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Good choice sets

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GOOD CHOICE SETS (*).

by J. C. E. DEKKER.

1. Introduction.

We are concerned with non-negative integers (*numbers*), collections of numbers (*sets*) and collections of sets (*classes*). The letters ε and o stand for the set of all numbers and the empty set of numbers respectively. We shall write $(\Gamma_0, \dots, \Gamma_n)$ or $[\Gamma_0, \dots, \Gamma_n]$ for the collection consisting of the entities (i. e., numbers, sets, classes, or ordered pairs of numbers) $\Gamma_0, \dots, \Gamma_n$. Brackets will often be used instead of parentheses if this makes it easier to read a formula. Let $\{a_0, a_1, \dots\}$ be a sequence of numbers. Then we shall use « a_n » and « $a(n)$ » in the same sense. We write $\subset$ for inclusion, proper or improper; proper inclusion is indicated by writing $\subset_+$. A mapping from a subcollection of ε^n into ε is called a *function*; if f is a function, we denote its domain and its range by δf and ϱf respectively. The sets α and β are *equivalent* [written: $\alpha \backsim \beta$], if there exists a one-to-one function f such that $\alpha \subset \delta f$ and $f(\alpha) = \beta$. Note that we may replace « $\alpha \subset \delta f$ » by « $\alpha = \delta f$ » without changing the meaning of « $\alpha \backsim \beta$ ». The sets α and β are *recursively equivalent* [written: $\alpha \simeq \beta$], if there exists a partial recursive one-to-one function p such that $\alpha \subset \delta p$ and $p(\alpha) = \beta$. Note that replacing « $\alpha \subset \delta p$ » by « $\alpha = \delta p$ » would change the meaning of « $\alpha \simeq \beta$ »; for $\sigma \simeq \sigma$ would become false for every set σ which is not r. e. (i. e., recursively enumerable), because δp is a r. e. set for every partial recursive function p of one variable. A possible definition of the cardinal number of a set α is: the class of all sets σ such that $\sigma \backsim \alpha$. Similarly we have defined the RET (i. e., *recursive equivalence type*) of a set α [written: Req (α)] as the class of all sets σ such that $\sigma \simeq \alpha$. For a study of RETs the reader is referred to [2] and [5].

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Let a class of mutually disjoint, non-empty sets be called an *md-class*; such a class is therefore countable, i. e., finite or denumerable. We wish to show how the notion of the RET of a class of sets can be introduced for certain (though not all) *md*-classes. Throughout this paper S stands for an *md*-class and σ for the union of all sets in S. For every $x \in \sigma$ we denote the unique set α such that $x \in \alpha$ and $\alpha \in S$ by α_x.

DEFINITION. A set γ is a *choice set* of S, if

(1) $\gamma \subset \sigma$,

(2) γ contains exactly one element of each set in S.

A possible definition of the cardinal number of S is: the class of all sets σ such that $\sigma \backsim \gamma$, for some choice set γ of S. Though any two choice sets of an *md*-class are equivalent, they need not be recursively equivalent. Let, for instance,

$$S = [(0, 1),\ (2, 3),\ (4, 5), \ldots].$$

Then S has c choice sets (c denoting the cardinality of the continuum), while every non-zero RET contains exactly $\aleph_0$ sets; the c choice sets of S can therefore not all be recursively equivalent.

DEFINITION. A set γ is a *good* choice set (abbreviated: *gc*-set) of S, if it is a choice set for which there exists a partial recursive function $p(x)$ such that

(3) $$\sigma \subset \delta p \qquad \text{and} \qquad (\forall x)\,[x \in \sigma \Longrightarrow p(x) \in \gamma \cdot \alpha_x].$$

We shall prove in sections 3 and 5:

(i) *any two good choice sets of an md-class are recursively equivalent,*

(ii) *among the c md-classes there are c which have a gc-set and c which have no gc-set.*

NOTATION. $\zeta(S)$ is the class of all gc-sets of S.

DEFINITION. An *md*-class S is a *gc-class* if $\zeta(S)$ is non-empty. If S is a *gc*-class,

$$\text{RET}(S) = \text{Req}(\gamma), \qquad \text{for any } \gamma \in \zeta(S).$$

In the trivial case that S is empty, $\zeta(S)$ contains exactly one set, namely o. The RET of the empty class is therefore 0. It is the purpose of this paper to prove a few propositions concerning *gc*-classes and their RET s.

While an *md*-class need not have an RET (since it need not be a *gc*-class), it is readily seen that every RET A is the RET of some *gc*-class. For let

$$\alpha \in A, \qquad S = \{(x) \mid x \in \alpha\},$$

then S has exactly one choice set, namely α. Using the identity function we conclude that α is also a *gc*-set of S, hence $\text{RET}(S) = A$. In this case all sets in S have the same RET, namely 1. It is not hard to see that for any two non-zero RETs A and B there is a *gc*-class S such that

$$\text{RET}(S) = A, \quad (\forall \sigma)[\sigma \in S \Longrightarrow \text{Req}(\sigma) = B].$$

For let $\alpha \in A$, $\beta \in B$. Suppose a_n is a one-to-one function ranging over α; if α is a finite set of cardinality $k \geqq 1$, we take $(0, \dots, k-1)$ as the domain of a_n, but if α is infinite, we take ε as the domain of a_n. Put

$$j(x, y) = x + \frac{1}{2}(x+y)(x+y+1), \qquad j(p, \beta) = \{j(p, y) \mid y \in \beta\},$$

$$S = [j(a_0, \beta), j(a_1, \beta), \dots].$$

Obviously, every set in S is recursively equivalent to β, i. e., has RET B. Let $b \in \beta$ and let γ be the range of the function $j(a_n, b)$. Then $p(x) = j[k(x), b]$ is a recursive function such that

$$\sigma \subset \delta p \qquad \text{and} \qquad (\forall x)[x \in \sigma \Longrightarrow p(x) \in \gamma \cdot \alpha_x].$$

This implies that γ is a *gc*-set of S and

$$\text{RET}(S) = \text{Req}(\gamma) = \text{Req}(\alpha) = A.$$

Note that $\text{Req}(\sigma) = A \cdot B$, because $\sigma = j(\alpha \times \beta)$. Hence

$$\text{Req}(\sigma) = \text{RET}(S) \cdot B. \tag{4}$$

The set σ is the union of all sets in the *gc*-class S, and all sets in S have the same RET, namely B. Relation (4) shows therefore that though our definition of $\text{RET}(S)$ may not be the only one possible, it is certainly natural, since in some sense

$$\underbrace{B + B + B + \dots}_{\text{« } A \text{ times »}} = A \cdot B.$$

2. Preliminaries.

The sets $\alpha_0, \dots, \alpha_n$ are *separable*, if there exist mutually disjoint r. e. sets $\beta_0, \dots, \beta_n$ such that $\alpha_i \subset \beta_i$, for $0 \leqq i \leqq n$. We write $\alpha_0 \mid \alpha_1$ if α_0 and α_1 are separable. It is readily seen that $\alpha_0, \dots, \alpha_n$ are separable if and only if there exists a partial recursive function $p(x)$ such that

$$\alpha_0 + \dots + \alpha_n \subset \delta p \qquad \text{and} \qquad \varrho p = (0, \dots, n), \tag{5}$$

$$\begin{cases} \text{for } x \in \alpha_0 + \dots + \alpha_n \quad \text{and} \quad 0 \leqq i \leqq n, \\ x \in \alpha_i \Longleftrightarrow p(x) = i. \end{cases} \tag{6}$$

NOTATIONS.

$$\varrho_0 = o,$$

$$\varrho_{x+1} = \begin{cases} [a(1), \dots, a(k)], \text{ where } a(1), \dots, a(k) \text{ are the} \\ \text{distinct numbers such that} \\ x + 1 = 2^{a(1)} + \dots + 2^{a(k)}, \end{cases}$$

$$r(x) = r_x = \text{card}(\varrho_x).$$

The class Q of all finite sets is enumerated without repetitions in the sequence $\varrho_0, \varrho_1, \dots$; the function r_x is clearly recursive.

A one-to-one function t_n from ε into ε is *regressive*, if there exists a partial recursive function $p(x)$ such that

$$\varrho t \subset \delta p, \tag{7}$$

$$p(t_0) = t_0 \qquad \text{and} \qquad (\forall n)[p(t_{n+1}) = t_n]. \tag{8}$$

A function from ε into ε is *retraceable*, if it is strictly increasing and regressive. A set is *regressive* (or *retraceable*) if it is finite or the range of a regressive (respectively, retraceable) function.

For every regressive function t_n there also exists a partial recursive function which satisfies besides (7) and (8) the conditions

$$\varrho p \subset \delta p, \tag{9}$$

$$(\forall x)\left[x \in \delta p \Longrightarrow (\exists y)[p^{y+1}(x) = p^y(x)]\right]. \tag{10}$$

Every partial recursive function $p(x)$ related to the regressive function t_n by (7), (8), (9) and (10) is called a function which *regresses* t_n or a *regressing* function of t_n; in the special case that t_n is strictly increasing, we call $p(x)$ a function which *retraces* t_n or a *retracing* function of t_n. If $p(x)$ is a regressing function of the regressive function t_n, then the function $p^*(x)$ defined by

(11) $$p^*(x) = (\mu y)\,[p^{y+1}(x) = p^y(x)], \qquad \text{for } x \in \delta p,$$

is a partial recursive extension of t_x^{-1}.

Consider the following proposition. Let the sets α and β and the partial recursive functions $f(x)$ and $g(x)$ be related by the conditions

(a) $$\alpha \subset \delta f \text{ and } f(\alpha) = \beta \text{ and } f \text{ is } 1-1 \text{ on } \alpha,$$

(b) $$\beta \subset \delta g \text{ and } g(\beta) = \alpha \text{ and } g \text{ is } 1-1 \text{ on } \beta,$$

(c) $$gf(x) = x, \qquad \text{for } x \in \alpha.$$

Then there exists a partial recursive one-to-one function $h(x)$ such that

(d) $$\alpha \subset \delta h \qquad \text{and} \qquad h(\alpha) = \beta,$$

(e) $$h(x) = f(x), \qquad \text{for } x \in \alpha.$$

The proof is almost immediate. Under the hypothesis,

$$\sigma = \{x \in \delta f \mid gf(x) = x\}$$

is a r. e. set; hence, if $h(x)$ is the restriction of $f(x)$ to σ, then $h(x)$ satisfies the requirements. This proposition will be used in the following form:

(12) $$\left\{\begin{array}{l} \textit{If } \alpha \textit{ and } \beta \textit{ are sets for which there exist partial} \\ \textit{recursive functions } f(x) \textit{ and } g(y) \textit{ which satisfy} \\ (a), (b), (c) \textit{ above, then } \alpha \simeq \beta. \end{array}\right.$$

3. Elementary properties.

PROPOSITION P1. *Every two gc-sets of an md-class are recursively equivalent.*

PROOF. Let γ and δ be *gc*-sets of the *md*-class S. If the class S is finite, $\gamma \simeq \delta$ because γ and δ are finite sets of the same cardinality. Now assume that S, γ and δ are infinite. There exist partial recursive functions p and q such that

(13) $$\sigma \subset \delta p \qquad \text{and} \qquad (\forall x)\,[x \in \sigma \Longrightarrow p(x) \in \gamma \cdot \alpha_x],$$

(14) $$\sigma \subset \delta q \qquad \text{and} \qquad (\forall x)\,[x \in \sigma \Longrightarrow q(x) \in \delta \cdot \alpha_x].$$

It follows that

$$(15)\qquad \delta \subset \delta p \text{ and } p(\delta) = \gamma \text{ and } p \text{ is } 1-1 \text{ on } \delta,$$

$$(16)\qquad \gamma \subset \delta q \text{ and } q(\gamma) = \delta \text{ and } q \text{ is } 1-1 \text{ on } \gamma,$$

$$(17)\qquad qp(x) = x, \qquad \text{for } x \in \delta.$$

The last three relations imply $\gamma \simeq \delta$ by (12).

P1 guarantees that the notion RET (S) is well-defined for any *gc*-class S. Let us consider the special case where S is a non-empty *md*-class which contains exactly k sets *all of which are finite.* It is readily seen that

(a) every choice set of S is a *gc*-set,

(b) S is a *gc*-class and RET $(S) = k$.

A finite *md*-class need not be a *gc*-class. For let $T = (\tau, \tau')$, where τ is any non-recursive set and τ' the complement of τ with respect to ε. If T had a *gc*-set, we would have $\tau \mid \tau'$, and τ would be recursive. Let us now take for τ an immune set with an immune complement. For every $k \geqq 3$ we can decompose τ' into $k-1$ immune sets $\tau_1, \dots, \tau_{k-1}$. Then $B = (\tau, \tau_1, \dots, \tau_{k-1})$ is an *md*-class which contains exactly k sets, but B is not a *gc*-class. We conclude that for every $k \geqq 2$, there exists an *md*-class of cardinality k which is not a *gc*-class.

PROPOSITION P2. *The non-empty finite md-class* $S = (\alpha_0, \dots, \alpha_n)$ *is a gc-class if and only if* $\alpha_0, \dots, \alpha_n$ *are separable; if S is a gc-class, each choice set of S is a gc-set and* RET (S) *equals the cardinality of S.*

PROOF. Let $S = (\alpha_0, \dots, \alpha_n)$.

(a) Let $\alpha_0, \dots, \alpha_n$ be separable, say $\alpha_i \subset \beta_i$, $0 \leqq i \leqq n$, for mutually disjoint r. e. sets $\beta_0, \dots, \beta_n$. Put $\beta = \beta_0 + \dots + \beta_n$. Let $\gamma = (c_0, \dots, c_n)$ with $c_i \in \alpha_i$, $0 \leqq i \leqq n$, be *any* choice set of S. Then the function p defined by

$$\delta p = \beta, \qquad (\forall x)(\forall i \leqq n)[x \in \beta_i \Longrightarrow p(x) = c_i],$$

is a partial recursive function which maps any element $x \in \sigma$ onto the number c_i such that $x \in \alpha_i$. Hence $\gamma \in \zeta(S)$ and S is a *gc*-class.

(b) Let S be a *gc*-class, $\delta = (d_0, \dots, d_n)$ with $d_i \in \alpha_i$, $0 \leqq i \leqq n$, a *gc*-set of S and q a partial recursive function such that

$$\sigma \subset \delta q, \qquad (\forall x)(\forall i \leqq n)[x \in \alpha_i \Longrightarrow q(x) = d_i].$$

Then the function h defined by

$$\delta h = \delta, \qquad (\forall i \leqq n)\,[h\,(d_i) = i],$$

is partial recursive, hence so is the function hq. Moreover,

$$\sigma \subset \delta\,(hq), \;\; \varrho\,(hq) = (0, \ldots, n) \;\text{ and}$$

$$(\forall x)\,(\forall i \leqq n)\,[x \in \alpha_i \Longleftrightarrow hq\,(x) = i].$$

We conclude that $\alpha_0, \ldots, \alpha_n$ are separable.

c) Let S be a *gc*-class. Then $\alpha_0, \ldots, \alpha_n$ are separable by (b) and every choice set of S is a *gc*-set by our proof of (a).

Each choice set γ of S has cardinality $n + 1$, hence $\text{RET}\,(S) = \text{Req}\,(\gamma) = n + 1$.

REMARK. It is readily seen that every subclass of a *gc*-class is again a *gc* class. For let S be a *gc*-class with union σ and *gc*-set γ and let $p\,(x)$ be a partial recursive function such that

$$\sigma \subset \delta p \qquad \text{and} \qquad (\forall x)\,[x \in \sigma \Longrightarrow p\,(x) \in \gamma \cdot \alpha_x].$$

Assume that $T \subset S$, where T has union τ. Then $\gamma \cdot \tau$ is a choice set of T, in fact a *gc*-set. For $\tau \subset \delta p$, since $\tau \subset \sigma \subset \delta p$; moreover, for $x \in \tau$, $p\,(x)$ is not only the unique element of $\gamma \cdot \alpha_x$, but also of $(\gamma \cdot \tau) \cdot \alpha_x$. Hence T is a *gc*-class.

Let S be a non-empty *md*-class. One of the basic propositions concerning such an *md*-class is: *σ is finite if and only if S is a finite class of finite sets.* This proposition will now be generalized.

DEFINITION. An *md*-class is *isolated* if it is a *gc*-class of which every (or, equivalently, at least one) *gc*-set is isolated. In other words: an *md*-class is *isolated* if it is a *gc*-class whose RET is an isol.

PROPOSITION P3. *Let S be a non-empty gc-class. Then σ is an isolated set if and only if S is an isolated class of isolated sets.*

PROOF. Let S be a non-empty *gc*-class.

(a) Assume that σ is isolated. Every set which belongs to S or $\zeta\,(S)$ is a subset of σ, hence again isolated. Thus S is an isolated class of isolated sets.

(b) Assume that S is an isolated class of isolated sets. Let $\gamma \in \zeta\,(S)$ and let p be a partial recursive function such that

$$\sigma \subset \delta p \qquad \text{and} \qquad (\forall x)\,[x \in \sigma \Longrightarrow p\,(x) \in \gamma \cdot \alpha_x].$$

We wish to prove that σ has no infinite r. e. subset. Let β be any r. e. subset of σ. If β is empty we are through; if β is non-empty, so is $p(\beta)$. Note that $p(\beta)$ is a r. e. subset of $p(\sigma)$, i. e., of γ. Thus, γ being isolated, $p(\beta)$ must be finite. The set $p(\beta)$ consists of all elements in γ which represent sets in S with which β has a non-empty intersection. Let T consist of all $\alpha \in S$ for which $\alpha \cdot \beta \neq o$. Since the set $p(\beta)$ is finite, but non-empty, T is a non-empty, finite subclass of S. However, S is a *gc*-class, hence so is T. The sets in T are separable, because T is a finite *gc*-class. Let

$$T = (\delta_0, \dots, \delta_k), \qquad \tau = \delta_0 + \dots + \delta_k.$$

The sets $\delta_0, \dots, \delta_k$ are isolated, since they belong to S; thus τ is isolated, because it is the union of $k+1$ separable, isolated sets. Recall that $\delta_0, \dots, \delta_k$ are the only sets in S with which β has a non-empty intersection. This implies $\beta \subset \tau$; hence β is finite, because β is r. e. and τ isolated.

DEFINITION. The classes S_1 and S_2 with unions σ_1 and σ_2 respectively are *separable* [written: $S_1 \mid S_2$], if $\sigma_1 \mid \sigma_2$.

PROPOSITION P4. *Let S_1 and S_2 be separable md-classes. Then $S_1 + S_2$ is an md-class and*

(a) *$S_1 + S_2$ is a gc-class if and only if both S_1 and S_2 are gc-classes,*
(b) *if $S_1 + S_2$ is a gc-class,*

$$\text{RET}(S_1 + S_2) = \text{RET}(S_1) + \text{RET}(S_2).$$

PROOF. Let S_1 and S_2 be separable *md*-classes with unions σ_1 and σ_2 respectively. Then $S_1 + S_2$ is an *md* class with union $\sigma_1 + \sigma_2$. Let $\sigma_1 \subset \tau_1$, $\sigma_2 \subset \tau_2$, where τ_1 and τ_2 are disjoint r. e. sets.

(1) Assume that S_1 and S_2 are *gc*-classes. Let $\gamma_1 \in \zeta(S_1)$, $\gamma_2 \in \zeta(S_2)$, then $\gamma_1 + \gamma_2$ is obviously a choice set of $S_1 + S_2$. For $x \in \sigma_1$ we denote the unique set α such that $x \in \alpha$ and $\alpha \in S_1$ by α_x; for $x \in \sigma_2$ we denote the unique set β such that $x \in \beta$ and $\beta \in S_2$ by β_x. Suppose p_1 and p_2 are partial recursive functions such that

$$\sigma_1 \subset \delta p_1 \quad \text{and} \quad (\forall x)[x \in \sigma_1 \Longrightarrow p_1(x) \in \gamma_1 \cdot \alpha_x],$$

$$\sigma_2 \subset \delta p_2 \quad \text{and} \quad (\forall x)[x \in \sigma_2 \Longrightarrow p_2(x) \in \gamma_2 \cdot \beta_x].$$

Let the function p_3 be defined by

$$\delta p_3 = \tau_1 \cdot \delta p_1 + \tau_2 \cdot \delta p_2,$$

$$\text{for } x \in \delta p_3, \quad p_3(x) = \begin{cases} p_1(x), \text{ if } x \in \tau_1 \cdot \delta p_1, \\ p_2(x), \text{ if } x \in \tau_2 \cdot \delta p_2. \end{cases}$$

The sets τ_1 and τ_2 are r. e. and disjoint, while the sets δp_1 and δp_2 are r. e. It follows that $\tau_1 \cdot \delta p_1$ and $\tau_2 \cdot \delta p_2$ are disjoint and r. e.: thus δp_3 is a r. e. set and $p_3(x)$ a partial recursive function. Clearly, $\sigma_1 + \sigma_2 \subset \delta p_3$ and for $x \in \sigma_1 + \sigma_2$,

$$x \in \sigma_1 \implies p_3(x) = p_1(x) \in \gamma_1 \cdot \alpha_x,$$

$$x \in \sigma_2 \implies p_3(x) = p_2(x) \in \gamma_2 \cdot \beta_x,$$

$$\gamma_1 \cdot \alpha_x \subset (\gamma_1 + \gamma_2) \cdot \alpha_x; \quad \gamma_2 \cdot \beta_x \subset (\gamma_1 + \gamma_2) \cdot \beta_x.$$

We conclude that $\gamma_1 + \gamma_2$ is a *gc*-set of $S_1 + S_2$.

(2) Assume that $S_1 + S_2$ is a *gc*-class. Then S_1 and S_2 are *gc*-classes, since they are subclasses of $S_1 + S_2$. Let $\gamma \in \zeta(S_1 + S_2)$ and let p be a partial recursive function such that

$$\sigma_1 + \sigma_2 \subset \delta p \quad \text{and} \quad (\forall x)[x \in \sigma_1 + \sigma_2 \implies p(x) \in \gamma \cdot (\sigma_1 + \sigma_2)].$$

Putting $\gamma_1 = \gamma \cdot \sigma_1$ and $\gamma_2 = \gamma \cdot \sigma_2$ we see that γ_1 and γ_2 are *gc*-sets of S_1 and S_2 respectively; moreover, p is a partial recursive function related to γ_1, S_1 and γ_2, S_2 in the desired manner.

(3) Let $S_1 + S_2$ be a *gc*-class. Then S_1 and S_2 are *gc*-classes by (a). Also, in view of our proof of (a),

$$\gamma_1 \in \zeta(S_1) \,\&\, \gamma_2 \in \zeta(S_2) \implies \gamma_1 + \gamma_2 \in \zeta(S_1 + S_2).$$

The relations $\gamma_1 \subset \sigma_1$, $\gamma_2 \subset \sigma_2$, $\sigma_1 \mid \sigma_2$ imply $\gamma_1 \mid \gamma_2$. Hence

$$\begin{aligned} \text{Ret}(S_1 + S_2) &= \text{Req}(\gamma_1 + \gamma_2) = \text{Req}(\gamma_1) + \text{Req}(\gamma_2) \\ &= \text{Ret}(S_1) + \text{Ret}(S_2). \end{aligned}$$

Notation. For any two classes A and B,

$$A \times B = \{j(\alpha \times \beta) \mid \alpha \in A \,\&\, \beta \in B\},$$

where $j(\alpha \times \beta) = \{j(x, y) \mid x \in \alpha \,\&\, y \in \beta\}$.

Remark. Let A and B have unions σ_A and σ_B respectively and let $\sigma_{A \times B}$ be the union of $A \times B$. Then it is readily seen that

$$\sigma_{A \times B} = j(\sigma_A \times \sigma_B).$$

We also note that for arbitrary sets $\sigma_1, \sigma_2, \tau_1, \tau_2$,

$$j(\sigma_1 \times \tau_1) \cdot j(\sigma_2 \times \tau_2) = j(\sigma_1 \cdot \sigma_2 \times \tau_1 \cdot \tau_2).$$

We finally observe that for any two non-empty countable classes A and B of non-empty sets, $A \times B$ is an *md*-class, if both A and B are *md*-classes.

PROPOSITION P5. *Let S_1 and S_2 be two non empty md-classes. Then $S_1 \times S_2$ is a non-empty md-class and*

(a) *$S_1 \times S_2$ is a gc-class if and only if both S_1 and S_2 are gc-classes,*

(b) *if $S_1 \times S_2$ is a gc-class,*

$$\text{RET}(S_1 \times S_2) = \text{RET}(S_1) \cdot \text{RET}(S_2).$$

PROOF. Let S_1 and S_2 be two non-empty *md*-classes with unions σ_1 and σ_2 respectively. We already know that $S_1 \times S_2$ is a non-empty *md*-class. Let for $x \in \sigma_1$, $y \in \sigma_2$,

$$\alpha_x = \text{the set } \alpha \text{ such that } x \in \alpha \text{ and } \alpha \in S_1,$$

$$\beta_y = \text{the set } \beta \text{ such that } y \in \beta \text{ and } \beta \in S_2.$$

Note that the union of $S_1 \times S_2$ is $j(\sigma_1 \times \sigma_2)$, while the relation $j(x, y) \in j(\sigma_1 \times \sigma_2)$ implies

$$j(\alpha_x \times \beta_y) = \text{the set } \delta \text{ such that } j(x, y) \in \delta \text{ and } \delta \in S_1 \times S_2.$$

(1) Assume that S_1 and S_2 are *gc*-classes with *gc*-sets γ_1 and γ_2 respectively. Let p_1 and p_2 be partial recursive functions such that

$$\sigma_1 \subset \delta p_1 \quad \text{and} \quad (\forall x)[x \in \sigma_1 \Longrightarrow p_1(x) \in \gamma_1 \cdot \alpha_x],$$

$$\sigma_2 \subset \delta p_2 \quad \text{and} \quad (\forall y)[y \in \sigma_2 \Longrightarrow p_2(y) \in \gamma_2 \cdot \beta_y].$$

Then the mapping p_3 defined by

$$p_3(z) = j[p_1 k(z), p_2 l(z)], \quad \text{for } z \in j(\delta p_1 \times \delta p_2),$$

is a partial recursive function such that $j(\sigma_1 \times \sigma_2) \subset \delta p_3$. Also,

$$\text{for } z = j(x, y) \in j(\sigma_1 \times \sigma_2),$$

$$p_3(z) = j[p_1(x), p_2(y)] \quad \in \quad j(\alpha_x \times \beta_y).$$

Since $p_1(x) \in \gamma_1 \cdot \alpha_x$ and $p_2(y) \in \gamma_2 \cdot \beta_y$, we have

$$p_3(z) \in j(\gamma_1 \cdot \alpha_x \times \gamma_2 \cdot \beta_y),$$

$$p_3(z) \in j(\gamma_1 \times \gamma_2) \cdot j(\alpha_x \times \beta_y).$$

Hence $j(\gamma_1 \times \gamma_2)$ is a *gc*-set of $S_1 \times S_2$.

(2) Assume that $S_1 \times S_2$ is a *gc*-class. Let for $z \in j(\sigma_1 \times \sigma_2)$ the set η such that $z \in \eta$ and $\eta \in S_1 \times S_2$ be denoted by η_z.

Suppose that δ is a *gc*-set of $S_1 \times S_2$ and that p is a partial recursive function such that

$$j(\sigma_1 \times \sigma_2) \subset \delta p,$$

$$j(x, y) \in j(\sigma_1 \times \sigma_2) \implies pj(x, y) \in \delta \cdot \eta_{j(x, y)}.$$

Let $\beta \in S_2$ and $b \in \beta$. Then $S_1 \times [\beta]$ is a *gc*-class, since it is a subclass of the *gc*-class $S_1 \times S_2$. Put

$$\delta_\beta = \{pj(x, b) \mid x \in \sigma_1\}, \text{ i. e,,}$$

$$\delta_\beta = \{pj(x, y) \mid x \in \sigma_1 \,\&\, y \in \beta\}.$$

Then δ_β is a *gc*-set of $S_1 \times [\beta]$ and $k(\delta_\beta)$ a choice set of S_1.
Now assume $x \in \sigma_1$. Since $b \in \beta \subset \sigma_2$ we have $j(x, b) \in j(\sigma_1 \times \sigma_2) \subset \delta p$, hence

$$x \in \sigma_1 \implies pj(x, b) \in \delta \cdot \eta_{j(x, b)}.$$

The last relation implies

$$\begin{aligned} x \in \sigma_1 &\implies pj(x, b) \in \delta_\beta \cdot j(\alpha_x \times \beta_b) \\ &\implies kpj(x, b) \in k(\delta_\beta) \,\&\, kpj(x, b) \in \alpha_x \\ &\implies kpj(x, b) \in k(\delta_\beta) \cdot \alpha_x . \end{aligned}$$

The set σ_1 is included in the domain of the partial recursive function $kpj(x, b)$ of x. Hence $k(\delta_\beta)$ is a *gc*-set of S_1. Similarly one can prove that S_2 has a *gc*-set. Hence S_1 and S_2 are *gc*-classes.

(3) Let $S_1 \times S_2$ be a *gc*-class. Then S_1 and S_2 are *gc*-classes by (a). Let γ_1 and γ_2 be *gc*-sets of S_1 and S_2 respectively. Then $j(\gamma_1 \times \gamma_2)$ is a *gc*-set of $S_1 \times S_2$ in view of our proof of (a). Thus

$$\begin{aligned} \text{RET}(S_1 \times S_2) &= \text{Req}\, j(\gamma_1 \times \gamma_2) \\ &= \text{Req}(\gamma_1) \cdot \text{Req}(\gamma_2) = \text{RET}(S_1) \cdot \text{RET}(S_2). \end{aligned}$$

4. The class Bin (α).

NOTATIONS. For any set α and any number k,

$$\gamma(\alpha, k) = \{n \mid \varrho_n \subset \alpha \ \& \ r_n = k\},$$

$$\mathrm{Bin}(\alpha) = \{\gamma(\alpha, k) \mid k \geqq 1\}.$$

The class Bin (α) is an *md*-class for any set α. If α is a finite set of cardinality $m \geqq 1$, then Bin (α) consists of m finite sets; in this case Bin (α) is a *gc*-class which has the number m as its cardinality and its RET. This is still true in case $m = 0$, for then Bin (α) is empty. For any infinite set α, Bin (α) is a denumerable *md*-class of infinite sets. The next proposition tells us when Bin (α) is a *gc*-class. We write R for Req (ε), i. e., for the RET which consists of all infinite r. e. sets.

PROPOSITION P6. *Let* $A = \mathrm{Req}(\alpha)$. *Then*

(a) *if* α *has an infinite* r. e. *subset,* Bin (α) *is a gc-class of* RET R,

(b) *if* α *is regressive,* Bin (α) *is a gc-class of* RET A,

(c) *if* α *is immune, but not regressive,* Bin (α) *is not a gc-class.*

PROOF. Let α be any set. For any number x such that $\varrho_x \neq o$ and $\varrho_x \subset \alpha$, we write γ_x for the unique set in Bin (α) which contains x. Hence $\gamma_x = \gamma(\alpha, r_x)$, since r_x denotes the cardinality of ϱ_x.

(a) Let α have an infinite r. e. subset, say β.

Suppose b_n is a one-to-one recursive function ranging over β, and $c_{n+1} = b_n$ for every number n. Then $\beta = (c_1, c_2, \ldots)$ and there exists a recursive function d such that $d(0) = 0$ and $\varrho_{d(n+1)} = (c_1, \ldots, c_{n+1})$. Let δ consist of the numbers $d(1), d(2), \ldots$. Since $rd(n) = n$ we see that δ is a choice set of Bin (α). Denoting the union of Bin (α) by σ we have for $x \in \sigma$,

$$d(r_x) \in \delta \cdot \gamma(\alpha, r_x), \quad \text{i. e.,} \quad d(r_x) \in \delta \cdot \gamma_x .$$

Thus δ is a *gc*-set of Bin (α) and the RET of Bin (α) is $\mathrm{Req}(\delta) = R$.

(b) Let the set α be regressive. If α is finite we are through. Now assume that α is infinite. Let a_n be a regressive function ranging over α and p a regressing function of a_n. Then there exists a partial recursive function q such that

$$\delta q = \delta p \quad \text{and} \quad (\forall x)(\forall n)[x = a_{n+1} \Longrightarrow \varrho_{q(x)} = (a_0, \ldots, a_n)].$$

Let δ consist of $q(a_1)$, $q(a_2)$, ..., then δ is a choice set of Bin (α), because $q(a_n) \in \gamma(\alpha, n)$, for $n \geqq 1$. To show that δ is a *gc*-set of Bin (α) we define a function f as follows:

first of all, $\delta f = \sigma$; secondly, let an element $x \in \sigma$ be given. Then the numbers $r(x)$, $i_0, \dots, i_{r(x)-1}$ such that

$$\varrho_x = [a(i_0), \dots, a(i_{r(x)-1})],$$
$$i_0 < i_1 < \dots < i_{r(x)-1},$$

can be computed. Clearly, $i_{r(x)-1} \geqq r(x) - 1$. By regressing the function a_n from $a(i_{r(x)-1})$ to $a_{r(x)-1}$ we can therefore compute the unique number y such that

$$\varrho_y = (a_0, \dots, a_{r(x)-1}).$$

This number y we call $f(x)$. For $x \in \sigma$,

$$f(x) = q(a_{r(x)}) \in \delta \cdot \alpha_x .$$

It is readily verified that f has a partial recursive extension. Thus δ is a *gc*-set of Bin (α). It remains to be shown that $\delta \simeq \alpha$. For every $y \in \alpha$ there exists a unique number n such that $y = a_n$; let us call this number n the *a-rank* of y; it can be effectively computed from y, since it equals $p^*(y)$. Let for $x \in \delta$,

$$g(x) = \text{the element of highest } a\text{-rank in } \varrho_x ,$$

then we have for $n \geqq 1$, $x \geqq 0$,

$$x = q(a_n) \implies \varrho_x = (a_0, \dots, a_{n-1}) \implies g(x) = a_{n-1} .$$

The function g therefore maps δ one-to-one onto α. It is readily proved that both g and g^{-1} have partial recursive extensions. Thus $\delta \simeq \alpha$ by (12).

c) Throughout this part of the proof α denotes an infinite set. We call a set *recursively infinite* (abbreviated: r. i.), if it has an infinite r. e. subset. Thus, if α is not r. i., α is immune. Consider the two statements:

(I) *if Bin* (α) *is a gc-class, there is a regressive gc-set* δ *of Bin* (α) *such that the function* d_n *defined by* « $d_n \in \delta \cdot \gamma(\alpha, n)$, *for* $n \geqq 1$ » *has the property*: $\varrho_{d(1)} \subset_+ \varrho_{d(2)} \subset_+ \cdots$,

(II) *if Bin* (α) *has a gc-set* δ *with the properties listed under* (I), *then either* α *is r. i. or* $\alpha \simeq \delta$.

These two statements imply

$$\text{Bin}(\alpha) \text{ a } gc\text{-class} \implies \alpha \text{ r. i. or } \alpha \text{ regressive},$$

i. e., the contrapositive of (c). It therefore suffices to establish (I) and (II).

Re (I). Let γ be a *gc*-set of Bin (α) and let for $n \geqq 1$, the unique element of $\gamma \cdot \gamma(\alpha, n)$ be denoted by $c(n)$. Since $\varrho_{c(n)}$ has cardinality n, there exist numbers $c_{11}, c_{21}, c_{22}, c_{31}, c_{32}, c_{33}, \ldots$ such that

$$\varrho_{c(1)} = (c_{11}),$$

$$\varrho_{c(2)} = (c_{21}, c_{22}), \text{ where } c_{21} < c_{22},$$

$$\varrho_{c(3)} = (c_{31}, c_{32}, c_{33}), \text{ where } c_{31} < c_{32} < c_{33},$$

$$\vdots$$

Put e_1 = the first number occurring in $c_{11}, c_{21}, c_{22}, c_{31}, \cdots$,
e_{n+1} = the first number occurring in $c_{11}, c_{21}, c_{22}, c_{31}, \cdots$, which does not belong to $(e_1, \ldots, e_n)$.

There clearly exists a one-to-one function d such that

$$\varrho_{d(0)} = o, \qquad \varrho_{d(n+1)} = (e_1, \cdots, e_{n+1}).$$

We recall that $\varrho_i \subset_+ \varrho_k$ implies $i < k$. Thus the function d is strictly increasing. Let δ consist of the numbers $d(1)$, $d(2), \ldots$, then $d(n)$ is the unique element of $\delta \cdot \gamma(\alpha, n)$, for $n \geqq 1$. The set δ is therefore a choice set of Bin (α). We now prove that γ is a regressive set. Let p be a partial recursive function such that $\sigma \subset \delta p$ and for $x \in \sigma$, $p(x) \in \gamma \cdot \gamma_x$, i. e., $p(x) \in \gamma \cdot \gamma(\alpha, r_x)$. Let any element of γ be given, say $c(n+1)$. Then we can compute the numbers $c_{n+1,1}, \cdots, c_{n+1,n+1}$ such that

$$\varrho_{c(n+1)} = (c_{n+1,1}, \cdots, c_{n+1,n+1}),$$

hence also the number i such that

$$\varrho_i = (c_{n+1,1}, \cdots, c_{n+1,n}).$$

Then $p(i) = c(n)$, since $i \in \gamma(\alpha, n)$. The number $c(n)$ can therefore be effectively computed from the number $c(n+1)$. Hence γ is the range of some regressive function, e. g., of the function $\bar{c}$ defined by $\bar{c}(n) = c(n+1)$, for $n \in \varepsilon$. We conclude that γ is a regressive set. We proceed to show that δ is a *gc*-set of Bin (α). Given any number $x \in \sigma$ we can compute the numbers $n = r(x)$ and $c(n) = p(x)$, hence also the numbers $c(1), \ldots, c(n)$ and the finite sequence

$$(\Sigma) \qquad c_{11}, c_{21}, c_{22}, c_{31}, c_{32}, c_{33}, \cdots, c_{n1}, \cdots, c_{nn}.$$

The last n elements of (Σ) are distinct, hence from (Σ) we can compute the numbers $e_1, \ldots, e_n$ and the number $d(n)$ such that $\varrho_{d(n)} = (e_1, \ldots, e_n)$. However, for $n \geqq 1$ we have $d(n) \in \delta \cdot \gamma_x$, i. e., $d(n) \in \delta \cdot \gamma(\alpha, n)$. Thus δ is a *gc*-set of $\text{Bin}(\alpha)$. It follows that $\gamma \simeq \delta$ by PI and that δ is regressive, because γ is regressive. This completes the proof of (I). We observe in passing that δ is a retraceable set. For first of all, $d(1) < d(2) < \ldots$. Secondly, given $d(n+1)$ we know an $(n+1)$-element subset of α, hence also an n-element subset of α, i. e., a number in $\gamma(\alpha, n)$, say t; from t we can compute the unique element of $\delta \cdot \gamma(\alpha, n)$, i. e., $d(n)$.

Re (II). Let δ be a *gc*-set of $\text{Bin}(\alpha)$ with the properties mentioned in (I). Define for $n \geqq 1$,

$$d(n) = \text{unique element of } \delta \cdot \gamma(\alpha, n),$$

$$e_1 \quad = \text{unique element of } \varrho_{d(1)},$$

$$e_{n+1} = \text{unique element of } \varrho_{d(n+1)} - \varrho_{d(n)}.$$

Thus $\varrho_{d(n)} = (e_1, \ldots, e_n)$, for $n \geqq 1$. Let the set η consist of $e_1, e_2, \ldots$, then $\eta \subset \alpha$, since

$$\eta = \sum_{n=1}^{\infty} \varrho_{d(n)} \quad \text{and} \quad (\forall n)[n \geqq 1 \Longrightarrow \varrho_{d(n)} \subset \alpha].$$

We distinguish two cases:

$$\text{(IIa)} \ \eta \subset_+ \alpha, \qquad \text{(IIb)} \ \eta = \alpha.$$

Re (IIa). Let $t \in \alpha - \eta$ and let q be a partial recursive function such that $\sigma \subset \delta q$ and $q(x) \in \delta \cdot \gamma(\alpha, r_x)$, for $x \in \sigma$. We may assume the number $d(1)$, hence also the number e_1, as known. We have $\varrho_{d(1)} = (e_1)$; here $e_1 \neq t$, since $t \in \alpha - \eta$. We now know a two-element subset of α, namely (e_1, t) and can compute its canonical index, say a and also the number $q(a)$, i. e., the canonical index of (e_1, e_2). Since $t \notin (e_1, e_2)$ we can compute the canonical index of a three-element subset of α, namely (e_1, e_2, t); let b be this canonical index. Then $q(b)$ is the canonical index of (e_1, e_2, e_3) etc. This effective procedure does not terminate, since $t \in \alpha - \eta$. Hence η is an infinite r. e. subset of α and α is r. i.

Re (IIb). Assume $\eta = \alpha$. We wish to prove $\alpha \simeq \delta$, i. e., $\eta \simeq \delta$. Let h be the mapping from δ onto η such that $h(d_n) = e_n$, for $n \geqq 1$. It follows from the definition of the function e in terms of the function d that h has a partial recursive extension. Let any number of η be given, say e_n; then $d_n = h^{-1}(e_n)$ can be computed in the following manner. From the one-element subset (e_n) of α we can compute d_1 and e_1. If upon comparing e_1 and e_n we find out that $e_n = e_1$, we know that $n = 1$ and we have found d_n, since in this case $d_n = d_1$. If, on the other hand, $e_n \neq e_1$, we know that $n \neq 1$ and we have a two-element subset of α, namely (e_1, e_n); this enables us to compute the numbers d_2 and e_2. We continue this procedure until it terminates, i. e., until we have found d_n and $(e_1, \dots, e_n)$; this must happen after a finite number of steps. It is readily proved that the function h^{-1} has a partial recursive extension. We conclude by (12) that $\delta \simeq \eta$, i. e., $\delta \simeq \alpha$. Since δ is regressive, so is α.

COROLLARY. *There exist exactly c md-classes; among these c are gc-classes and c are not.*

PROOF. There are at most c *md*-classes, since every *md*-class is countable. Let A be called a *Bin*-class, if $A = Bin(\alpha)$, for some α. There are exactly c immune sets; among these c are regressive and c are not. Thus there are exactly c *Bin*-classes of immune sets; among these c are *gc*-classes and c are not. It readily follows that there exist c *md*-classes of immune sets; among these c are *gc*-classes and c are not. This is slightly stronger than the corollary.

An isol is called *regressive*, if it consists entirely of regressive sets, (or equivalently, if it contains at least one regressive set). Let Λ_R denote the collection of all regressive isols. It is proved in [3] that Λ_R is neither closed under addition nor under multiplication, but that the $\min(x, y)$ function from ε^2 into ε can be extended in a natural manner to a $\min(X, Y)$ function from Λ_R^2 into Λ_R. However, it is not true that $\min(X, Y) = X$ or $\min(X, Y) = Y$, for any two regressive isols X and Y.

PROPOSITION P7. *Let α, β be two non-empty, isolated sets, $A = \mathrm{Req}(\alpha)$, $B = \mathrm{Req}(\beta)$ and*

$$S = \{j(\xi \times \eta) \mid (\exists n)[n \geqq 1 \ \& \ \xi = \gamma(\alpha, n) \ \& \ \eta = \gamma(\beta, n)]\}.$$

If α and β are regressive, i. e., if $A, B \in \Lambda_R$, then S is a gc-class and $\mathrm{RET}(S) = \min(A, B)$.

PROOF. Assume the hypothesis. If A or B is finite, so is $\min(A, B)$ and the desired conclusion holds. From now on we assume that α and β are infinite regressive sets. Suppose γ and δ are *gc*-sets of $\mathrm{Bin}(\alpha)$ and $\mathrm{Bin}(\beta)$ respectively and that for $n \geqq 1$, $c_n \in \gamma \cdot \gamma(\alpha, n)$ and $d_n \in \delta \cdot \gamma(\beta, n)$. Let the unions of the classes $\mathrm{Bin}(\alpha)$, $\mathrm{Bin}(\beta)$, S be denoted by σ_1, σ_2, σ_3 respectively. Consider partial recursive functions p and q and the set μ such that

$$\sigma_1 \subset \delta p \quad \text{and} \quad (\forall x)[x \in \sigma_1 \Longrightarrow p(x) = c_{r(x)}],$$

$$\sigma_2 \subset \delta q \quad \text{and} \quad (\forall y)[y \in \sigma_2 \Longrightarrow q(y) = d_{r(y)}],$$

$$\mu = [j(c_1, d_1), j(c_2, d_2), \ldots].$$

The set μ is a choice set of S, because

$$S = [j[\gamma(\alpha, 1) \times \gamma(\beta, 1)], j[\gamma(\alpha, 2) \times \gamma(\beta, 2)], \ldots].$$

Assume $j(x, y) \in \sigma_3$. Then

$$j[p(x), q(y)] = j[c_{r(x)}, d_{r(y)}] \quad \in \quad j[\gamma(\alpha, r_x) \times \gamma(\beta, r_y)],$$

where the third set mentioned in the last formula is the unique set in S which contains $j(x, y)$. Put

$$h(z) = j[pk(z), ql(z)], \quad \text{for} \quad z \in j(\delta p \times \delta q),$$

then h is a partial recursive function related to σ_3 and μ in the usual manner. Thus μ is a gc-set of S and

$$\text{RET}(S) = \text{Req}(\mu) = \min(A, B).$$

REMARK. Under the hypothesis of P7, RET (S) is a regressive isol, since Λ_R^2 is closed under the minimum function. Note that Bin (α), Bin (β) and S are all gc-classes. The isolated sets α and β can, however, be chosen in such a manner that S is a gc-class, while only one of the two classes Bin (α) and Bin (β) is a gc-class. This can be shown by the following example due to J. Barback. Let τ_1 be an immune, regressive set and τ_2 an immune, indecomposable set. Put

$$\alpha = \{2x \mid x \in \tau_1\}, \quad \eta = \{2x + 1 \mid x \in \tau_2\}, \quad \beta = \alpha + \eta.$$

Then $\alpha \simeq \tau_1$, $\eta \simeq \tau_2$, hence α is also immune and regressive, while η is also immune and indecomposable. The set β is immune, because it is the sum of two separable, immune sets. Note that $\eta \subset \beta$ and $\eta \mid \beta - \eta$. Thus, if β were regressive, η would be regressive by [3, P5]; however, η is indecomposable, while every infinite regressive set is decomposable. Thus β is immune, but not regressive. While Bin (α) is a gc-class, Bin (β) is not a gc-class. Let γ be a gc-set of Bin (α) and let p be a partial recursive function related to γ and σ_1 in the usual manner. Let $c_n \in \gamma \cdot \gamma(\alpha, n)$, for $n \geqq 1$. Put

$$\theta = [j(c_1, c_1), j(c_2, c_2), \ldots],$$

$$h(z) = j[pk(z), pk(z)], \quad \text{for} \quad z \in j(\delta p \times \varepsilon).$$

Then θ is a choice set of S and for $n \geqq 1$,

$$j(x, y) \in j[\gamma(\alpha, n) \times \gamma(\beta, n)] \implies hj(x, y) = j(c_n, c_n) \in j[\gamma(\alpha, n) \times \gamma(\beta, n)].$$

Moreover, $\sigma_3 \subset \delta h$, hence h is a partial recursive function related to θ and σ_3 in the usual manner. Thus θ is a *gc*-set of S. We conclude that of the three classes Bin (α), Bin (β) and S, exactly two are *gc*-classes, namely Bin (α) and S. We observe in passing that

$$\text{RET}(S) = \text{Req}(\theta) = \text{Req}(\gamma) = \text{Req}(\alpha) = A.$$

5. Characterization of *gc*-classes.

DEFINITIONS. Let $p(x)$ be a partial recursive function and S a *gc*-class. Then $p(x)$ is a *gc-function of S*, if

(18) $\sigma \subset \delta p$ and $p(\sigma) \in \zeta(S)$,

(19) $(\forall x)[x \in \sigma \implies p(x) \in p(\sigma) \cdot \alpha_x]$,

(20) $\varrho p \subset \delta p$ and $(\forall x)[x \in \delta p \implies p^2(x) = p(x)]$.

A *gc-function* is a partial recursive function which is a *gc*-function of at least one *gc*-class.

Every *gc*-class has at least one *gc*-function. For let S be a *gc*-class. Then every partial recursive function $p(x)$ related to S by (18) and (19) has a partial recursive restriction $p_1(x)$ such that

$$\varrho p_1 \subset \delta p_1 \quad \text{and} \quad (\forall x)[x \in \delta p_1 \implies p_1^2(x) = p_1(x)].$$

NOTATION. For any partial recursive function $p(x)$,

$$\text{Gen}(p) = \{p^{-1}(y) \mid y \in \varrho p\}.$$

For every partial recursive function $p(x)$, $\mathrm{Gen}(p)$ is an *md*-class; it is empty if and only if $p(x)$ is nowhere defined; moreover, it is a r. e. class of r. e. sets; in fact, it is r. e. without repetitions.

PROPOSITION P8. *A partial recursive function $p(x)$ is a gc function if and only if it satisfies* (20). *Moreover, if $p(x)$ satisfies* (20), *it is a gc-function of the class $S = \mathrm{Gen}(p)$ with $\sigma = \delta p$ and $p(\sigma) = \varrho p \in \zeta(S)$.*

PROOF. One direction of the biconditional is trivial. Let $p(x)$ be a partial recursive function which satisfies (20). Observe that (20) is equivalent to

(21) $$\varrho p \subset \delta p \quad \text{and} \quad (\forall y)[y \in \varrho p \Longrightarrow p(y) = y].$$

For let $p(x)$ satisfy (20). Assume $y_1 \in \varrho p$, say $y_1 = p(x_1)$. Then $p(y_1) = p^2(x_1) = p(x_1) = y_1$. Conversely, assume that $p(x)$ satisfies (21). Let $x_1 \in \delta p$ and put $y_1 = p(x_1)$. Then $y_1 \in \varrho p$ and $p^2(x_1) = p(y_1) = y_1 = p(x_1)$. We may therefore assume that $p(x)$ satisfies both (20) and (21). Let $S = \mathrm{Gen}(p)$. Then $\sigma = p^{-1}(\varrho p) = \delta p$, and $p(\sigma) = p(\delta p) = \varrho p$. We claim

(i) $\varrho p \subset \sigma$,

(ii) $\alpha \in S \Longrightarrow \varrho p \cdot \alpha$ contains exactly one element,

(iii) $(\forall x)[x \in \sigma \Longrightarrow p(x) \in \varrho p \cdot \alpha_x]$.

Note that (i) and (ii) imply that ϱp is a choice set of S, while (i), (ii), (iii) and $\sigma = \delta p$ imply that ϱp is a *gc*-set of S.

Re (i). $\varrho p \subset \delta p$ and $\delta p = \sigma$; thus $\varrho p \subset \sigma$.

Re (ii). Let $\alpha \in S$, say $\alpha = p^{-1}(y_1)$, for some $y_1 \in \varrho p$. Then $y_1 \in \varrho p$ implies $p(y_1) = y_1$, hence $y_1 \in p^{-1}(y_1) = \alpha$; thus $y_1 \in \varrho p \cdot \alpha$. Moreover, y_1 is the only element of $\varrho p \cdot \alpha$. For assume $y_2 \in \varrho p \cdot \alpha$. Then $p(y_2) = y_2$ because $y_2 \in \varrho p$, and $p(y_2) = y_1$ because $y_2 \in \alpha$; thus $y_1 = y_2$.

Re (iii). Let $s \in \sigma$. Then $s \in \delta p$; put $y_1 = p(s)$. Hence $s \in p^{-1}(y_1)$ and $\alpha_s = p^{-1}(y_1)$. We now have $s \in \alpha_s$, $\alpha_s \in S$ and $\alpha_s = p^{-1}(y_1)$. According to (ii), y_1 is the only element in $\varrho p \cdot \alpha_s$. However, $y_1 = p(s)$, hence $p(s) \in \varrho p \cdot \alpha_s$.

PROPOSITION P9. *Let $p(x)$ be a gc-function of the gc-class S. Then $\sigma = \delta p$ if and only if $S = \mathrm{Gen}(p)$.*

PROOF. Let $p(x)$ be a *gc*-function of the *gc*-class S. The « if part » is immediate, for $S = \mathrm{Gen}(p)$ implies

$$\sigma = \Sigma\,\{p^{-1}(y) \mid y \in \varrho p\} = \delta p.$$

Now assume $\sigma = \delta p$. Let $T = \mathrm{Gen}(p)$; denote the union of all sets in T by τ. We know by P8 that $p(x)$ is a *gc*-function of T with $\tau = \delta p$. Thus

$\sigma = \tau$, since both σ and τ are equal to δp. It clearly suffices to prove $S = T$, i. e., (i) $S \subset T$, and (ii) $T \subset S$.

Re (i). Let $\alpha \in S$. Let x_0 be any element of α; put $y_0 = p(x_0)$. Then

$$x \in \alpha \implies p(x) = p(x_0) \implies p(x) = y_0 ,$$

i. e., $\alpha \subset p^{-1}(y_0)$. Denoting $p^{-1}(y_0)$ by β we see that $\alpha \subset \beta$ and $\beta \in T$. The inclusion $\alpha \subset \beta$ must be improper. For suppose $b \in \beta - \alpha$. Then $b \in \sigma$, since $b \in \beta$ and $\beta \in T$, while $\sigma = \tau$. We claim

$$[\bar{\alpha} \in S \,\&\, \bar{\alpha} \neq \alpha] \implies b \notin \bar{\alpha}. \tag{22}$$

For assuming the hypothesis of (22),

$$[x \in \bar{\alpha} \,\&\, \bar{\alpha} \neq \alpha] \implies p(x) \neq y_0 ,$$

$$b \in \beta \implies b \in p^{-1}(y_0) \implies p(b) = y_0 ,$$

so that $x \neq b$. Combining (22) with the hypothesis $b \notin \alpha$ we obtain

$$b \notin \Sigma \{\alpha \mid \alpha \in S\}, \text{ i. e., } \quad b \notin \sigma.$$

The assumption $\alpha \subset_+ \beta$ leads therefore to the contradiction: $b \in \sigma$ and $b \notin \sigma$. Hence $\alpha = \beta$, and $\alpha \in T$ because $\beta \in T$.

Re (ii). Let $\beta \in T$, say $\beta = p^{-1}(y_1)$, where $y_1 \in \varrho p$. Then $\beta \subset \sigma$, since $\beta \subset \tau$ and $\tau = \sigma$. Note that $\beta = p^{-1}(y_1)$ implies $p(\beta) = (y_1)$; combining this with $\beta \subset \sigma$, we see that β must be included in some set of S, say α. The set β is non-empty, for it belongs to the *md*-class T; let $b \in \beta$. We claim that $a \in \alpha$ implies $a \in \beta$. For assume $a \in \alpha$. Then $a, b \in \alpha$, since $b \in \beta$, $\beta \subset \alpha$; this implies $p(a) = p(b)$. On the other hand, $\beta = p^{-1}(y_1)$, hence $b \in \beta$ implies $p(b) = y_1$. Thus $p(a) = y_1$ and $a \in p^{-1}(y_1)$, i. e., $a \in \beta$. We have therefore proved that $\alpha \subset \beta$. Since we also have $\beta \subset \alpha$, we conclude that $\alpha = \beta$. Hence $\beta \in S$, since $\alpha \in S$. We have proved that $T \subset S$.

Definition I. A class S is *primitive,* if it satisfies one of the following three conditions:

(i) S is empty, (ii) S is a non-empty, finite *md*-class of r. e. sets, (iii) S is a denumerable *md*-class of r. e. sets and there exists a recursive function $a(n, x)$ such that if $\alpha_n = \varrho a(n, x)$, for $n \in \varepsilon$, then $\alpha_0, \alpha_1, \ldots$ are distinct and $S = (\alpha_0, \alpha_1, \ldots)$.

Definition II. A class S is *primitive,* if it is a *gc*-class with a *gc*-function $p(x)$ such that $S = \text{Gen}(p)$.

DEFINITION III. A class S is *primitive,* if $S = \text{Gen}(p)$, for some partial recursive function $p(x)$.

PROPOSITION P10. *The three definitions of a primitive class are equivalent.*

PROOF. Let S be an *md*-class. We shall establish the three conditionals

(a) S I-primitive $\Longrightarrow$ S II-primitive,

(b) S II-primitive $\Longrightarrow$ S III-primitive,

(c) S III-primitive $\Longrightarrow$ S I-primitive.

Since (b) is trivial we shall restrict our attention to (a) and (c).

Re (a). Let S be I-primitive. We distinguish three cases.

Case 1. S is empty. Let $p(x)$ be the partial recursive function which is nowhere defined. Then $S = \text{Gen}(p)$ and $p(x)$ is a *gc*-function of S.

Case 2. S is a non-empty, finite *md*-class of r. e. sets, say $S = (\alpha_0, \dots, \alpha_n)$. Note that $\alpha_0, \dots, \alpha_n$ are non empty and mutually disjoint. Let $a_i \in \alpha_i$, for $0 \leqq i \leqq n$. Define a function $p(x)$ by

$$\delta p = \alpha_0 + \dots + \alpha_n, \quad (\forall x)(\forall i \leqq n)[x \in \alpha_i \Longrightarrow p(x) = a_i].$$

Then $p(x)$ is partial recursive and $\varrho p = (a_0, \dots, a_n)$. Hence

$$\text{Gen}(p) = \{p^{-1}(y) \mid y \in \varrho p\} = [p^{-1}(a_0), \dots, p^{-1}(a_n)] = (\alpha_0, \dots, \alpha_n) = S.$$

It also follows from the definition of $p(x)$ that

$$p(\sigma) = p(\alpha_0 + \dots + \alpha_n) = (a_0, \dots, a_n) \in \zeta(S),$$

$$(\forall x)[x \in \sigma \quad \Longrightarrow \quad p(x) \in p(\sigma) \cdot \alpha_x],$$

$$\varrho p \subset \sigma, \qquad \text{hence} \qquad \varrho p \subset \delta p,$$

$$(\forall x)[x \in \sigma \quad \Longrightarrow \quad p^2(x) = p(x)].$$

Hence $p(x)$ is a *gc*-function of S.

Case 3. S is a denumerable *md*-class of r. e. sets and there exists a recursive function $a(n, x)$ such that the sets $\alpha_0 = \varrho a(0, x)$, $\alpha_1 = \varrho a(1, x), \dots$ are distinct and $S = (\alpha_0, \alpha_1, \dots)$. Define a function $p(x)$ by

$$\delta p = \alpha_0 + \alpha_1 + \dots, \text{ and } \quad (\forall x)(\forall i)[x \in \alpha_i \Longrightarrow p(x) = a(i, 0)].$$

The set δp is therefore r. e., and given any $x \in \delta p$ we can compute the unique number i such that $x \in \alpha_i$. Thus $p(x)$ is a partial recursive function with $[a(0, 0), a(1, 0), \dots]$ as its range. Also,

$$\text{Gen}(p) = \{p^{-1}(y) \mid y \in \varrho p\} = [p^{-1} a(o, o), p^{-1} a(1, o), \dots] = (\alpha_0, \alpha_1, \dots) = S.$$

We can verify as we did in Case 2 that $p(x)$ is a *gc*-function of S. In each of the three cases, $S = \text{Gen}(p)$, where $p(x)$ is a *gc*-function of S, i. e., S is II-primitive.

Re (c). Let S be III-primitive, say $S = \text{Gen}(p)$ for some partial recursive function p. We distinguish three cases.

Case 1. $p(x)$ is nowhere defined. Then S is empty.

Case 2. ϱp is non empty, but finite, say $\varrho p = (c_0, \dots, c_k)$, where $c_0, \dots, c_k$ are distinct. Then

$$S \quad = \quad \text{Gen}(p) \quad = \quad [p^{-1}(c_0), \dots, p^{-1}(c_k)],$$

where $p^{-1}(c_0), \dots, p^{-1}(c_k)$ are mutually disjoint, because $c_0, \dots, c_k$ are distinct. For $0 \leqq i \leqq k$, $p^{-1}(c_i)$ is a r. e. set, since $p(x)$ is a partial recursive function. Hence S is a finite class which consists of $k+1$ mutually disjoint r. e. sets.

Case 3. ϱp is infinite. Since ϱp is also r. e., there exists a one-to-one recursive function which ranges over ϱp, say c_n. Thus

$$S \quad = \quad \text{Gen}(p) \quad = \quad [p^{-1}(c_0), \; p^{-1}(c_1), \dots].$$

Given any number n we can effectively find a (definition of a) recursive function $a_n(x)$ of x which ranges over $p^{-1}(c_n)$. The sets $p^{-1}(c_0)$, $p^{-1}(c_1), \dots$ are mutually disjoint, since $c_0, c_1, \dots$ are distinct. Put $a(n, x) = a_n(x)$, for $n \in \varepsilon$, then $a(n, x)$ is a recursive function; also, the sets $\varrho a(0, x)$, $\varrho a(1, x), \dots$ are distinct and S consists of $\varrho a(0, x)$, $\varrho a(1, x), \dots$. In each of the three cases S is I-primitive.

COROLLARY. *A class S is primitive if and only if it is a gc-class with a gc-function $p(x)$ such that $\delta p = \sigma$.*

PROOF. By P9 and P10.

DEFINITION. An *md*-class T is a *restriction* of a *gc*-class S, if

(a) for every $\beta \in T$, there is an α_β such that $\beta \subset \alpha_\beta$ and $\alpha_\beta \in S$,

(b) there is a $\gamma \in \zeta(S)$ such that $\beta \in T$ implies $\gamma \cdot \alpha_\beta \subset \beta$.

REMARK. Let the *md*-class T be a restriction of the *gc*-class S. Then every set $\beta \in T$ uniquely determines the set α_β such that $\beta \subset \alpha_\beta$ and $\alpha_\beta \in S$, since β is non-empty and the sets in S are mutually disjoint.

It is clear that every subclass of a *gc*-class S is a restriction of S. We observed in section 3 that every subclass of a *gc* class is again a *gc*-class. This last statement will now be generalized.

PROPOSITION P11. *Every restriction of a gc-class is again a gc-class.*

PROOF. Let $\gamma \in \zeta(S)$ and let the *md*-class T be related to S and γ by (a) and (b). Suppose $p(x)$ is a partial recursive function related to σ and γ in the usual manner. Let S_0 be the class of all sets α_β, for $\beta \in T$. Denote the unions of S_0 and T by σ_0 and τ respectively, and let $\gamma_0 = \gamma \cdot \sigma_0$. The relation $S_0 \subset S$ implies first of all that $\gamma_0 \in \zeta(S_0)$ and secondly that $p(x)$ is related to σ_0 and γ_0 in the usual manner. The class T can now be obtained from S_0 by replacing every $\alpha \in S_0$ by a set β such that $\gamma_0 \cdot \alpha \subset \beta \subset \alpha$. Thus

$$(\forall x)[x \in \alpha \;\&\; \alpha \in S_0 \Longrightarrow p(x) \in \gamma_0 \cdot \alpha] \qquad \text{implies}$$

$$(\forall x)[x \in \beta \;\&\; \beta \in T \Longrightarrow p(x) \in \gamma_0 \cdot \beta].$$

Hence $\gamma_0 \in \zeta(T)$ and $p(x)$ is related to τ and γ_0 in the usual manner. Note that $\gamma \cdot \sigma_0 = \gamma \cdot \tau$; we could therefore also have defined γ_0 as $\gamma \cdot \tau$.

PROPOSITION P12. *Let T be a gc-class. For every gc-function $p(x)$ of T, T is a restriction of the primitive gc-class* Gen (p).

PROOF. Let T be a *gc*-class and let $p(x)$ be one of its *gc*-functions. Put $S = \text{Gen}(p)$. In view of P8 the class S is a primitive class with $p(x)$ as a *gc*-function; also $\sigma = \delta p$ and $p(\sigma) = \varrho(p) \in \zeta(S)$. We wish to prove that T is a restriction of S, i. e.,

(a) for every $\beta \in T$, there is an α_β such that $\beta \subset \alpha_\beta$ and $\alpha_\beta \in S$,

(b) there is a $\gamma \in \zeta(S)$ such that $\beta \in T \Longrightarrow \gamma \cdot \alpha_\beta \subset \beta$.

Re (a). Let $\beta \in T$. Then $\beta \neq o$; let $b \in \beta$, $c = p(b)$ and $\alpha_\beta = p^{-1}(c)$. Also, $c \in \varrho p$, hence

$$\alpha_\beta = p^{-1}(c) \quad \in \quad \text{Gen}(p) = S.$$

Since the element b of β is mapped by p onto c, p maps every element of β onto c. Hence $\beta \subset p^{-1}(c)$, i. e., $\beta \subset \alpha_\beta$.

Re (b). Put $\gamma = p(\sigma)$, then $\gamma \in \zeta(S)$. Let $\beta \in T$. Define b, c, α_β as in the proof of (a). Then $c \in \varrho p = p(\sigma)$, hence $c \in \gamma$. Moreover, since $p(x)$ is a *gc*-function, we have

$$c \in \varrho p \quad \Longrightarrow \quad p(c) = c \quad \Longrightarrow \quad c \in p^{-1}(c) \quad \Longrightarrow \quad c \in \alpha_\beta .$$

We conclude that $c \in \gamma \cdot \alpha_\beta$. Since $\gamma \in \zeta(S)$ and $\alpha_\beta \in S$, the set $\gamma \cdot \alpha_\beta$ contains only one element, hence $\gamma \cdot \alpha_\beta = (c)$. Finally, $b \in \beta$ and $\beta \in T$ imply $p(b) \in \beta$, hence $c \in \beta$. Thus $\gamma \cdot \alpha_\beta \subset \beta$.

It follows from P11 and P12 that: *an md-class is a gc-class if and only if it is a restriction of some primitive gc-class.* Let us therefore compare *gc*-classes in general with primitive *gc*-classes. We have seen in section 4 that there are exactly c *gc*-classes and in section 1 that for every RET A, there is a *gc*-class with A as its RET. On the other hand, we immediately see from the definition of a primitive class that there are exactly $\aleph_0$ primitive classes and that a primitive class can only have one of $0, 1, \ldots$ or R as its RET.

The restrictions of a primitive class can be simply described. For let S be a primitive class. Then every restriction T of S can be obtained as follows: choose a $\gamma \in \zeta(S)$ and form a subclass T of S by treating each $\alpha \in S$ in the following manner: either delete α altogether or replace α by any set β such that $\gamma \cdot \alpha \subset \beta \subset \alpha$.

It remains to characterize the *gc*-sets of any primitive class P. If P is finite, the *gc*-sets of P are simply the choice sets of P. Now assume P is infinite; let $a(n, x)$ be a recursive function such that

(i) $n \neq m \implies \varrho a(n, x)$ disjoint from $\varrho a(m, x)$,

(ii) S consists of $\varrho a(0, x), \varrho a(1, x), \ldots$.

Then γ is a *gc*-set of P if and only if γ equals $\varrho a(f_n, u_n)$, for some recursive permutation f_n and some recursive function u_n.

6. Miscellaneous remarks.

(A). We have not yet introduced a relation of recursive equivalence between *md*-classes. This can, however, be done in a natural manner.

NOTATION. For every *md*-class S,

$$R(S) = \{(x, y) \in \sigma^2 \mid \alpha_x = \alpha_y\}.$$

DEFINITION. Let S_1 and S_2 be *md*-classes with unions σ_1 and σ_2 respectively. Let $R_1 = R(S_1)$ and $R_2 = R(S_2)$. Then S_1 is *recursively equivalent* to S_2 [written: $S_1 \simeq S_2$], if there exists a partial recursive one-to-one function $p(x)$ such that

(a) $\sigma_1 \subset \delta p$ and $p(\sigma_1) = \sigma_2$,

(b) $x R_1 y \iff p(x) R_2 p(y)$, for $x, y \in \sigma_1$.

This $\simeq$ relation between *md*-classes is clearly reflexive, symmetric and transitive. We also have for *md*-classes S_1 and S_2: $S_1 \simeq S_2$ implies $\sigma_1 \simeq \sigma_2$. The following five properties of the $\simeq$ relation between *md*-classes are readily verified.

(i) *Let $S_1 \simeq S_2$. Then S_1 is a gc-class if and only if S_2 is a gc-class.*

(ii) *Let $S_1 \simeq S_2$, where S_1 and S_2 are gc-classes. Then*

$$\gamma_1 \in \zeta(S_1) \;\&\; \gamma_2 \in \zeta(S_2) \implies \gamma_1 \simeq \gamma_2.$$

(iii) *Let* $S_1 \simeq S_1^*$, $S_2 \simeq S_2^*$, $S_1 | S_2$ *and* $S_1^* | S_2^*$. *Then* $S_1 + S_2 \simeq S_1^* + S_2^*$.

(iv) $S_1 \simeq S_1^* \ \& \ S_2 \simeq S_2^* \implies S_1 \times S_2 \simeq S_1^* \times S_2^*$.

(v) $\alpha \simeq \beta \implies \mathrm{Bin}(\alpha) \simeq \mathrm{Bin}(\beta)$.

(*B*) Let us say that an *md*-class *has property* π, if there exists a partial recursive function $q(x, y)$ such that

$$(23) \qquad \sigma^2 \subset \delta q \quad \text{and} \quad \varrho q \subset (0, 1),$$

$$(24) \qquad \begin{cases} \alpha_x = \alpha_y \implies q(x, y) = 1, & \text{for} \quad x, y \in \sigma, \\ \alpha_x \neq \alpha_y \implies q(x, y) = 0, & \text{for} \quad x, y \in \sigma. \end{cases}$$

Intuitively speaking S has property π, if there is an effective procedure which enables us to decide for any two numbers in σ whether or not they belong to the same set in S. It is readily seen that

$$(25) \qquad S \ \textit{a gc-class} \implies S \ \textit{has property} \ \pi.$$

For assume the hypothesis. Let $\gamma \in \zeta(S)$ and let $p(x)$ be a partial recursive function related to γ and σ in the usual manner. Then the partial recursive function $q(x, y)$ defined by

$$q(x, y) = \overline{sg}\, | p(x) - p(y) |, \qquad \text{for} \qquad x, y \in \delta p,$$

satisfies (23) and (24). We claim that the converse of (25) is false. For let $S = \mathrm{Bin}(\alpha)$ for a set α which is immune, but not regressive. Then S is not a *gc*-class by P6. On the other hand, S has property π, since the recursive function

$$q(x, y) = \overline{sg}\, | r_x - r_y |, \qquad \text{for} \qquad x, y \in \varepsilon,$$

satisfies (23) and (24).

(*C*) We recall the definition of $\Phi_f(T)$.

NOTATION. Let $f(x)$ be any one-to-one function from ε into ε and let $T \in \Lambda_R - \varepsilon$. Then

$$\Phi_f(T) = \mathrm{Req}\ \varrho t_{f(n)},$$

where t_n is any regressive function ranging over any set in T.

It is readily seen that if $f(x)$ is a strictly increasing, recursive function, Φ_f maps $\Lambda_R - \varepsilon$ into itself. Several other properties of the mapping Φ_f are discussed in [6] and [7]. Let us assume that $f(x)$ is a strictly increasing recursive function such that $f(0) = 0$. We wish to show how one can associate with every $T \in \Lambda_R - \varepsilon$ a simple *gc*-class of finite sets which has

$\Phi_f(T)$ as its RET. Let $\tau \in T$ and let t_n be a regressive function ranging over τ. Put

$$\tau_n = \{t_x \mid f(n) \leqq x < f(n+1)\}, \qquad \text{for} \qquad n \in \varepsilon,$$

$$S = (\tau_0, \tau_1, \ldots), \quad \gamma = (t_{f(0)}, t_{f(1)}, \ldots).$$

Then S is an *md*-class of finite sets with τ as its union and γ as a choice set. Let

$$g(x) = f(\mu n)[f(n) \leqq x < f(n+1)], \qquad \text{for} \qquad x \in \varepsilon,$$

$$q(x) = tgt^{-1}(x), \qquad \text{for} \qquad x \in \tau.$$

The function $g(x)$ is recursive and

$$(\forall x)(\forall n)[x \in \tau_n \Longrightarrow q(x) \in \gamma \cdot \tau_n].$$

Given any $x \in \tau$ we can compute the unique number n such that $x = t_n$, i. e., the number $n = t^{-1}(x)$, hence also the number $g(n)$. However, $g(n)$ is *less than or equal to* n; this enables us to compute $q(x) = t_{g(n)}$ from t_n. It is readily proved that $q(x)$ has a partial recursive extension. Thus $\gamma \in \zeta(S)$ and

$$\text{RET}(S) = \text{Req}(\gamma) = \text{Req}\, \varrho t_{f(n)} = \Phi_f(T).$$

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