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NONLOCAL ELLIPTIC BOUNDARY VALUE PROBLEMS (*)

MARTIN SCHECHTER

1. Introduction.

In [13, 14] Beals considers boundary value problems of the form

$$Au = f \quad \text{in} \quad \Omega \tag{1.1}$$

$$B_j u = \sum_{k=1}^{r} M_{jk} C_k u \quad \text{on} \quad \partial\Omega, \qquad 1 \leq j \leq r, \tag{1.2}$$

where A is an elliptic operator of order $m = 2r$ in a domain $\Omega \subset E^n$, $\{B_j\}_{j=1}^r$, $\{C_j\}_{j=1}^r$ are sets of differential boundary operators and the M_{jk} are arbitrary linear operators bounded in a certain sense. He considers the problem for those $u \in L^2(\Omega)$ for which $Au \in L^2(\Omega)$ and all derivatives $< m$ are in $L^2(\partial\Omega)$ and such that (1.2) holds. Under suitable hypotheses he proves that the operator $A(M)$ thus defined is closed (and a Fredholm operator for Ω bounded) and that its adjoint is of the form

$$A' v = g \quad \text{in} \ \Omega. \tag{1.3}$$

$$B_j' v = \sum_{k=1}^{r} M_{kj}' C_j' v \quad \text{on} \quad \partial\Omega, \qquad 1 \leq j \leq r, \tag{1.4}$$

where A' is the formal adjoint of A and the B_j', C_k', M_{kj}' are related to the B_j, C_k, M_{jk} by integration by parts. Such problems are called « nonlocal », since the M_{jk} need not be differential operators.

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For second order, self-adjoint operators, previous work along these lines was done by Calkin [15], Bade and Freeman [16] and Freeman [17]. When the M_{jk} are singular integral operators, problem (1.1,2) was studied by Dynin [18] and Agronovich and Dynin [19]. Abstract boundary value problems were considered by Višik [20], Hörmander [21], Browder [22-25], Peetre [26], Phillips [27], and Schechter [2,28].

In this paper we extend some of the results of Beals (but restrict ourselves to bounded domains). We consider the problem (1.1,2) in a slightly more general framework, namely the $H^{t,p}(\Omega)$ spaces for t real and $1 < p < \infty$. The boundary values of the functions and their derivatives of order $< m$ are taken in the $W^{\sigma,p}(\partial\Omega)$ spaces, with $t \leq \sigma + 1/p \leq t + m$. This allows a bit more latitude in applications. It also allows one to relax the assumptions on the M_{jk}.

We also consider the more general type of problem where (1.2) is replaced by

$$B_j u = \sum_{k=1}^{m} L_{jk} \frac{\partial^{k-1} u}{\partial n^{k-1}} \quad \text{on} \quad \partial\Omega, \qquad 1 \leq j \leq r, \tag{1.5}$$

where $\partial^l u/\partial n^l$ denotes the normal derivative of order l on $\partial\Omega$. We investigate the regularity of solutions of (1.1) (1.5) under varying assumptions on the L_{jk} and obtain a priori estimates. Our methods make use of L^p estimates of [2, 4, 6].

As an example, let A be a second order operator. Consider the closure in $L^p(\Omega)$ of A acting on those $u \in C^\infty(\bar{\Omega})$ satisfying boundary conditions of the form

$$\frac{\partial u}{\partial n} = R_1 u + R_2 \frac{\partial u}{\partial n} \quad \text{on} \quad \partial\Omega, \tag{1.6}$$

where the R_i are arbitrary linear operators. Sufficient conditions are given for the operator $A(R)$ thus defined to be Fredholm (and therefore closed) and for the solutions of $A(R)u = 0$ to be smooth. When $R_2 = 0$, we give sufficient conditions for the adjoint of $A(R)$ to be the closure in $L^{p'}(\Omega)$, $p' = p/(1-p)$, of the formal adjoint A' of A restricted to those $v \in C^\infty(\bar{\Omega})$ such that

$$\frac{\partial v}{\partial \nu} = R_1^* v \tag{1.7}$$

where R_1^* is the adjoint of R_1 and $\partial v/\partial\nu$ is a first order derivative obtained by integration by parts. Similar results hold if (1.6) is replaced by

$$u = S_1 u + S_2 \frac{\partial u}{\partial n} \quad \text{on} \quad \partial\Omega. \tag{1.8}$$

Next consider A defined on those $u \in L^p(\Omega)$ such that $Au \in L^p(\Omega)$ and all derivatives of u of orders < 2 are in $L^p(\partial\Omega)$ and satisfy (1.6). If R_1 maps $W^{j,p}(\partial\Omega)$ into itself and R_2 maps $W^{j-1,p}(\partial\Omega)$ into $W^{j,p}(\partial\Omega)$ for each integer $j \geq 0$, then $Au \in C^\infty(\bar{\Omega})$ implies $u \in C^\infty(\bar{\Omega})$ (assuming that $\partial\Omega$ and the coefficients of A are infinitely differentiable).

2. Main results.

Let Ω be a bounded domain in Euclidean n-dimensional space E^n with boundary $\partial\Omega$ of class C^∞. Throughout the paper we shall assume that A is a properly elliptic operator of order $m = 2r$ with coefficients in $C^\infty(\bar{\Omega})$, where $\bar{\Omega}$ denotes the closure of Ω (for definitions for all terms we refer to [1]). $\{B_j\}_{j=1}^r$ will denote a normal set of differential operators of orders $< m$ with coefficients in $C^\infty(\partial\Omega)$. This means that the orders of the B_j are distinct and that $\partial\Omega$ is nowhere characteristic for any of them. We shall also assume that $\{B_j\}_{j=1}^r$ covers A.

It is convenient to discuss boundary value problems for A within the framework of the $H^{s,p}(\Omega)$ spaces, s, p real, $1 < p < \infty$. We give brief definitions here; for further details we refer to [2]. The space $H^{s,p}(E^n)$ is the completion of $C_0^\infty(E^n)$ with respect to the norm given by

$$\| u \|_{s,p}^{E^n} = \left[\int | \mathcal{F}^{-1} (1 + | \xi |^2)^{s/2} \mathcal{F} u |^p \, dx \right]^{1/p}$$

where $\mathcal{F}$ denotes the Fourier transform and $C_0^\infty(E^n)$ is the set of infinitely differentiable functions on E^n with compact supports. For $s \geq 0$ we let $H^{s,p}(\Omega)$ denote the restrictions to Ω of functions in $H^{s,p}(E^n)$ with the norm

$$\| u \|_{s,p} = glb \, \| v \|_{s,p}^{E^n} , \qquad v = u \quad \text{on} \quad \Omega.$$

For $s < 0$, $u \in C^\infty(\bar{\Omega})$ we set

$$\| u \|_{s,p} = \underset{v \in C^\infty(\bar{\Omega})}{lub} \frac{| (u, v) |}{\| v \|_{-s,p'}},$$

where (u, v) is the scalar product in $L^2(\Omega)$ and $p' = p/(p-1)$. We then complete $C^\infty(\bar{\Omega})$ with respect to this norm to obtain $H^{s,p}(\Omega)$.

For $s > 0$ we let $W^{s,p}(\partial\Omega)$ denote the restrictions to $\partial\Omega$ of functions in $H^{s+1/p,p}(\Omega)$ with norm

$$\langle g \rangle_{s,p} = glb \, \| v \|_{s+1/p,p}, \qquad v = g \quad \text{on} \quad \partial\Omega.$$

It follows from the results of [3] that they are Banach spaces (cf. [2,4]). Moreover for $g \in C^\infty(\partial\Omega)$ we set

$$\langle g \rangle_{0,p} = \lim_{0 < s \to 0} \langle g \rangle_{s,p}.$$

This limit exists and gives a norm (cf. [4]). For $s < 0$ we set

$$\langle g \rangle_{s,p} = \underset{h \in C^\infty(\partial\Omega)}{lub} \frac{|\langle g, h \rangle|}{\langle h \rangle_{-s,p'}},$$

where $\langle g, h \rangle$ denotes the $L^2(\partial\Omega)$ scalar product. For $s \leq 0$ we let $W^{s,p}(\partial\Omega)$ denote the completions of $C^\infty(\partial\Omega)$ with respect to these norms.

For $u \in C^\infty(\overline{\Omega})$ and s real we introduce the norm

$$||| u |||_{s,p} = \| u \|_{s,p} + \| Au \|_{s-m,p}$$

and denote the completion of $C^\infty(\overline{\Omega})$ with respect to this norm by $H_A^{s,p}(\Omega)$.

Let γ_l denote the normal derivative of order l on $\partial\Omega$. We shall show that for $l < m$, γ_l can be defined for elements of $H_A^{s,p}(\Omega)$.

LEMMA 2.1. *For each s there is a constant K such that*

$$\sum_{l=0}^{m-1} \langle \gamma_l u \rangle_{s-l-1/p,p} \leq K \, ||| u |||_{s,p}$$

for all $u \in C^\infty(\overline{\Omega})$.

From the lemma we see that the mapping $\gamma = \{\gamma_0, \dots, \gamma_{m-1}\}$ can be extended by continuity to a bounded mapping from $H_A^{s,p}(\Omega)$ to the space

$$E_{s,p} \equiv \prod_{l=0}^{m-1} W^{s-l-1/p,p}(\partial\Omega).$$

Similarly the mapping $B = \{B_1, \dots, B_r\}$ can be extended to a bounded mapping from $H_A^{s,p}(\Omega)$ to

$$F_{s,p} \equiv \prod_{j=1}^{r} W^{s-m_j-1/p,p}(\partial\Omega),$$

where m_j is the order of B_j. We also extend A to be a mapping from $H_A^{s,p}(\Omega)$ to $H^{s-m,p}(\Omega)$.

Suppose we are given an rxm matrix $L = (L_{jl})$ of operators such that L_{jl} is a linear operator from $W^{s-l-1/p,p}(\partial\Omega)$ to $W^{s-m_j-1/p,p}(\partial\Omega)$. Then the (matrix) operator L is a linear map from $E_{s,p}$ to $F_{s,p}$. Let t be a number satisfying $s - m \leq t \leq s$. We define the operator $A_{s,p}(L)$ as the restriction of A to those $u \in H_A^{s,p}(\Omega)$ such that $Au \in H^{t,p}(\Omega)$ and

$$Bu = L\gamma u. \tag{2.1}$$

We consider $A_{s,p}(L)$ as an operator in $H^{t,p}(\Omega)$.

Let T be a linear operator on a Banach space X. It is called a *semi-Fredholm* operator if

1) the domain $D(T)$ of T is dense in X
2) T is closed
3) the null space $N(T)$ of T has finite dimension
4) the range $R(T)$ of T is closed in X.

It is called a *Fredholm* operator if, in addition,

5) the codimension of $R(T)$ in X is finite

THEOREM 2.1. *If there are constants $\varepsilon < 1$, c, such that*

$$\| L\gamma u \|_{F_{s,p}} \leq \varepsilon \| Bu \|_{F_{s,p}} + c(\| Au \|_{t,p} + \| u \|_{s-m,p}) \tag{2.2}$$

holds for all $u \in C^\infty(\bar{\Omega})$, then $A_{s,p}(L)$ is a semi-Fredholm operator.

The proof of Theorem 2.1 can be made to depend on

THEOREM 2.2. *If* (2.2) *holds, then*

$$\| u \|_{s,p} \leq C(\| Au \|_{t,p} + \| u \|_{s-m,p} + \| (B - L\gamma) u \|_{F_{s,p}}) \tag{2.3}$$

for all $u \in C^\infty(\bar{\Omega})$.

Another criterion is given by

THEOREM 2.3. *If L is a compact operator from $E_{s,p}$ to $F_{s,p}$, then* (2.3) *holds and hence $A_{s,p}(L)$ is a semi-Fredholm operator. In particular, this is true if L maps $E_{s-\varepsilon,p}$ boundedly into $F_{s,p}$ or $E_{s,p}$ boundedly into $F_{s+\varepsilon,p}$ for some $\varepsilon > 0$.*

THEOREM 2.4. *Assume that L maps $E_{s,p}$ into $F_{\varrho,q}$, $1 < q < \infty$. If $u \in H_A^{s,p}(\Omega)$, $Au \in H^{\varrho-m,q}(\Omega)$ and $Bu = L\gamma u$ then $u \in H_A^{\varrho,q}(\Omega)$.*

THEOREM 2.5. *If there is an* $\varepsilon > 0$ *such that* L *maps* $E_{\sigma,p}$ *into* $F_{\sigma+\varepsilon,p}$ *for* $s \leq \sigma \leq t + m - \varepsilon$, *then* $A_{s,p}(L) = A_{t+m,p}(L)$.

The dual space of $F_{s,p}$ is

$$F_{s,p}^* \equiv \prod_{j=1}^{r} W^{m_j - s + 1/p, p'}(\partial\Omega)$$

(cf. [4,6]). A similar formula gives $E_{s,p}^*$.

THEOREM 2.6. *Suppose* $\varrho > s$ *and that for some* $\varepsilon > 0$ L *maps* $E_{\sigma,p}$ *boundedly into* $F_{\sigma+\varepsilon,p}$ *for* $s \leq \sigma \leq \varrho - \varepsilon$. *Assume further that* $f \in H^{m-\varrho,p'}(\Omega)$, $G = g_1, \dots, g_r \in F_{\varrho,p}^*$ *and*

$$|(f, Au) + \langle G, Bu - L\gamma u\rangle| \leq c_0 \,|||u|||_{s,p} \tag{2.4}$$

for all $u \in C^\infty(\overline{\Omega})$, *where* $\langle , \rangle$ *denotes duality between* $F_{s,p}$ *and* $F_{s,p}^*$. *Then* $f \in H^{m-s,p'}(\Omega)$, $G \in F_{s,p}^*$ *and*

$$\|f\|_{m-s,p'} + \|G\|_{F_{s,p}^*} \leq \text{const.}\,(c_0 + \|f\|_{m-s,p'} + \|G\|_{F_{s,p}^*}) \tag{2.5}$$

COROLLARY 2.1. *If* L *maps* $E_{\sigma,p}$ *boundedly into* $F_{\sigma+\varepsilon,p}$ *for some* $\varepsilon > 0$ *and each real* σ *and*

$$(f, Au) + \langle G, Bu - L\gamma u\rangle = 0 \tag{2.6}$$

for all $u \in C^\infty(\overline{\Omega})$, *then* $f \in C^\infty(\overline{\Omega})$ *and each* $g_j \in C^\infty(\partial\Omega)$, $1 \leq j \leq r$. *Moreover the set of all such* f, G *is finite dimensional.*

For Theorems 2.7-2.10 we assume that there is an $\varepsilon > 0$ such that L is a bounded mapping from $E_{\sigma,p}$ to $F_{\sigma+\varepsilon,p}$ for each real σ.

Let $V(L)$ denote the set of those $u \in C^\infty(\overline{\Omega})$ satisfying (2.1) and $V(L)'$ the set of those $v \in C^\infty(\overline{\Omega})$ satisfying

$$(Au, v) = (u, A'v)$$

for all $u \in V(L)$, where A' is the formal adjoint of A. By $N(A(L))$ [resp. $N(A(L)')$] we shall denote the set of those $u \in V(L)$ [resp. $v \in V(L)'$] which satisfy $Au = 0$ [resp. $A'v = 0$] We have

TEOREM 2.7. *For each real* σ

$$N(A_{\sigma,p}(L)) = N(A(L)).$$

Let $\tilde{N}$ denote the set of those $h \in C^\infty(\bar{\Omega})$ for which there is a G $C^\infty(\partial\Omega)$ such that

$$(h, Au) + \langle G, Bu - L\gamma u \rangle = 0 \tag{2.7}$$

for all $u \in C^\infty(\bar{\Omega})$. By Corollary 2.1, $\tilde{N}$ is finite dimensional. Clearly $\tilde{N} \subseteq N(A(L)')$. We shall prove

THEOREM 2.8. $\tilde{N} = N(A(L)')$. *Hence the latter is finite dimensional.*

In proving Theorem 2.8 we shall make use of

THEOREM 2.9. $R(A_{s,p}(L))$ *consists of those* $f \in H^{t,p}(\Omega)$ *which are orthogonal to* $\tilde{N}$, *i.e., which satisfy* $(f, h) = 0$ *for all* $h \in \tilde{N}$.

THEOREM 2.10. *If* $v \in H^{\sigma,p}(\Omega)$ *for some* σ *and*

$$(v, Au) = 0 \tag{2.8}$$

for all $u \in V(L)$, *then* $v \in \tilde{N}$.

COROLLARY 2.2. *If* $v \in H^{s,p'}(\Omega)$, $f \in H^{t,p'}(\Omega)$ *and*

$$(v, Au) = (f, u) \tag{2.9}$$

for all $u \in V(L)$, *then* $v \in H^{t+m,p'}(\Omega)$.

COROLLARY 2.3. $A_{s,p}(L)$ *is a Fredholm operator.*

Let $\nu_1, \dots, \nu_r$ be the complementary set of the m_j among the integers $0, \dots, m-1$. Let $\{C_k\}_{k=1}^r$ be any normal set of differential boundary operators with coefficients in $C^\infty(\partial\Omega)$ and such that the order of C_k is ν_k. Then there are normal sets $\{B'_j\}_{j=1}^r$ and $\{C'_k\}_{k=1}^r$ such that

$$(Au, v) - (u, A'v) + \sum_{j=1}^{r} \langle B_j u, C'_j v \rangle - \sum_{k=1}^{r} \langle C_k u, B'_k v \rangle = 0 \tag{2.10}$$

holds for $u, v \in C^\infty(\bar{\Omega})$ (cf. [5,1]). The order of B'_j in $m - \nu_j - 1$, while that of C'_j is $m - m_j - 1$. We set $B' = \{B'_1, \dots, B'_r\}$, $C = \{C_1, \dots, C_r\}$, $C' = \{C'_1, \dots, C'_r\}$.

Let $M = (M_{jk})$ be an $r \times r$ matrix of linear operators such that M_{jk} maps

$$W^{s-\nu_k-1/p,p}(\partial\Omega) \quad \text{into} \quad W^{s-m_j-1/p,p}(\partial\Omega).$$

Thus M maps $J_{s,p}$ into $F_{s,p}$, where

$$J_{s,p} = \prod_{k=1}^{r} W^{s-\nu_k-1/p,\,p}(\partial\Omega).$$

By expressing each C_j in terms of normal and tangential derivatives on $\partial\Omega$, we obtain a unique operator L_1 from $E_{s,p}$ to $F_{t,p}$ such that

$$L_1\,\gamma = MC\,.$$

We have

PROPOSITION 2.1. *If $s = t + m$ and (2.3) holds, then $A_{s,p}(L_1)$ is the closure of A in $H^{t,p}(\Omega)$ defined for those $u \in C^\infty(\overline{\Omega})$ satisfying $(B - MC)\,u = 0$ on $\partial\Omega$.*

REMARK. By Theorem 2.3, the inequality (2.3) holds when M is a compact operator from $J_{s,p}$ to $F_{s,p}$.

We now assume that there is a number τ such that $t \le \tau \le s$ and such that M is a bounded operator from $J_{\tau,p}$ to $F_{s,p}$. If $\tau < s$, it follows that M is compact from $J_{s,p}$ to $F_{s,p}$. If $\tau = s$, we assume this. We set

$$L_1' = M^*\,C',$$

where M^* is the adjoint of M. It follows that L_1 is a bounded operator from $E_{\tau,p}$ to $F_{s,p}$ while L_1' is bounded from $F^*_{s,p}$ to $E^*_{\tau,p}$. For $-t \le \sigma \le$ $\le m - t$, $1 < q < \infty$, we let $A'_{\sigma,q}(L_1')$ denote the restriction of A' to those $v \in H^{\sigma,q}_{A'}(\Omega)$ for which $A'\,v \in H^{-t,q}(\Omega)$ and

$$B'\,v = M^*\,C'\,v. \tag{2.11}$$

We have

THEOREM. 2.11. *Under the above hypotheses,*

$$(A_{s,p}(L_1))^* = A'_{m-\tau,p'}(L_1') \tag{2.12}$$

and

$$(A'_{m-\tau,p'}(L_1'))^* = A_{s,p}(L_1). \tag{2.13}$$

The case considered by Beals [13, 14] is $t = 0$, $p = 2$, $s = m - 1/p'$, $\tau = 1/p$ (we have been able to avoid the consideration of the operator S of his papers). By known interpolation theorems (cf. [11, 6]) the assumption that L maps $E_{\sigma,p}$ boundedly into $F_{\sigma+\varepsilon,p}$ for all σ need only be verified for sequences $\sigma_k' \to \infty$, $\sigma_k'' \to -\infty$.

3. Background Material.

We now list those known results which will be used in our proofs.

THEOREM 3.1. *For each number* ϱ *there is a constant* C *such that*

$$\| u \|_{\varrho, p} \leq C (\| Au \|_{\varrho - m, p} + \| u \|_{\varrho - m, p} + \| Bu \|_{F_{\varrho, p}}) \tag{3.1}$$

for all $u \in C^\infty (\overline{\Omega})$.

Inequality (3.1) is a weaker form of Theorem 2.1 of [6].

THEOREM 3.2. *For* $\sigma > \varrho$, *the unit sphere in* $H^{\sigma, p} (\Omega)$ *is conditionally compact in* $H^{\varrho, p} (\Omega)$. *The same is true for the spaces* $W^{\varrho, p} (\Omega)$.

For $\varrho = 0$ and σ positive and large, Theorem 3.2 follows easily from Sobolev's Lemma. For the other cases one applies an abstract interpolation result of Lions Peetre [7, Theorem 2.3, p. 38].

THEOREM 3.3. *If* $f \in H^{m-\varrho, p'} (\Omega)$, $G \in F^*_{\varrho, p}$ *and*

$$| (f, Au) + \langle G, Bu \rangle | \leq c_1 ||| u |||_{s, p} \tag{3.2}$$

for all $u \in C^\infty (\overline{\Omega})$, *then* $f \in H^{m-s, p'} (\Omega)$, $G \in F^*_{s, p}$ *and*

$$\| f \|_{m-s, p'} + \| G \|_{F^*_{s, p}} \leq \text{const.} \; (c_1 + \| f \|_{m - \varrho, p'}) \tag{3.3}$$

THEOREM 3.3. follows from Theorem 2.1 of [4]. (The term $\| Au \|_{s-m, p}$ was missing from the right hand side of the inequality corresponding to (3.2). However, one checks easily from the proof given there that it could have been included.)

THEOREM 3.4. *For each set* $\Phi = \{\Phi_1, \dots, \Phi_r\}$ *of functions in* $C^\infty (\partial \Omega)$ *there is a* $u \in C^\infty (\overline{\Omega})$ *such that*

$$Bu = \Phi \tag{3.4}$$

and for each real ϱ

$$K^{-1} \| Au \|_{\varrho + m, p} \leq \| u \|_{\varrho, p} \leq K \| \Phi \|_{F_{\varrho, p}}, \tag{3.5}$$

where the constant K *does not depend on* Φ *or* u.

PROOF. Consider the boundary problem

$$(3.6)\qquad (A' A + 1)\, u = 0\,;$$

$$(3.7)\qquad Bu = \Phi, \qquad B' Au = 0\,.$$

By (2.10)

$$(3.8)\qquad (Aw, Av) - (w, A' Av) = \langle Bw, C' Av \rangle - \langle Cw, B' Av \rangle$$

for all $w, v \in C^\infty(\bar{\Omega})$. From this one easily checks that the problem (3.6, 7) is self-adjoint. Moreover, it is a well posed elliptic boundary value problem. (Here we make use of the fact that B' covers A' [8, 1]). In addition, when $\Phi = 0$ we have by (3.8).

$$\| Au \|^2 + \| u \|^2 = (u, (A' A + 1)\, u) = 0,$$

showing that $u = 0$. Applying the theory of such problems, we see that for each $\Phi \subset C^\infty(\partial\Omega)$ there is a unique solution $u \in C^\infty(\bar{\Omega})$ of (3.6, 7) (cf. [1]). We can also apply Theorem 3.1 to this problem, taking into consideration the fact that the term $\| u \|_{\varrho-m, p}$ may be dropped in (3.1) when there is uniqueness. Thus we have for each ϱ

$$\| u \|_{\varrho, p} \leq K \| \Phi \|_{F_{\varrho, p}},$$

where the constant K does not depend on Φ or u. We now note that Au is a solution of

$$A' w = - u$$

$$B' w = 0.$$

Applying Theorem 3.1 to this problem we obtain

$$(3.9)\qquad \| Au \|_{m+\varrho, p} \leq C\,(\| u \|_{\varrho, p} + \| Au \|_{\varrho, p})$$

for each ϱ. We claim that this implies

$$(3.10)\qquad \| Au \|_{m+\varrho, p} \leq K \| u \|_{\varrho, p}\,.$$

For otherwise there would be a sequence $\{u_k\}$ of functions $u_k \in C^\infty(\bar{\Omega})$ satisfying $(A' A + 1)\, u_k = 0$, $B' Au_k = 0$ such that

$$\| Au_k \|_{m+\varrho, p} = 1, \qquad \| u_k \|_{\varrho, p} \to 0.$$

By Theorem 3.2 there is a subsequence (also denoted by $\{u_k\}$) for which Au_k converges in $H^{\varrho,p}(\Omega)$. By (3.9) Au_k converges in $H^{m+\varrho,p}(\Omega)$. On one hand the limit must be zero, since for $w \in C_0^\infty(\Omega)$

$$(Au_k\,,\, w) = (u_k\; A'\, w) \longrightarrow 0$$

while on the other, the limit must have norm 1. This gives a contradiction and (3.10) holds. This completes the proof.

THEOREM 3.5. *For each real* ϱ

$$\| \gamma u \|_{E_{\varrho,p}} \leq C\,(\| Au \|_{\varrho-m,p} + \| u \|_{\varrho-m,p} + \| Bu \|_{F_{\varrho,p}})$$

holds for all $u \in C^\infty(\overline{\Omega})$.

This is just Theorem 2.3 of [6].

THEOREM 3.6. *If* $f \in H^{\varrho,p}(\Omega)$ *and*

$$|\,(f,\, A'\, \psi)\,| \leq C \,\|\, \psi \,\|_{m-\varrho,p'} \tag{3.11}$$

for all $\psi \in C_0^\infty(\Omega)$ *(the set of infinitely differentiable functions with compact supports in* Ω*), then* $f \in H_A^{\varrho,p}(\Omega)$.

PROOF. We follow the reasoning of [10, p. 14]. We consider A as an operator in $H^{\varrho,p}(\Omega)$ with domain $H_A^{\varrho,p}(\Omega)$. Let A_w be the extension of A to those $f \in H^{\varrho,p}(\Omega)$ satisfying (3.11). For such an f there is an $h \in H^{\varrho-m,p}$ such that

$$(f,\, A'\, \psi) = (h,\, \psi)$$

for all $\psi \in C_0^\infty(\Omega)$. We then define $A_w f = h$.
Clearly $A \subseteq A_w$. We now show that $A^* \subseteq A_w^*$.
This will mean that $A = A_w$ and the theorem will follow.

Suppose $v \in D(A^*)$. Then

$$(v,\, Au) = (w,\, u)$$

for some $w \in H^{-\varrho,p'}(\Omega)$ and all $u \in C^\infty(\overline{\Omega})$. In particular, this holds for all u satisfying zero Dirichlet or Neumann data on $\partial\Omega$. From this it follows that $v \in H_0^{m-\varrho,p'}(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $(H^{m-\varrho,p'}(\Omega)$ (cf., e. g., [2]). Hence there is a sequence $\{v_k\}$ $C_0^\infty(\Omega)$ converging to v in $H^{m-\varrho,p'}(\Omega)$. If $f \in D(A_w)$, then

$$(f,\, A'\, v_k) = (A_w f,\, v_k).$$

But

$$\| A' (v_k - v_l) \|_{-\varrho, p'} \leq C \| v_k - v_l \|_{m-\varrho, p'}$$

and hence $A' v_k \to g \in H^{-\varrho, p'}(\Omega)$. Thus

$$(f, g) = (A_w f, v)$$

for all $f \in D(A_w)$. Thus $v \in D(A_w^*)$. This completes the proof.

COROLLARY 3.1. *If $f \in H^{m-\varrho, p'}(\Omega)$, $G \in F^*_{\varrho, p}$ and*

$$| (f, Au) + \langle G, Bu \rangle | \leq c_2 (\| u \|_{s, p} + \| Cu \|_{J_{s, p}}) \tag{3.12}$$

for all $u \in C^\infty(\bar{\Omega})$, then $f \in H_{A'}^{m-s, p'}(\Omega)$, $G = C' f \in F^*_{s, p}$ and

$$||| f \|_{m-s, p'} \leq C (c_2 + \| f \|_{m-\varrho, p'})$$

PROOF. The only thing which does not follow immediately from Theorems 3.3 and 3.6 is the fact that $G = C' f$. By (3.12) there are $h \in H^{-s, p'}(\Omega)$ and $\Phi \in J^*_{s, p}$ such that

$$(f, Au) + \langle G, Bu \rangle = (h, u) + \langle \Phi, Cu \rangle$$

for all $u \in C^\infty(\bar{\Omega})$. By (2.10) this becomes

$$(A' f - h, u) + \langle G - C' f, Bu \rangle + \langle B' f - \Phi, Cu \rangle = 0.$$

Since this is true for all $u \in C^\infty(\bar{\Omega})$, it follows that $G = C' f$. This completes the proof.

THEOREM 3.7. *Let $\{Q_j\}_{j=1}^k$, $k \leq m$, be a normal set of boundary operators of orders $\mu_j < m$. Then for each set $\{\Phi_j\}_{j=1}^k$ of functions in $C^\infty(\partial\Omega)$ there is $u \in C^\infty(\bar{\Omega})$ such that*

$$Q_j u = \Phi_j \text{ on } \partial\Omega, \qquad 1 \leq j \leq k,$$

and for each ϱ

$$||| u |||_{\varrho, p} \leq C \sum_{j=1}^{k} \langle \Phi_j \rangle_{\varrho - \mu_j - 1/p, p}, \tag{3.13}$$

where the constant C does not depend on u or the Φ_j.

PROOF. By adding appropriate operators to the Q_j and taking the corresponding Φ_j to be zero, we may assume that $k = m$. Consider the boundary value problem

$$(\Delta - 1)^m u = 0 \text{ in } \Omega,$$

$$Q_j u = \Phi_j \text{ on } \partial\Omega, \qquad 1 \leq j \leq m,$$

where Δ is the Laplacian. This problem is equivalent to the Dirichlet problem, and hence we know that there always exists a unique solution. Applying (3.1) to this problem, we obtain

$$\| u \|_{\varrho, p} \leq C \sum_{j=1}^{m} \langle \Phi_j \rangle_{\varrho-\mu_j-1/p, p}.$$

For $\varrho \geq m$ we have

$$\| Au \|_{\varrho-m, p} \leq C \| u \|_{\varrho, p}.$$

For $\varrho \leq 0$ we have

$$\| Au \|_{\varrho-m, p} = \text{l. u. b.} \frac{|(Au, v)|}{\| v \|_{m-\varrho, p'}}$$

$$\leq \text{l. u. b.} \| v \|_{m-\varrho, p'}^{-1} |(u, A' v) + \Sigma \langle Q_j u, Q'_{m-j+1} v \rangle|$$

$$\leq C (\| u \|_{\varrho, p} + \Sigma \langle Q_j u \rangle_{\varrho-\mu_j-1/p, p}),$$

where Q'_{m-j+1} is an appropriate boundary operator of order $m - \mu_j - 1$. We now apply an abstract interpolation theorem due to Calderón [11, 10.1] to the spaces considered (cf. [6, Theorem 3.1]) to conclude that (3.13) holds for all real ϱ.

4. Proofs.

Proof of Lemma 2.1. By (2.10) there is a normal set $\{N_j\}_{j=1}^m$ of boundary operators such that

$$(4.1) \qquad (Aw, v) - (w, A' v) = \sum_{j=0}^{m-1} \langle \gamma_j w, N_{m-j} v \rangle$$

for $w, v \in C^\infty(\overline{\Omega})$, where the order of N_j is $j - 1$.

By Theorem 3.7, for each set $\Phi_1, \dots, \Phi_m$ of functions in $C^\infty(\partial\Omega)$ there is a function $v \in C^\infty(\overline{\Omega})$ such that

$$(4.2) \qquad N_j v = \Phi_j \text{ on } \partial\Omega, \qquad 1 \leq j \leq m,$$

while for each ϱ, $1 < q < \infty$,

$$\| v \|_{\varrho, q} + \| A' v \|_{\varrho-m, q} \leq C \sum_{j=1}^{m} \langle \Phi_j \rangle_{\varrho-j+1-1/q, q} , \tag{4.3}$$

where the constant C does not depend on v or the Φ_j. Now

$$|(Au, v)| \leq \| Au \|_{s-m, p} \| v \|_{m-s, p'} , \tag{4.4}$$

$$|(u\ A' v)| \leq \| u \|_{s, p} \| A' v \|_{-s, p'} . \tag{4.5}$$

Setting $\varrho = m - s$, $q = p'$ in (4.3) and applying (4.1) we have

$$| \Sigma \langle \gamma_j u, \Phi_{m-j} \rangle | \leq C' ||| u |||_{s, p} \Sigma \langle \Phi_{m-j} \rangle_{j-s+1/p, p'} . \tag{4.6}$$

Taking all of the Φ_j but one to be zero in (4.6), we obtain estimates for each $\gamma_j u$, namely

$$\Sigma \langle \gamma_j u \rangle_{s-j-1/p, p} \leq C'' ||| u |||_{s, p} . \tag{4.7}$$

This completes the proof.

PROOF OF THEOREM 2.2. By (2.2)

$$\| Bu \|_{F_{s, p}} \leq \| (B - L\gamma) u \|_{F_{s, p}} + \| L\gamma u \|_{F_{s, p}}$$

$$\leq \| (B - L\gamma) u \|_{F_{s, p}} + \varepsilon \| Bu \|_{F_{s, p}} + C (\| Au \|_{t, p} + \| u \|_{s-m, p}).$$

Thus

$$\| Bu \|_{F_{s, p}} \leq C (\| B - L\gamma u \|_{F_{s, p}} + \| Au \|_{t, p} + \| u \|_{s-m, p}). \tag{4.8}$$

Combining this with (3.1) we obtain (2.3).

PROOF OF THEOREM 2.1. Since smooth functions with compact support in Ω are in $D(A_{s,p}(L))$ and they are dense in $H^{t,p}(\Omega)$, 1) holds. By completion, (2.3) holds for functions in $H_A^{s,p}(\Omega)$. This gives immediately that $A_{s,p}$ is closed. Moreover

$$||| u |||_{s, p} \leq C \| u \|_{s-m, p}$$

holds for all $u \in N(A_{s,p}(L))$. A standard argument using Theorem 3.2 shows that this set must be finite dimensional. Another application of Theorem

3.2 shows that

$$|||\, u \,|||_{s,p} \leq C \,||\, Au \,||_{t,p}$$

holds for all $u \in D(A_{s,p}(L))/N(A_{s,p}(L))$.

This gives immediately that the range of $A_{s,p}$ is closed.

PROOF OF THEOREM 2.3. We show that (2.3) holds. If it did not, there would be a sequence $\{u_k\}$ of functions in $C^\infty(\bar{\Omega})$ such that

$$|||\, u_k \,|||_{s,p} = 1$$

while

$$||\, Au_k \,||_{t,p} + ||\, u_k \,||_{s-m,p} + ||\, (B - L\gamma)\, u_k \,||_{F_{s,p}} \to 0.$$

By Lemma 2.1

$$||\, \gamma u_k \,||_{E_{s,p}} \leq \text{const.}$$

and hence these is a subsequence (also denoted by $\{u_k\}$) for which $L\gamma\, u_k$ converges in $F_{s,p}$. Thus Bu_k converges in the same space. If we now make use of (3.1) we see that u_k converges in $H_A^{s,p}(\Omega)$. Since it converges in $H^{s-m,p}(\Omega)$ to zero, il must converge to the same limit in $H_A^{s,p}(\Omega)$. But this is impossible, since the $H_A^{s,p}(\Omega)$ norm of the limit must be unity. The last part of the theorem follows from Theorem 3.2.

PROOF OF THEOREM 2.4. Since $\gamma u \in E_{s,p}$, $L\gamma\, u \in F_{\varrho,q}$. By Theorem 3.7 there is a $v \in H_A^{\varrho,q}(\Omega)$ such that $Bv = L\gamma\, u$. Set $w = u - v$. Then $Bw = 0$ while $Aw \in H^{\varrho-m,q}(\Omega)$. Thus by (2.10)

$$|\, (w, A'\, g) + \langle\, Cw, B'\, g \,\rangle \,| = |\, (Aw, g) \,| \leq C \,||\, g \,||_{m-\varrho,q'}$$

for all $g \in C^\infty(\bar{\Omega})$. Thus $w \in H_A^{\varrho,q}(\Omega)$ by Corollary 3.1 (applied to A', B', where we use the fact [8,1] that B' covers A'). Thus $u = w + v \in H_A^{\varrho,q}(\Omega)$.

PROOF OF THEOREM 2.5. By Theorem 2.4 $u \in H_A^{\varrho,p}(\Omega)$, where $\varrho = \min(t + m, s + \varepsilon)$. If $\varrho = t + m$, the theorem is proved. Otherwise we replace s by ϱ and repeat the process as many times as needed to reach $t + m$.

PROOF OF THEOREM 2.6. If (2.4) holds

$$|\, (f, Au) + G, Bu \,| \leq c_0 \,|||\, u \,|||_{s,p} + C \,||\, G \,||_{F^*_{\varrho,p}} |||\, u \,|||_{\varrho-\varepsilon,p}$$

$$\leq (c_0 + C \,||\, G \,||_{F^*_{\varrho,p}})\, |||\, u \,|||_{\tau,p}$$

where $\tau = \max(s, \varrho - \varepsilon)$. Thus $f \in H^{m-\tau, p}(\Omega)$, $G \in F^*_{\tau, p}$ and

$$\|f\|_{m-\tau, p'} + \|G\|_{F^*_{\tau, p}} \leq C(c_0 + \|G\|_{F^*_{\varrho, p}} + \|f\|_{m-\varrho, p'}). \tag{4.9}$$

If $\tau = s$, we are finished. Otherwise we continue the process until we obtain the desired result.

PROOF OF COROLLARY 2.1. Taking $c_0 = 0$ in (2.4), we have by Theorem 2.6 that $f \in H^{m-\sigma, p'}(\Omega)$ and $G \in F^*_{\sigma, p}$ for each real s. By Sobolev's lemma $f \in C^\infty(\overline{\Omega})$ and each $g_j \in C^\infty(\partial\Omega)$. Moreover for any σ, ϱ, we have by (2.5)

$$\|f\|_{m-\sigma, p'} + \|G\|_{F^*_{\sigma, p}} \leq C(\|f\|_{m-\varrho, p'} + \|G\|_{F^*_{\varrho, p}})$$

where the constant C does not depend on f, G. An application of Theorem 3.2 shows that the set of such f, G is finite dimensional.

THEOREM 2.7 follows immediately from Theorem 2.5 and Sobolev's Lemma.

PROOF OF THEOREM 2.9. By Theorem 2.5 we may assume that $s = t + m$. Suppose $f \in H^{t, p}(\Omega)$ is orthogonal to $\widetilde{N}$. Then

$$(f, h) + \langle 0, G \rangle = 0$$

for all $h \in \widetilde{N}$, where G is any vector corresponding to h. This shows that

there is a sequence $\{u_k\}$ of functions in $C^\infty(\overline{\Omega})$ such that $Au_k \rightarrow f$ in $H^{t, p}(\Omega)$ and $(B - L\gamma)u_k \rightarrow 0$ in $F_{s, p}$. Moreover, we may take the u_k to be orthogonal to $N(A(L))$. Thus by Theorem 2.2, u_k converges in $H^{s, p}_A(\Omega)$ to some element u. Thus $Au = f$ and $Bu = L\gamma u$. Hence $f \in R(A_{s, p}(L))$. Conversely, if $f \in R(A_{s, p}(L))$, such a sequence exists. If $h \in \widetilde{N}$ and G is any corresponding vector, then

$$(h, Au_k) + \langle G, Bu_k - L\gamma u_k \rangle = 0$$

for each k. Taking the limit, we have

$$(h, f) = 0.$$

Since h was any element of $\widetilde{N}$, f is orthogonal to $\widetilde{N}$.

PROOF OF THEOREM 2.10. Write $v = v' + v''$, where $v'' \in \tilde{N}$ and v' is orthogonal to $\tilde{N}$ (cf. [9]). Let w be any function in $C^\infty(\bar{\Omega})$ and write $w = w' + w''$, where $w'' \in \tilde{N}$ while w' is orthogonal to it. By Theorem 2.9 there is a $u \in V(L)$ such that $Au = w'$. Now $(v', w'') = 0$, while $(v', w') = (v', Au) = (v, Au) - (v'', Au) = 0$.

Hence $(v', w) = 0$. Since this is true for all $w \in C^\infty(\bar{\Omega})$, $v' = 0$. Thus $v = v'' \in \tilde{N}$.

PROOF OF COROLLARY 2.2. If $u \in V(L)$ is orthogonal to $N(A(L))$, then by Theorem 2.2

$$|(f, u)| \leq \|f\|_{s-m, p'} \|u\|_{m-s, p} \leq C \|Au\|_{-s, p}.$$

Hence there is a $v_0 \in H^{s, p'}(\Omega)$ orthogonal to $N(A(L))$ and such that

$$(f, u) = (v_0, Au) \tag{4.10}$$

for all $u \in V(L)$ orthogonal to $N(A(L))$. By (2.9) we see that f itself is orthogonal to $N(A(L))$.

Thus (4.10) holds for all $u \in V(L)$. Subtracting (2.9) from (4.10), we have

$$(v - v_0, Au) = 0$$

for all $u \in V(L)$. Thus $v - v_0 \in \tilde{N} \subseteq C^\infty(\bar{\Omega})$. Hence $v \in H^{s, p'}(\Omega)$.

PROOF OF THEOREM 2.8. Clearly $\tilde{N} \subseteq N(A(L)')$.
Conversely, if $v \in N(A(L)')$, then $(v, Au) = (A'v, u) = 0$ for all $u \in V(L)$. But then by Theorem 2.10, we have $v \in \tilde{N}$.

COROLLARY 2.3 follows from Theorems 2.3 and 2.9 and Corollary 2.1.

PROOF OF PROPOSITION 2.1. If u is in the domain of the closure of A as described, then there is a sequence $\{u_k\}$ of functions in $C^\infty(\bar{\Omega})$ such that $u_k \to u$, $Au_k \to Au$ in $H^{t, p}(\Omega)$ while $(B - MC) u_k = 0$. By (2.3) we see that $u \in H_A^{s, p}(\Omega)$ and $(B - MC) u = 0$.

Conversely if $u \in D(A_{s, p}(L_1))$, there is a sequence $\{u_k\}$ of functions in $C^\infty(\bar{\Omega})$ such that $u_k \to u$ in $H_A^{s, p}(\Omega)$ while $(B - MC) u_k \to 0$ in $F_{s, p}$. By

Theorem 3.7 there is a linear mapping W from $F_{s,p}$ to $H_A^{s,p}(\Omega)$ such that

$$BW\Phi = \Phi, \quad CW\Phi = 0, \tag{4.11}$$

$$|||\, W\Phi \,|||_{s,p} \leq C \,\|\, \Phi \,\|_{F_{s,p}}, \tag{4.12}$$

for all $\Phi \in F_{s,p}$. Set $v_k = W(B - MC)\, u_k$, $w_k = u_k - v_k$.

Then $(B - MC)\, w_k = 0$ while by (4.12)

$$|||\, w_k - u \,|||_{s,p} \leq |||\, u_k - u \,|||_{s,p} + \|\,(B - MC)\, u_k \,\|_{F_{s,p}}$$

$$\to 0.$$

Hence u is in the domain of the closure of A as described.

PROOF OF THEOREM 2.11. If $u \in D(A_{s,p}(L_1))$ and $v \in D(A'_{m-\tau,p'}(L'_1))$, then by (2.10)

$$(u, A'v) - (Au, v) = \langle Bu, C'v \rangle - \langle Cu, B'v \rangle$$

$$- \langle MCu, C'v \rangle - \langle Cu, M^* C'v \rangle = 0.$$

Thus $A'_{m-\tau,p'}(L'_1) \subseteq (A_{s,p}(L_1))^*$. Next suppose $v, f \in H^{-t,p'}(\Omega)$ satisfy

$$(u, f) = (Au, v) \tag{4.13}$$

for all $u \in D(A_{s,p}(L_1))$. By Theorem 3.4 there is a mapping U from $F_{s,p}$ to $H_A^{s,p}(\Omega)$ such that

$$BU\Phi = \Phi, \qquad AU\Phi \in H^{s,p}(\Omega)$$

$$\|\, U\Phi \,\|_{s,p} + \|\, AU\Phi \,\|_{s,p} \leq K \,\|\, \Phi \,\|_{F_{s,p}}, \tag{4.14}$$

for all $\Phi \in F_{s,p}$. Consider the operator $(B - MC)\, U = I - MCU$ or $F_{s,p}$. Since MCU is compact, this operator is Fredholm. It thus has a bounded inverse from its range $K_{s,p}$ onto a complement of its null space. Let $L_{s,p}$ be some finite dimensional complement of $K_{s,p}$. The set S of those $\Phi \in L_{s,p}$ for which there is a $u \in H_A^{s,p}(\Omega)$ such that

$$Au \in H^{s,p}(\Omega), \qquad (B - MC)\, u = \Phi$$

is thus finite dimensional. Thus there is a mapping U_1 from S to $H_A^{s,p}(\Omega)$ such that

$$A U_1 \Phi \in H^{s,p}(\Omega), \qquad (B - MC) U_1 \Phi = \Phi$$

$$\| U_1 \Phi \|_{s,p} + \| A U_1 \Phi \|_{s,p} \leq K \| \Phi \|_{F_{s,p}} \tag{4.15}$$

for all $\Phi \in S$. Now let u be any function in $C^\infty(\bar{\Omega})$ and set $\Phi = (B - MC) u$. We decompose Φ in the form $\Phi = \Phi' + \Phi''$, where $\Phi' \in K_{s,p}$ and $\Phi'' \in S$. Set $u_0 = u' + u''$, where

$$u' = U(I - MCU)^{-1} \Phi', \qquad u'' = U_1 \Phi''.$$

Then $(B - MC) u_0 = \Phi$ and hence $u - u_0$ is in $D(A_{s,p}(L_1))$. Thus

$$(u, f) - (Au, v) = (u_0, f) - (Au_0, v), \tag{4.16}$$

showing that the expression on the right depends only on Φ. Denoting it by $F\Phi$ we see by (4.14) and (4.15) that it is a bounded linear functional defined on a subspace of $F_{s,p}$ (actually, this subspace is the whole of $F_{s,p}$, but we need not know this fact here). Thus, by the Hahn-Banach theorem, there is a $G \in F_{s,p}^*$ such that

$$F\Phi = \langle \Phi, G \rangle$$

for all Φ in the domain of definition of F. Thus

$$(u, f) - (Au, v) = \langle (B - MC) u, G \rangle \tag{4.17}$$

for all $u \in C^\infty(\bar{\bar{\Omega}})$. In particular,

$$| (Au, v) + \langle Bu, G \rangle | \leq c (\| u \|_{t,p} + \| Cu \|_{J_{\tau,p}})$$

for all such u. This allows us to apply Corollary 3.1 to obtain that $v \in H_{A'}^{m-\tau,p'}(\Omega)$ and $G = C' v \in F_{\tau,p}^*$. From (4.13) it is clear that $f = A' v$. Thus $(A_{s,p}(L_1))^* \subseteq A'_{m-\tau,p'}(L'_1)$, and (2.12) is proved. Since L_1 is compact from $E_{s,p}$ to $F_{s,p}$, it follows from Theorem 2.3 that $A_{s,p}(L_1)$ is closed. Since $H^{t,p}(\Omega)$ is reflexive, (2.13) follows from the fact that

$$(A_{s,p}(L_1))^{**} = A_{s,p}(L_1).$$

After this paper was completed, R. S. Freeman sent us a copy of his work [29] which treats similar problems. He considers the L^2 theory for bounded or unbounded domains. Although not explicitly staded in his paper, his methods also apply to boundary conditions of the form (1.2) as considered here.

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