HANS LEWY

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ON THE DEFINITENESS OF QUADRATIC FORMS
WHICH OBEY CONDITIONS OF SYMMETRY

by HANS LEWY

According to classical Algebra a real quadratic form is positive defi-
nite whenever certain determinants of its coefficient matrix are positive.
This test is in general non-linear in the coefficients and its application is
made cumbersome by the difficulty of evaluating determinants. Any hope
of simplification must depend therefore on special properties of the coeffi-
cient matrix. In the course of work on differential equations the A. was
led to consider quadratic forms of the following type.

Denote by V_k or V_k^n the set of arbitrary combinations of k distinct
elements of the set $\{1, 2, \dots, n\}$, counting as identical two combinations which
differ only by the arrangement of their elements. Let $u(\omega)$, $\omega \in V_k$ be a
real function of ω. We consider a real quadratic form $Q[u]$ subject to the
only condition that it remain invariant when the numbers of $\{1, 2, \dots, n\}$
undergo an arbitrary permutation π, i. e. if $u(\omega)$ is replaced by $u(\pi\omega)$.
Then the form $Q[u]$ is positive definite if and only if certain (explicit)
inequalities hold which are linear in the coefficients; in fact it suffices to
establish that $Q[u] > 0$ for certain finitely many choices of $u(\omega)$ which do
not depend on the coefficients of Q.

NOTATION: In the sequel k, n are fixed positive integers and $n \geq k$;
ω, ω_0 are elements of V_k^n; $\eta \supset \varepsilon$, $\eta \in V_r^n$, $\varepsilon \in V_s^n$ if the set η contains the set
ε; $|\eta| = r$; $\eta \cap \varepsilon$ is that combination which consists of the elements of
$\{1, \dots, n\}$ in common to η and ε.

In writing $\underset{\eta \supset \varepsilon}{\Sigma}$ or $\underset{\eta \subset \varepsilon}{\Sigma}$ we always understand summation over the first
combination mentioned in the subscript, in this case η, the second, ε, being
fixed, we also make this explicit by writing $\underset{\eta,\, \eta \supset \varepsilon}{\Sigma}$ etc... $\underset{\omega}{\Sigma}$ is an abbreviation

for $\underset{\omega,\ \omega\epsilon V_k}{\Sigma}$; $\underset{\omega\supset\eta}{\Sigma}$ an abbreviation for $\underset{\omega,\ \omega\epsilon V_k,\ \omega\supset\eta}{\Sigma}$ where η is fixed; $\underset{\omega\supset\eta\epsilon V_s}{\Sigma}$ means 0 if $s > k$, and means $\underset{\omega}{\Sigma}$ if $s = 0$.

1. **LEMMA 1.1.** *Suppose* $1 < k, 2k - 1 \leq n$. *Given real* $v(\eta)$, $\eta \in V_{k-1}$, *the equations*

$$\underset{\omega\supset\eta}{\Sigma} u(\omega) = v(\eta), \ \eta \in V_{k-1}$$

have a real solution $u(\omega)$, $\omega \in V_k$.

PROOF. If the Lemma were false there would exist real non-trivial $\lambda(\eta)$, $\eta \in V_{k-1}$ such that identically in $u(\omega)$

$$\underset{\eta\epsilon V_{k-1}}{\Sigma} \lambda(\eta) \underset{\omega\supset\eta}{\Sigma} u(\omega) = 0.$$

Choosing an arbitrary $\omega_0 \in V_k$ put $u(\omega_0) = 1$, $u(\omega) = 0$ for $\omega \neq \omega_0$ and find

(1.1)
$$\underset{\eta\epsilon V_{k-1},\ \eta\subset\omega_0}{\Sigma} \lambda(\eta) = 0, \ \omega_0 \in V_k.$$

We shall prove that (1.1) implies that all $\lambda(\eta)$ vanish; whence the Lemma.

Take for ω_0 all elements of V_k contained in $\{1, 2, \dots, k, k+1\}$ with $\omega_0 \supset \{1\}$, and add the equations (1.1) corresponding :

$$0 = \underset{\omega_0,\ 1\subset\omega_0\subset\{1,\dots,k+1\}}{\Sigma}\ \underset{\eta\subset\omega_0,\ \eta\epsilon V_{k-1}}{\Sigma} \lambda(\eta) = p \underset{\eta\epsilon V_{k-1},\ 1\subset\eta\subset\{1,2,\dots,k+1\}}{\Sigma} \lambda(\eta)$$

$$+ p' \underset{\eta\epsilon V_{k-1},\ 1\subset\eta\subset\{1,2,\dots,k+1\}}{\Sigma} \lambda(\eta)$$

with $p > 0$. The second sum on the right is

$$\underset{\eta\epsilon V_{k-1},\ \eta\subset\{2,\dots,k+1\}}{\Sigma} \lambda(\eta) = 0$$

by (1.1). Hence

$$\underset{\eta\epsilon V_{k-1},\ 1\subset\eta\subset\{1,2,\dots,k+1\}}{\Sigma} \lambda(\eta) = 0$$

or generally for $\{j_1, \dots, j_{k+1}\} \in V_{k+1}$

(1.2)
$$\underset{\eta\epsilon V_{k-1},\ j_1\subset\eta\subset\{j_1,\dots,j_{k+1}\}}{\Sigma} \lambda(\eta) = 0.$$

Suppose we have proved for $r = r_0 - 1$, $r_0 \leq k - 1$ that given $\varepsilon \in V_r$, $\delta \in V_{r+k}$, $\delta \supset \varepsilon$

$$(1.3) \qquad \sum_{\eta \in V_{k-1}, \, \varepsilon \, \subset \, \eta \, \subset \, \delta} \lambda(\eta) = 0,$$

which coincides with (1.2) for $r = 1$. We shall prove the same relation with r_0 instead of r. Take $\varepsilon = \{1, 2, \ldots, r\}$, $\delta' = \{1, 2, \ldots, r + k + 1\} \in V_{r+k}$ and obtain from the last relation

$$\sum_{\delta \in V_{r+k}, \, \varepsilon \, \subset \, \delta \, \subset \, \delta'} \quad \sum_{\eta \in V_{k-1}, \, \varepsilon \, \subset \, \eta \, \subset \, \delta} \lambda(\eta) = 0$$

Here left hand can be split thus:

$$p \sum_{\{1, 2, \ldots, r+1\} \, \subset \, \eta \, \in \, V_{k-1}, \, \eta \, \subset \, \delta'} \lambda(\eta) + p' \sum_{\{1, 2, \ldots r\} \, \subset \, \eta \, \in \, V_{k-1}, \, \eta \, \subset \, \delta', \, \eta \, \supset \, r+1} \lambda(\eta) = 0$$

with p a positive integer, while the factor of p'

$$\sum_{\{1, \ldots, r\} \, \subset \, \eta \, \in \, V_{k-1}, \, \eta \, \subset \, \{1, \ldots, r, r+2, \ldots, r+k+1\}} \lambda(\eta) = 0$$

by (1.3). Hence

$$\sum_{\{1, 2, \ldots, r+1\} \, \subset \, \eta \, \in \, V_{k-1}, \, \eta \, \subset \, \{1, \ldots, r+1+k\}} \lambda(\eta) = 0,$$

or, more generally, relation (1.3) with r replaced by $r + 1 = r_0$. Note that we made use of the inequality $r_0 + k \leq n$ in granting the existence of $\delta' \in V_{r_0+k}$. Writing (1.3) for $r_0 = k - 1$ we obtain with $\varepsilon \in V_{k-1}$, $\delta \in V_{2k-1}$, $\delta \supset \varepsilon$

$$0 = \sum_{\eta \in V_{k-1}, \, \varepsilon \, \subset \, \eta \, \subset \, \delta} \lambda(\eta) = \lambda(\varepsilon), \quad \varepsilon \in V_{k-1}$$

which proves Lemma 1.1.

Note that there is an obvious interpretation of the statement of Lemma 1.1 which makes it correct even for $k = 1$

LEMMA 1.2. Suppose $1 \leq r < k$, $2k - 1 \leq n$. For any given real $v(\eta)$, $\eta \in V_r$ the equations

$$\sum_{\omega \supset \eta} u(\omega) = v(\eta), \quad \eta \in V_r$$

have a real solution $u(\omega)$, $\omega \in V_k$.

PROOF. By Lemma 1.1 the assertion holds for $r = k - 1$. Suppose it proved for $k - 1 \geq r \geq r_0 > 1$ we prove it for $r = r_0 - 1$. The previous Lemma with k replaced by r_0 shows that for given real $v(\eta)$, $\eta \in V_{r_0-1}$ there are real $v'(\eta')$, $\eta' \in V_{r_0}$ such that

$$\sum_{\eta', \, V_{r_0} \ni \eta' \supset \eta} v'(\eta') = v(\eta), \ \eta \in V_{r_0-1}.$$

We now solve by induction

$$\sum_{\omega \supset \eta'} u(\omega) = v'(\eta'), \ \eta' \in V_{r_0}.$$

Then for these $u(\omega)$

$$\sum_{\eta', \, V_{r_0} \ni \eta' \supset \eta} \sum_{\omega \supset \eta'} u(\omega) = v(\eta), \ \eta \in V_{r_0-1}.$$

But left hand amounts to $p \sum_{\omega \supset \eta} u(\omega)$ where $p > 0$. The Lemma follows.

LEMMA 1.3. *Let* $2k \leq n$. *There are real* $u(\omega)$, $\omega \in V_k$ *such that*

$$\sum_{\omega \supset \eta} u(\omega) = 0, \ \eta \in V_r,$$

with fixed r *in* $0 \leq r < k$, *but for at least one* K *in* V_{r+1}

$$\sum_{\omega \supset K} u(\omega) \neq 0.$$

PROOF. Solve non-trivially $\sum_{K, \, V_{r+1} \ni K \supset \eta} v(K) = 0, \eta \in V_r$; and $\sum_{\omega \supset K} u(\omega) = v(K)$ according to Lemma 1.2.

The first is possible since $\binom{n}{r+1}$ is the number of unknowns and the number of equations is $\binom{n}{r}$ which, for $0 \leq r < k \leq \dfrac{n}{2}$, is $< \binom{n}{r+1}$. Then at least one $v(K) \neq 0$, but

$$p \sum_{\omega \supset \eta} u(\omega) \equiv \sum_{K \supset \eta} \sum_{\omega \supset K} u(\omega) = \sum_{K \supset \eta} v(K) = 0, \ \eta \in V_r$$

where p is a positive integer.

2. Let $Q[u]$ be a quadratic form in real $u(\omega)$, $\omega \in V_k$, such that Q remains invariant if the $u(\omega)$, $\omega \in V_k$ are replaced by $u(\pi \omega)$ where π is an arbitrary permutation of $(1, 2, \dots, n)$. Then we can write, with $|\omega_i \cap \omega_j| =$ $=$ number of digits in common to ω_i and ω_j,

$$Q[u] = \Sigma \, c_{ij} \, u(\omega_i) \, u(\omega_j)$$

$$= \sum_{r=0}^{k} \sum_{|\omega_i \cap \omega_j| = r} c_{ij} \, u(\omega_i) \, u(\omega_j)$$

Applying to $Q[u]$ all permutations π we obtain

(2.1)
$$Q[u] = \sum_{r=0}^{k} c_r \sum_{|\omega_i \cap \omega_j| = r} u(\omega_i) \, u(\omega_j)$$

where we intend, interpreting the $\Sigma \, u(\omega_i) \, u(\omega_j)$ for $\omega_i \neq \omega_j$, that the sum should contain both terms $u(\omega_i) \, u(\omega_j)$ and $u(\omega_j) \, u(\omega_i)$.

It is possible to write, for $2k \leq n$,

(2.2)
$$Q[u] = \sum_{r=0}^{k} d_r \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} u(\omega) \right)^2.$$

In fact we find that (2.2) is of form (2.1) with

$$c_k = d_0 + \binom{k}{1} d_1 + \binom{k}{2} d_2 + \dots + d_k$$

$$c_{k-1} = d_0 + \binom{k-1}{1} d_1 + \binom{k-1}{2} d_2 + \dots + d_{k-1}$$

. . . .

(2.3)
$$c_1 = d_0 + d_1$$

$$c_0 = d_0$$

whose resolution with respect to d_j evidently exists; it is precisely

(2.4) $$d_r = c_r - \binom{r}{1} c_{r-1} + \binom{r}{2} c_{r-2} - + \dots + (-)^r c_0, \; 0 \leq r \leq k$$

It suffices to note that relations (2.3) yield if $d_\nu = \alpha^\nu$, the $\nu - th$ power of an indeterminate α, that $c_\nu = (1 + \alpha)^\nu = \beta^\nu$; in this case $\alpha^r = (\beta - 1)^r$ which then coincides with (2.4); but the powers of β are linearly indepen-

dent, hence the solution in this special case must have the same coefficients as in the general case.

THEOREM 2.1. *If* $n \geq 2k$, *the quadratic form*

$$Q[u] = \sum_{r=0}^{k} d_r \sum_{\eta \, \epsilon \, V_r} \left(\sum_{\omega \, \supset \, \eta} u(\omega) \right)^2,$$

in real $u(\omega)$, $\omega \, \epsilon \, V_k$ *is positive definite if and only if the linear inequalities hold*

(2.5) $$\sum_{r=\sigma}^{k} d_r \binom{k-\sigma}{r-\sigma} \binom{n-r-\sigma}{k-r} > 0, \quad \sigma = 0, \quad 1, \ldots, k.$$

PROOF. If all d_r are positive, the statement is trivial. We may therefore assume that for some r, $d_r < 0$. Note, however, that $d_k > 0$ is necessary for positive definiteness of Q. For (by Lemma 1.3) we can choose real $u(\omega)$, not all zero, such that

$$\sum_{\omega \, \supset \, \eta} u(\omega) = 0, \quad \eta \, \epsilon \, V_{k-1}.$$

Then $\sum_{\omega \, \supset \, \eta} u(\omega) = 0$, $\eta \, \epsilon \, V_r$, $0 \leq r < k-1$ follows; $Q[u] = d_k \sum_{\omega} u^2(\omega) > 0$, whence $d_k > 0$, which is the inequality (2.5) for $\sigma = k$. Set

$$d_r^- = \begin{cases} - d_r, & \text{if} \quad d_r < 0, \\ 0, & \text{if} \quad d_r \geq 0, \end{cases}$$

$$d_r^+ = \begin{cases} d_r, & \text{if} \quad d_r \geq 0, \\ 0, & \text{if} \quad d_r < 0. \end{cases}$$

The proof of the Theorem involves maximizing the quotient

(2.6) $$\lambda = \frac{\sum_{r=0}^{k-1} d_r^- \sum_{\eta \, \epsilon \, V_r} \left(\sum_{\omega \, \supset \, \eta} u(\omega) \right)^2}{\sum_{r=0}^{k} d_r^+ \sum_{\eta \, \epsilon \, V_r} \left(\sum_{\omega \, \supset \, \eta} u(\omega) \right)^2}.$$

The existence of a maximum is certain in view of $d_k > 0$; for it suffices to restrict the competition to the compact set of those $u(\omega)$ which

make the denominator equal to 1. Now if $\lambda = \lambda_{\max} \geq 1$ then $Q[u] =$

$$= (1 - \lambda_{\max}) \sum_{r=0}^{k} d_r^{+} \sum_{\eta \,\epsilon\, V_r} \left(\sum_{\omega \supset \eta} u(\omega) \right)^2 \leq 0 \quad \text{and} \quad Q[u] \text{ is not positive definite.}$$

On the other hand if $Q[u]$ can assume non-positive values, the quotient λ can become 1 or ≥ 1, whence certainly $\lambda_{\max} \geq 1$. Hence $\lambda_{\max} < 1$ implies, and is implied by, the positive definiteness of $Q[u]$ (notation $Q \gg 0$).

We shall now indicate $k + 1$ ways of choosing non-trivial $u_i(\omega)$, $i = 0, 1, \ldots, k$ such that the corresponding inequalities $Q[u_i(\omega)] > 0$ are our necessary conditions for $Q \gg 0$. These choices will afterwards be shown to certainly contain one for which $\lambda = \lambda_{\max}$ and for which then $\lambda_{\max} < 1$.

First choice is all $u(\omega) = u_0(\omega) = 1$. Then $\sum_{\omega \supset \eta \,\epsilon\, V_r} u_0(\omega) = \binom{n-r}{k-r}$ and

$$0 < Q[u_0] = \sum_{r=0}^{k} d_r \binom{n}{r} \binom{n-r}{k-r}^2 = \binom{n}{k} \sum_{r=0}^{k} d_r \binom{k}{n} \binom{n-r}{k-r}$$

or

$$0 < \sum_{r=0}^{k} d_r \binom{k}{r} \binom{n-r}{k-r}$$

and

$$\lambda_0 = \sum_{r} d_r^{-} \binom{k}{r} \binom{n-r}{k-r} \Big/ \sum_{r} d_r^{+} \binom{k}{r} \binom{n-r}{k-r} < 1$$

The next choice, $u_1(\omega)$, is made as follows: we single out the numbers 1 and 2 and set

$$u_1(\omega) = 0, \quad \text{if} \quad \omega \supset (1 \cup 2) \quad \text{or if} \quad \omega \cap (1 \cup 2) = \varnothing.$$

We put

$$u_1(\omega) = \begin{cases} 1, & \text{if} \quad \omega \supset (2), \quad \omega \not\supset (1 \cup 2) \\ -1, & \text{if} \quad \omega \supset (1), \quad \omega \not\supset (1 \cup 2). \end{cases}$$

Then we easily verify

$$\sum_{\omega \supset \eta \,\epsilon\, V_r} u_1(\omega) = \begin{cases} 0, & \text{if} \quad \eta \supset (1 \cup 2) \quad \text{or} \quad \eta \cap (1 \cup 2) = \varnothing \\ \binom{n-1-r}{k-r}, & \text{if} \quad \eta \supset (2), \quad \text{but} \quad \eta \not\supset (1 \cup 2) \\ -\binom{n-1-r}{k-r}, & \text{if} \quad \eta \supset (1), \quad \text{but} \quad \eta \not\supset (1 \cup 2). \end{cases}$$

Accordingly

$$0 < Q[u_1] = 2 \sum_{r=1}^{k} d_r \binom{n-2}{r-1}\binom{n-1-r}{k-r}^2 =$$

$$= 2\binom{n-2}{k-1} \sum_{1}^{k} d_r \binom{k-1}{r-1}\binom{n-1-r}{k-r}$$

or

$$0 < \sum_{r=1}^{k} d_r \binom{k-1}{r-1}\binom{n-1-r}{k-r}$$

and

$$\lambda_1 \doteq \sum_{r=1}^{k} d_r^- \binom{k-1}{r-1}\binom{n-1-r}{k-r} \Big/ \sum_{r=1}^{k} d^+ \binom{k-1}{r-1}\binom{n-1-r}{k-1} < 1.$$

We choose $u_i(\omega)$ for $1 \leq i \leq \dfrac{n}{2}$ as follows. We separate the numbers $1, \ldots, 2i$ and define

$$u_i(\omega) = 0 \quad \text{if} \quad |\omega \cap \{1, 2, \ldots, 2i\}| > i \quad \text{or} \quad |\omega \cap \{1, 2, \ldots, 2i\}| < i.$$

Furthermore, $u_i(\omega) = 0$ even though $|\omega \cap \{1, 2, \ldots, 2i\}| = i$ unless ω contains exactly one element, say a_1, of the pair $\{1, 2\}$, one, say a_2, of the pair $\{3, 4\}, \ldots,$ and one, say a_i, of the pair $\{2i-1, 2i\}$. In this case we put

$$u_1(\omega) = (-1)^{a_1+a_2+\ldots a_i}.$$

Now let $\eta \in V_r$, $|\eta \cap \{1, 2, \ldots, 2i\}| \lessgtr i$. Then $\sum\limits_{\omega \supset \eta} u_i(\omega) = 0$, either trivially because η contains more than one element of any pair above, whence $u(\omega) = 0$ for $\omega \supset \eta$; or if η does not contain any element of one such pair $\{a, a'\}$ then each ω with $u_i(\omega) \neq 0$ must contain either a or a' and then the pairing of an $u_i(\omega)$ containing a with one containing a' will give zero sum. There remains the possibility that $r \geq i$ and η contains of each pair exactly one element, say $\eta \supset \{a_1, a_2, \ldots, a_i\}$. Then

$$\sum_{\omega \supset \eta \in V_r} u_i(\omega) = (-1)^{a_1+a_2+\ldots+a_i}\binom{n-i-r}{k-r}$$

and

$$Q[u_i] = \sum d_r \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} u_i(\omega)\right)^2$$

$$= 2^i \sum_{r=i}^{k} d_r \binom{n-2i}{r-i}\binom{n-i-r}{k-r}^2$$

$$= 2^i \binom{n-2i}{k-i} \sum_{r=i}^{k} d_r \binom{k-i}{r-i}\binom{n-i-r}{k-r} > 0,$$

$$(2.7) \qquad \lambda_i = \sum_{r \geq i}^{k} d_r^- \binom{k-i}{r-i} \binom{n-i-r}{k-r} \Big/ \sum_{r \geq i}^{k} d_r^+ \binom{k-i}{r-i} \binom{n-i-r}{k-r} < 1$$

with $i = 0, 1, \dots, k-1$. These inequalities, together with $d_k > 0$, are therefore consequences of $Q \gg 0$.

We now turn the proof of sufficiency. For it we need the derivative of λ in (2.6) with respect to $u(\omega_0)$ where ω_0 is a fixed element of V_k. We have

$$\frac{\partial}{\partial u(\omega_0)} \sum_\eta \left(\sum_{\omega \supset \eta \, \epsilon \, V_r} u(\omega) \right)^2 = 2 \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \, \epsilon \, V_r} u(\omega) ;$$

hence λ is stationary if and only if

$$(2.8) \qquad \lambda \sum d_r^+ \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \, \epsilon \, V_r} u(\omega) = \sum d_r^- \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \, \epsilon \, V_r} u(\omega), \qquad \omega_0 \, \epsilon \, V_k .$$

Summing over all $\omega_0 \, \epsilon \, V_k$ we obtain

$$\lambda \sum_r d_r^+ \sum_{\omega_0 \, \epsilon \, V_k} \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \, \epsilon \, V_r} u(\omega) = \sum_r d_r^- \sum_{\omega_0 \, \epsilon \, V_k} \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \, \epsilon \, V_r} u(\omega).$$

The factor of $\lambda \, d_r^+$ is the special case $\sigma = 0$ of

$$\sum_{\omega_0 \supset \delta_\sigma} \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \, \epsilon \, V_r} u(\omega) \equiv \sum f_r(\omega) \, u(\omega)$$

where σ is an integer $0 \leq \sigma \leq k$, ω_0, ω are in V_k, and δ_σ is an element of V_σ.

Let $|\omega \cap \delta_\sigma| = t$ so that $0 \leq t \leq \sigma$. In ω are contained $\binom{t}{\tau} \binom{k-t}{r-\tau}$ distinct $\eta \, \epsilon \, V_r$ with $|\eta \cap \delta_\sigma| = \tau$ for $0 \leq \tau \leq t$. For each such η, the number of ω_0 containing η and also containing δ_σ is

$$\binom{n-r-\sigma+\tau}{k-r-\sigma+\tau}.$$

Hence

$$(2.9) \qquad f_r(\omega) = \sum_{\tau \geq 0}^{t} \binom{t}{\tau} \binom{k-t}{r-\tau} \binom{n-r-\sigma+\tau}{k-r-\sigma+\tau}.$$

We now formulate the following hypothesis H_σ for $\sigma \geq 1$:

$$(2.10) \qquad \sum_{\omega \supset \eta} u(\omega) = 0, \qquad \eta \, \epsilon \, V_r, \qquad 0 \leq r \leq \sigma - 1,$$

but there is a $\delta \in V_\sigma$, called δ_σ, such that

$$\sum_{\omega \supset \delta_\sigma} u(\omega) \neq 0.$$

Note that (2.10) implies by summation

$$\sum f_s(\omega)\, u(\omega) = \sum_{\omega_0 \supset \delta_\sigma} \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \,\in\, V_s} u(\omega) = 0, \quad \text{for} \quad 0 \leq s \leq \sigma - 1.$$

We have the

PROPOSITION: Under H_σ

$$\sum f_r(\omega)\, u(\omega) = \binom{k-\sigma}{r-\sigma} \binom{n-r-\sigma}{k-r} \sum_{\omega \supset \delta_\sigma} u(\omega).$$

LEMMA 2.1. *As $f_r(\omega)$ depends by (2.9) on $|\omega \cap \delta_\sigma| = t$ rather than on ω, we put $f_r(\omega) = A_t$ and have*

$$A_\sigma - \binom{\sigma}{1} A_{\sigma-1} + \binom{\sigma}{2} A_{\sigma-2} - + \ldots + (-)^\sigma A_0 = \binom{k-\sigma}{r-\sigma} \binom{n-r-\sigma}{k-r}.$$

PROOF. Consider left hand, expressed with the aid of (2.9); for fixed τ, the coefficient of $\binom{n-r-\sigma+\tau}{k-r-\sigma+\tau}$ is

$$\binom{\sigma}{\tau}\binom{k-\sigma}{r-\tau} - \binom{\sigma}{1}\binom{\sigma-1}{\tau}\binom{k-\sigma+1}{r-\tau} + \binom{\sigma}{2}\binom{\sigma-2}{\tau}\binom{k-\sigma+2}{r-\tau} - + \ldots$$

$$= \binom{\sigma}{\tau}\left[\binom{k-\sigma}{r-\tau} - \binom{\sigma-\tau}{1}\binom{k-\sigma+1}{r-\tau} + \binom{\sigma-\tau}{2}\binom{k-\sigma+2}{r-\tau} - + \ldots\right] =$$

$$= (-)^{\sigma-\tau}\binom{\sigma}{\tau}\binom{k-\sigma}{r-\sigma},$$

in view of the binomial relation

$$(2.11) \qquad \binom{a}{b} - \binom{c}{1}\binom{a+1}{b} + \binom{c}{2}\binom{a+2}{b} - + \ldots = (-)^c \binom{a}{b-c}.$$

Consequently

$$A_\sigma - \binom{\sigma}{1} A_{\sigma-1} + \binom{\sigma}{2} A_{\sigma-2} - + \ldots = \binom{k-\sigma}{r-\sigma} \sum_{\tau=0}^{\sigma} (-)^{\sigma-\tau}\binom{\sigma}{\tau}\binom{n-r-\sigma+\tau}{k-r-\sigma+\tau}.$$

Now we have the binomial relation

$$(2.12) \qquad \binom{a}{b} - \binom{c}{1}\binom{a+1}{b+1} + \binom{c}{2}\binom{a+2}{b+2} \cdots = (-)^c \binom{a}{b+c}.$$

Thus

$$A_\sigma - \binom{\sigma}{1}A_{\sigma-1} + \binom{\sigma}{2}A_{\sigma-2} + \ldots = \binom{k-\sigma}{r-\sigma}\binom{n-r-\sigma}{k-r} \text{ q.e.d.}$$

LEMMA 2.2. Under H_σ,

$$(2.13) \qquad \underset{\omega}{\Sigma}\, t^\varrho\, u\,(\omega) = 0, \quad \varrho = 0, 1, \, , \ldots, \sigma - 1, \; t = |\,\omega \cap \delta_\sigma\,|, \quad \text{for} \quad \sigma \leq k \leq \frac{n}{2}.$$

PROOF. By (2.9), $f_s(\omega)$ is a polynomial in t of degree s; the highest coefficient, (i.e. that of t^s), is

$$\overset{s}{\underset{\tau \geq 0}{\Sigma}} (-)^{\tau+s} \frac{1}{\tau!(s-\tau)!}\binom{n-\sigma+\tau-s}{k-\sigma+\tau-s} =$$

$$= \frac{(-1)^s}{s!} \overset{s}{\underset{\tau \geq 0}{\Sigma}} \binom{s}{\tau}\binom{n-\sigma+\tau-s}{k-\sigma+\tau-s}(-)^\tau = \frac{1}{s!}\binom{n-\sigma-s}{k-\sigma}$$

in wiew of (2.12).

For $s < \sigma \leq k \leq \frac{n}{2}$, this coefficient is $\neq 0$. Moreover, (2.13) holds for $\varrho = 0$, since $\sigma > 0$. Suppose it to hold for $0, 1, \ldots, \varrho_0 < \sigma - 1$, then it holds for $\varrho_0 + 1$. In fact from H_σ we know $\Sigma f_{\varrho_0+1}(\omega)\,u(\omega) = 0$ and $f_{\varrho_0+1}(\omega)$ is a polynomial in t of degree $\varrho_0 + 1$ whose highest coefficient does not vanish. Thus subtracting a suitable linear combination of $\Sigma f_j(\omega)\,u(\omega) = 0$, $j = 0, 1, \ldots, \varrho_0$, we obtain (2.13) for $\varrho = \varrho_0 + 1$, q.e.d.

LEMMA. 2.3. Under H_σ,

$$\underset{\omega}{\Sigma}\, t^\sigma\, u\,(\omega) = \sigma!\, \underset{\omega \supset \delta_\sigma}{\Sigma}\, u\,(\omega) \quad \text{if} \quad \sigma \leq k \leq \frac{n}{2}.$$

PROOF. From the previous Lemma we conclude

$$\frac{1}{\sigma!}\, \underset{\omega}{\Sigma}\, t^\sigma\, u\,(\omega) = \underset{\omega}{\Sigma} \binom{t}{\sigma} u\,(\omega), \quad t = |\,\omega \cap \delta_\sigma\,|.$$

But $\binom{t}{\sigma} = 0$ for $t = 0, 1, \ldots, \sigma - 1$

whence

$$\Sigma\, t^\sigma\, u\,(\omega) = \sigma\,! \qquad \underset{|\,\omega\,\cap\,\delta_\sigma\,|=\sigma}{\Sigma}\, u\,(\omega) = \sigma\,!\, \underset{\omega\,\supset\,\delta_\sigma}{\Sigma}\, u\,(\omega)\,.$$

PROOF OF PROPOSITION: Let $A\,(t)$ be a polynomial in t of degree σ, such that $A\,(\nu) = f_\nu\,(\omega)$ for $|\,\omega\,\cap\,\delta_\sigma\,| = \nu$, $\nu = 0, 1, \dots, \sigma$.

Put $A\,(t) = \tilde{A}\,\dbinom{t}{\sigma} + p\,(t)$ where the degree of the polynomial $p\,(t)$ is $\leq \sigma - 1$. It follows from Lemma 2.3. that under H_σ

$$\underset{\omega}{\Sigma}\, u\,(\omega) f_r\,(\omega) = \tilde{A}\, \underset{\omega\,\supset\,\delta_\sigma}{\Sigma}\, u\,(\omega)\,.$$

But the σ-th difference

$$A\,(\sigma) - \binom{\sigma}{1} A\,(\sigma - 1) + \binom{\sigma}{2} A\,(\sigma - 2) + \dots = \tilde{A}$$

since this difference of a polynomial of degree $\leq \sigma - 1$ vanishes. Hence the Proposition follows from Lemma 2.1. Returning to the proof of sufficiency in Theorem 2.1, let us consider $\lambda = \lambda_{\max}$ and (2.8) with maximizing $u\,(\omega)$. Either $\underset{\omega}{\Sigma}\, u\,(\omega) \neq 0$ or there is a $\sigma \leq k$ for with hypothesis H_σ applies; for certainly not all $u\,(\omega) = 0$, $\omega \,\epsilon\, V_k$. In case $\Sigma\, u\,(\omega) \neq 0$ we sum (2.8) over all $\omega_0 \,\epsilon\, V_k$ and obtain

$$\Sigma\, f_r\,(\omega)\, u\,(\omega) = \binom{k}{r}\binom{n-r}{k-r}\,\Sigma\, u\,(\omega)\,,$$

$$\left[\lambda_{\max}\, \Sigma\, d_r^{+} \binom{k}{r}\binom{n-r}{k-r} - \Sigma\, d_r^{-} \binom{k}{r}\binom{n-r}{k-r}\right]\Sigma\, u\,(\omega) = 0$$

and $\lambda_{\max} = \lambda_0 < 1$ by hypothesis (2.5) for $\sigma = 0$. But if $\Sigma\, u\,(\omega) = 0$, H_σ applies for some $\sigma > 0$. Summing (2.8) over all ω_0 containing δ_σ we obtain by the Proposition

$$\left(\lambda_{\max}\, \Sigma\, d_r^{+} \binom{k-\sigma}{r-\sigma}\binom{n-r-\sigma}{k-r} - \Sigma\, d_r^{-} \binom{k-\sigma}{r-\sigma}\binom{n-r-\sigma}{k-r}\right)\underset{\omega\,\supset\,\delta_\sigma}{\Sigma}\, u\,(\omega) = 0$$

and $\underset{\omega\,\supset\,\delta_\sigma}{\Sigma}\, u\,(\omega) \neq 0$ yields that $\lambda_{\max} = \lambda_\sigma < 1$ by (2.5). Hence $Q \gg 0$. The theorem is proved.

3. If the inequality $n \geq 2k$ of Theorem 2.1 does not apply we can nevertheless obtain a corresponding theorem by the following simple device.

Let $n \leq 2k \leq 2n$. Denote by ω' the complement of ω in $\{1, 2, \ldots, n\}$ so that $\omega' \in V_{n-k}$. If we put

$$(3.1) \qquad\qquad u(\omega) \equiv u'(\omega')$$

then a quadratic form in $u(\omega)$ becomes a quadratic form in $u'(\omega')$; invariance under permutation of $\{1, 2, \ldots, n\}$ of one form implies that of the other. If we put $k' = n - k$, we have $2k' \leq n$ so that we can apply to $Q'[u']$, defined by $Q'[u'] \equiv Q[u]$, the theorem 2.1. This requires, of course, evaluation of the coefficients d'_r, $0 \leq r \leq k'$, of Q'. We can also obtain a more direct version of Theorem if we recall the definition of $u_i(\omega)$. Observe that the definition of $u_i(\omega)$ is such that

$$u_i(\omega) = u'_i(\omega')(-1)^i$$

We have therefore the

THEOREM 3.1. A quadratic form $Q[u]$ of real $u(\omega)$, $\omega \in V_k$, $1 \leq k \leq n$ is positive definite if and only if

$$(3.2) \qquad\qquad Q[u_i(\omega)] > 0$$

for $i = 0, 1, \ldots, k$ when $2k \leq n$; and $Q >> 0$ if and only if (3.2) holds for $i = 0, 1, \ldots, n - k$ when $2k \geq n$.

The conditions of positive definiteness of $Q[u]$, if $2k \geq n$, may be made explicit in an analogous manner as for $2k \leq n$. With

$$(3.3) \qquad\qquad Q[u] = \sum_{r=0}^{k} c_r \sum_{|\omega_j \cap \omega_j| = r} u(\omega_i)\, u(\omega_j)\,, \quad 2k \geq n$$

we have necessarily $c_r = 0$ for $r < 2k - n$. With the aid of (3.1) we find

$$Q[u] = Q'[u'] = \sum_{r'=0}^{n-k} c'_{r'} \sum_{|\omega'_i \cap \omega'_j| = r'} u'(\omega'_i)\, u'(\omega'_j)\,.$$

Between c'_r and c_r we have the relation

$$c'_{r'} = c_r \quad \text{for} \quad r - r' = 2k - n\,.$$

For this relation holds certainly if $r = k$, $r' = n - k$; and a reduction of r by one unit entails a like reduction of r'. By (2.4) we can construct

the quantities $d'_{r'}$ of $Q'[u']$, replacing k by $k' = n - k$; i.e. $r = r' + 2k - n$ and

(3.4) $d'_{r'} = c_r - \binom{r'}{1} c_{r-1} + \binom{r'}{2} c_{r-2} - + \dots + (-)^{r'} c_{r-r'}$, $\quad 0 \le r' \le n - k$

THEOREM 3.2. For $2k \ge n$, $Q[u]$ of (3.3) is positive definite if and only if

$$\sum_{r'=\sigma}^{k'} d'_{r'} \binom{k' - \sigma}{r' - \sigma} \binom{n - r' - \sigma}{k' - r'} > 0, \quad \sigma = 0, 1, \dots, k' = n - k$$

where $d'_{r'}$ and c_r are related by (3.4).

4. If n is allowed to be infinite, i.e. if $\omega \in V_k^\infty$ is a k-combination of arbitrary k distinct positive integers, the quadratic form $Q[u]$ of (2.1) will certainly converge if $\{u(\omega)\}$ is an element of the Banach space of real $u(\omega)$ with norm

(4.1) $$\| u \| = \sum_\omega | u(\omega) |.$$

For $0 \le r \le k$ we have

(4.2) $$\binom{k}{r} \| u \| = \sum_{\eta \in V_r} \sum_{\omega \supset \eta} | u(\omega) |$$

whence

$$\sum_{\eta \in V_r} \sum_{\omega \supset \eta} u^2(\omega) \le \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} | u(\omega) | \right)^2 < \infty$$

the last inequality being a consequence of (4.2). It is thus possible to reduce $Q[u]$ to the form (2.2) with d_k and c_k related through (2.3). Clearly the conditions (2.5) are necessary conditions of $Q \gg 0$ for every $n \ge 2k$ since $V_n^k \subset V_k^\infty$ and an obvious specialization of $u(\omega)$ reduces to the finite case. Now the σ-th inequality in (2.5) is a polynomial in n of degree $k - \sigma$ and highest coefficient d_σ. Dividing by $n^{k-\sigma}$ and letting $n \to \infty$ we obtain from (2.5)

$$d_\sigma \ge 0, \quad \sigma = 0, 1, \dots, k$$

as necessary conditions for the positive definiteness of $Q[u]$.

Taking n finite and $\ge 2k$ we find as before that $d_k > 0$ is necessary for $Q \gg 0$.

THEOREM 4.1. The quadratic form $Q[u]$ of (2.1) on the Banach space of norm (4.1) is positive definite if and only if

(4.3) $$d_k > 0, \ d_\sigma \geq 0, \ \sigma = 0, 1, \ldots, k-1$$

with d_r as in (2.4).

PROOF. Necessity having been proved, we must only prove sufficiency. By (2.2), (4.3), $Q[u] \geq d_k \sum_\omega u^2(\omega)$; if $\|u\| = \sum |u(\omega)| > 0$, then $\sum u^2(\omega) > 0$, $Q[u] > 0$, q.e.d.

University of California
Berkeley