Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 19, nº 4 (1965), p. 593-608

<http://www.numdam.org/item?id=ASNSP_1965_3_19_4_593_0>

SOME FUNCTION SPACES DEFINED USING THE MEAN OSCILLATION OVER CUBES

SVEN SPANNE

0. Introduction.

The purpose of this paper is to extend some recent results by John and Nirenberg [2], Campanato [1] and Meyers [3]. These authors study spaces of functions on a set Q in R^n, defined by conditions on the mean oscillation on cubes I contained in Q, i. e. the quantity

$$Of(I) = \inf_{\sigma \in R} (\text{meas } I)^{-1} \int_I |f(y) - \sigma| \, dy. \tag{1}$$

In John and Nirenberg [2] is given a characterization of the space $\mathcal{L}_{(1)}$, defined by the condition

$$\sup_{I \subset Q} Of(I) < \infty \tag{2}$$

where the set Q is a cube.

As a generalization one can study the spaces $\mathcal{L}_{(t^\alpha)}$, which are defined by

$$\sup_{I \subset Q} Of(I)/(\text{meas } I)^{\frac{\alpha}{n}} < \infty \tag{3}$$

where α is a given real number. When $0 < \alpha \leq 1$ (Meyers [3], Campanato [1]), one gets the spaces Lip_α and when $-1 \leq \alpha < 0$ (Campanato [1]), spaces first studied by Morrey.

In the present paper we shall consider a generalization of these spaces, obtained by replacing the functions 1 and t^α, $0 < \alpha \leq 1$ by arbitrary

Pervenuto alla Redazione il 9 Giugno 1965.

positive non-decreasing functions φ, i. e. the spaces $\mathcal{L}_\varphi(Q)$ defined by

$$\sup_{I \subset Q} O f(I)/\varphi(\text{meas } I)^{\frac{1}{n}}) < \infty \tag{4}$$

where Q is a given cube.

The main results are the theorems 1 and 2, which contain a characterization of the spaces $\mathcal{L}_\varphi$, implying, e. g. the following results.

(a) If the function φ satisfies the Dini condition $\int_0^\delta \varphi(t)\frac{dt}{t} < \infty$ for some $\delta > 0$, then every function f in $\mathcal{L}_\varphi$ is continuous (modulo a null function) and its modulus of continuity satisfies the inequality $\omega(f, r) \leq \leq C \int_0^r \varphi(t)\frac{dt}{t}$, where C depends on f, for sufficiently small r.

(b) If $\varphi(t)/t$ is non-increasing and $\int_0^\delta \varphi(t)\frac{dt}{t}$ is not convergent, then there exists a function in $\mathcal{L}_\varphi$ that is neither bounded nor continuous, not even modulo nullfunctions.

The result (a) is a generalization of some results in Meyers [3] and Campanato [1], where it is proved that the spaces $\mathcal{L}_{(t^\alpha)}$, $0 < \alpha \leq 1$ coincide with the spaces Lip_α. The result (b) generalizes a remark by John and Nirenberg that the function $\log|x|$ belongs to $\mathcal{L}_{(1)}(Q)$ for each Q in R^n.

Our characterization of $\mathcal{L}_\varphi$ is an extension of the one given by John and Nirenberg for $\mathcal{L}_{(1)}$, using upper bounds for the measure of the sets $\{y \mid y \in I, |f(y) - f(I)| > \sigma\}$, where $f(I)$ is the mean value of f on I.

To obtain these results, we shall use a combination of the methods used by John and Nirenberg and of the one used by Campanato. We shall employ the main result of John and Nirenberg in the form of lemma 4, while the main idea in Campanato's proof corresponds to our lemma 3.

My thanks are due to Professor Jaak Peetre for suggesting the problem of proving the result (b) and for all his encouragement and help during my work.

1. Some preliminaries.

We use, for $x = (x_1, \dots, x_n)$ in R^n, the norm $|x| = \max_{1 \leq i \leq n} |x_i|$ and denote by $I(x, r)$ the closed cube $\{y \mid |x - y| \leq r/2\}$ of centre x, edgelength r and the edges parallel to the coordinate axes. In the sequel, all cubes are supposed to have their edges parallel to the coordinate axes.

If f is a locally integrable function, defined on the cube $I = I(x, r)$, we denote by

$$f(I) = f(x, r) = r^{-n} \int_I f(y)\, dy \tag{1.1}$$

the mean value of f on I.

For technical reasons we shall use, instead of the mean oscillation $Of(I)$, the quantity

$$\Omega f(I) = \Omega f(x, r) = r^{-n} \int_I |f(y) - f(I)|\, dy. \tag{1.2}$$

This is justified by the following lemma.

LEMMA 1: Let f be an integrable function on I. Then we have

$$Of(I) \leq \Omega f(I) \leq 2 Of(I) \tag{1.3}$$

PROOF.: The inequality to the left is evident and the one to the right follows since

$$\Omega f(I) = r^{-n} \int_I |f(y) - f(I)|\, dy \leq r^{-n} \int_I |f(y) - \sigma|\, dy +$$

$$+ |\sigma - f(I)| = r^{-n} \left(\int_I |f(y) - \sigma|\, dy + \left| \int_I (f(y) - \sigma)\, dy \right| \right) \leq$$

$$\leq 2 r^{-n} \int_I |f(y) - \sigma|\, dy \text{ for every } \sigma.$$

Let Q be a given cube in R^n. We write

$$\psi(f, r) = \psi(f, r; Q) = \sup_{I(x, r) \subset Q} \Omega f(x, r). \tag{1.4}$$

Then we have the following restrictions on the multiplicative variation of ψ.

LEMMA 2: We have that

(1.5) (a) $\Omega f(x', r') \leq 2\,(r/r')^n\, \Omega f(x,r)$ if $I(x', r') \subset I(x, r)$

(1.6) (b) $\psi(f, r') \leq 2\,(r/r')^n\, \psi(f, r)$ if $r' \leq r$

(1.7) (c) $\psi(f, r) \leq (8n + 1)\, \psi(f, r/2)$.

REMARK. A function $\theta(t)$ is called almost decreasing (almost increasing) if there is a constant A such that $\theta(t) \leq A\theta(t')$ if $t \geq t'$ ($t \leq t'$). It follows from the lemma that there are constants α and β, depending only on n, such that $\psi(f, t)/t^\alpha$ is almost decreasing and $\psi(f, t)/t^\beta$ is almost increasing, and moreover, the constants A may be taken depending only on n.

PROOF:

(a) We have

$$\Omega f(x', r') = r'^{-n} \int_{I(x', r')} |f(y) - f(x', r')|\, dy \leq$$

$$\leq r'^{-n} \int_{I(x', r')} |f(y) - f(x, r)|\, dy + |f(x, r) - f(x', r')| =$$

$$= r'^{-n} \int_{I(x', r')} |f(y) - f(x, r)|\, dy + \left| r'^{-n} \int_{I(x', r')} (f(x, r) - f(y))\, dy \right| \leq$$

$$2r'^{-n} \int_{I(x', r')} |f(y) - f(x, r)|\, dy \leq 2r'^{-n} \int_{I(x', r')} |f(y) - f(x, r)|\, dy =$$

$$= 2\left(\frac{r}{r'}\right)^n \Omega f(x, r) \text{ since } I(x', r') \subset I(x, r).$$

(b) This is an immediate consequence of (a).

(c) Take $I = I(x, r)$ in Q. We divide I (by halving each edge) into 2^n equal cubes I_k. Let I_k and I_l be two of these cubes, of centres x_k and x_l, having a full $(n-1)$ dimensional face in common, and let I' be a congruent cube of centre $\frac{1}{2}(x_k + x_1)$. Then

$$|f(I_k) - f(I_l)| \leq |f(I_k) - f(I')| + |f(I_l) - f(I')|$$

and

$$|f(I_k) - f(I')| \leq |f(y) - f(I_k)| + |f(y) - f(I')|.$$

Integrating this inequality over $I_k \cap I'$, we obtain

$$\frac{1}{2}\left(\frac{r}{2}\right)^n |f(I_k) - f(I')| \leq \int_{I_k \cap I'} (|f(y) - f(I_k)| + |f(y) - f(I')|)\, dt \leq$$

$$\leq \int_{I_k} |f(y) - f(I_k)|\, dy + \int_{I'} |f(y) f(I')|\, dy \leq$$

$$\leq \left(\frac{r}{2}\right)^n (\Omega f(I_k) + \Omega f(I')) \leq \left(\frac{r}{2}\right)^n 2\psi(f, r/2).$$

Hence

$$|f(I_k) - f(I_l)| \leq 8\psi(f, r/2).$$

Any two of the subcubes may be connected by a chain of at most n of adjacent subcubes. Hence, for two arbitrary subcubes, we obtain

$$|f(I_k) - f(I_l)| \leq 8n\psi(f, r/2)$$

and since $f(I) = 2^{-n} \sum_k f(I_k)$ we get

$$|f(I) - f(I_j)| \leq 2^{-n} \sum_k |f(I_k) - f(I_j)| \leq 8n\psi(f, r/2).$$

This implies

$$\Omega f(x, r) \leq r^{-n} \int_I |f(y) - f(I)|\, dy = r^{-n} \sum_j \int_{I_j} |f(y) - f(I)|\, dy \leq$$

$$\leq r^{-n} \sum_j \int_{I_j} (|f(y) - f(I_j)| + 8n\psi(f, r/2))\, dy \leq$$

$$\leq r^{-n} \sum_j (r/2)^n (\psi(f, r/2) + 8n\psi(f, r/2)) = (8n + 1)\, \psi(f, r/2)$$

which completes the proof.

We shall need some further properties.

LEMMA 3:

$$(1.8)\ (a) \qquad |f(x', r') - f(x, r)| \leq (r/r')^n \, \Omega f(x, r) \text{ if } I(x', r') \subset I(x, r)$$

$$(1.9)\ (b) \qquad |f(x', r2^{-j}) - f(x, r)| \leq 2^n \sum_{k=0}^{j-1} \psi(f, r \cdot 2^{-k}) \text{ if } I(x', r \cdot 2^{-j}) \subset I(x, r).$$

The proof of (a) is contained in the proof of lemma 2 (a) and (b) is an immediate consequence of (a).

We introduce the following notations

$$(1.10) \qquad \varphi(f, r) = \sup_{0 < s \leq r} \psi(f, s) = \sup_{\substack{I(x, s) \subset Q \\ 0 < s \leq r}} \Omega f(x, s).$$

The conclusion of lemma 2 (b) and (c) and the remark are also valid for the function φ.

REMARK: Obviously $\Omega f(x, r) \leq \omega(f, r)$, where $\omega(f, r) = \operatorname{ess\,sup}_{|x-y| \leq r} |f(x) - f(y)|$ is the modulus of continuity of f. Then we also get $\psi(f, r) \leq \omega(f, r)$ and $\varphi(f, r) \leq \omega(f, r)$.

$$(1.11) \qquad m(f;\, x, r;\, \sigma) = \text{meas}\, \{y | y \in I(x, r),\ |f(y) - f(x, r)| > \sigma\}$$

$f^*(x, r;\, s) =$ the non-increasing rearrangement on $(0, \infty)$ of the function $|f(y) - f(x, r)|$ restricted to $I(x, r)$. This is the function that is non-increasing, positive and continuous from the right, and such that

$$(1.12) \qquad m(f;\, x, r;\, f^*(x, r;\, s)) = s.$$

LEMMA 4: (John-Nirenberg)

There are constants a_n, depending only on n, such that

$$(1.13) \qquad m(f;\, x, r;\, a_n \lambda \varphi(f, r)) \leq 2^{-\lambda} \cdot r^n, \ \lambda \geq 1.$$

PROOF: According to John and Nirenberg [2], Lemma 1, page 415, we have

$$(1.14) \qquad m(f;\, x, r;\, \sigma) \leq B \exp\left(-\frac{b\sigma}{\varphi(f, r)}\right) r^n, \ \sigma > 0$$

where B and b depend only on n.

Choosing $a_n = (|\log B| + \log 2)/b$, the result follows.

If φ is an arbitrary function, we write

$$(1.15)\qquad \Phi_r(s) = \int_s^r \varphi(t)\,\frac{dt}{t}, \quad \text{if } 0 < s \leq r, \ = 0 \text{ if } s \geq r,$$

in particular

$$(1.16)\qquad \Psi_r(f, s) = \int_s^r \psi(f, t)\,\frac{dt}{t}$$

and

$$(1.17)\qquad \Phi_r(f, s) = \int_s^r \varphi(f, t)\,\frac{dt}{t}.$$

LEMMA 5: There are constants c_n, depending only on n, such that

$$(1.18)\qquad \frac{1}{c_n}\sum_{k=0}^{j-1}\psi(f, r\cdot 2^{-k}) \leq \Psi_{2r}(f, r\cdot 2^{-j+1}) \leq c_n \sum_{k-0}^{j-1}\psi(f, r\cdot 2^{-k})$$

and analoguously for Φ.

PROOF: According to the remark following lemma 2 there are constants α and β such that $\psi(f, r)/r^\alpha$ is almost decreasing and $\psi(f, r)\cdot r^\beta$ is almost increasing. Thus

$$\sum_{k=0}^{j-1}\psi(f, r\cdot 2^{-k}) = \frac{1}{\log 2}\sum_{k=0}^{j-1}\int_{r\cdot 2^{-k}}^{r\cdot 2^{-k+1}} \psi(f, r\cdot 2^{-k})\,\frac{dt}{t} \leq$$

$$\leq \frac{1}{\log 2} A\cdot 2^\beta \int_{r\cdot 2^{-j+1}}^{2r} \psi(f, t)\,\frac{dt}{t} = \frac{A\cdot 2^\beta}{\log 2}\Psi_{2r}(f, r\cdot 2^{-j+1})$$

and

$$\int_{r\cdot 2^{-j+1}}^{2r} \psi(f, t)\,\frac{dt}{t} = \sum_{k=0}^{j-1}\int_{r\cdot 2^{-k}}^{r\cdot 2^{-k+1}} (f, t)\,\frac{dt}{t} \leq$$

$$\leq A\cdot 2^\alpha \sum_{k=0}^{j-1}\int_{r\cdot 2^{-k}}^{r\cdot 2^{-k+1}} \psi(f, r\cdot 2^{-k})\,\frac{dt}{t} = A\cdot 2^\alpha \log 2 \sum_{k=0}^{j-1}\psi(f, r\cdot 2^{-k})$$

where, by the remark, A, α and β depend only on the dimension n.

2. The spaces $\mathcal{L}_\varphi$.

Let φ be a strictly positive, non-decreasing function, defined on some interval $0 < t < \delta$. Moreover, we suppose that $\varphi(t)/t^\alpha$ is almost decreasing for some α. According to our earlier remarks, this is no essential restriction.

DEFINITION: A locally integrable function f belongs to the class $\mathcal{L}_\varphi(Q)$ if

$$\sup_{I(x,r)\subset Q} \Omega f(x,r)/\varphi(r) < \infty \tag{2.1}$$

and we define

$$|f|_\varphi = |f;\ Q, \varphi| = \sup_{I(x,r)\subset Q} \Omega f(x,r)/\varphi(r). \tag{2.2}$$

Then $\|f\|_\varphi = |f|_\varphi + \left|\int_I f(y)\,dy\right|$ is a norm on $\mathcal{L}_\varphi$ modulo nullfunctions, and it is not difficult to show that with this norm, $\mathcal{L}_\varphi$ is a Banach space.

Above we have supposed that φ is defined at least for $0 < t \leq r_0$, but $\mathcal{L}_\varphi$ depends, up to equivalent norms, only on the values of φ in an arbitrary right neighbourhood of the origin. It is evident that if $\varphi(t) \leq C\psi(t)$ in some neighbourhood of 0, then $\mathcal{L}_\varphi$ is contained in $\mathcal{L}_\psi$. We shall later prove, in theorem 3, that if $\varphi(t)/t$ is non-increasing, then the converse is also true.

3. Formulation of the main results.

THEOREM 1: There are constants B_n, C_n, depending only on the dimension n, such that

(a) If f belongs to $\mathcal{L}_\varphi(Q)$, then

$$f^*(x, r;\ s^n) \leq C_n\, \Phi_{2r}(s)\, |f|_\Phi \text{ for all } s > 0 \text{ and } I(x,r) \subset Q. \tag{3.1}$$

(b) If there is a constant k such that

$$f^*(x, r;\ s^n) \leq k\, \Phi_{2r}(s) \text{ for all } s \geq 0 \text{ and } I(x,s) \subset Q \tag{3.2}$$

then f belongs to $\mathcal{L}_\varphi(Q)$ and $|f|_\varphi \leq B_n k$.

This theorem is in some sense the best possible as the following theorem shows.

THEOREM 2: (a) If $\varphi(t)/t$ is almost decreasing, then $f(x) = \Phi_r(|x_1|)$ belongs to $\mathcal{L}_\varphi(Q)$ for each cube Q in R^n. (b) If $\varphi(t)/t$ is non-increasing and $f(x) = \Phi_r(|x_1|)$, then

$$\frac{1}{c}\varphi(t) \leq \varphi(f, t) \leq c\,\varphi(t) \tag{3.3}$$

in some neighbourhood of the origin for each cube Q containing the origin.

THEOREM 3: Suppose that $\varphi_1(t)/t$ is non-increasing. Then the inclusion $\mathcal{L}_{\varphi_1} \subset \mathcal{L}_{\varphi_2}$ is valid if and only if there are constants C and δ such that

$$\varphi_1(r) \leq C\varphi_2(r), \quad 0 < r < \delta. \tag{3.4}$$

All such inclusions are continuous.

Theorem 1 has the following

COROLLARY 1: If $\int_0^\delta \varphi(t)\frac{dt}{t} < \infty$, then every function f in $\mathcal{L}_\varphi$ is continuous (modulo the nullfunctions) and its modulus of continuity satisfies the inequality

$$\omega(f, r) \leq c\left(\int_0^r \varphi(t)\frac{dt}{t}\right)|f|_\varphi \tag{3.5}$$

REMARK: Corollary 1 may be proved without supposing that $\varphi(t)$ is non-decreasing and without using the lemma 4. The result is that for every function such that

$$\int_0^\delta \psi(f, t)\frac{dt}{t} < \infty, \text{ we have}$$

$$\omega(f, r) \leq C_n\int_0^r \psi(f, t)\frac{dt}{t} \tag{3.6}$$

Theorem 2 implies immediately

COROLLARY 2: If $\varphi(t)/t$ is almost decreasing and $\int_0^\delta \varphi(t)\frac{dt}{t} = +\infty$, then there exists a function in $\mathcal{L}_\varphi$ that is neither essentially bounded nor continuous

EXEMPLE: (1) Taking $\varphi(t) = 1$, we get the result of John and Nirenberg that is used in the proof.

(2) Let $\varphi(t) = t^\alpha$, $0 < \alpha \leq 1$. Then $\int_0^r \varphi(t) \frac{dt}{t} = \frac{r^\alpha}{\alpha}$, hence $\omega(f, r) \leq$ $\leq c\,|f|_\varphi \cdot r$ by corollary 1. This proves that $\mathcal{L}_{(t^\alpha)}$ is included in Lip_α, and the opposite inclusion is obvious, since $\varphi(f, r) \leq \omega(f, r)$. This gives the result of Meyers and Campanato.

(3) Let $\varphi(t) = (1-\alpha)\left(\log \frac{1}{t}\right)^{-\alpha}$, $0 < \alpha < 1$.

Then $\Phi_1(t) = \left(\log \frac{1}{t}\right)^{1-\alpha}$ and hence

$$f^*(x, r; t^n) \leq C_n \left(\log \frac{1}{t}\right)^{1-\alpha} |f|_\varphi \quad \text{if } f \in \mathcal{L}_\varphi(Q) \tag{3.7}$$

where Q is a small cube. (Actually, φ does not satisy our conditions in the interval $0 < t < 1$, but this does not matter, since only the behaviour of φ near 0 is important. Our choice of interval simplifies the notations). As f^* and $|f - f(I)|$ are equimeasurable, we get

$$\int_I F(|f(y) - f(I)|)\, dy = \int_0^{r^n} F(f^*(s))\, ds \tag{3.8}$$

for $I = I(x, r) \subset Q$ and F a measurable function.

Choose $F(u) = \exp\left(c\,|u|^{\frac{1}{1-\alpha}}\right)$. Then

$$F(f^*(s)) \leq \exp\left(c\left(C_n\left(\log \frac{1}{s^{\frac{1}{n}}}\right)^{1-\alpha} |f|_\varphi\right)^{\frac{1}{1-\alpha}}\right) = s^{-\frac{1}{n} c\,(C_n |f|_\varphi)^{\frac{1}{1-\alpha}}}$$

which is integrable if $|f|_\varphi < \frac{1}{C_n}\left(\frac{n}{c}\right)^{1-\alpha}$. This shows, since Q is a bounded cube, that f belongs to the Orlicz space defined by the function $M(u) =$ $= \exp\left(|u|^{\frac{1}{1-\alpha}}\right) - 1$.

4. Proof of the theorems:

PROOF OF THEOREM 1 (a)

Let $I = I(x_0, r_0)$ and $f \in \mathcal{L}_\varphi(I)$. We divide I into 2^{jn} congruent subcubes $I_{jk} = I(x_{jk}, r_0 \cdot 2^{-j})$ by repeated halving all edges and put $r_j = r_0 \cdot 2^{-j}$.

Then

$$(4.1)\qquad m_0(\sigma) = m(f; x_0, r_0; \sigma) = \sum_{k=1}^{2^{jn}} \text{meas}\,\{y \mid y \in I_{jk}, |f(y) - f(x_0, r_0)| > \sigma\}$$

According to lemma 3 have

$$|f(x_{jk}, r_j) - f(x_0, r_0)| \leq 2^n \sum_{k=0}^{j-1} \varphi(r_0 \cdot 2^{-k}) |f|_\varphi \text{. Choosing}$$

$$\sigma = (2^n + na_n) \sum_{k=0}^{j-1} \varphi(r_0 \cdot 2^{-k}) |f|_\varphi \text{ with } a_n \text{ as in lemma 4. Then}$$

$$\sigma - 2^n \sum_{k=0}^{j-1} \varphi(r_0 \cdot 2^{-k}) |f|_\varphi \geq na_n \sum_{k=0}^{j-1} \varphi(r_0 \cdot 2^{-k}) |f|_\varphi \geq$$

$\geq na_n j \varphi(r_j) |f|_\varphi$, as φ is increasing. (This is the point where this condition on φ is essential). Hence,

$$\text{meas}\,\{y \mid y \in I_{jk}, |f(y) - f(x_0, r_0)| > \sigma\} \leq$$

$$\leq \text{meas}\,\{y \mid y \in I_{jk}, |f(y) - f(x_{jk}, r_j)| > a_n n j \varphi(r_j)\} \leq$$

$\leq 2^{-nj} r_j^n$ by lemma 4 and thus

$$(4.2)\qquad m_0(\sigma) \leq \sum_{k=1}^{2^{jn}} 2^{-nj} r_j^n = r_j^n = (2^{-j} r_0)^n.$$

$$\text{But } \sigma = (2^n + na_n) \sum_{k=0}^{j-1} \varphi(r_0 \cdot 2^{-k}) |f|_\varphi \leq$$

$$\leq (2^n + na_n) c_n \Phi_{2r_0}(r_0 \cdot 2^{-j+1}) |f|_\varphi \text{ by lemma 5.}$$

Put $d_n = (2^n + na_n) c_n$, and suppose that $r_0 \cdot 2^{-j} \leq r \leq r_0 \cdot 2^{-j+1}$, $j \geq 1$. Then we have $m_0(d_n \Phi_{2r_0}(r)) \leq m_0(d_n \Phi_{2r_0}(r_0 \cdot 2^{-j+1})) \leq m_0(\sigma) \leq (2^{-j} r_0)^n \leq r^n$, as m_0 and Φ_{2r_0} are nonincreasing functions. The result is that

$$(4.3)\qquad m(f; x_0, r_0; \sigma) \leq \left(\Phi_{2r_0}^{-1}\left(\frac{\sigma}{d_n |f|_\varphi}\right)\right)^n$$

$$\text{if } \sigma > d_n \Phi_{2r_0}(r_0) |f|_\varphi$$

and, equivalently

$$(4.4)\qquad f^*(x_0, r_0; r^n) \leq d_n \Phi_{2r_0}(r) |f|_\varphi, \quad r \leq r_0$$

which gives the theorem (*a*) with $C_n = d_n$, since $f^*(x_0, r_0; r^n) = 0$ for $r > r_0$.

PROOF OF THEOREM 1 (b)

Suppose that $f^*(x, r; s^n) \leq k\, \Phi_{2r}(s)$, $s > 0$.
Then

$$r^{-n} \int_{I(x,r)} |f(y) - f(x, r)|\, dy = r^{-n} \int_0^{r^n} f^*(x, r; t)\, dt \leq$$

$$= k r^{-n} \int_0^r \Phi_{2r}(t)\, d(t^n) \leq k r^{-n} \int_0^r \int_t^{2r} \varphi(u) \frac{du}{u}\, d(t^n) \leq$$

$$\leq k r^{-n} \int_0^{2r} \left(\int_t^{2r} \frac{\varphi(u)}{u} du \right) d(t^n) = k r^{-n} \int_0^{2r} \frac{\varphi(u)}{u} \left(\int_0^u d(t^n) \right) du$$

$$= k r^{-n} \int_0^{2r} u^{n-1} \varphi(u)\, du \leq k \cdot r^{-n} \varphi(2r) \int_0^{2r} u^{n-1}\, du$$

as φ is nondecreasing. Hence

$$\Omega f(x, r) \leq k \cdot \frac{2^n}{n} \varphi(2r) \leq A \cdot k \cdot \frac{2^n}{n} \cdot 2^\alpha \varphi(r) \tag{4.5}$$

as $\varphi(t)/t^\alpha$ is almost decreasing, and the conditions in the theorem imply that f belongs to $\mathcal{L}_\varphi$ and $|f|_\varphi \leq \frac{1}{n} 2^{\alpha+n} \cdot k$.

PROOF OF THEOREM 2

(a) Suppose that $\varphi(t)/t$ is an almost decreasing function.
We put

$$\psi(t) = \begin{matrix} \varphi(t) & 0 < t < r_0 \\ 0 & t \geq r_0 \end{matrix}$$

and

$$g(x) = \int_{|x|}^{r_0} \psi(t) \frac{dt}{t}.$$

Then we have the inequality

$$\inf_{\sigma} \frac{1}{h} \int_{x}^{x+h} | g(y) - \sigma | \, dy \leq \varphi(h), \; x > 0, \; 0 < h < r_0$$

since

$$\frac{1}{h} \int_{x}^{x+h} | g(y) - g(x+h) | \, dy = \frac{1}{h} \int_{x}^{x+h} \int_{y}^{x+h} \psi(t) \frac{dt}{t} \, dy =$$

$$= \frac{1}{h} \int_{x}^{x+h} \frac{\psi(t)}{t} \left(\int_{x}^{t} dx \right) dt \leq \frac{A}{h} \int_{x}^{x+h} \frac{\psi(t-x)}{t-x} \cdot (t-x) \, dt$$

$$= \frac{A}{h} \int_{x}^{x+h} \psi(t-x) \, dt = \frac{A}{h} \int_{0}^{h} \psi(t) \, dt = \frac{A}{h} \int_{0}^{h} \varphi(t) \, dt \leq A\varphi(h).$$

If $x < -h$, the inequality is still valid, and if $-h \leq x \leq 0$ we have

$$\int_{x}^{x+h} | g(y) - \sigma | \, dy = \int_{0}^{|x|} + \int_{0}^{|h-x|} | g(y) - \sigma | \, dy \leq 2 \int_{0}^{h} | g(y) - \sigma | \, dy.$$

Hence

$$\inf_{\sigma} \frac{1}{h} \int_{x-\frac{h}{2}}^{x+\frac{k}{2}} | g(y) - \sigma | \, dy \leq 2A\varphi(h), \; | h | \leq r_0. \tag{4.6}$$

Now define

$f(y) = f(y_1, \dots, y_n) = g(y_1)$. Then

$$\Omega f(x, r) \leq 2 \inf_{\sigma} r^{-n} \int_{I(x,r)} | f(y) - \sigma | \, dy \text{ by lemma 1}$$

$$\leq 2 \inf_{\sigma} r^{-n} \int_{I(x,r)} | g(y_1) - \sigma | \, dy_1 \dots dy_n =$$

$$= 2 \inf_{\sigma} \int_{x_1 - \frac{r}{2}}^{x_1 + \frac{r}{2}} | g(y_1) - \sigma | \, dy_1 \leq 2.2A\varphi(r), \; r \leq r.$$

We have proved that $f \in \mathcal{L}_{\varphi}$ and $| f |_{\varphi} \leq 4A$.

(b) Suppose that $\varphi(t)/t$ is non-increasing. We start by the case $n = 1$. Then $\Omega g(x, r) = r^{-1} \int_{x-\frac{r}{2}}^{x+\frac{r}{2}} |g(y) - g(x, r)| \, dy$ and hence

$$\Omega g(x, r) \geq r^{-1} \int_{x+\frac{r}{4}}^{x+r/2} |g(y) - g(x, r)| \, dy \text{ if } x > r/2. \tag{4.7}$$

As $g'(x) = \frac{-\psi(x)}{x}$ is non-decreasing for $x > 0$, we have

$$g(y) = g(x) + \int_x^y g'(t) \, dt \geq g(x) + (y - x) g'(x), \; y > 0. \tag{4.8}$$

Hence

$$g(x) = \frac{1}{r} \int_{x-\frac{r}{2}}^{x+\frac{r}{2}} g(x) \, dy \leq \frac{1}{r} \int_{x-\frac{r}{2}}^{x+\frac{r}{2}} (g(y) - (y - x) g'(x)) \, dy = \tag{4.9}$$

$$= \frac{1}{r} \int_{x-\frac{r}{2}}^{x+\frac{r}{2}} g(y) \, dy = g(x, r).$$

Moreover, if $x + \frac{r}{4} \leq y \leq x + \frac{r}{2}$, $g(y) \geq g\left(x + \frac{r}{4}\right) \geq g(x)$ and we obtain

$$g(x, r) \geq \frac{1}{r} \cdot \frac{r}{4} \left| g\left(x + \frac{r}{4}\right) - g(x) \right| = \frac{1}{4} \int_x^{x+\frac{r}{4}} \psi(t) \frac{dt}{t} \geq$$

$$\geq \frac{1}{4} \frac{\psi\left(x + \frac{r}{4}\right)}{x + \frac{r}{4}} \cdot \frac{r}{4}, \text{ and if we choose } x = \frac{3r}{4},$$

the result is

$$\Omega g\left(\frac{3r}{4}, r\right) \geq \frac{1}{16} \psi(r) = \frac{1}{16} \varphi(r), r \leq r_0 .$$

Hence

$$\varphi(g, r) \geq \psi(g, r) \geq \frac{1}{16} \varphi(r), r \leq r_0 . \tag{4.10}$$

The extension to n dimensions is obvious.

PROOF OF COROLLARY 1:

Obviously, $\operatorname{ess\,sup}_{y \in I} |f(y) - f(I)| = \sup_s f^*(x, r; s)$, as the functions f^* and $|f - f(I)|$ are equimeasurable. But $\sup_s f^*(x, r; s) = \lim_{s \to 0} f^*(x, r; s) \leq$

$$C_n \lim_{s \to 0} \Phi_{2r}(s) |f|_\varphi = C_n \int_0^{2r} \varphi(t) \frac{dt}{t} f = C_n \int_0^r \varphi(2t) \frac{dt}{t} |f|_\varphi \leq C_n A . 2^a \int_0^r \varphi(t) \frac{dt}{t}$$

if the integral converges, and

$$\omega(f, r) = \operatorname{ess\,sup}_{|x-y| \leq r} |f(x) - f(y)| \leq 2 \sup_I \sup_{y \in I} |f(y) - f(I)|$$

where the first supremum is taken over cubes I of edgelength $\leq r$.

PROOF OF THEOREM 3:

Trivially, the inequality implies the inclusion. Conversely we may suppose that Q contains the origin. If $\mathcal{L}_{\varphi_1} \subset \mathcal{L}_{\varphi_2}$ and $\varphi_1(t)/t$ is non-increasing, then by theorem 2 (b), $\Phi_{1, r_0} \in \mathcal{L}_{\varphi_1} \subset \mathcal{L}_{\varphi_2}$, i. e., there is a $\delta > 0$ such that

$$\varphi(\Phi_{1, r_0}, r) \leq |\Phi_{1, r_0}|_{\varphi_2} \cdot \varphi_2(r), \text{ if } r, r_0 < \delta.$$

But $\varphi_1(r) \leq c\varphi(\Phi_{1, r_0}, r)$, still by theorem 2. Hence

$$\varphi_1(r) \leq c\, |\Phi_{1, r_0}|_{\varphi_2} \cdot \varphi_2(r) \text{ if } r \text{ is small enough.}$$

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