g. quasielliptic polynomials), and we investigate the connection between the shape of $F(P)$ and the growth of $P(\xi)$ in various directions.

In section 3 is given the definition of multi-quasiellipticity. Essentially a hypoelliptic polynomial $P(\xi)$ is multi-quasielliptic (mqe) if it is of the same strength as a function $\varrho(\xi) = \sum_1^N |\xi^{\alpha^i}|$, for a set of non-negative integer-valued vectors $\{\alpha^i\}$ ⁽¹⁾.

Consequently, products of quasielliptic polynomials are mqe (Cf. (0.2)), and we can prove that, conversely, a multi-quasielliptic polynomial is of the same strength as such a product when $n = 2$, but not in general when $n > 2$. As an example of the more-dimensional case, we study a class of hypoelliptic operators, introduced by Cattabriga [2], which are shown to be mqe, of a strongly degenerate type ⁽²⁾. Finally we prove the hypoellipticity of mqe polynomials, find estimates for their hypoellipticity parameters a and b , and give a definition of a principal part for an arbitrary mqe polynomial.

1. Hypoellipticity of products of quasielliptic polynomials.

The notion of $\binom{k}{j}$ -hypoellipticity was introduced by Gorin [7]. Let $P(D)$, $D = i^{-1}(\partial/\partial x_1, \dots, \partial/\partial x_n)$, be a differential operator with constant coefficients, $P(\xi)$ its characteristic polynomial, $\xi = (\xi_1, \dots, \xi_n)$. Let, for $1 \leq k \leq n$,

$$N^k(P) = \{\xi + i\eta \in C^n; P(\xi + i\eta) = 0, \eta = (0, \dots, \eta_k, \dots, 0)\}.$$

⁽¹⁾ After submitting the note [4], I became aware that a similar criterion had been used by Nikolskii [11], for definition of a class of differential operators connected with certain boundary value problems.

⁽²⁾ The present work originated in a discussion with L. Cattabriga and G. C. Barozzi about some problems raised by their papers [1], [2].

Then $P(D)$ is $\binom{k}{j}$ -hypoelliptic of type $a_{jk} > 0$, if $1 \leq j \leq n$ and the characteristic polynomial $P(\xi)$ satisfies the condition.

$$(1.1) \quad |\xi_j| \leq C(1 + |\eta_k|)^{a_{jk}} \quad \text{for all } \xi + i\eta \in N^k(P).$$

If we assume that always $a_{jj} \geq 1$, then (1.1) is equivalent to the condition

$$(1.2) \quad |(\partial/\partial \xi_k)^l P(\xi)| \leq C(1 + |P(\xi)|)(1 + |\xi_j|)^{-l a_{jk}} \quad \text{for all } \xi \in R^n, \text{ all } l.$$

If $P(\xi)$ satisfies (1.1) for $j = 1, \dots, n$, then every tempered elementary solution E for $P(D)$ is infinitely differentiable outside every strip $|x_k| \leq e$, and satisfies there locally an estimate of the type (see Grusin [8])

$$|D^\alpha E(x)| \leq C^{|\alpha|+1} \Gamma(\langle a^k, \alpha \rangle) |x_k|^{m_k - \langle a^k, \alpha + e \rangle} + C_1(1 + |x|)^N,$$

for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$. Here $e = (1, \dots, 1)$, $a^k = (a_{1k}, \dots, a_{nk})$, and $m_k = \deg P(0, \dots, \xi_k, \dots, 0)$. It follows, in particular, that if $P(D)$ is $\binom{k}{j}$ -hypoelliptic of type a_{jk} for $1 \leq j, k \leq n$, then it is hypoelliptic in the sense of Hörmander [10], all solutions of $P(D)u = 0$ being infinitely differentiable and satisfying, locally, estimates of Gevrey type:

$$|D^\alpha u(x)| \leq C^{|\alpha|+1} \max_k \Gamma(\langle a^k, \alpha \rangle) \quad \text{for all } \alpha.$$

In other words P is then hypoelliptic of type $a = (a_1, \dots, a_n)$, $a_j = \max_k a_{jk}$.

Consequently, the estimate (1.2) can be extended to (Hörmander [10])

$$(1.2') \quad |P^{(\beta)}(\xi)| \leq C(1 + |P(\xi)|)(1 + \sum |\xi_j|^{1/a_j})^{-|\beta|}, \quad \text{all } \beta, \xi \in R^n.$$

Here $P^{(\beta)}(\xi) = (\partial/\partial \xi)^{\beta} P(\xi)$, $\beta = (\beta_1, \dots, \beta_n)$, and $|\beta| = \sum \beta_j$. It follows from (1.2') that

$$(1.3) \quad (\sum |\xi_j|^{1/a_j})^m \leq C(1 + |P(\xi)|), \quad \text{real } \xi, m = \deg P(\xi).$$

Besides (1.2') we also have the following necessary and sufficient condition for hypoellipticity (Cf. Hörmander [9]):

$$(1.4) \quad |P^{(\beta)}(\xi)| \leq C(1 + |P(\xi)|)^{1 - \langle b, \beta \rangle}, \quad |P(\xi)| \rightarrow \infty \text{ as } |\xi| \rightarrow \infty, \xi \text{ real},$$

for all β , and for some $b = (b_1, \dots, b_n) > 0$ such that $\langle b, \beta \rangle \leq 1$ when $P^{(\beta)}(\xi) \neq 0$, hence in particular $0 < b_j \leq 1/m_j, j = 1, \dots, n$.

For instance, if P is elliptic of order m , i. e. if

$$P(\xi) = \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha = P_0(\xi) + \sum_{|\alpha| < m} c_\alpha \xi^\alpha \text{ with } P_0(\xi) \neq 0 \text{ for real } \xi \neq 0$$

then (1.2'), (1.4) hold, with $a = (a_1, \dots, a_n) = (1, \dots, 1), b = (1/m, \dots, 1/m)$. Slightly more general are the quasielliptic polynomials of weight $q = (q_1, \dots, q_n)$, defined as follows (Hörmander [10]):

i) If $P(\xi) = \sum c_j \xi_j^{m_j} +$ mixed or lower order terms, then $q_j = m/m_j$ with $m = \max m_j$, and $P(\xi) = \sum_{\langle q, \alpha \rangle \leq m} c_\alpha \xi^\alpha = P_0(\xi) + \sum_{\langle q, \alpha \rangle < m} c_\alpha \xi^\alpha$.

ii) $P_0(\xi) \neq 0$ for real $\xi \neq 0$.

For such polynomials, due to the quasi-homogeneity of $P_0(\xi)$,

$$(1.5) \quad C^{-1} \leq |P_0(\xi)| / \sum_1^n |\xi_j^{m_j}| \leq C, \quad \text{real } \xi \neq 0.$$

It follows that, if $\langle q, \alpha \rangle \leq m$, then

$$|\xi^\alpha| = \prod |\xi_j^{m_j}|^{\alpha_j/m_j} \leq \varepsilon (\sum |\xi_j^{m_j}|)^{\langle q, \alpha \rangle / m}, \quad \xi \text{ real, } |\xi| > c(\varepsilon),$$

where ε can be arbitrarily small when $\langle q, \alpha \rangle < m$. Hence (1.5) still holds if P_0 is replaced by P . But

$$|P^{(\beta)}| = \sum_{\langle q, \alpha \rangle \leq m - \langle q, \beta \rangle} c_{\alpha, \beta} \xi^\alpha \leq C \varepsilon (\sum |\xi_j^{m_j}|)^{1 - \langle q, \beta \rangle / m}$$

and it follows that, for $|\xi| > c(\varepsilon), \beta \neq 0$,

$$(1.6) \quad |P^{(\beta)}(\xi)| \leq C \varepsilon |P_0(\xi)|^{1 - \langle q, \beta \rangle / m} \leq C_1 \varepsilon |P_0(\xi)| (\sum |\xi_i^{m_i}|)^{-\langle q, \beta \rangle / m}.$$

Hence P is hypoelliptic of type $a_{jk} = m_k/m_j, a_j = m/m_j = q_j$, and $b_j = q_j/m = 1/m_j$. That the derived values for the a_{jk} are the best possible will be shown in Theorem 4.1. (Cf. also Grušin [8]).

Next, consider P a product of N quasielliptic polynomials Q^i , of weight $q^i, 1 \leq i \leq N$. Then P is hypoelliptic. (A product of polynomials is hypoelliptic if and only if each factor is hypoelliptic). Moreover,

$$(1.7) \quad a_{jk}(P) = \max_i a_{jk}(Q^i) = \max_i (m_k^i / m_j^i).$$

It follows that

$$a_{jk}(P) a_{kj}(P) = \max_i (m_k^i / m_j^i) / \min_i (m_k^i / m_j^i) > 1,$$

except when $a_{jk}(Q^i) = m_k^i/m_j^i$ does not depend on i . But evidently $a_{kj}(P)a_{jk}(P) \leq 1$ if P is quasielliptic. Hence we see that a product of quasielliptic polynomials is quasielliptic (if and) only if all the factors are of proportional weights. (Cf. Barozzi [1]). In contrast to the quasielliptic polynomials, the product P may very well have no $a_{jk} \leq 1$, for $j \neq k$.

If $P = \Pi Q_i$, $Q^i = Q_0^i + R^i$ quasielliptic, and if $P_0 = \Pi Q_0^i$, then it follows from (1.5) applied to each Q_0^i , that

$$(1.8) \quad C^{-1} \leq |P_0(\xi)| / \Pi \sum_{i=1}^n |\xi_j^{m_j^i}| \leq C, \quad \text{real } \xi \neq 0.$$

Since

$$(P - P_0)/P_0 = \Pi (1 + R^i/Q_0^i) - 1 = O(1) |\xi|^{-c} \quad \text{for some } c > 0,$$

as $|\xi| \rightarrow \infty$, ξ real, it follows that $P_0(\xi)$ can be replaced by $P(\xi)$ in (1.8) for $|\xi|$ big enough. (Thus the lower estimate (1.3) is best possible only if P is quasielliptic). Let us put $b^i = (1/m_1^i, \dots, 1/m_n^i) = q^i/m^i$, so that $Q_0^i = \sum c_\alpha \xi^\alpha$ with $\langle b^i, \alpha \rangle = 1$. Then

$$(1.9) \quad P(\xi) = \sum_{(P)} c'_\alpha \xi^\alpha, \quad \text{with } \langle b^i, \alpha \rangle \leq \sum_k \max_j (m_j^k/m_j^i) = h^i,$$

when α belongs to the index set (P) of P , and for every $\alpha \in (P_0)$ there is at least one value of i for which $\langle b^i, \alpha \rangle = h^i$. Hence, for all i ,

$$(1.10) \quad |P(\xi)| \leq C \{ \sum_j |\xi_j^{m_j^i}| \}^{h^i} \leq C' |Q^i(\xi)|^{h^i}, \quad \xi \text{ real}, \quad |\xi| \geq K.$$

Now, the ratio $P^{(\alpha)}(\xi)/P(\xi)$ is a sum of terms of the type $\Pi (Q^i)^{(\alpha^i)}/Q^i$, where, in view of (1.6) and (1.10),

$$|(Q^i)^{(\alpha^i)}/Q^i| \leq C |Q^i|^{-\langle b^i, \alpha^i \rangle} \leq C' |P|^{-\langle b^i, \alpha^i \rangle/h^i} \quad \text{for } |\xi| \geq K, \quad \xi \text{ real}.$$

Consequently, if we set $b = b(P) = (b_1, \dots, b_n)$, with $b_i = \min_j b_j^i/h^i$, then $|P^{(\alpha)}(\xi)/P(\xi)| \leq C \sum |P|^{-\langle b, \alpha \rangle}$, $|\xi| \geq K$. This follows from the estimate $\sum_i \langle b^i, \alpha^i \rangle/h^i \geq \sum \langle b, \alpha^i \rangle = \langle b, \alpha \rangle$, which is valid because $\sum_i \alpha^i = \alpha$ for every one of the terms in the expansion of $P^{(\alpha)}/P$. In other words, in general the estimate

$$|P^{(\alpha)}(\xi)| \leq C(1 + |P(\xi)|)^{1-\langle b, \alpha \rangle},$$

ξ real, can hold with optimal b only if we choose

$$(b)_l^{-1} = \{ \min_i (b_i/h_i) \}^{-1} = \max_i \{ m_l^i \sum_k \max_j (m_j^k/m_j^i) \}.$$

In view of (1.7), it follows that for $P = \Pi Q^i$,

$$(b_i)^{-1} = \sum_k \max_j (m_j^k \max_i (m_i^i/m_j^i)) = \sum_k \max_j (m_j^k a_{jl}),$$

or, finally, since $a_{jl}(Q^k) = m_l^k/m_j^k$, $b_l(Q^k) = 1/m_l^k$, that

$$\{b_l(P)\}^{-1} = \sum_k \{b_l(Q^k)\}^{-1} \max_j \{a_{jl}(P)/a_{jl}(Q^k)\}.$$

In particular, because $a_{jl}(P) \geq a_{jl}(Q^k)$,

$$\{b_l(P)\}^{-1} \geq \sum_k \{b_l(Q^k)\}^{-1} = m_l(P),$$

with equality if and only if, for all k and j , we have $a_{jl}(Q^k) = a_{jl}(P)$. As we have seen above, this is the case if and only if all the Q^i have proportional weights (so that P itself becomes quasielliptic).

2. The Newton polyhedron of a hypoelliptic polynomial.

Let $P(\xi) = \sum c_\alpha \xi^\alpha$ be a given polynomial, with index-set $(P) = \{\gamma \in N^n; c_\gamma \neq 0\}$. Denote by $(P)^*$ the convex hull in R^n of $P \cup \{0\}$, by $F = F(P) = \bigcup F^i$ the union of those $(n-1)$ -dimensional flat faces F^i of the polyhedron $(P)^*$ that are not contained in a coordinate (hyper-)plane, and by α^i , $\alpha^i \in (P)$, the vertices of F . If $n=2$, then F is a polygon with corners α^i , the upper Newton polygon for P . So when $n \geq 3$ we may call F the *Newton polyhedron* for P . We notice that always $(P_1 P_2) = (P_1) + (P_2)$, so that

$$(2.1) \quad (P_1 P_2)^* \subset (P_1)^* + (P_2)^*, \quad \{\alpha^k(P_1 P_2)\} \subset \{\alpha^i(P_1)\} + \{\alpha^j(P_2)\}.$$

LEMMA 2.1. Let $F(P)$ be the Newton polyhedron of a hypoelliptic polynomial P . Then every face F^k of F has a normal $\nu^k > 0$, i. e. with all components strictly positive.

Proof. When $n=2$, it is sufficient to recall the convexity of F , and the well-known property of hypoelliptic polynomials that if

$$(2.2) \quad P(\xi) = \sum_{j \leq m_k} R_j(\xi') \xi_k^j, \quad \xi' = (\xi_1, \dots, \xi_{k-1}, \xi_{k+1}, \dots, \xi_n),$$

then $R_{m_k}(\xi') = \text{constant}$. If $n > 2$, we first observe that, due to the convexity of F , it is sufficient to consider the normals of faces F^i abutting

on a coordinate plane $x_k = 0$. Since $P(\xi', 0)$ is hypoelliptic if $P(\xi', \xi_k)$ is so, we may take as our induction hypothesis that F^i has a normal $\nu^i = (\nu', \nu_k)$, with $\nu' > 0$. Now, if R_j is defined by (2.2), then $j!$. $R_j(\xi') = P^{(\beta)}(\xi', 0), P^{(\beta)} = (\partial/\partial \xi_k)^j P$. Hence we have, for $t > 0$, if we set

$$(2.3) \quad t^\nu \xi = (t^{\nu_1} \xi_1, \dots, t^{\nu_n} \xi_n) = (t^{\nu'} \xi', t^{\nu_k} \xi_k),$$

that

$$P^{(\beta)}(t^{\nu'} \xi', 0)/P(t^{\nu'} \xi', 0) = j! R_j(t^{\nu'} \xi')/R_0(t^{\nu'} \xi').$$

In view of the hypoellipticity condition (1.2'), it follows that

$$(2.4) \quad R_j(t^{\nu'} \xi')/R_0(t^{\nu'} \xi') \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad \xi' \text{ real, } j \neq 0.$$

But if $\alpha' \in (R_j)$, $j \geq 0$, then $\nu' \alpha' + \nu_k j \leq 1$, and there is at least one $j_0 > 0$ and one $\alpha' \in (R_{j_0})$ such that $\nu' \alpha' + \nu_k j_0 = 1$. On the other hand, (2.4) implies that $\nu' \alpha' < 1$ for all $\alpha \in (R_{j_0})$. Hence we must have $\nu_k > 0$, and then also $\nu = (\nu', \nu_k) > 0$.

In the two-dimensional case, if $\alpha^0, \alpha^1, \dots, \alpha^l$ are the vertices of the Newton polygon $F(P)$, for P hypoelliptic, we can find by recursion numbers $m_j^i > 0$, $i = 0, 1, \dots, l$, $j = 1, 2$, such that

$$(2.5) \quad \alpha^k = (\alpha_1^k, \alpha_2^k) = (\sum_{i \leq k} m_1^i, \sum_{i > k} m_2^i).$$

With

$$(2.6) \quad \beta^k = \alpha^k - (m_1^k, 0) = \alpha^{k-1} - (0, m_2^k) = (\sum_{i < k} m_1^i, \sum_{i > k} m_2^i),$$

$$b^k = (1/m_1^k, 1/m_2^k),$$

and F^k the segment of F between α^{k-1} and α^k , we find that

$$\alpha = \beta^k + \alpha', \quad \langle b^k, \alpha' \rangle = 1, \quad \text{for all } \alpha \in F^k.$$

LEMMA 2.2. Consider a hypoelliptic polynomial $P(\xi) = \sum c_\alpha \xi^\alpha$, $\xi \in R^2$, and define α^k, β^k, b^k as above. Then, for $k = 1, \dots, l$, as $t \rightarrow \infty$, $t^{b^k} \xi = (t^{b_1^k} \xi_1, t^{b_2^k} \xi_2)$, we have

$$(2.7) \quad t^{-\langle b^k, \alpha^k \rangle} P(t^{b^k} \xi) = \xi^{\beta^k} Q^k(\xi) + o(1),$$

with

$$(2.8) \quad Q^k(\xi) = c^k \xi_1^{m_1^k} + \dots + c^{k-1} \xi_2^{m_2^k}, \quad c^k = c_\alpha \text{ with } \alpha = \alpha^k.$$

If b is between b^k and b^{k+1} , i. e. if $b_2^k/b_1^k > b_2/b_1 > b_2^{k+1}/b_1^{k+1}$, then

$$(2.9) \quad t^{-\langle b^k, \alpha^k \rangle} P(t^b \xi) = c^k \xi^{\alpha^k} + o(1).$$

Finally,

$$(2.10) \quad cP(\xi) = \prod_k Q^k(\xi) + R(\xi), \quad c = \prod_1^{l-1} c^i,$$

with $(R) \subset (P)^*$, $(R) \cap F(P) = \emptyset$.

Proof. The relations (2.7)-(2.9) are a direct consequence of the definition of F^k as part of a line of support for the convex set $(P)^*$. To prove (2.10) we notice that, as $t \rightarrow \infty$,

$$(2.11) \quad \begin{aligned} t^{-b_1^k m_1^i} Q^i(t^{b^k} \xi) &= c^i \xi_1^{m_1^i} + o(1) \quad \text{if } i < k, \\ t^{-1} Q^k(t^{b^k} \xi) &= Q^k(\xi), \quad \text{and} \\ t^{-b_2^k m_2^i} Q^i(t^{b^k} \xi) &= c^{i-1} \xi_2^{m_2^i} + o(1) \quad \text{if } i > k. \end{aligned}$$

It follows that

$$(2.12) \quad t^{-\langle b^k, \alpha \rangle} \prod_i Q^i(t^{b^k} \xi) = c \xi^\beta Q^k(\xi) + o(1), \quad \text{all } k,$$

with $\alpha = (\sum_{i \leq k} m_1^i, \sum_{i > k} m_2^i) = \alpha^k$, $c = \prod_{i < k} c^i \prod_{i > k} c^{i-1} = \prod_1^{l-1} c^i$, and $\beta = \alpha - (m_1^k, 0) = \beta^k$. This means, in view of (2.7), that cP and $\prod Q^i$ have identical coefficients for all ξ^α with $\alpha \in F = \bigcup F^k$, which clearly implies (2.10) and completes the proof of the lemma.

If $\xi \in R^n$, $n > 2$, then Lemma 2.2 does not apply. Instead, if we call $F(Q)$ simple when it consists of a single flat face, then we have

LEMMA 2.3. Let a hypoelliptic polynomial $P(\xi)$, $\xi \in R^n$, be a product $\prod Q^i$ of polynomials with simple Newton polyhedrons, and let $b^i = b(Q^i) = (b_1^i, \dots, b_n^i) = (1/m_1^i, \dots, 1/m_n^i)$, $i = 1, \dots, l$, with the b^i non-proportional. Suppose there is given a vector $b > 0$, diagonal matrices ε^i with all elements = 0 or 1, and numbers $\lambda^i > 0$, such that

$$b - \lambda^i b^i \leq \varepsilon^i (b - \lambda^i b^i) = 0, \quad i = 1, \dots, l.$$

Then we have, as a generalization of (2.11)-(2.12),

$$(2.13) \quad t^{-\sum \lambda^i} \prod Q^i(t^b \xi) = \prod Q^i(\varepsilon^i \xi) + o(1), \quad \text{as } t \rightarrow \infty.$$

It follows that, except for degenerate cases, all the faces of $F(P)$ are « products » of faces of the $F(Q^i)$ of lower or the same dimension.

The proof of the lemma being immediate, we may omit it in favor of some simple examples. If $n = 2$, then $\varepsilon^i \xi = (\xi_1, 0)$ or $(0, \xi_2)$ or, for at most one i , $\varepsilon^i \xi = \xi$, and so (2.13) implies either (2.7) or (2.9). If $n = 3$, then the two-dimensional faces of $F(P)$ correspond to one of the two cases (after a renumbering of the coordinate axes)

$$P(\varepsilon, \xi) = \prod Q^i(\varepsilon^i \xi) = \xi^\beta Q^k(\xi); \quad P(\varepsilon, \xi) = \xi^\beta Q^k(0, \xi_2, \xi_3) Q^l(\xi_1, 0, \xi_3).$$

This means that, for $n = 3$, the Newton polyhedron $F(P)$ is built up of triangles (translated copies of the $F(Q^i)$) and parallelograms (whose sides are translated copies of sides of the $F(Q^i)$). In particular, when $P = Q^1 Q^2$, then in a typical case $F(P)$ consists of two triangular faces, given by

$$P(\varepsilon, \xi) = Q^1(\xi) Q^2(\xi_1, 0, 0); \quad P(\varepsilon, \xi) = Q^1(0, \xi_2, 0) Q^2(\xi),$$

and of one parallelogram, corresponding to

$$P(\varepsilon, \xi) = Q^1(0, \xi_2, \xi_3) Q^2(\xi_1, 0, \xi_3).$$

(See Fig. 1). A degenerate case can appear if $F(Q^1)$ and $F(Q^2)$ have two parallel sides. Then we find one face of $F(P)$ in form of a parallel-trapeze (a sum of a triangle and a parallelogram) given by, e. g.

$$P(\varepsilon, \xi) = Q^1(0, \xi_2, \xi_3) Q^2(\xi).$$

(See Fig. 2).

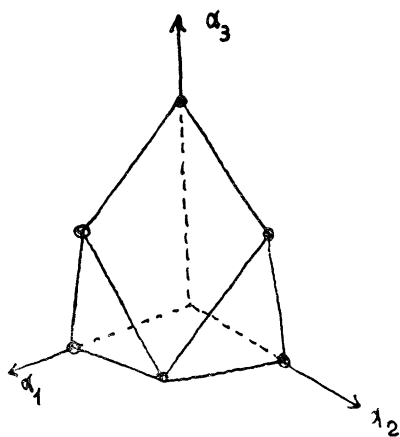


Fig. 1

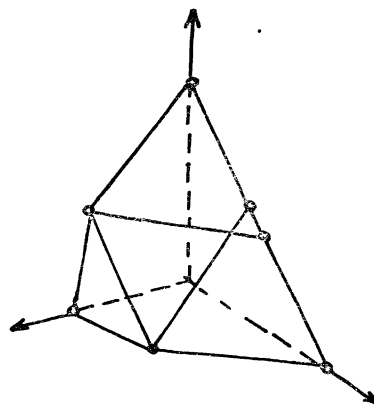


Fig. 2

Next, if $n = 2$, $P = Q^1 Q^2 Q^3$, then in the general case $F(P)$ consists of exactly three triangles and three parallelograms in any one of five essentially different combinations.

3. Multi-quasielliptic polynomials.

DEFINITION. A polynomial $P(\xi)$ is called *multi-quasielliptic* (mqe.), if P has a Newton polyhedron $F(P)$ with vertices $\{\alpha^k\}$ and normals $\{\nu^i\}$ such that

- i) $\nu^i > 0$, all i
- ii) $\sum |\xi^{\alpha^k}| \leq C(1 + |P(\xi)|)$, real ξ .

Notice that i) is a necessary condition for hypoellipticity, (Lemma 2.1). We have now

THEOREM 3.1. In the two-dimensional case, $P(\xi)$ is multi-quasielliptic if and only if it is (strictly) of the same strength, hence hypoelliptic with the same parameters, as a product of quasielliptic polynomials. In more dimensions every product of quasielliptic polynomials is mqe., but there are mqe. polynomials that are not of the same strength as any such product.

Proof. If $P = \prod Q^i$, Q^i quasielliptic, then, as we have seen in section 1, P is hypoelliptic (hence i) is satisfied) and has the lower estimate

$$|P(\xi)| \geq c \prod_i \sum_j |\xi_j^{m_j^i}|, \quad \text{for } \xi \text{ real, } |\xi| \geq K.$$

Since every $\alpha^k(P)$ is of the form $\sum_i m_{j(i)}^k$, it follows that P satisfies ii).

If $n > 2$, let $M(\xi')$ be multi-quasielliptic, but not quasielliptic, with real coefficients, $\xi' \in R^{n-1}$, and set $P(\xi) = P(\xi', \xi_n) = M(\xi') + i \xi_n$. Then evidently P is also multi-quasielliptic, but it can be neither quasielliptic nor a product of two or more quasielliptic polynomials.

Now let $n = 2$, so that we can use the notations and results of Lemma 2.2. We have

$$\begin{aligned} t^{-\langle b^k, \alpha^k \rangle} \sum |(t^{b^k} \xi)^{\alpha^j}| &= (|\xi^{\alpha^{k-1}}| + |\xi^{\alpha^k}|) + o(1) = \\ &= |\xi^{\beta^k}| (|\xi_1^{m_1^k}| + |\xi_2^{m_2^k}|) + o(1), \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Comparing with (2.7), we see that P can be mqe. only if $Q^k(\xi)$ is quasielliptic, for every k . Finally, let $0 < c < 1$ be chosen such that $(R) \subset c(P)^*$, R defined by (2.10). Then $|R(\xi)| \leq C(1 + |P(\xi)|)^c$ for all real ξ , and the proof of Theorem 3.1 is complete.

As an example, for $n = 2$, consider the Gorčakov polynomial (0.1),

$$P(\xi_1, \xi_2) = A\xi_1^{2p} + B\xi_2^{2q} + C\xi_1^{2r}\xi_2^{2s}, \quad A, B, C > 0,$$

$p > r, q > s, r/p + s/q > 1$. Here $Q_1 = C\xi_1^{2r} + B\xi_2^{2(q-s)}$ and $Q_2 = A\xi_1^{2(p-r)} + C\xi_2^{2s}$, clearly quasielliptic, and hence

$$C \cdot P = Q_1 Q_2 - AB \xi_1^{2(p-r)} \xi_2^{2(q-s)} \asymp Q_1 Q_2 \quad \text{as } |\xi| \rightarrow \infty, \quad \xi \text{ real.}$$

(Notice that $(p-r)/p + (q-s)/q = 1 - (r/p + s/q - 1) < 1$).

As a further example, for the case $n > 2$, let us study a class of polynomials introduced by Cattabriga [2]. Consider a polynomial

$$(3.1) \quad P(\tau, \xi) = \tau^m + \sum_{j < m} \tau^j M_j(\xi), \quad \tau \in R^1, \quad \xi \in R^{n-1}.$$

Suppose there are positive integers c_0, \dots, c_{m-1} , and a quasielliptic polynomial $Q(\xi)$, with $b(Q) = (1/m_1, \dots, 1/m_n)$, such that

$$(3.2) \quad (M_j) \subset c_j(Q)^*, \quad j = 0, 1, \dots, m-1.$$

Set $P'(\tau, \sigma) = \tau^m + \sum_{j < m} \tau^j \sigma^{c_j}$, $\sigma \in R^1$, let

$$f^k = \{j \mid (j, c_j) \in F^k(P')\} = \{j_{k-1}, \dots, j_k\},$$

and write

$$(3.3) \quad N_k(\tau, \xi) = \tau^{-j_{k-1}} \sum_{j \in f^k} \tau^j M_{j_0}(\xi),$$

where M_{j_0} is the principal part of M_j in the sense of quasielliptic polynomials. We shall assume, in addition to (3.2), that

$$(3.2') \quad (M_j)^* = c_j(Q)^* \quad \text{for } j = j_k, \quad \text{all } k.$$

Then $P(\tau, \xi)$ is multi-quasielliptic if and only if, for every k ,

$$N_k(\tau, \xi) \neq 0 \quad \text{for } (\tau, \xi) \text{ real, } \neq 0.$$

The necessity of the condition is trivial. To prove the sufficiency, we observe, following Cattabriga, that the assumption implies that

$$N_{k+1}(0, \xi) = M_{j_k 0}(\xi) \neq 0 \quad \text{for real } \xi \neq 0, \text{ all } k.$$

Hence M_{j_0} is quasielliptic, of the same weight as $(Q_0)^{j_0}$, if $j = j_k$. Thus, if $\tau = \tau'_i(\zeta)$, $i = j_{k-1}, \dots, j_k$, are the zeros of $N_k(\tau, \zeta)$, then $\text{Im } \tau'_i(\xi) \neq 0$ for ξ real, $\neq 0$, and all i . Hence $C \geq |\text{Im } \tau'_i(\zeta)| \geq c > 0$, for $\zeta = \xi + i\eta \in C^n$ with $Q(\xi) = 1$, $Q(\eta) < \varepsilon$, ε small enough.

Suppose that ε is so small that also $M_{j_k 0}(\zeta) \neq 0$ for $Q(\xi) = 1$, $Q(\eta) < \varepsilon$, and set $\varrho^k = (c_{j_{k-1}} - c_{j_k}) / (j_k - j_{k-1})$, $b = b(Q)$. Then we get, uniformly in ξ ,

$$t^{-e^k} \tau_i(t^b \xi) = \tau'_i(\xi) + o(1) \quad \text{as } t \rightarrow \infty, \quad \xi \text{ real}, \quad Q(\xi) = 1.$$

With $t = Q(\xi)$ we then obtain the second of the following estimates

$$(3.4) \quad |\text{Re } \tau_i(\zeta)| \leq C(1 + |Q(\zeta)|^{e_k}) \quad \text{for all } \zeta \in C^n,$$

$$(1 + |\text{Im } \tau_i(\zeta)|) \geq c(1 + |Q(\xi)|^{e_k}), \quad \text{for } |Q(\eta)| < \varepsilon |Q(\xi)|,$$

while the first of the estimates is trivially true. Since $\sum_k (j_k - j_{k-1}) = m$, all zeros of $P(\tau, \zeta)$ are of this type. It follows that, when $P(\tau, \zeta) = 0$, $\zeta = \xi + i\eta$,

$$\text{Im } (\tau, \zeta) = \eta_s \implies |\text{Re } \tau|^{1/e} + \sum |\xi_i|^{m_i} \leq A(1 + |\eta_s|^{m_s}),$$

$$\text{Im } (\tau, \zeta) = \text{Im } \tau \implies |\text{Re } \tau| + (\sum |\xi_i|^{m_i})^{1/e'} \leq A_1(1 + |\text{Im } \tau|),$$

where $\varrho = \max_k \varrho_k$, $\varrho' = \max_k (1/\varrho_k)$. In other words, if $\tau = \zeta_n$, then we have proved that P is hypoelliptic, with

$$(3.5) \quad a_{is}(P) = m_s/m_i = a_{is}(Q),$$

$$a_{ns}(P) = \varrho m_s, \quad a_{in}(P) = \varrho'/m_i, \quad \text{for } i, s < n;$$

$$a_{nn} = 1.$$

Finally, writing $|P(\tau, \xi)| = \prod_i |\tau - \tau_i(\xi)| \geq c \prod_i (|\tau - \text{Re } \tau_i(\xi)| + |\text{Im } \tau_i(\xi)|)$, for ξ real, and using (3.4), we see that

$$(3.6) \quad |P(\tau, \xi)| \geq c_1 \prod_k (|\tau| + |Q(\xi)|^{e_k})^{j_k - j_{k-1}}, \quad \text{for } |(\tau, \xi)| \geq K > 0.$$

Hence P is in fact multi-quasielliptic. But in addition we have proved that, if $Q(\xi)$ is a real polynomial, then $P(\tau, \xi)$ is of the same strength, and (consequently) hypoelliptic of the same type, as a polynomial $P_0(\tau, Q(\xi))$, with

$$P_0(\tau, \sigma) = \sum_{j=0}^m h_j \tau^j \sigma^{c_j}$$

quasielliptic. So finally, using (3.5) or the two-dimensional part of Theorem 3.1, we see that $P(\tau, \xi)$ is of the same strength as a certain product

$$\prod_k \{(Q(\xi))^{r_k} + i \tau^{s_k}\}$$

of quasielliptic polynomials, if Q is real. However, in contrast to the two-dimensional case, $P(\tau, \xi)$ is not necessarily strictly of the same strength as a product of quasielliptic operators. In fact, repeating the proof of Theorem 3.1, we see that P is strictly of the same strength as such a product if and only if, for every k , $M_{j_k}(\xi)$ is a factor in M_j for $j = j_{k-1}, \dots, j_k - 1$. The situation is exemplified in Fig. 2 of section 2.

As for the general mqe. polynomials, we have

THEOREM 3.2. Every multi-quasielliptic polynomial is hypoelliptic, but the converse is not true.

Proof. The second half of the theorem follows from the existence of counterexamples, given by Pini [12], who has studied non-quasielliptic polynomials $P(\xi)$, with simple (i. e. single-faced) Newton polyhedrons.

Suppose then that $P(\xi)$ is mqe., and that the Newton polyhedron $F(P)$ consists of flat faces $F^k(P)$, with normals $\nu^k > 0$, such that

$$(3.7) \quad \langle \nu^k, \alpha \rangle h^k \text{ for } \alpha \in F^k(P), \text{ all } k.$$

Let $\beta \in (P)^*$, $\notin F(P)$. Then there is a c , $0 < c < 1$, such that $\beta = c \beta'$, $\beta' \in F^k(P)$ for some k , or else $\beta = 0$. Now, if $\alpha^1, \dots, \alpha^r$ are the vertices of F^k , then $\beta' = \sum_1^r \lambda_i \alpha^i$, with $\lambda_i > 0$, $\sum \lambda_i = 1$. It follows that

$$|\xi^\beta| = \prod_1^r |\xi^{\alpha^i}|^{c \lambda_i} \leq \left(1 + \sum_1^r |\xi^{\alpha^i}| \right)^c \leq C (1 + |P(\xi)|)^c, \quad \xi \text{ real,}$$

if $\beta \neq \theta$, and the same estimate holds if $\beta = 0$, $\xi^\beta = 1$. Thus, if we set $P_F(\xi) = \sum_{\alpha \in F(P)} c_\alpha \xi^\alpha$, when $P(\xi) = \sum_{(P)} c_\alpha \xi^\alpha$, then $P(\xi) = P_F(\xi) + R(\xi)$, where

$R(\xi)$ is strictly weaker than $P_F(\xi)$. Hence we may call P_F the *principal part* of $P(\xi)$, when $P(\xi)$ is mqe. (If not only $P(\xi)$ but also $\operatorname{Re} P(\xi)$ is multi-quasielliptic, then we may call P *properly mqe*. Generalizing an observation by Barozzi [1] about properly quasielliptic polynomials, we notice that if $P = \sum c_\alpha \xi^\alpha$ is properly mqe., and if $\{\alpha^i\}$ are the vertices of $F(P)$, then all the $\operatorname{Re} c_{\alpha^i}$ are of the same sign, and the components of the α^i are all even integers. Consequently, for P properly mqe., we may introduce $\sum_i c_{\alpha^i} \xi^{\alpha^i}$ as a *reduced principal part*).

In particular, if $\beta \in (P^{(\alpha)}(\xi))$, $\alpha \neq 0$, then

$$\langle \nu^k, \beta \rangle \leq h^k - \langle \nu^k, \alpha \rangle, \quad c(\beta) \leq \max_k (h^k - \langle \nu^k, \alpha \rangle) / h^k,$$

and it follows that

$$(3.8) \quad |P^{(\alpha)}(\xi)| \leq C(1 + |P(\xi)|)^{1 - \langle b, \alpha \rangle}, \quad b = (b_1, \dots, b_n),$$

$$b_i = \min_k \nu_i^k / h^k.$$

Since (3.8), together with the estimate $\sum |\xi^{\alpha^k}| \leq C(1 + |P(\xi)|)$, implies that P is hypoelliptic, the proof of theorem 3.2 is complete.

A simple geometric rule for finding the values of the hypoellipticity types of a mqe. polynomial is given by

THEOREM 3.3. Let $F = \bigcup F^i$ be the Newton polyhedron of a multi-quasielliptic polynomial P , and denote by $x_k(F^i)$ the coordinate of the intersection between the x_k -axis and the hyperplane through F^i . Then

$$(3.9) \quad \{b_k(P)\}^{-1} = \max_i x_k(F^i), \quad k = 1, \dots, n;$$

$$(3.10) \quad a_{jk}(P) = \max_i [x_k(F^i) / x_j(F^i)], \quad j, k = 1, \dots, n.$$

Proof. If e_k is the k -th coordinate vector, then it follows from (3.8) that

$$b_k = \min_i (\nu_i^k / h^i) = \min_i \langle \nu^i, e_k \rangle / \langle \nu^i, x_k(F^i) e_k \rangle = \min_i [x_k(F^i)]^{-1},$$

which proves (3.9). To prove (3.10) recall that $a_{jk}(P)$ is defined as the smallest real number for which

$$(3.11) \quad |(\partial/\partial \xi_k)^s P(\xi)| \leq C(1 + |P(\xi)|)(1 + |\xi_j|)^{-s/a_{jk}}, \quad \text{real } \xi, \quad s = 1, 2, \dots$$

In other words, when P is multiquasielliptic, $a_{jk}(P)$ is the smallest real number for which

$$\alpha - se_k + (s/a_{jk}) e_j \in (P)^*, \quad \text{for all } \alpha \in (P) \quad \text{with } \alpha_k \geq s.$$

This means that

$$\max_{\alpha \in (P)} \max_i [\langle \nu^i, \alpha \rangle - s\nu_k^i + (s/a_{jk}) \nu_j^i / h^i] = 1.$$

But $\max_i \langle \nu^i, \alpha \rangle / h^i \leq 1$ when $\alpha \in (P)$, $= 1$ when $\alpha \in F^i(P)$. Thus, for all i , and for $s = 1, 2, \dots$,

$$\langle \nu^i, \alpha \rangle - s\nu_k^i + (s/a_{jk}) \nu_j^i / h^i \leq 1 - s \{ \nu_k^i - (1/a_{jk}) \nu_j^i / h^i \},$$

with equality if $\alpha \in F^i(P)$. It follows that (3.11) can be valid only if we choose

$$a_{jk} = \max_i (\nu_j^i / \nu_k^i) = \max_i \{ (\nu_j^i / h^i) / (\nu_k^i / h^i) \},$$

a condition that is equivalent to (3.10). (Notice that we get the same value for a_{jk} if we keep $s = 1$ in (3.11)). That (3.9), (3.10) give the best possible values for b_k, a_{jk} will be proved in Theorem 4.1.

The exceptional role played by the quasielliptic polynomials in the class of all multiquasielliptic polynomials is clearly indicated by

COROLLARY 3.1. Let P be multiquasielliptic. Then, since obviously

$$m_k(P) = \min_i x_k(F^i), \quad k = 1, \dots, n,$$

it follows from Theorem 3.3, for $j, k = 1, \dots, n$, that

$$(3.12) \quad b_k(P) \leq [m_k(P)]^{-1},$$

or, more generally, that

$$(3.13) \quad m_k(P) b_j(P) \leq a_{jk}(P) \leq [m_j(P) b_k(P)]^{-1},$$

that

$$(3.14) \quad a_{jk}(P) a_{kj}(P) \geq 1 \quad \text{if } j \neq k; \quad a_{jj}(P) = 1,$$

and that (Cf. (1.4))

$$(3.15) \quad \max_{\alpha \in (P)} \langle b, \alpha \rangle \leq 1, \quad \text{when } b = (b_1(P), \dots, b_n(P)).$$

Moreover, in each of the relations (3.12) to (3.15), the inequalities can be replaced by identities if and only if P is quasielliptic.

Proof. The inequalities are trivially true, so it only remains to prove the last part of the corollary. But if (3.12) or (3.13) can be replaced by identities, then we must have $\max_i x_k(F^i) = \min_i x_k(F^i)$ for all k , which means that $F(P)$ consists of a single face F^1 . Similarly equality in (3.14) means that $\max_i (x_k(F^i)/x_j(F^i)) = \min_i (x_k(F^i)/x_j(F^i))$, and the conclusion is the same, $F = F^1$. (It may be noted here that for a general hypoelliptic P , as has been shown by Grusin [8], P is quasielliptic if (and only if) there are numbers $c_1, \dots, c_n > 0$ such that $a_{jk}(P) \leq c_j/c_k$ for all j, k . Obviously Grusin's condition is stronger than the condition $a_{jk} a_{kj} \leq 1$ for all j, k). Finally, if we set $\nu^i = (\nu_1^i, \dots, \nu_n^i)$ and recall that $b_k = [\max_i x_k(F^i)]^{-1} = \min_i (\nu_k^i/h^i)$, then we see that

$$\langle b, \alpha \rangle \leq \min_i (\langle \nu^i, \alpha \rangle / h^i) \leq 1,$$

with strict inequality if P is not quasielliptic.

In the case when a mqe. polynomial P is in fact a product $\prod Q^s$ of quasielliptic polynomials, we have already, in section 1, proved a stronger result than Theorem 3.3, namely that (3.9) and (3.10) are valid, with the maximum in each taken only over the faces F^i of $F(P)$ that are translates of some $F(Q^s)$. (Cf. Fig. 1 of section 2).

It is also easy to prove again by use of Theorem 3.3 the formula (3.5) derived before for $a_{jk}(P)$, when P is a polynomial (3.1) of the type introduced by Cattabriga. Furthermore, we can use (3.10) to compute $b_k(P)$ for such polynomials. But instead let us consider the related case of a polynomial

$$(3.16) \quad P(\xi_0, \xi') = P_0(\xi_0, M(\xi')), \quad \xi' = (\xi_1, \dots, \xi_{n-1}),$$

where $M(\xi)$ is real mqe. of type $b(M) = (b_1(M), \dots, b_{n-1}(M))$, while $P_0(\tau, \sigma)$ is mqe. of type $(b_0(P_0), b_1(P_0))$. Then obviously P is itself multi-quasielliptic. Let $F(M) = \bigcup F^i(M)$, where $F^i(M)$ is part of the hyperplane $\langle \nu^i(M), \alpha' \rangle = 1$, $\alpha' \in R^{n-1}$, and let $F(P_0) = \bigcup F^j(P_0)$, where $F^j(P_0)$ is part of the line $\nu_0^j \beta_0 + \nu_1^j \beta_1 = 1$. Then $F(P) = \bigcup F^{ij}(P)$, where $F^{ij}(P)$ is part of the hyperplane $\langle \nu^{ij}, \gamma \rangle = \nu_0^j \gamma_0 + \langle \nu_1^j \nu^i(M), \gamma' \rangle = 1$, $\gamma = (\gamma_0, \gamma') \in R^n$. Since $x^{ij}(P) = [\nu^{ij}(P)]^{-1}$, it follows that

$$(3.17) \quad b_k(P) = \min_{j,i} \nu_1^j \nu_k^i(M) = b_1(P_0) b_k(M) \quad \text{for } k = 1, \dots, n-1,$$

$$b_0(P) = \min_j \nu_0^j = b_0(P_0),$$

while

$$(3.18) \quad \begin{aligned} a_{ks}(P) &= \max_{i,j} (\nu_k^i(M)/\nu_s^i(M)) = a_{ks}(M), \quad k, s \neq 0, \\ a_{0s}(P) &= \max_{i,j} (\nu_0^j(P_0)/\nu_1^j(P_0) \nu_s^i(M)) = a_{01}(P_0)/b_s(M), \quad s \neq 0, \\ a_{k0}(P) &= \max_{i,j} (\nu_1^j(P_0) \nu_k^i(M)/\nu_0^j(P_0)) = a_{10}(P_0)/m_k(M), \quad k \neq 0, \\ a_{00} &= 1. \end{aligned}$$

4. Extremal character of multiquasielliptic polynomials.

The purpose of this chapter is to find out to what extent Theorem 3.3 (and its corollary) can be generalized to the case of an arbitrary hypoelliptic polynomial.

THEOREM 4.1. Let $F(P) = \bigcup F^i$ be the Newton polyhedron of a hypoelliptic polynomial P , and define $x_k(F^i)$ as in Theorem 3.3. Then for all k

$$(4.1) \quad \max_i x_k(F^i) \leq [b_k(P)]^{-1}$$

while for all j, k

$$(4.2) \quad a_{jk}(P) \geq \max_i [x_k(F^i)/x_j(F^i)].$$

Thus, if there exists a multiquasielliptic polynomial $P'(\xi)$ with $F(P') = F(P)$, then

$$(4.3) \quad b_k(P) \leq b_k(P'), \quad a_{jk}(P) \geq a_{jk}(P').$$

Proof. Let $\nu^i = \nu(F^i) = (1/x_1(F^i), \dots, 1/x_n(F^i))$, and denote by P_F^i the polynomial $\sum c_\alpha \xi^\alpha$, $\alpha \in F^i$, where $\{c_\alpha\}$ are the coefficients of $P(\xi)$. Then $P(t^{\nu^i} \xi) = t \{P(\xi) + o(1)\}$, and $\partial/\partial \xi_k P(t^{\nu^i} \xi) = t^{1-\nu_k^i} \{\partial/\partial \xi_k P(\xi) + o(1)\}$, as $t \rightarrow \infty$. Thus if we choose $\xi \in R^n$ such that $\partial/\partial \xi_k P(\xi) \neq 0$, it follows from the definition (1.4) of $b(P)$ that $b_k \leq \nu_k^i = 1/x_k(F^i)$, for all i .

Set now $\xi = (\xi', \xi_n)$, say, and choose a real $\xi' = (\xi_1, \dots, \xi_{n-1})$, with all components $\neq 0$, and such that

$$(4.4) \quad P_F^i(\xi) = A(\xi') \Pi(\xi_n - \Phi(\xi')), \quad A(\xi') \neq 0.$$

Then, for each $\Phi(\xi')$ and each real t big enough, $P(\zeta)$, $\zeta \in C^n$, has a zero

$$(4.5) \quad \zeta = (t^{\nu_1} \xi_1, \dots, t^{\nu_{n-1}} \xi_{n-1}, t^{\nu_n} \psi(\xi', t)), \quad \nu = \nu(F^i),$$

with $\psi(\xi', t) = \Phi(\xi') + o(1)$ as $t \rightarrow \infty$. Clearly $|\operatorname{Re} \zeta_j| = |\zeta_j| t^{\nu_j}$ for $j < n$, while $|\operatorname{Im} \zeta_n| = o(1) t^{\nu_n}$. It follows that $a_{jn} \geq x_j(F^i)/x_n(F^i)$ for $j < n$, all i . Estimates for the other a_{jk} , with $j \neq k$, are proved in the same way.

It follows from Theorem 4.1 that for a general hypoelliptic P , the inequalities

$$b_k \leq m_k^{-1}; \quad m_k b_j \leq a_{jk}; \quad a_{jk} a_{kj} \geq 1$$

of Corollary 3.1 are still valid. The second part of (3.14), however, $a_{jj}(P) = 1$ for all j , is characteristic for the mqe. polynomials.

THEOREM 4.2. If $P(\xi)$ is hypoelliptic, then $a_{jj}(P) = 1$ for all j if and only if P is multi-quasielliptic.

Proof. Let $F(P) = \cup F^i$, and set as before $P_F^i(\xi) = \sum c_\alpha \xi^\alpha$, $\alpha \in F^i$. Suppose that $P_F^i(\xi) = 0$ for a real ξ with $\xi_1, \dots, \xi_n \neq 0$, and set $\xi = (\xi', \xi_n)$. Then either $P_F^i(\xi', s_n) = 0$ for all s_n , or we can write P_F^i as in (4.4), in which case we can prove, via (4.5), that $a_{nn} > 1$. Thus if $a_{nn} = 1$, we must have $P_F^i(\xi', s_n) \equiv 0$. We can now repeat the procedure, writing $\xi' = (\xi'', \xi_{n-1})$, $\xi'' \in R^{n-2}$, and concluding that either $a_{n-1, n-1} > 1$ or $P_F^i(\xi'', s_{n-1}, s_n) \equiv 0$, etc. Since $P_F^i \not\equiv 0$, we can prove in this way that $a_{kk}(P) > 1$ for at least one k , if $P_F^i(\xi) = 0$ for some i and some real ξ with $\xi_1, \dots, \xi_n \neq 0$.

Similarly, if $F^{\mu, i}(P)$ is any one of the μ -dimensional faces of $F(P)$, $\mu = 1, 2, \dots, n-1$, and if $P_F^{\mu, i}$ is defined accordingly, we can prove that $a_{kk}(P) > 1$ for at least one k , if $P_F^{\mu, i}(\xi) = 0$ for some μ, i and some real ξ with $\xi_1, \dots, \xi_n \neq 0$. (Cf. Grusin [8], Theorem 2.2, where it is proved, essentially, that $a_{jk}(P) > x_k(F^{\mu, i})/x_j(F^{\mu, i})$ for at least one couple j, k with $j \neq k$, if $P_F^{\mu, i}(\xi) = 0$ for some μ, i and some real $\xi \neq (0, \dots, 0)$). The second half of Theorem 4.1 is now a consequence of

THEOREM 4.3 ⁽⁴⁾. Let $P(\xi)$ be a given polynomial, and define $P_F^{\mu, i}$ as above. Then P is multi-quasielliptic if (and only if)

- i) every face $F^j(P)$ of $F(P)$ has a normal $\nu^j > 0$,
- ii) $P_F^{\mu, i}(\xi) \neq 0$ for all μ, i and all real $\xi = (\xi_1, \dots, \xi_n)$ with $\xi_1, \dots, \xi_n \neq 0$.

⁽⁴⁾ For the case of a real polynomial $P(\xi)$, a slightly more general result was recently announced, without indication of the proof, by Mihailov in his note [15]. We include a proof here for the sake of completeness.

Proof. Let us first assume that $P(\xi)$ is a real polynomial. Then the sign of $P(\xi)$ is constant, say positive, for $|\xi|$ big enough. Consequently, if $\{\alpha^j\}$ are the vertices of $F(P)$, the components of each α^j must be even integers ≥ 0 , and $c_{\alpha^j} > 0$. Let $\eta_j = \xi^{\alpha^j}$, $j = 1, 2, \dots, N$. Then all $\eta_j \geq 0$, and

$$(4.6) \quad P_F(\xi) = B(\xi^{\alpha^1}, \dots, \xi^{\alpha^N})$$

with

$$B = B(\eta_1, \dots, \eta_N) = \sum b_\theta \eta^\theta; \quad \theta = (\theta_1, \dots, \theta_N), \quad \theta_i \geq 0, \quad \sum \theta_i = 1.$$

Similarly, for every μ, i , we let $P_F^{\mu, i}(\xi) = B^{\mu, i}(\xi^{\alpha^1}, \dots, \xi^{\alpha^N})$ define a function $B^{\mu, i}(\eta)$. Finally, we denote by D the set $\{\eta \in R_+^N; \eta = (\xi^{\alpha^1}, \dots, \xi^{\alpha^N}), \xi \in R^n\}$. Then if P satisfies condition ii) of Theorem 4.2, it follows that every $B^{\mu, i}(\eta)$ is continuous, homogeneous of degree 1, and > 0 on the closed subset D of R_+^N . We want to show that $\xi^{\alpha^1} + \dots + \xi^{\alpha^N} \leq C(1 + P_F(\xi))$ for ξ real, in other words that $\beta(\eta) = \sum_1^N \eta_j \leq C(1 + B(\eta))$ for $\eta \in D$, some $C > 0$.

Suppose first that, for some k , we have $\beta(\eta) = \sum \eta_j \leq (1 + \varepsilon)\eta_k$, and let $B^{0, k}(\eta) = b_k \eta_k$. Then $b_k > 0$, and since $\eta^\theta = \prod \eta_i^{\theta_i}$, $\sum \theta_i = 1$, $\theta_k < 1$ if $\eta^\theta \neq \eta_k$, it follows that

$$B(\eta) \geq b_k \eta_k - \sum' |b_\theta| \eta^\theta \geq \frac{1}{2} b_k \eta_k \geq C_0 \beta(\eta), \quad C_0 > 0,$$

if only ε is small enough, $\varepsilon \leq \varepsilon_0$, say. Now, if $\beta(\eta) > (1 + \varepsilon_0)\eta_k$ for all k , then we suppose instead that $\beta(\eta) \leq (1 + \varepsilon)\beta^{1, k}(\eta)$, for some k . (Let us take, by definition, $\beta^{\mu, k}(\eta) = \sum \eta_j$, summed over all j with $\alpha^j \in F^{\mu, k}(P)$). If $\beta^{1, k}(\eta) = \eta_i + \eta_j$, it follows that $(\varepsilon_0 - \varepsilon)\eta_i < (1 + \varepsilon)\eta_j$, $(\varepsilon_0 - \varepsilon)\eta_j < (1 + \varepsilon)\eta_i$. Thus, if $\varepsilon < \varepsilon_0$, we have $\eta_i \asymp \eta_j \asymp \beta(\eta)$, in the sense that the ratio of any two of $\eta_i, \eta_j, \beta(\eta)$ is bounded away from zero. But then it follows from the mentioned properties of every $B^{\mu, i}(\eta)$ that

$$B^{1, k}(\eta) \geq c \beta^{1, k}(\eta), \quad c > 0, \quad \text{for all } \eta \in D.$$

Hence if ε is small enough, then also

$$B(\eta) \geq B^{1, k}(\eta) - \sum'' |b_\theta| \eta^\theta \geq \frac{1}{2} c \beta^{1, k}(\eta) \geq c_1 \beta(\eta), \quad \text{for } \eta \in D,$$

with $c_0 \geq c_1 > 0$. It is now clear that we can prove, by induction, that

$$(4.7) \quad B(\eta) \geq c_\mu \beta(\eta), \quad \text{if } \eta \in D, \quad \beta(\eta) \leq (1 + \varepsilon_\mu) \beta^{\mu, k}(\eta)$$

for some k . (The only trouble is to show that if $\eta \in D$, $\beta(\eta) \leq (1 + \varepsilon) \beta^{\mu, k}(\eta)$, with ε small enough, then either $\beta(\eta) \leq (1 + \varepsilon_{\mu-1}) \beta^{\mu-1, i}(\eta)$ for some i or $\eta_i \subset \beta(\eta)$ for all i with $\alpha^i \in F^{\mu, k}(P)$. Suppose for instance that $\beta^{\mu, k}(\eta) = \eta_1 + \dots + \eta_m$, $m \geq \mu + 1$, but that only $\eta_1, \dots, \eta_j \subset \beta(\eta)$, $j \leq \mu$. Then $\alpha^1, \dots, \alpha^j$ all belong to one of the faces $F^{\mu-1, i}$ of $F^{\mu, k}$, and so $\beta(\eta) \leq (1 + \varepsilon_{\mu-1}) \beta^{\mu-1, i}$ for some i , if $\varepsilon \leq \varepsilon_\mu < \varepsilon_{\mu-1}$. Otherwise we would be able to find $\theta = (\theta_1, \dots, \theta_j)$, $\theta' = (\theta'_1, \dots, \theta'_m)$, with $\theta_i, \theta'_s > 0$, $\sum \theta_i = \sum \theta'_s = 1$, and some $\theta'_s \neq 0$, $s > j$, such that $\eta^\theta \equiv \eta^{\theta'}$, when $\eta \in D$, and this would lead to a contradiction. Finally if $\eta_1, \dots, \eta_j \subset \beta(\eta)$, $j > \mu$, then we can write every η_i , $i \leq m$, in the form η^θ , $\theta = (\theta_1, \dots, \theta_j)$, $\sum \theta_s = 1$, which implies that $\eta_i \subset \beta(\eta)$ for all i with $\alpha^i \in F^{\mu, k}$.

Let now $\varrho_F(\xi) = \beta(\xi^{\alpha^1}, \dots, \xi^{\alpha^N})$, and $\varrho_F^k(\xi) = \beta^{n-1, k}(\xi^{\alpha_1}, \dots, \xi^{\alpha^N})$, $k = 1, 2, \dots$. Then, in view of (4.6), (4.7), we know there are constants $c, \varepsilon > 0$ such that $P_F(\xi) \geq c \varrho_F(\xi)$ when ξ is real, and

$$(4.8) \quad \varrho_F(\xi) \leq (1 + \varepsilon) \varrho_F^k(\xi) \quad \text{for some } k.$$

But (4.8) is valid for any given $\varepsilon > 0$, if ξ is real and $|\xi|$ is big enough. To prove this, let $\nu \in S_+ = \{\nu \in R^n; \nu \geq 0, |\nu| = 1\}$, and set $h(\nu) = \max_{F(P)} \langle \nu, \alpha^i \rangle$, $F(\nu) = \{\alpha^i \in F(P); \langle \nu, \alpha^i \rangle = h(\nu)\}$. Then $F(\nu) = F^{\mu, k}$ for some μ, k , and $\langle \nu, \alpha^i \rangle < h(\nu)$ for all $\alpha^j \notin F^{\mu, k}$. It is easy to see that, for given μ, k , the part $S^{\mu, k}$ of S_+ , where $F(\nu) = F^{\mu, k}$ is a simply connected subset of the intersection of S_+ with some linear manifold of dimension $n - \mu$. In particular, $S^{n-1, k} = \nu(F^k) = \nu^k(F)$. (Cf. Fig. 3, where the subdivision of S_+ corresponds to the case when $F(P)$ is given by Fig. 1 of section 2).

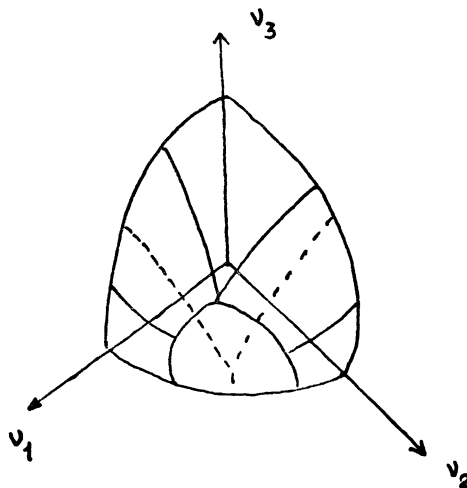


Fig. 3

Let finally to every $\nu^k(F)$ correspond a neighborhood S^k on S_+ given by $S^k = \{\nu \geq 0; |\nu - \nu^k| = \min_j |\nu - \nu^j|\}$. Then it follows, by a continuity argument, that $\max_{\nu \in S^k} [\langle \nu, \alpha^j \rangle / h(\nu)] \leq \delta < 1$ for all $\alpha^j \notin F^k$, all k . Suppose now that $\xi = (\xi_1, \dots, \xi_n) \in R^n$, $\xi_1, \dots, \xi_n \geq 1$. Then we can determine uniquely $t > 0$ and $\nu \in S_+$ such that $\xi = (t^{\nu_1}, \dots, t^{\nu_n}) = t^\nu e$. If $\nu \in S^k$, it follows that $\varrho_F^k(\xi) \geq \max_{F^k} (t^\nu e)^{\alpha^i} = t^{h(\nu)}$. On the other hand, there is a constant K , depending only on F , such that $\varrho_F(\xi) - \varrho_F^k(\xi) \leq K t^{\delta h(\nu)}$. But $h(\nu)$ is bounded away from zero, and $\delta < 1$. Consequently (4.8) holds for $\xi = t^\nu e$, $t \geq t_0$. The same type of argument can be used when $\xi = (\pm t^{\nu_1}, \dots, \pm t^{\nu_n})$, and if $|\xi_i| < 1$ for some components of ξ , we can argue in the same way, using t, ν as parameters on a ray parallel to some of the coordinate planes. Hence (4.8) is valid, for given ε , if ξ is real, $|\xi|$ big enough. We have then proved that $P_F(\xi) \geq c \varrho_F(\xi)$ for such ξ , and it follows that P is multiquasielliptic.

If $P(\xi)$ is a polynomial satisfying the conditions of Theorem 4.2, but not necessarily real, we observe that $|P(\xi)|^2$ is a real polynomial, with $F(|P|^2) = 2F(P)$, and $(|P|^2)_F^{\alpha^i} = |P_F^{\alpha^i}|^2$. Hence $|P|^2$ is multiquasielliptic, and it follows that also P is multiquasielliptic.

From the proof of Theorem 4.3 it is easy to derive

COROLLARY 4.1. Let $P(\xi)$ be multi-quasielliptic, $F(P) = \bigcup_1^M F^i$. Then there is a constant $C > 0$, and a division of the set $|\xi| \geq C$ into simply connected subsets V^i , $1 \leq i \leq M$, such that

$$P(\xi) \asymp P_F^i(\xi) \asymp \varrho_F^i(\xi) \asymp \varrho_F(\xi) \quad \text{for } \xi \in V^i.$$

REFERENCES

- [1] BAROZZI, G. C., *Sul prodotto di polinomi quasi-ellittici*. Boll. Un. Mat. Ital. 20, 1965, 169-176.
- [2] CATTABRIGA, L., *Su una classe di polinomi ipoellittici*. Rend. Sem. Mat. Univ. Padova, 36, 1966, 285-309; 37, 1967, 60-74.
- [3] FRIBERG, J., *Estimates for partially hypoelliptic operators*. Medd. Lund Univ. Mat. Sem. 17, 1963.
- [4] FRIBERG, J., *Asymptotic behavior of spectral functions for multi-quasielliptic operators*. Sémin. Mat. Sup. Montréal, 19, 1965, 63-72.
- [5] FRIBERG, J., *Principal parts and canonical factorizations of hypoelliptic polynomials in two variables*. Rend. Sem. Mat. Univ. Padova 37, 1967, 112-132.
- [6] GORCAKOV, V. N., *Asymptotic behavior of spectral functions for hypoelliptic operators of a certain class*. Soviet Math. Dokl. 4, 1963, 1328-1331.
- [7] GORIN, E. A., *Partially hypoelliptic differential equations with constant coefficients*. (Russian). Sibirskii Mat. Ž. 3, 1962, 500-526.
- [8] GRUSIN, V. V., *A connection between local and global properties of hypoelliptic operators with constant coefficients*. (Russian). Mat. Sbornik 66: 4, 1965, 525-550.
- [9] HÖRMANDER, L., *On the interior regularity of the solutions of partial differential equations*. Comm. Pure Appl. Math. 11, 1958, 197-218.
- [10] HÖRMANDER, L., *Linear differential operators*. Springer, Berlin, 1963.
- [11] NIKOLSKII, S. M., *The first boundary problem for a general linear equation*. Soviet Math. Dokl. 3, 1962, pp. 1388-1390.
- [12] PINI, B., *Osservazioni sulla ipoellitticità*. Boll. Un. Mat. Ital. (3) 18, 1963, 420-432.
- [13] PINI, B., *Sulla classe di Gevrey delle soluzioni di certe equazioni ipoellittiche*. Boll. Un. Mat. Ital. (3) 18, 1963, 260-269.
- [14] VOLEVIC, L. R., *Local properties of solutions of quasielliptic systems*. (Russian). Mat. Sbornik 59 (101) (supplement), 1962, 500-526.
- [15] MIHAILOV, V. P., *The behavior of certain classes of polynomials at infinity*. Soviet. Math. Dokl. 164 (1965), 1256-1259.