MOSHE MARCUS

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LOCAL BEHAVIOUR OF SINGULAR SOLUTIONS OF ELLIPTIC EQUATIONS

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1. Introduction.

Let $L(x, D_x)$ be a differential operator of order m, defined in a domain of the n-dimensional Euclidean space E_n, of the form:

$$(1.1) \qquad L(x, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha, \qquad x = (x_1, \dots x_n),$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index (α_i being a non-negative integer, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and D_x^α is the partial derivative:

$$(1.2) \qquad D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}, \qquad D_{x_i} = \frac{\partial}{\partial x_i}.$$

The coefficients $a_\alpha(x)$ are in general complex functions of x.

Suppose that L is elliptic in a sphere $|x - x^0| < R$. Then if $n > 2$, L is necessarily of even order. We shall suppose that m is even also in the case $n = 2$.

Let $u(x)$ be a solution of $Lu = 0$ in the deleted sphere $0 < |x - x^0| < R$, with a singularity of finite order at x^0.

In the case that the coefficients of L are analytic at x^0, F. John [6] proved the following results:

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(i) If $u(x)$ satisfies the condition:

$$D_x^{\alpha} u(x) = o(r^{-n-j}), \quad (r = |x - x^0| \to 0). \tag{1.3}$$

for every α such that $|\alpha| = m - 1, j$ being a non-negative integer, then in a neighborhood of x^0, $u(x)$ may be represented as the sum of a linear combination of derivatives of a fundamental solution of L, up to order j, and a function which is analytic at x^0.

(ii) If (1.3) holds with $j = -1$ then $u(x)$ is regular analytic at x^0, i. e. the singularity is removable.

Later L. Bers [2] studied the local behaviour of $u(x)$ in the neighborhood of x^0, assuming only that the coefficients of L are Hölder continuous with exponent ε) in $|x - x^0| < R$. He obtained the following results:

(iii) If $u(x)$ satisfies the condition:

$$u(x) = O(r^{m-n-\delta}), \qquad (r = |x - x^0| \to 0), \tag{1.4}$$

where $0 < \delta < \varepsilon$, then $u(x)$ is asymptotic to $cJ(x - x^0)$, where c is a constant and $J(x)$ is a fundamental solution of the osculating operator:

$$L^{(0)}(x^0, D_x) = \sum_{|\alpha| = m} a_\alpha(x) D_x^\alpha . \tag{1.5}$$

Moreover, the derivatives of $u(x)$ up to order m are asymptotic to the corresponding derivatives of $cJ(x - x^0)$.

(iv) If $u(x)$ satisfies the condition:

$$u(x) = o(r^{m-n}), \tag{1.6}$$

then the singularity at x^0 is removable. If the coefficients of L are real, then in the case that n is even and $n \leq m$, (1.6) may be replaced by the weaker condition:

$$u(x) = o(r^{m-n} |\log r|) \text{ (1)}. \tag{1.6)'$$

Using a theorem of Douglis and Nirenberg [4] it may be shown that, as a result of (1.4), the solution $u(x)$ in (iii) satisfies also the following condition:

$$D_x^\alpha u(x) = O(r^{1-n-\delta}) = o(r^{-n}), \qquad |\alpha| = m - 1. \tag{1.4)'$$

Therefore it is seen that (iii) is parallel to (i) with $j = 0$.

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(1) In [2] L. BERS dealt with real elliptic equations. But it is easily verified that result (iii) and the first part of (iv) remain valid for elliptic equation with complex coefficients.

In the present paper we generalize result (iii) to solutions $u(x)$ with singularities of any finite order. More precisely we obtain:

(v) Let L be an elliptic operator with Hölder continuous coefficients (with exponent ε) in a sphere $|x - x^0| < R$. Let $u(x)$ be a solution of $Lu = 0$ in $0 < |x - x^0| < R$ such that:

$$(1.7) \qquad u(x) = O(r^{m-n-j-\delta}), \qquad (r = |x - x^0| \to 0),$$

where $0 \leq j$ is an integer and $0 < \delta < \varepsilon$.

Then $u(x)$ is asymptotic to a linear combination of derivatives of order j of the function $J(x - x^0)$, where $J(x)$ is a fundamental solution of (1.5). Moreover, the derivatives of $u(x)$ up to order m are asymptotic to the corresponding derivatives of this linear combination.

A similar result is obtained for solutions of the inhomogeneous equation $Lu = f$, where $f(x)$ is a Hölder continuous function (with exponent ε) in $0 < |x - x^0| < R$ which may have a certain singularity of finite order at x^0.

It is also shown that by imposing stronger regularity conditions on the coefficients of L and the function $f(x)$ the asympotic estimates may be accordingly improved.

These results are further generalized to the case of an elliptic operator L whose coefficients depend not only on x, but also on a vector-parameter $t = (t_1, \dots, t_k)$. This generalization will be needed for certain applications in a paper [8] on the Dirichlet problem in a domain whose boundary is partly degenerated. In order to obtain it, we have to find first the dependance of the coefficients of the asymptotic formula and the remainder function on L, $u(x)$ and $f(x)$. To this purpose we derive explicit formulas of the coefficients, and the remainder function in terms of L, u and f and some other functions directly related to these.

2. **Definitions, notations and basic results.**

The study of the local behaviour of singular solutions of elliptic equations is closely related to the concept of a fundamental solution. In the case of an elliptic operator with analytic coefficients, the local existence of a fundamental solution and its basic properties were proved by F. John [6,7]. These results will be summed up in this section since they will be frequently required in the sequel. But first we need some definitions and notations.

Let $\mathcal{D}$ be a domain in E_n. We shall denote its boundary by $\partial\mathcal{D}$ and its closure by $\overline{\mathcal{D}}$. If K is a bounded set such that $\overline{K} \subset \mathcal{D}$, we shall write $K \subset\subset \mathcal{D}$.

Let x^0 be a fixed point and R a positive number. We shall denote by $Z(x^0, R)$ the sphere $|x - x^0| < R$ and by $Z'(x^0, R)$ the deleted sphere $0 < |x - x^0| < R$. We shall also write $Z(0, R) = Z(R)$ and $Z'(0, R) = Z'(R)$.

The family of real or complex functions $\{f(x)\}$, such that $f(x)$ together with its partial derivatives up to order j are continuous in $\mathcal{D}$ will be denoted as usual by $C_j(\mathcal{D})$. The set of Hölder continuous functions with exponent ε $(0 < \varepsilon < 1)$ in $\mathcal{D}$ will be denoted by $C_\varepsilon(\mathcal{D})$. Finally, the family of functions $\{f(x)\}$ such that $f(x) \in C_j(\mathcal{D})$ and $D_x^\alpha f(x) \in C_\varepsilon(\mathcal{D})$ for $|\alpha| = j$ will be denoted by $C_{j+\varepsilon}(\mathcal{D})$.

If $f(x) \in C_j(\mathcal{D})$ we define:

$$M_{\mathcal{D}}^j[f] = \underset{x \in \mathcal{D}, |\alpha| = j}{\text{l. u. b.}} |D_x^\alpha f(x)|, \tag{2.1}$$

and if $f(x) \in C_{j+\varepsilon}(\mathcal{D})$ we define also:

$$H_{\mathcal{D}}^j[f] = \underset{|\alpha| = j;\, x, y \in \mathcal{D}}{\text{l. u. b.}} [|D_x^\alpha f(x) - D_y^\alpha f(y)| \cdot |x - y|^{-\varepsilon}]. \tag{2.2}$$

Suppose that the coefficients of the operator L defined in (1.1) belong to $C_m(\mathcal{D})$. Then the operator

$$\overline{L}u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D_x^\alpha (a_\alpha(x) u) \tag{2.3}$$

is defined in $\mathcal{D}$. $\overline{L}$ is the *formally adjoint differential operator of* L.

Let $u, v \in C_m(\mathcal{D})$ and let $\mathcal{D}'$ be a compact subdomain of $\mathcal{D}$ whose boundary is sufficiently smooth according to the conditions of Green's theorem. Then by this theorem we have:

$$\int_{\mathcal{D}'} (uLv - v\overline{L}u)\, dx = \int_{\partial\mathcal{D}'} M[u, v]\, dS_x, \tag{2.4}$$

where dS_x is a surface element of $\partial\mathcal{D}'$ and $M[u, v]$ is a bilinear form:

$$M[u, v] = \sum_{|\alpha| + |\beta| \leq m-1} a_{\alpha\beta}^*(x, \overline{n_x})\, D_x^\alpha u D_x^\beta v, \quad x \in \partial\mathcal{D}', \tag{2.5}$$

$\overline{n_x}$ being the unit normal vector on $\partial\mathcal{D}'$ at the point x, directed to the outside of $\mathcal{D}'$. This bilinear form is not unique, but for every operator L we choose one of the possible forms, which will be fixed throughout the discussion.

DEFINITION 1.

Let $v(x)$ be a solution of $Lv = 0$ with an isolated singular point y in the domain $\mathcal{D}$. We shall say that $v(x)$ is a fundamental solution of L with pole y in $\mathcal{D}$ if for every function $u(x) \in C_m(\mathcal{D})$ and any open sphere $Z \subset\subset \mathcal{D}$ such that $y \in Z$, we have

$$u(y) = \int_Z v(x)\, \bar{L}u(x)\, dx + \int_{\partial Z} M[u, v]\, dS_x . \tag{2.6}$$

Suppose that $V(x, y)$ (as a function of x) is a fundamental solution of L with pole y in $\mathcal{D}$, for every $y \in \mathcal{D}$. Then we shall say that $V(x, y)$ is a general fundamental solution of L in $\mathcal{D}$. (Here we shall use the abbreviation: $V(x, y)$ is a *g.f.s. of L in* $\mathcal{D}$). If moreover $V(x, y)$ is analytic in the domain $\{(x, y) : x, y \in \mathcal{D}, x \neq y\}$ then we shall say that it is an *analytic g.f.s. of L in* $\mathcal{D}$.

EXISTENCE THEOREM FOR FUNDAMENTAL SOLUTIONS, (F. John [7], ch. III).

Suppose that (1.1) is an elliptic operator with analytic coefficients in a sphere $Z(x^0, R_0)$. Let j be a positive integer such that $n + j$ is even. Then, there exists a function $W_j(x, y)$ and a sphere $Z(x^0, R_1)$, $0 < R_1 < R_0$, such that $W_j(x, y)$ is analytic in the domain

$$\{(x, y) : x, y \in Z(x^0, R_1), x \neq y\}$$

and the function

$$K_j(x, y) = \Delta_y^{(n+j)/2}\, W_j(x, y) \tag{2.7}$$

is a g.f.s. of L in $Z(x^0, R_1)$.

Let $0 < R_2 < R_1$. Then the function $W_j(x, y)$ satisfies the following inequalities for $x, y \in Z(x^0, R_2)$:

$$|D^i_{xy}\, W_j(x, y)| \leq \begin{cases} \text{con. } r^{m+j-i}, & (n \text{ odd}), \\ \text{con. } r^{m+j-i}(1 + |\log r|), & (n \text{ even}), \end{cases} \tag{2.8}$$

where $r = |x - y|$ and the constant depends on R_2, j and i. ($D^i_{xy} W_j$ represents any partial derivative with respect to $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ of order i).

The fundamental solutions (2.7) depend on j, but they are essentially equivalent as it is seen from statement IV below. In the following we shall write $K(x, y)$ for any fundamental solution of the type (2.7).

In addition to the above it may be shown that $K(x, y)$ possesses the following property:

I Denote $\underline{r} = |x - y|$ and $\xi = (x - y)/\underline{r}$. Then for $\underline{r}$ and $|y - x^0|$ sufficiently small we have:

$$K(x, y) = \underline{r}^{m-n} \sum_{\nu=0}^{\infty} c_\nu (y, \xi) \underline{r}^\nu + w(x, y) \log \underline{r}, \tag{2.9}$$

where the coefficients c_ν are analytic functions of (y, ξ) and $w(x, y)$ is analytic in (x, y) and satisfies the equation $L(x, D_x) w(x, y) = 0$. Moreover if n is odd then $w(x, y) \equiv 0$. If n is even and $n \leq m$ then the limit

$$\lim \underline{r}^{n-m} w(x, y) = c(y, \xi) \tag{2.10}$$

exists and is attained uniformly with respect to ξ. In the case that L is real or, more generally, strongly elliptic, it may be shown that $c(y, \xi) \neq 0$. (John [7], pp. 61-65).

We note that F. John dealt in his works with a real elliptic operator. But his treatment is valid also for general elliptic equations with complex analytic coefficients. An exception is the last statement of result I, the proof of which is valid only for strongly elliptic equations.

As a consequence of (2.7) and (2.8) we obtain:

II Let $0 < R_2 < R_1$. Then, for $x, y \in Z(x^0, R_2)$, $K(x, y)$ satisfies the inequalities:

$$|D_{xy}^i K(x, y)| \leq \begin{cases} \text{con. } \underline{r}^{m-n-i} & , (n \text{ odd}), \\ \text{con. } \underline{r}^{m-n-i} (1 + |\log \underline{r}|) & , (n \text{ even}), \end{cases} \tag{2.11}$$

where the constant depends on R_2 and i.

The following property of $K(x, y)$ follows from those already mentioned above:

III Let $\mathcal{D}'$ be a compact subdomain of $Z(x^0, R_1)$. If $f(x) \in C_1(\mathcal{D}') \cap C_0(\overline{\mathcal{D}'})$, then:

$$L(x, D_x) \left[\int_{\mathcal{D}'} K(x, y) f(y)\, dy \right] = f(x), \qquad x \in \mathcal{D}'. \tag{2.12}$$

(John [7] pp. 54-55).

An analytic g.f.s. of L is essentially unique in the following sense:

IV Let $V(x, y)$ [respectively $\overline{V}(x, y)$] be an analytic g.f.s. of L [respectively $\overline{L}$] in a domain in which L is elliptic with analytic coefficients. Then we have:

$$\overline{V}(x, y) = V(y, x) + \mu(y, x), \tag{2.13}$$

where $\mu(y, x)$ is analytic in $\mathcal{D} \times \mathcal{D}$. If $Z(x^0, R) \subset\subset \mathcal{D}$ then for $x, y \in Z(x^0, R)$ $\mu(y, x)$ is given by the formula:

$$\mu(y, x) = \int_{|\xi - x^0| = R} M[\overline{V}(\xi, y), V(\xi, x)]\, dS_\xi . \tag{2.14}$$

From this result and the existence theorem we conclude that two analytic g.f.s. of L in $\mathcal{D}$ differ only by a function $w^*(x, y)$ which is analytic in $\mathcal{D} \times \mathcal{D}$ and satisfies the equation $L(x, D_x)\, w^*(x, y) = 0$. (John [6], pp. 297).

The following two results of John [6] will also be needed in the sequel.

V Let L and $\mathcal{D}$ be as above. Then every solution $u(x) \in C_m(\mathcal{D})$ of $Lu = 0$ in $\mathcal{D}$ is analytic in $\mathcal{D}$.

VI Let L be the operator mentioned in the existence theorem. Suppose that $u(x)$ is a solution of $Lu = 0$ in the deleted sphere $Z'(x^0, R_0)$, such that (1.3) holds. Then for $x \in Z(x^0, R_1)$ we have:

$$u(x) = \sum_{|\beta| \leq j} c_\beta [D_y^\beta K(x, y)]_{y = x^0} + w(x), \tag{2.15}$$

where c_β are constants and $w(x)$ is analytic in $Z(x^0, R_1)$. (This is the complete formulation of result (i) of section 1.).

In connection with result III we note that formula (2.9) holds also under the following weaker assumptions:

$$f(x) \in C_\varepsilon(\mathcal{D}') \cap L_1(\mathcal{D}'). \tag{2.16}$$

Indeed, using some well-known lemmas of potential theory, (see Appendix A), it follows from (2.11) and (2.16) that the function

$$F(x) = \int_{\mathcal{D}'} K(x, y) f(y)\, dy, \tag{2.17}$$

belongs to $C_m(\mathcal{D}')$ and

$$D_x^\alpha F(x) = \int_{\mathcal{D}'} D_x^\alpha K(x, y)[f(y) - f(x)]\, dy + f(x)\, D_x^\alpha \int_{\mathcal{D}'} K(x, y)\, dy, \tag{2.18}$$

for $|\alpha| \leq m$. Hence:

$$L(x, D_x)\, F(x) = f(x)\, L(x, D_x) \int_{\mathcal{D}'} K(x, y)\, dy = f(x). \tag{2.19}$$

We note also that inequality (2.11) may be slightly improved in the case that n is even and $m < n + i$. It follows from results II and IV, that in this case, the factor $(1 + |\log r|)$ on the right side of (2.11) may be deleted.

Another basic result that we shall need in the sequel is a theorem on interior estimates for elliptic equations of Douglis and Nirenberg [4]. We shall refer to Theorems 1 and 4 of [4] (pp. 517 and 529) as the Douglis-Nirenberg theorem.

Let $Lu = f$ be an uniformly elliptic equation of order m in $\mathcal{D}$, such that the coefficients of L and the function f belong to $C_{p+\varepsilon}(\mathcal{D})$, where $0 \leq p$ is an integer and $0 < \varepsilon < 1$. Suppose that $u(x) \in C_{m+\varepsilon}(\mathcal{D})$. Then the Douglis-Nirenberg theorem asserts that $u(x) \in C_{m+p+\varepsilon}(\mathcal{D})$ and it provides estimates for the derivatives of $u(x)$ up to order $m + p$ at interior points of $\mathcal{D}$.

It was shown later, in the paper of Agmon-Douglis-Nirenberg [1], (pp. 719), that under the above mentioned assumptions on L and f, if $u(x) \in C_m(\mathcal{D})$ then it follows already that $u(x) \in C_{m+\varepsilon}(\mathcal{D})$.

The following lemma is a consequence of the Douglis-Nirenberg theorem.

LEMMA 1.

Suppose that (1.1) is an uniformly elliptic operator in the sphere $Z(R_0)$ and let $u(x)$ be a solution of $Lu = f$ in the deleted sphere $Z'(R_0)$ such that $u(x) \in C_m(Z'(R_0))$. If the coefficients of L belong to $C_{p+\varepsilon}(Z(R_0))$ and if $f(x) \in C_{p+\varepsilon}(Z'(R_0))$, where $0 \leq p$ is an integer and $0 < \varepsilon < 1$, then:

(i) $u(x) \in C_{m+p+\varepsilon}(Z'(R_0))$.

(ii) Let $K_r = \{y : r/2 < |y| < 3r/2\}$ and denote

$$A(r) = M^0_{K_r}[u] + r^m \left\{ \sum_{j=0}^{p} r^j M^j_{K_r}[f] + r^{p+\varepsilon} H^j_{K_r}[f] \right\} \tag{2.20}$$

With this notation we have:

$$|D^\alpha_x u(x)| \leq \bar{c}_1 r^{-|\alpha|} A(r), \tag{2.21}$$

for $|\alpha| \leq m + p$ and $0 < |x| = r < R_0/2$, and

$$H^{m+p}_{K'_r}[u] \leq \bar{c}_1 r^{-m-p-\varepsilon} A(r), \qquad (0 < r < R_0/2), \tag{2.22}$$

where $K'_r = \{y : 3r/4 < |y| < 5r/4\}$.

The constant $\bar{c}_1$ depends only on n, m, p, ε, on the ellipticity constant of L in Z_0 and on certain norms μ_α of the coefficients $a_\alpha(x)$ of L. These norms may be defined as follows:

$$\mu_a = R_0^{m-|a|}\left\{\sum_{j=0}^{p} R_0^j M_{Z_1}^j[a_a] + R_0^{p+\varepsilon} H_{Z_1}^p[a_a]\right\}, \tag{2.23}$$

where $Z_1 = Z(3R_0/4)$.

This lemma is obtained by applying the Douglis-Nirenberg theorem to the equation $Lu = f$ in the domain $r/2 < |x| < 3r/2$.

We end this section with a few additional notations which will be required in the following sections.

If $g(x) \in C_j(Z'(R))$ then the notation:

$$g(x) = O_j(r^s |\log r|^\sigma) \tag{2.24}$$

where $|x| = r$, means that:

$$D_x^\alpha g(x) = O(r^{s-|\alpha|}|\log r|^\sigma), \qquad (r \to 0), \quad |\alpha| \leq j. \tag{2.25}$$

Similarly if $g(x) \in C_{j+\varepsilon}(Z'(R))$, $0 < \varepsilon < 1$, then the notation

$$g(x) = O_{j+\varepsilon}(r^s |\log r|^\sigma) \tag{2.24'}$$

means that:

$$\begin{cases} g(x) = O_j(r^s |\log r|^\sigma) \\ H_{K_r'}^j[g] = O_j(r^{s-j-\varepsilon} |\log r|^\sigma), \end{cases} \tag{2.25'}$$

where K_r' is defined as in (2.22).

In a similar way we define o_j and $o_{j+\varepsilon}$.

Finally, if $g(x) \in C_j$ in a neighborhood of the origin we define:

$$(g)_j = \sum_{|\alpha| \leq j}\left[\frac{1}{\alpha!} D_x^\alpha g(0)\right] x^\alpha, \tag{2.26}$$

where $\alpha! = \alpha_1! \dots \alpha_n!$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$.

The following is an immediate consequence of Lemma 1:

Lemma 2.

In addition to the assumptions of Lemma 1, suppose that:

$$u(x) = O(r^s |\log r|^\sigma), \tag{2.27}$$

$$f(x) = O_{p+\varepsilon}(r^{s-m} |\log r|^\sigma), \tag{2.28}$$

where σ is positive and s is any real number. Then:

$$u(x) = O_{m+p+\varepsilon}(r^{s} \mid \log r \mid^{\sigma}). \tag{2.29}$$

These results hold also if O is replaced everywhere by o.

3. Existence of special singular solutions.

Suppose that the differential operator (1.1) is elliptic with analytic coefficients in a neighborhood of the origin.

Let $f(x)$ be a function with a finite singularity at the origin, such that

$$f(x) = O_a(r^{-s} \mid \log r \mid^{\sigma}), \qquad \mid x \mid = r \rightarrow 0, \tag{3.1}$$

where $0 < a$ is not an integer, $0 \leq s$ is an arbitrary real number and $0 \leq \sigma$ is an integer.

In this section we shall construct a solution $u(x)$ of $Lu = f$ in a deleted neighborhood of the origin, such that:

$$u(x) = O_{m+a}(r^{m-s} \mid \log r \mid^{\sigma''}) + P(x), \tag{3.2}$$

where $P(x)$ is a polynomial.

The main result is based on a number of lemmas which we now proceed to proove. In the following we shall denote by D_x^j a general derivative of the form $D_x^{\alpha}, \mid \alpha \mid = j$, and similarly by D_{xy}^j a derivative of the form $D_x^{\alpha} D_y^{\alpha'}$ with $\mid \alpha \mid + \mid \alpha' \mid = j$.

LEMMA 3.

Let $G(x, y)$ be an analytic function of $2n$ variables $(x_1, \dots, x_n, y_1, \dots, y_n)$ in the domain

$$\{(x,y) : x, y \in Z(R_0),\ x \neq y\}, \tag{3.3}$$

and suppose that it satisfies there the inequalities:

$$\mid D_{xy}^i G(x, y) \mid \leq \text{con. } r^{p'-i}(1 + \mid \log r \mid^{\sigma'}), \qquad i = 0, 1, 2, \dots \tag{3.4}$$

where $1 - n < p'$ and $0 \leq \sigma'$ are integers and $r = \mid x - y \mid$.

Let q be an integer, $0 \leq q < p' + n - 1$ and put $p = p' + n - (1 + q)$. Let $f(x)$ be a function belonging to $C_{q+\varepsilon}(Z'(R_0))$, $0 < \varepsilon < 1$, such that:

$$f(x) = O_{q+\varepsilon}(r^{-s} \,|\log r\,|^{\sigma}), \quad |\,x\,| = r \to 0, \tag{3.5}$$

s being a real number, $-q \leq s < n$, and σ a non-negative integer.

Denote

$$F(x, y) = D_y^{\beta}\, G(x, y), \quad (|\,\beta\,| = q + 1) \tag{3.6}$$

and

$$u(x) = \int_{Z(R)} F(x, y) f(y)\, dy, \quad (0 < R < R_0) \tag{3.7}$$

Then $u(x) \in C_{p+q}(Z'(R))$ and if $s < p$ then $u(x)$ belongs also to $C_{s'}(Z(R))$, s' being the greatest integer which is smaller than $p - s$. Moreover we have:

$$u(x) = O_{p+q}(r^{p-s} \,|\log r\,|^{\sigma''}) + P(x), \tag{3.8}$$

where $\sigma'' = \sigma + \sigma' + 1$ if s in an integer and $\sigma'' = \sigma + \sigma'$ otherwise, an

$$P(x) = \begin{cases} (u)_{s'}\,, & \text{if } s < p, \\ 0 & \text{if } s \geq p. \end{cases}$$

PROOF. In order to prove the lemma it is sufficient to show that:

$$u(x) \in C_{p+q}(Z'(R)) \tag{3.9}$$

and

$$D_x^{p+q}\, u(x) = O(r^{-s-q} \,|\log r\,|^{\sigma^*}), \quad r \to 0, \tag{3.10}$$

where $\sigma^* = \sigma + \sigma' + 1$ if $s + q = 0$ and $\sigma^* = \sigma + \sigma'$ otherwise. The remaining statements are easily obtained from (3.9) and (3.10) by integration.

Let $x^0 \in Z'(R)$ and $0 < h < |\,x^0\,|/4$. We divide the sphere $Z(R)$ into three disjoint subsets:

$$\begin{aligned} &\mathcal{D}_1 = \{y : |\,y\,| < h\}, \quad \mathcal{D}_2 = \{y : |\,y - x^0\,| < 2h,\ |\,y\,| < R\}, \\ &\mathcal{D}_3 = Z(R) - (\mathcal{D}_1 \cup \mathcal{D}_2)\,; \end{aligned} \tag{3.11}$$

and accordingly we write:

$$(3.12) \qquad u(x) = \left\{ \int_{\mathcal{D}_1} + \int_{\mathcal{D}_2} + \int_{\mathcal{D}_3} \right\} F(x, y) f(y)\, dy$$

$$\equiv I_1(x) + I_2(x) + I_3(x).$$

Clearly, $I_1(x)$ and $I_3(x)$ belong to $C_\infty(\mathcal{D}_2)$. Now, let $2h < (R - |x^0|)$ and denote

$$\mathcal{D}_2^\varepsilon = \{y : \varepsilon < |y - x^0| < 2h\}, \qquad (0 < \varepsilon).$$

By (3.6) we have

$$(3.13) \qquad F(x, y) = D_y^\gamma D_{y_i} G(x, y) \equiv D_y^\gamma G^*(x, y), \quad (|\gamma| = q),$$

for suitable γ and i. Hence, by integration by parts we obtain:

$$\int_{\mathcal{D}_2^\varepsilon} F(x, y) f(y)\, dy = (-1)^q \int_{\mathcal{D}_2^\varepsilon} G^*(x, y)\, D_y^\gamma f(y)\, dy$$

$$+ \int_{|y - x^0| = 2h} N[G^*, f]\, dS_y - \int_{|y - x^0| = \varepsilon} N[G^*, f]\, dS_y$$

where $N[G^*, f]$ is a bilinear form in the derivatives of G^* and f with respect to y, such that the sum of the orders of the derivatives in each term is $\leq q - 1$. If now we let ε tend to zero we get

$$(3.14) \qquad I_2(x) = (-1)^q \int_{\mathcal{D}_2} G^*(x, y)\, D_y^\gamma f(y)\, dy + \int_{|y - x^0| = 2h} N[G^*, f]\, dS_y$$

$$\equiv I_2'(x) + I_2''(x).$$

Clearly $I_2''(x)$ is an analytic function in $\mathcal{D}_2$. Also, from (3.4) and (3.5) it follows, by known lemmas of potential theory (see Appendix A), that $I_2'(x) \in C_{p+q}(\mathcal{D}_2)$.

Summing up we conclude that $u(x) \in C_{p+q}(\mathcal{D}_2)$ and since x^0 was an arbitrary point of $Z'(R)$, statement (3.9) is proved.

In order to prove (3.10) we again use the partition of $Z(R)$ defined by (3.11), with $0 < |x^0| < \frac{1}{2} \min.(1, R)$ and $h = |x^0|/4$. For $|x - x^0| < h$ we have:

$$D_x^{p+q} I_1(x) = \int_{\mathcal{D}_1} D_x^{p+q} F(x, y) f(y)\, dy \tag{3.15}$$

$$D_x^{p+q} I_3(x) = \int_{\mathcal{D}_3} D_x^{p+q} F(x, y) f(y)\, dy \tag{3.16}$$

$$D_x^{p+q} I_2''(x) = \int_{|y - x^0| = 2h} N[D_x^{p+q} G^*(x, y), f(y)]\, dS_y \tag{3.17}$$

and finally:

$$D_x^{p+q} I_2'(x) = \int_{\mathcal{D}_2} D_x^{p+q} G^*(x, y)[f^*(y) - f^*(x)]\, dy + f^*(x) D_x^{p+q} \int_{\mathcal{D}_2} G^*(x, y)\, dy, \tag{3.18}$$

where $f^*(y) = D_y^{\gamma} f(y)$.

Estimating the first three integrals with the aid of (3.4) and (3.5) we obtain (for $|x - x^0| < h$):

$$|D_x^{p+q} I_1(x)| + |D_x^{p+q} I_3(x)| + |D_x^{p+q} I_2''(x)| \leq \text{con.}\, h^{-s-q} |\log h|^{\sigma^*}. \tag{3.19}$$

Similarly we obtain:

$$\left| \int_{\mathcal{D}_2} D_x^{p+q} G^*(x, y)[f^*(y) - f^*(x)]\, dy \right| \leq \text{con.}\, h^{-s-q} |\log h|^{\sigma+\sigma'}. \tag{3.20}$$

Using integration by parts it is immediately seen that the function

$$\int_{\mathcal{D}_2} G^*(x, y)\, dy = \int_{\mathcal{D}_2} D_{y_i} G(x, y)\, dy$$

is analytic for $x \in \mathcal{D}_2$. Therefore it follows, from (3.5) and (3.20), that

$$|D_x^{p+q} I_2'(x)| \leq \text{con.}\, h^{-s-q} |\log h|^{\sigma+\sigma'}, \quad (|x - x^0| < h). \tag{3.21}$$

In conclusion we obtain the inequality:

$$|D_x^{p+q} u(x^0)| \leq \text{con.}\, h^{-s-q} |\log h|^{\sigma^*}, \tag{3.22}$$

for $|x^0| < \frac{1}{2}\min.(1, R)$, where $h = |x^0|/4$ and the constant depends only on G, f and n. This completes the proof of (3.10).

As a consequence of Lemma 3 we obtain the following result.

LEMMA 4.

Suppose that the differential operator (1.1) is elliptic with analytic coefficients in a sphere $Z(R_0)$. Let $f(x)$ be a function of $C_{q+\varepsilon}(Z'(R_0))$, $(0 < \varepsilon < 1,\ 0 \leq q$ an integer) satisfying condition (3.5), with s and σ as in Lemma 3.

Let $K(x, y)$ be a g. f. s. of L of the form (2.7), with $n + j = 2([q/2] + 1)$, in a sphere $Z(R_1)$, $0 < R_1 < R_0$. Define:

$$u(x) = \int_{Z(R)} K(x, y) f(y)\, dy, \quad (0 < R < R_1). \tag{3.23}$$

Then $u(x)$ is a solution of $Lu = f$ in $Z'(R)$, possessing the following properties:

(*a*) $u(x) \in C_{m+q+\varepsilon}(Z'(R))$.

(*b*) If $0 < m - s$ then $u(x) \in C_{s'}(Z(R))$, s' being the greatest integer which is smaller than $m - s$.

(*c*) $u(x) = O_{m+q+\varepsilon}(r^{m-s} |\log r|^{\sigma'}) + P(x)$, where

$$P(x) = \begin{cases} (u)_{s'}, & \text{if} \quad s < m, \\ 0, & \text{if} \quad m \leq s, \end{cases}$$

and σ' is an integer $\geq \sigma$. (If n is odd and s is not an integer, then $\sigma = \sigma'$).

PROOF. From the assumptions on $f(x)$ it follows immediately that $f(x) \in C_\varepsilon \cap L_1$ in $Z'(R)$. Hence it follows that $u(x)$ is a solution of $Lu = f$ in $Z'(R)$.

Properties (*a*), (*b*) and (*c*), except for the Hölder continuity of the derivatives of $u(x)$ of order $m + q$, are a direct consequence of Lemma 3 and the formulas (2.7) and (2.8) concerning $K(x, y)$.

Let $P(x)$ be the polynom mentioned in (*c*) and put $v_\beta(x) = D_x^\beta(u(x) - P(x))$ where $|\beta| = q$. Using property (*c*) and formula (3.5) it is easily verified that:

$$Lv_\beta(x) = D_x^\beta f(x) - D_x^\beta L\, P(x) + f_\beta^*(x) \equiv f_\beta(x), \quad x \in Z'(R), \tag{3.24}$$

where $f_\beta^*(x) = O_1(r^{-s-q+1} \,|\log r\,|^{\sigma'})$, so that:

$$f_\beta(x) = O_\varepsilon(r^{-s-q} \,|\log r\,|^{\sigma'}). \tag{3.25}$$

(We use also the fact that $s + q \geq 0$).

Applying Lemma 2 to the equation $Lv_\beta = f_\beta$ we find that $v_\beta \in C_{m+\varepsilon}(Z'(R))$ and

$$v_\beta(x) = O_{m+\varepsilon}(r^{m-s} \,|\log r\,|^{\sigma'}). \tag{3.26}$$

Hence we obtain properties (a) and (c) in the complete form.

In order to prove a similar result for functions $f(x)$ satisfying (3.5) with $n \leq s$, we shall deal first with the special case $L = \Delta_x^\nu$, $(0 < \nu$ an integer). In this case we prove the following result:

LEMMA 5.

Let $f(x)$ be a function of $C_{q+\varepsilon}(Z'(R_0))$, $(0 < \varepsilon < 1$, q a non-negative integer) which satisfies condition (3.5) for a certain real number $s \geq n$ and a non-negative integer σ.

Let $0 < R < R_0$ and let ν be a positive integer. Then there exists a solution $u_\nu(x)$ of the equation $\Delta_x^\nu u_\nu(x) = f(x)$ in $Z'(R)$ possessing the properties (a), (b), (c) of Lemma 4, with $m = 2\nu$.

PROOF. First we shall deal with the case $\nu = 1$. A fundamental solution of the operator Δ is given by:

$$J(x-y) = \begin{cases} |\,x-y\,|^{2-n}/(2-n)\,\omega_n\,, & n > 2, \\ \dfrac{1}{2\pi}\log|\,x-y\,|\,, & n = 2, \end{cases} \tag{3.27}$$

where ω_n is the surface area of the n-dimensional unit sphere.

Let θ be a fixed number, $0 < \theta < 1$. Then for $x \neq 0$ and $|\,y\,|/|\,x\,| \leq \theta$ the function $J(x-y)$ may be written in the following form:

$$J(x-y) = \begin{cases} \displaystyle\sum_{j=0}^{\infty} |\,y\,|^j\,|\,x\,|^{-j-n+2}\,P_j\left(\frac{x}{|\,x\,|}\cdot\frac{y}{|\,y\,|}\right), & n > 2, \\ \displaystyle\sum_{j=0}^{\infty} |\,y\,|^j\,|\,x\,|^{-j}\,T_j\left(\frac{x}{|\,x\,|}\cdot\frac{y}{|\,y\,|}\right) + \frac{1}{2\pi}\log|\,x\,|, & n = 2, \end{cases} \tag{3.28}$$

Where $P_j(\varrho)$ and $T_j(\varrho)$ are the Legendre and Tchebysheff polynomials, respectively. Every term of the above series is a harmonic function in x and y, provided $x \neq 0$, $y \neq 0$. Also if $-1 \leq \varrho \leq 1$ then $|\,P_j(\varrho)\,| \leq 1$, $|\,T_j(\varrho)\,| \leq 1$. Therefore it is clear that the series (3.25) converge absolutely and uniformly in x and y for $|\,y\,|/|\,x\,| \leq \theta$ and $r_0 < |\,x\,|$, r_0 being a fixed positive number.

We now define the function:

$$(3.29)\quad J^p(x,y)=\begin{cases} J(x-y)-\sum_{j=0}^{p}|y|^j\,|x|^{-j-n+2}P_j\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right), & n>2,\\ J(x-y)-\sum_{j=0}^{p}|y|^j\,|x|^{-j}\,T_j\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)-\frac{1}{2\pi}\log|x|, & n=2,\end{cases}$$

For $x \neq 0$ and $|y|/|x| \leq \theta$ we have:

$$(3.30)\qquad |J^p(x,y)| \leq \frac{1}{1-\theta}\,|y|^{p+1}\,|x|^{1-n-p}$$

$J^p(x,y)$ differs from $J(x-y)$ by a function $w(x,y)$ which is analytic in (x,y), (provided $x \neq 0$, $y \neq 0$) and harmonic in $x\,(x \neq 0)$ for any fixed $y \neq 0$. Therefore $J^p(x,y)$ is also an analytic g.f.s. of Δ in the whole space E_n, except for the origin.

Let us denote:

$$(3.31)\qquad u(x)=\int_{Z(R)} J^p(x,y)\,f(y)\,dy, \qquad (p=[s]-n).$$

From (3.5) and (3.30) it follows that the integral converges absolutely for $0<|x|<R$. We shall show that $u(x)$ satisfies all the conditions which are required in the case $\nu=1$. First, we shall prove that $u(x)$ is a solution of $\Delta u=f$ in $0<|x|<R$. This statement is not evident in the present case since, generally, $f(x)\notin L_1(Z(R))$.

Let R' be a fixed number, $0<R'<R/4$ and let us write:

$$(3.32)\qquad u(x)=\int_{|y|<R'} J^p(x,y)\,f(y)\,dy+\int_{R'<|y|<R} J^p(x,y)\,f(y)\,dy=I'(x)+I''(x).$$

Clearly $I''(x)\in C_2$ and

$$(3.33)\qquad \Delta_x I''(x)=f(x),$$

for $R'<|x|<R$. Now, let y be a fixed point, $0<|y|<R'$. Applying the Douglis-Nirenberg theorem to the equation $\Delta_x J^p(x,y)=0$ in the domain $3R'/2<|x|<R$, and using (3.30) we obtain:

$$(3.34)\qquad |D_x^i J^p(x,y)| \leq 3\,|y|^{p+1}\cdot(R')^{1-n-p}\cdot\left(\frac{R'}{2}\right)^{-i} \equiv c\,|y|^{p+1},$$

for $2R' < |x| < R - R'$. Hence it follows that $I'(x) \in C_\infty$ in the domain $2R' < |x| < R - R'$ and that the derivatives of $I'(x)$ are given by the formula:

$$D_x^\alpha I'(x) = \int_{|y| < R'} D_x^\alpha J^p(x, y) f(y)\, dy.$$

Since $\Delta_x J^p(x, y) = 0$ we conclude that:

$$(3.35) \qquad \Delta_x I'(x) = 0, \qquad 2R' < |x| < R - R'.$$

This together with (3.33) proves our statement.

It will now be shown that:

$$(3.36) \qquad u(x) \in C_{2+q}(Z'(R))$$

and

$$(3.37) \qquad u(x) = O(r^{2-s} |\log r|^{\sigma_1}),$$

where σ_1 is an integer $\geq \sigma$; ($\sigma_1 = \sigma$ if $n > 2$ and s is not an integer).

Let $I'(x)$ and $I''(x)$ be the functions defined in (3.32) with $0 < R' < \frac{1}{4}\min(1, R)$. We already know that $I'(x) \in C_\infty$ for $2R' < |x| < R - R'$; it is easily verified that

$$(3.38)_1 \qquad |I'(x)| \leq \text{con.}\,(R')^{2-s} |\log R'|^\sigma,$$

for $2R' < |x| < \frac{1}{2}\min.(1, R) \equiv R^*$.

By (3.29) the function $I''(x)$ (for $n > 2$) may be written in ...

$$I''(x) = F(x) - \sum_{j=0}^{p} |x|^{-j-n+2} F_j(x),$$

where

$$F(x) = \int_{R' < |y| < R} J(x - y) f(y)\, dy\,; \quad F_j(x) = \int_{R' < |y| < R} |y|^j P_j\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right) f(y)\, dy.$$

(In the case $n = 2$ we have to replace P_j by T_j and to add a term containing $\log|x|$. All the following remarks hold also in this case.). It is easily verified, by the method employed in the proof of Lemma 3, that $F(x) \in C_{2+q}$ for $R' < |x| < R$ and that

$$|F(x)| \leq \text{con.}\,(R')^{2-s} |\log R'|^{\sigma_1}, \qquad (\text{for } 2R' < |x| < R^*).$$

Clearly, $F_j(x) \in C_\infty$ for $x \neq 0$ and

$$| F_j(x) | \leq \text{con.}\, (R')^{j-s+n} \, | \log R' |^{\sigma_1} . \qquad (\text{for } x \neq 0)$$

Hence it follows that $I''(x) \in C_{2+q}$ for $R' < | x | < R$ and

$$(3.38)_2 \qquad | I''(x) | \leq \text{con.}\, (R')^{2-s} \, | \log R' |^{\sigma_1} , \qquad (\text{for } 2R' < | x | < R^*).$$

Summing up these results we obtain (3.36) and (3.37). Hence it follows, by Lemma 2, that the solution $u(x)$ possesses properties (a) and (c) with $m = 2$. (in regard to (b) we remark that $m - s = 2 - s \leq 0$). This completes the proof of the lemma in the case $\nu = 1$.

From this result it follows immediately, by induction, that the statement of the lemma is valid for any positive integer ν such that $s - 2\nu \geq n - 2$.

Suppose now that $s - n + 2 < 2\nu$ and put $j = \left[\frac{s-n}{2}\right] + 1$.

Then there exists a solution $u_j(x)$ of the equation

$$(3.39) \qquad \Delta^j u_j(x) = f(x), \qquad x \in Z'(R^*), \qquad R < R^* < R_0 ,$$

which satisfies properties (a), (b), (c) (with $m = 2j$, $R = R^*$ and $P(x) \equiv 0$ in (c)). Since $n - 2 \leq s - 2j \leq n - 1$, it follows by lemma 4, that there exists a solution $u(x)$ of the equation

$$(3.40) \qquad \Delta^{\nu - j} u(x) = u_j(x), \qquad x \in Z'(R),$$

which satisfies properties (a), (b), (c) with $m = 2\nu$.

This completes the proof of the lemma.

We come now to the main result of this section:

THEOREM 1.

Let (1.1) be an elliptic operator with analytic coefficients in a sphere $Z(R_0)$. Let $f(x) \in C_a(Z'(R_0))$ (where $0 < a$ is not an integer) and suppose that:

$$(3.41) \qquad f(x) = O_a(r^{-s} \, | \log r |^{\sigma}), \qquad | x | = r \to 0$$

with $-[a] \leq s$ a real number and $0 \leq \sigma$ an integer.

Let R_1 be a number in the open interval (o, R_0) such that there exists an analytic g.f.s. of L in $Z(R_1)$, and let $0 < R < R_1$.

Then there exists a solution $u(x)$ of $Lu = f$ in $Z'(R)$ which satisfies properties (a), (b), (c) of Lemma 4 with $q + \varepsilon = a$.

PROOF. The theorem has already been proved for the case $s < n$ (Lemma 4), therefore we have to deal now only with the case $n \leq s$. The proof will be by induction on s. Suppose that the theorem was proved for $-[a] \leq s < s_0$, where $n \leq s_0$ is an integer. We shall now prove the theorem for $s_0 \leq s < s_0 + 1$.

Let $\nu = \left[\frac{s-n}{2}\right] + 1$. By Lemma 5 there exists a function $u_\nu(x) \in C_{2\nu+a}(Z'(R'))$, $R < R' < R_1$, such that:

$$\begin{cases} \Delta^\nu u_\nu(x) = f(x), & x \in Z'(R') \\ u_\nu(x) = O_{2\nu+a}(r^{2\nu-s} \,|\log r|^{\sigma_1}), \end{cases} \tag{3.42}$$

where $\sigma \leq \sigma_1$ is an integer and if n is odd and s is not an integer then $\sigma_1 = \sigma$.

Since $n - 2 \leq s - 2\nu < n$ it follows by Lemma 4 that there exists a function $v(x) \in C_{m+2\nu+a}(Z'(R''))$, $R < R'' < R'$, such that:

$$\begin{cases} Lv(x) = u_\nu(x), & x \in Z'(R''), \\ v(x) = O_{m+2\nu+a}(r^{m+2\nu-s} \,|\log r|^{\sigma_2}) + P_2(x), \end{cases} \tag{3.43}$$

where $P_2(x)$ is a polynom and $\sigma_1 \leq \sigma_2$ is an integer. (Again if n is odd and s is not an integer, then $\sigma_1 = \sigma_2$).

Now we have:

$$\Delta^\nu Lv(x) = L\Delta^\nu v(x) + Av(x), \tag{3.44}$$

where A is a linear differential operator of order $2m + 2\nu - 1$ at most. Therefore by (3.42) and (3.43) it follows that:

$$L\Delta^\nu v(x) = f(x) - Av(x) \tag{3.45}$$

and

$$\begin{cases} Av(x) \in C_{a+1}(Z'(R'')), \\ Av(x) = O_{a+1}(r^{1-s} \,|\log r|^{\sigma_2}). \end{cases} \tag{3.46}$$

But by the assumption on which we based the induction there exists a function $v'(x) \in C_{m+a+1}(Z'(R))$, such that:

$$\begin{cases} Lv'(x) = Av(x), & x \in Z'(R) \\ v'(x) = O_{m+a+1}(r^{m+1-s} \,|\log r|^{\sigma_3}) + P_3(x), \end{cases} \tag{3.47}$$

where $P_3(x)$ is a polynom and $\sigma_2 \leq \sigma_3$ is an integer; if n is odd and s is not an integer then $\sigma_2 = \sigma_3$.

Summing up it follows that the function:

$$u(x) = \Delta^{\nu} v(x) + v'(x) \tag{3.48}$$

satisfies all the required properties.

This completes the proof of the theorem.

4. Asymptotic estimates in the neighborhood of a singular point.

Using John's theorem (VI, section 2) and Theorem 1 we shall obtain some results concerning the behaviour of a solution $u(x)$ of an elliptic equation $Lu = f$ whose coefficients are not necessarily analytic, in the neighborhood of a singular point of finite order. For simplicity we shall suppose that the differential operator L which is defined by (1.1) is elliptic in a sphere $Z(R_0)$ and that the singular point of the solution is the origin.

First we need two definitions.

DEFINITION 2.

Let j be a non-negative integer and suppose that the coefficients of L, defined by (1.1), satisfy the following regularity condition:

$$a_\alpha(x) \in C_{j+|\alpha|-m}, \qquad m - j \leq |\alpha|. \tag{4.1}$$

We define:

$$L^{(j)} = \sum_{m-j \leq |\alpha| \leq m} (a_\alpha(x))_{j+|\alpha|-m} D_x^\alpha. \tag{4.2}$$

$L^{(0)}$ is the osculating operator of L. Accordingly we shall say that $L^{(j)}$ is the osculating operator of L of order j.

DEFINITION 3.

Let L be the operator (1.1). We shall say that L satisfies the condition $\mathcal{R}_{p,\, j+\varepsilon}(\mathcal{D})$, where p and j are non-negative integers and $0 \leq \varepsilon < 1$, if:

$$\begin{cases} a_\alpha(x) \in C_{\nu(\alpha)}(\mathcal{D}), \\ \nu(\alpha) = \max.(|\alpha| - 2m + j, p) + \varepsilon. \end{cases} \tag{4.3}$$

We prove now the following:

LEMMA 6.

Let $u(x) \in C_{m+p+\varepsilon}(Z'(R))$, where $0 \leq p$ is an integer and $0 < \varepsilon < 1$, and suppose that:

$$u(x) = O_{m+p+\varepsilon}(r^s), \tag{4.4}$$

s being any real number. Further, suppose that the operator L satisfies condition $\mathcal{R}_{p,\, j+\varepsilon}(Z(R))$, ($0 \leq j$ an integer).

Under these assumptions, if $L^{(j)}$ is the osculating operator of order j, then:

$$(L - L^{(j)})\, u(x) = O_{p+\varepsilon}(r^{s-m+j+\varepsilon}). \tag{4.5}$$

PROOF. By Definition 2 we have:

$$(L - L^{(j)}) = \sum_{|\alpha| \leq m} b_\alpha(x)\, D_x^\alpha \tag{4.6}$$

where

$$\begin{cases} b_\alpha(x) = a_\alpha - (a_\alpha)_{|\alpha|-m+j}\,, & m - j \leq |\alpha|, \\ b_\alpha(x) = a_\alpha\,, & m - j > |\alpha|. \end{cases} \tag{4.7}$$

Further, by (4.7) and Definition 3 it follows that:

$$\begin{cases} b_\alpha(x) \in C_{\nu(\alpha)}(Z(R)), \\ b_\alpha(x) = O_{|\alpha|-m+j+\varepsilon}\,, \quad (r^{|\alpha|-m+j+\varepsilon})\; m - j \leq |\alpha|. \end{cases} \tag{4.8}$$

Using (4.4) and (4.8) it is readily verified that

$$b_\alpha(x)\, D_x^\alpha\, u(x) = O_{p+\varepsilon}\,(r^{s-m+j+\varepsilon}), \tag{4.9}$$

for $m - j \leq |\alpha|$.

In the case $|\alpha| < m - j$, (4.9) follows immediately from the fact that $b_\alpha(x) \in C_{p+\varepsilon}(Z(R))$ and

$$D_x^\alpha\, u(x) = O_{p+j+1}\,(r^{s-m+j+1}). \tag{4.10}$$

The required estimate follows from (4.6) and (4.9).

Using Theorem 1 and Lemma 6 we prove now the following asymptotic estimates:

THEOREM 2.

Let L be an elliptic operator in $Z(R_0)$, defined by (1.1). Let $u(x) \in C_m(Z'(R_0))$ be a solution of $Lu = f$ in $Z'(R_0)$ such that:

$$u(x) = O(r^{m-s}), \tag{4.11}$$

where $n \leq s$ is a real number.

Let j be a non-negative integer. Put $\mu = s - [s]$ and $p = \max.(0, 1 + j - [s])$. Suppose that L satisfies condition $\mathcal{R}_{p,\, j+\varepsilon}(Z(R_0))$, with $\mu < \varepsilon < 1$. The osculating operator of order j, $L^{(j)}$, is elliptic in a neighborhood of the origin. Therefore there exists an analytic g. f. s $K^{(j)}(x, y)$ of $L^{(j)}$ in a sphere $Z(R_1)$ such that $0 < R_1 < R_0$.

If $f(x) \in C_{p+\varepsilon}(Z'(R_0))$ and satisfies the condition:

$$f(x) = O_{p+\varepsilon}(r^{-s+j+\varepsilon}), \tag{4.12}$$

then we have (for $|x| < R < R_1$):

$$u(x) = \sum_{q_1 \leq |\beta| \leq q} c_\beta D_y^\beta K^{(j)}(x, 0) + w(x) + O_{m+p+\varepsilon}(r^{m-s+j+\varepsilon} |\log r|^{\sigma'}), \tag{4.13}$$

where $q = [s] - n$, $q_1 = \max.(0, q - j)$, c_β are constants, $w(x)$ is an analytic function in $Z(R)$, and σ' is a non-negative integer.

PROOF. By Lemma 2 we have:

$$u(x) = O_{m+p+\varepsilon}(r^{m-s}). \tag{4.14}$$

Hence by Lemma 6:

$$(L - L^{(j)})\, u(x) = f'(x) = O_{p+\varepsilon}(r^{-s+j+\varepsilon}), \tag{4.15}$$

and therefore:

$$L^{(j)} u(x) = f(x) - f'(x) = f''(x) = O_{p+\varepsilon}(r^{-s+j+\varepsilon}). \tag{4.16}$$

Since $-s + j + \varepsilon < p$, it follows, by Theorem 1, that there exists a function $v(x) \in C_{m+p+\varepsilon}(Z'(R))$ such that:

$$\begin{cases} L^{(j)} v(x) = f''(x), \quad x \in Z'(R), \\ v(x) = O_{m+p+\varepsilon}(r^{m-s+j+\varepsilon} |\log r|^{\sigma'}) + P(x), \end{cases} \tag{4.17}$$

where σ' is a non-negative integer and $P(x)$ is a polynom. From (4.14), (4.16) and (4.17) it follows that the function $u^*(x) = u(x) - v(x)$ is a solution of $L^{(j)} u^* = 0$ satisfying the condition:

$$D_x^\alpha u^*(x) = O(r^{-s+1}) = o(r^{-[s]}), \quad \text{for } |\alpha| = m - 1. \tag{4.18}$$

Hence, by John's Theorem (VI, section 2) it follows that:

$$u^*(x) = u(x) - v(x) = \sum_{|\beta| \leq q} c_\beta D_y^\beta K^{(j)}(x, 0) + w'(x), \tag{4.19}$$

where $q = [s] - n$, c_β are constants and $w'(x)$ is an analytic function in $Z(R)$.

The required result follows now immediately from (4.17) and (4.19). (If $j < q$, then the sum

$$\sum_{|\beta| < q_1} c_\beta D_y^\beta K^{(j)}(x, 0),$$

is covered by the last term on the right in formula (4.13).).

REMARK. If in addition to the assumptions of Theorem 2 we suppose that s is an integer and

$$u(x) = o(r^{m-s}), \tag{4.20}$$

then by Lemma 2 it follows that:

$$u(x) = o_{m+p+\varepsilon}(r^{m-s}) \tag{4.21}$$

and on the basis of John's Theorem VI it may be shown, exactly as in the proof of Theorem 2, that

$$u(x) = \sum_{q_1 \leq |\beta| \leq q-1} c_\beta D_y^\beta K^{(j)}(x, 0) + w(x) + O_{m+p+\varepsilon}(r^{m-s+j+\varepsilon} |\log r|^{\sigma'}) \tag{4.22}$$

where $q, q_1, c_\beta, w(x)$ and σ' are as in (4.13). If $q = 0$ or $j = 0$ then the first term on the right must be deleted.

Hence we obtain:

COROLLARY 2.1. Under the assumptions of Theorem 2, if $n \leq s$ is an integer and if $u(x)$ satisfies condition (4.20), then:

$$\begin{cases} u(x) = O_{m+p+\varepsilon}(r^{m-s+\varepsilon} |\log r|^{\sigma'}), & (\text{if } j = 0) \\ u(x) = O_{m+p+\varepsilon}(r^{m-s+1} |\log r|), & (\text{if } j \geq 1). \end{cases} \tag{4.23}$$

If n is odd or n is even but $m - s + 1 < 0$ then the factor $|\log r|$ in the second formula may be deleted.

A further result of Theorem 2 is the following:

COROLLARY 2.2. Suppose that n is even. Let L, u and f be as in Theorem 2, with $n \leq s < m + n/2 + 1$ and $j = 0$.

Under these assumptions, if the derivatives of $u(x)$ of order $\nu = m + n/2 - [s]$ belong to $L_2(Z(R_0))$, then:

$$D_x^{\nu} u(x) = O(r^{-n/2-\mu+\varepsilon} \,|\log r|^{\sigma'}), \quad (\mu = s - [s]). \tag{4.24}$$

In the case that $[s] \leq m$, the derivatives of $u(x)$ of order $m - [s]$ are continuous at the origin.

PROOF. By Theorem 2 we have (for $|x| < R < R_1$):

$$u(x) = v(x) + w(x) + O_{m+\varepsilon}(r^{m-s+\varepsilon} \,|\log r|^{\sigma'}), \tag{4.25}$$

where

$$v(x) = \sum_{|\beta| = [s] - n} c_\beta D_y^\beta K^{(0)}(x, 0). \tag{4.26}$$

(All the notations above are exactly as in Theorem 2).

By (4.25) and the assumptions on $u(x)$ it follows that $D_x^\alpha v(x) \in L_2(Z(R_0))$ for $|\alpha| = \nu$. The derivatives $D_x^\alpha v(x)$ are linear combinations of derivatives of $K^{(0)}(x, y)$ of order $m - n/2$.

By (2.9) and (2.10):

$$D_y^{m-n/2} K^{(0)}(x, y) = [h_1(\zeta) + g(x) h_2(\zeta)] \, r^{-n/2} + O(r^{-n/2+1} \,|\log r|), \tag{4.27}$$

where $r = |x|$, $\zeta = x/r$; h_1 and h_2 are analytic functions for $|\zeta| = 1$ and $g(x)$ is an analytic function for $|x| < R$. Hence:

$$D_y^{m-n/2} K^{(0)}(x, 0) = [h_1(\zeta) + g(0) h_2(\zeta)] \, r^{-n/2} + O(r^{-n/2+1} \,|\log r|), \tag{4.28}$$

and therefore:

$$D_x^{\nu} v(x) = h_3(\zeta) \, r^{-n/2} + O(r^{-n/2+1} \,|\log r|), \tag{4.29}$$

where $h_3(\zeta)$ is analytic for $|\zeta| = 1$.

Since $D_x^\nu v(x) \in L_2(Z(R_0))$, it follows immediately from (4.29) that $h_3(\zeta) \equiv 0$. Hence we obtain, by integration:

$$D_x^{\nu-i} v(x) = O(r^{-n/2+1+i} \,|\log r|^2), \tag{4.30}$$

for $i = 0, 1, \ldots$, min. $\left(\frac{n}{2} - 1, \nu\right)$. Also if $n/2 \leq \nu$ it follows that the derivatives of $v(x)$ of order $\nu - n/2 = m - [s]$ are continuous at the origin.

The assertions of Corollary 2.2 follow immediately from these results together with (4.24).

5. Explicit formulas for the coefficients of the asymptotic estimate (4.13).

In this section we derive formulas for the coefficients c_β and the remainder function $w(x)$ of (4.13), in terms of L, u and f and some other functions directly related to these. These formulas will be required in the next section in which we generalize the results of section 4 to the case of an elliptic operator L of the form (1.1), whose coefficients depend not only on x but also on a parameter $t = (t_1, \ldots, t_k)$.

First we shall deal with the case of an operator L with analytic coefficients. It this case, instead of (4.13) we have the formula (2.15) of John's Theorem VI. We shall bring here a proof of this theorem, wich will enable us to calculate the coefficients e^β and the function $w(x)$ of (2.15) in terms of L, $u(x)$ and $K(x, y)$. For simplicity we shall suppose that the point x^0 in VI is the origin.

As in the original proof of John [6] we begin with the following definition:

Definition 4.

Let $v(x)$ be an analytic function in the neighborhood of the origin such that the following limit exists:

$$\lim_{r \to 0} \int_{|x| = r} M[v, u]\, dS_x \equiv H[v], \tag{5.1}$$

where $M[v, u]$ is the bilinear form of formula (2.4) and the normal on the surface of the sphere $|x| = r$ is directed outwards.

Denote by $\Sigma = \Sigma(u)$ the space $\{v(x)\}$ of all functions satisfying the above conditions. Clearly Σ is a linear space and H is a linear operator defined on Σ.

Lemma 7.

If $v(x)$ is an analytic function in the neighborhood of the origin, such that:

$$D_x^\beta v(0) = 0, \qquad \text{for } |\beta| \leq j, \tag{5.2}$$

then $H[v] = 0$.

Let $v'(x)$ be any function analytic in the neighborhood of the origin. Then $v' \in \Sigma$ if and only if $(v')_j \in \Sigma$. If $v' \in \Sigma$ then

$$H[v'] = H[(v')_j]. \tag{5.3}$$

PROOF. From (1.3) it follows by integration that

$$D_x^i u(x) = \begin{cases} o(r^{-n-j+m-1-i}), & m-1-n-j < i \le m-1, \\ o(\log r), & i = m-1-n-j, \\ O(1) & 0 \le i < m-1-n-j. \end{cases} \tag{5.4}$$

Therefore, if $i + k \le m - 1$ then:

$$D_x^i u(x) \cdot D_x^k v(x) = o(r^{-n+1}). \tag{5.5}$$

Hence it follows that $H[v] = 0$.

The second assertion of the lemma is an immediate consequence of the first.

Denote by Σ_j the subspace of Σ which consist of all the polynomials of order $\le j$ belonging to Σ.

LEMMA 8.

If $0 \le j < m$ then Σ_j contains all the polynomials of order $\le j$. Therefore if $v(x)$ is any analytic function in the neighborhood of the origin, then $H[v]$ is defined and we have:

$$H[v] = \sum_{|\beta| \le j} \frac{1}{\beta!} H[x^\beta] D_x^\beta v(0). \tag{5.6}$$

PROOF. Let $v(x)$ be an analytic function in the sphere $|x| \le R < R_0$. Then by formula (2.4), (with v and u interchanged) we have:

$$-\int_{r<|x|<R} u(x)\, \overline{L}\, v dx = \int_{|x|=R} M[v,u]\, dS_x - \int_{|x|=r} M[v,u]\, dS_x, \tag{5.7}$$

the normals on each sphere being directed outwards. From (5.7) it follows that $v(x) \in \Sigma$ if and only if the integral

$$\int_{|x|<R} u(x)\, \overline{L} v(x)\, dx \tag{5.8}$$

exists and if this is the case then:

$$H[v] = \int_{|x|=R} M[v,u]\, dS_x + \int_{|x|<R} u\bar{L}v dx . \tag{5.9}$$

Let $v = x^\beta$ where $|\beta| \leq j$. Then:

$$\int_{r<|x|<R} u(x)\, \bar{L}(x^\beta)\, dx = \int_{r<|x|<R} \{ \sum_{|\alpha| \leq |\beta|} a'_\alpha(x)\, u(x)\, D_x^\alpha(x^\beta)\}\, dx , \tag{5.10}$$

where $a'_\alpha(x)$ are linear combinations of derivatives of the coefficients $a_\alpha(x)$ of L, (see (2.3)). Using integration by parts we obtain:

$$\int_{r<|x|<R} a'_\alpha(x)\, u(x)\, D_x^\alpha(x^\beta)\, dx = \left\{ \int_{|x|=R} - \int_{|x|=r} \right\} M_{\alpha,\beta}[a'_\alpha u, x^\beta]\, dS_x , \tag{5.11}$$

where $M_{\alpha,\beta}$ is a differential bilinear form such that the sum of the orders of the derivatives in each term is $\leq (|\alpha| - 1)$. If $i + k \leq |\alpha| - 1$ $(|\alpha| \leq \leq |\beta| \leq j)$ then by (5.4) it follows:

$$D_x^i(x^\beta)\, D_x^k(a'_\alpha u) = o(r^{-n+1}) . \tag{5.12}$$

Hence, if we let r tend to zero in (5.11) we obtain:

$$\int_{|x|<R} a'_\alpha u D_x^\alpha(x^\beta)\, dx = \int_{|x|=R} M_{\alpha,\beta}[a'_\alpha u, x^\beta]\, dS_x \tag{5.13}$$

and therefore, by (5.10) it follows:

$$\int_{|x|<R} u\bar{L}(x^\beta)\, dx = \int_{|x|=R} M_\beta[u, x^\beta]\, dS_x , \tag{5.14}$$

where

$$M_\beta[u, x^\beta] = \sum_{|\alpha| \leq |\beta|} M_{\alpha,\beta}[a'_\alpha u, x^\beta] . \tag{5.15}$$

Hence it follows that $x^\beta \in \Sigma$ (for $|\beta| \leq j$) and

$$H[x^\beta] = \int_{|x|=R} M[x^\beta, u]\, dS_x + \int_{|x|<R} u\bar{L}(x^\beta)\, dx \tag{5.16}$$

$$= \int_{|x|=R} M[x^\beta, u]\, dS_x + \int_{|x|=R} M_\beta[u, x^\beta]\, dS_x .$$

This proves the first assertion of the lemma. The second assertion follows immediately from the first together with Lemma 7.

We shall prove now formula (2.15) in the case of $j < m$.

Let $\bar{K}(x, y)$ be an analytic g.f.s. of $\bar{L}$ in a neighborhood of the origin containing the sphere $|x| \leq R_1$. Then by Definition 1 we have:

$$u(y) = \int_{r<|x|<R_1} \bar{K}(x, y)\, Lu(x)\, dx + \left\{ \int_{|x|=R} - \int_{|x|=r} \right\} \bar{M}[u(x), \bar{K}(x, y)]\, dS_x \tag{5.17}$$

where $\bar{M}$ is a bilinear form which belongs to $\bar{L}$ in the sense of formula (2.4). Clearly if $M[u, v]$ is a linear form belonging to L then $-M[v, u]$ is a bilinear form belonging to $\bar{L}$, so that in (5.17) we may substitute $-M[\bar{K}, u]$ for $\bar{M}[u, \bar{K}]$. Hence we obtain (remembering that $Lu = 0$):

$$u(y) = \left\{ \int_{|x|=r} - \int_{|x|=R_1} \right\} M[\bar{K}(x, y), u(x)]\, dS_x . \tag{5.18}$$

For a fixed y $(0 < |y| < R_1)$ the function $\bar{K}(x, y)$ is analytic at the origin so that it belongs to Σ. Now if we let r tend to zero in (5.18), we obtain by (5.6):

$$u(y) = \sum_{|\beta| \leq j} \frac{1}{\beta!} H[x^\beta]\, D_x^\beta \bar{K}(0, y) + w_0(y), \tag{5.19}$$

where

$$w_0(y) = -\int_{|x|=R_1} M[\bar{K}(x, y), u(x)]\, dS_x \tag{5.20}$$

is analytic in $Z(R_1)$.

Let $K(x, y)$ be an analytic g.f.s. of L in $Z(R_2)$ and let $0 < R < \min.(R_1, R_2)$. Then by result IV of section 2 we have (for $x, y \in Z(R)$):

$$\bar{K}(x, y) = K(y, x) + \mu(y, x) \tag{5.21}$$

where $\mu(y, x)$ is analytic in $Z(R)$ and:

$$\mu(y, x) = \int_{|\xi| = R} M[\bar{K}(\xi, y), K(\xi, x)] \, dS_\xi , \quad x, y \in Z(R). \tag{5.22}$$

Summing up these results we obtain (2.15) for $|x| < R$, in the case $j < m$, with the following formulas for c_β and $w(x)$:

$$c_\beta = \frac{1}{\beta!} H[x^\beta] = \frac{1}{\beta!} \left\{ \int_{|\xi| = R} M[\xi^\beta, u(\xi)] \, dS_\xi + \int_{|\xi| = R} M_\beta [u(\xi), \xi^\beta] \, dS_\xi \right\}, \tag{5.23}$$

$$\begin{aligned} w(x) &= w_0(x) + \sum_{|\beta| \leq j} c_\beta [D_y^\beta \mu(x, y)]_{y=0} \\ &= - \int_{|\xi| = R} M[\bar{K}(\xi, x), u(\xi)] \, dS_\xi \\ &+ \sum_{|\beta| \leq j} c_\beta \left\{ \int_{|\xi| = R} M[\bar{K}(\xi, x), D_y^\beta K(\xi, y)] \, dS_\xi \right\}_{y=0} \\ &= - \int_{|\xi| = R} M[\bar{K}(\xi, x), w(\xi)] \, dS_\xi . \end{aligned} \tag{5.24}$$

We now proceed to the proof of (2.15) for $m \leq j$. In this case we obtain by integration:

$$u(x) = O(r^{m-1-n-j}), \tag{5.25}$$

and hence by Lemma 2 it follows:

$$u(x) = O_\infty(r^{m-1-n-j}). \tag{5.26}$$

Let $\nu = \left[\frac{j+1}{2}\right] - \frac{m}{2} + 1$. Then $2\nu \leq -m + 1 + n + j$ and by Lemma 5 there exists a function $u_\nu(x) \in C_\infty(Z'(R'))$, $0 < R' < R_0$, such that:

$$\begin{cases} \Delta^\nu u_\nu(x) = u(x), & x \in Z'(R') \\ u_\nu(x) = O_\infty(r^{m-1-n-j+2\nu} |\log r|^{\sigma'}). \end{cases} \tag{5.27}$$

Denote by L^* the operator $L\Delta^\nu$ and let $K^*(x, y)$ be an analytic g.f.s. of L^* in a neighborhood of the origin containing the spere $|x| \leq R^* < R'$.

The function $u_\nu(x)$ is a solution of $L^* u(x) = 0$ for $x \in Z'(R')$ and satisfies the condition:

$$D_x^{m+2\nu-1} u_\nu(x) = o(r^{-n-j-1}) . \tag{5.28}$$

Moreover, since $j + 1 < m + 2\nu$, it follows by the first part of the proof that:

$$u_\nu(x) = \sum_{|\beta| \leq j+1} c_\beta^* D_y^\beta K^*(x, 0) + w^*(x), \tag{5.29}$$

in a certain sphere, say $Z(R_3)$, $(0 < R_3 < R^*)$, where $w^*(x)$ is an analytic function. Hence we obtain:

$$u(x) = \sum_{|\beta| \leq j+1} c_\beta^* D_y^\beta K'(x, 0) + w'(x), \tag{5.30}$$

where $K'(x, y) = \Delta_x^\nu K^*(x, y)$ and $w'(x) = \Delta_x^\nu w^*(x)$.

It is easily verified that $K'(x, y)$ is a g. f. s. of L in $Z(R_3)$. Moreover by result IV of section 2 it is clear that if $K(x, y)$ is another analytic g. f. s. of L in $Z(R_3)$ a formula similar to (5.30), with $K(x, y)$ instead of $K'(x, y)$, holds.

By (5.30) and (1.3) we have:

$$g(x) \equiv D_x^{m-1} \sum_{|\beta| = j+1} c_\beta^* D_y^\beta K'(x, 0) = o(r^{-n-j}). \tag{5.31}$$

Now by (2.9) and (2.10) it follows that:

$$g(x) = h(\xi) r^{-n-j} + O(r^{-n-j+1}), \tag{5.32}$$

where $h(\xi)$ is analytic for $|\xi| = 1$. From these two estimates we conclude that $h(\xi) \equiv 0$ and so:

$$g(x) = O(r^{-n-j+1}). \tag{5.33}$$

Hence, by (5.30):

$$D_x^{m-1} u(x) = O(r^{-n-j+1}), \tag{5.34}$$

and by integration we obtain:

$$u(x) = O(r^{m-n-j}). \tag{5.35}$$

If we replace (5.25) by (5.35) and repeat the argument $\left(\text{with } \nu = \left[\frac{j}{2}\right] - \frac{m}{2} + \frac{1}{2}\right)$ we obtain (2.15), which is the required result. Following this argument, the formulas for c_β and $w(x)$ are easily derived on the basis of (5.23) and (5.24). Of course these formulas will now depend not only on L, K and u but also on L^*, K^*, and u_ν. We remark that by the proof of Lemma 5, $u_\nu(x)$ may be constructed in terms of $u(x)$ and a fundamental solution of Δ.

It remains now to deal with the more general case described in Theorem 2. But from the proof of this theorem it is clear that the coefficients c_β and the function $w(x)$ of (4.13) may be calculated exactly as in the former case with $L^{(j)}$ and $K^{(j)}(x, y)$ instead of L and K and $u^*(x) = u(x) - v(x)$ instead of $u(x)$. Moreover by the proof of the Theorem 1 and the lemmas leading to it, it is clear that the function $v(x)$ mentioned in Theorem 2, may be constructed in terms of L, $L^{(j)}$, $K^{(j)}$, $u(x)$ and $f(x)$.

6. Operators whose coefficients depend on a parameter t.

In this section we shall be concerned with an elliptic operator of the form :

$$L(x, t, D_x) = \sum_{|\alpha| \leq m} a_\alpha(x, t)\, D_x^\alpha\,, \tag{6.1}$$

where $t = (t_1, \dots, t_k)$. We shall assume that the coefficients are defined in a domain $\mathcal{D} \times \mathcal{T}$, $\mathcal{D}$ and $\mathcal{T}$ being domains in E_n and E_k respectively. We shall also assume that the operator L is uniformly elliptic in $\mathcal{D} \times \mathcal{T}$, i. e. that L is elliptic in $\mathcal{D}$ uniformly with respect to (x, t) in $\mathcal{D} \times \mathcal{T}$.

We begin with a few definitions and general remarks.

We denote by x, y points in E_n and by t a general point in E_k. Also we denote by $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ a vector with complex components.

DEFINITION 5.

Let $f(x, t) \in C_0(\mathcal{D} \times \mathcal{T})$ and suppose that all the derivatives of the form $D_x^\alpha D_t^\varrho f(x, t)$ with $|\alpha| \leq p$ and $|\varrho| \leq q$ exist and are continuous in $\mathcal{D} \times \mathcal{T}$. Then we shall say that $f(x, t) \in C_{p,q}(x\,; t)$ in $\mathcal{D} \times \mathcal{T}$.

If moreover the derivatives of the form $D_x^\alpha D_t^\varrho f(x, t)$, with $|\alpha| = p$ and $|\varrho| \leq q$, are Hölder continuous with exponent ε in $\mathcal{D}$, uniformly with respect to t in every compact subdomain of $\mathcal{T}$, we shall say that $f(x, t) \in C_{p+\varepsilon, q}(x\,; t)$ in $\mathcal{D} \times \mathcal{T}$.

If $f(x, t) \in C_{a,q}(x\,;t)$ in $\mathcal{D} \times \mathcal{T}$, where a is a non-negative real number and q is a non-negative integer, and if $\mathcal{D}$ contains the origin, then the notation:

$$(6.2) \qquad f(x, t) = O_{a,q}(r^s\,|\log r\,|^\sigma), \quad (t \in \mathcal{T},\ r = |\,x\,| \rightarrow 0)$$

means that for every fixed t in $\mathcal{T}$:

$$(6.3) \qquad D_t^\varrho f(x, t) = O_a(r^s\,|\log r\,|^\sigma), \quad (r = |\,x\,| \rightarrow 0)$$

and that (6.3) is uniform with respect to t in every compact subdomain of $\mathcal{T}$.

The notation $o_{a,q}$ is similarly defined.

DEFINITION 6.

Let $f(x, t) \in C_{\infty,q}(x\,;t)$ in $\mathcal{D} \times \mathcal{T}$ and suppose that the functions $D_t^\varrho f(x, t)$ $(|\,\varrho\,| \leq q)$, are analytic in $\mathcal{D}$ for every fixed t in $\mathcal{T}$. Let $x^0 \in \mathcal{D}$ and let $f^{(\varrho)}(\tilde{x}, t)$ be the analytic continuation of $D_t^\varrho f(x, t)$ $(|\,\varrho\,| \leq q)$ for fixed $t \in \mathcal{T}$, in a complex neighborhood of x^0, possibly depending on t. Suppose that for every $x^0 \in \mathcal{D}$ and every compact subset $\mathcal{T}'$ of $\mathcal{T}$ there exists a positive R such that the functions $f^{(\varrho)}(\tilde{x}, t)$ $(|\,\varrho\,| \leq q)$ satisfy the following conditions:

(i) If $t \in \mathcal{T}'$ then $f^{(\varrho)}(\tilde{x}, t)$ is analytic in the complex domain $|\tilde{x}_i - x_i^0| < R$, $(i = 1, \ldots, n)$.

(ii) $f^{(\varrho)}(\tilde{x}, t)$ is continuous in $(\tilde{x}, t)$ for $\tilde{x}$ in the above complex domain and $t \in \mathcal{T}'$.

Then we shall say that $f(x, t) \in \mathcal{A}_q(x\,;t)$ in $\mathcal{D} \times \mathcal{T}$.

Let $R(x^0, \mathcal{T}')$ be the l. u. b. of the set of numbers $\{R\}$ for which conditions (i) and (ii) are fulfilled. We shall say that $R(x^0, \mathcal{T}')$ is the convergence radius of $f(x, t)$ about x^0, with respect to $\mathcal{T}'$.

REMARKS.

(I) Let $h_i(x)$ and $g_i(t)$, $(i = 1, \ldots, \nu)$ be two sets of functions such that $h_i(x)$ is analytic in $\mathcal{D}$ and $g_i(t) \in C_q(\mathcal{T})$. Then:

$$(6.4) \qquad f(x, t) = \sum_{i=0}^{\nu} g_i(t)\, h_i(x) \in \mathcal{A}_q(x\,;t) \quad \text{in} \quad \mathcal{D} \times \mathcal{T}.$$

(II) Suppose that $f(x, t) \in \mathcal{A}_q(x\,;t)$ in $\mathcal{D} \times \mathcal{T}$ and let $x^0 \in D$. Then we may expand $f(x, t)$ in a power series about x^0.

$$(6.5) \qquad f(x, t) = \sum_{0 \leq |\alpha|} \frac{1}{\alpha!} D_x^\alpha f(x^0, t) \cdot (x - x^0)^\alpha = \sum_{0 \leq |\alpha|} b_\alpha(t)\,(x - x^0)^\alpha .$$

Let $\mathcal{J}'$ be a compact subset of $\mathcal{J}$ and let $0 < R' < R$, where $R = R(x^0, \mathcal{J}')$ is the convergence radius. Then by Cauchy's formula:

$$b_\alpha(t) = \frac{1}{(2\pi i)^n} \oint_{K_1} \dots \oint_{K_n} \frac{f(\tilde{x}, t)}{(\tilde{x} - x^0)^{\alpha'}} d\tilde{x}, \qquad t \in \mathcal{J}' \tag{6.6}$$

where K_i is the circle $|\tilde{x}_i - x_i^0| = (R + R')/2$, $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\alpha' = (\alpha_1 + 1, \dots, \alpha_n + 1)$. From this formula and Definition 6 it is clear that there exist fixed numbers B_α such that:

$$|b_\alpha(t)| \leq B_\alpha \quad \text{for} \quad t \in \mathcal{J}', \tag{6.7}$$

and the series

$$\sum_{0 \leq |\alpha|} B_\alpha (x - x^0)^\alpha \tag{6.8}$$

converges in the complex domain $|\tilde{x}_i - x_i^0| < R'$, $(i = 1, \dots, n)$. This series will be called a majorant of $f(x, t)$ at x^0 with respect to $\mathcal{J}'$.

(III) Clearly we have:

$$D_t^\varrho f(x, t) = \sum_{0 \leq |\alpha|} D_t^\varrho b_\alpha(t) (x - x^0)^\alpha, \qquad |\varrho| \leq q \tag{6.9}$$

and these functions also possess a majorant at x^0 with respect to every compact subset of $\mathcal{J}$. It is now evident that the derivatives $D_t^\varrho f(\tilde{x}, t)$ $(|\varrho| \leq q)$ exist and are identical with the functions $f^{(\varrho)}(\tilde{x}, t)$ mentioned in Definition 6.

(IV) Suppose that the coefficients of (6.1) belong to $\mathcal{A}_q(x; t)$ in $\mathcal{D} \times \mathcal{J}$ and let $K(x, y, t)$ be a g. f. s. of $L(x, t, D_x)$ in $\mathcal{D}$, of the type constructed by John (see section 2). John's proof of the existence of a fundamental solution depends strongly on the Cauchy-Kowalevsky theorem for systems of linear partial differential equations. If we examine the proof of this theorem in the case of a system whose coefficients depend not only on x but also on a parameter $t = (t_1, \dots, t_k)$, then in the light of the former remarks we conclude the following:

If the coefficients of the system of equations mentioned above belong to $\mathcal{A}_q(x; t)$ in a domain $\mathcal{D} \times \mathcal{J}$ and if $x^0 \in \mathcal{D}$, then the solution $u(x, t)$ of Cauchy's problem for this system with initial conditions on a non-characteristic hyperplane through x^0, belongs to $\mathcal{A}_q(x; t)$ in a domain $\mathcal{D}' \times \mathcal{J}$ where $\mathcal{D}'$ is some neighborhood of x^0.

If now we follow John's construction of the fundamental solution, taking into account the above remark we obtain:

THEOREM 3.

Let L be the operator defined by (6.1). Suppose that the coefficients belong to $\mathcal{A}_q(x;t)$ in $\mathcal{D}\times\mathcal{T}$ and that L is uniformly elliptic in $\mathcal{D}\times\mathcal{T}$. Let j be a positive integer tuch that $n+j$ is even. Suppose that $Z(x^0, R_0)\subset\mathcal{D}$. Then there exists a function $W_j(x, y, t)$ and a sphere $Z(x^0, R_1)$, $0<R_1<R_0$, such that W_j belongs to $\mathcal{A}_q(x, y; t)$ in the domain

$$\{(x, y, t): \ x, y \in Z(x^0, R_1), \ x \neq y, \ t \in \mathcal{T}\} \tag{6.10}$$

and possesses the following properties:

(i) If $\mathcal{T}'$ is a compact subset of $\mathcal{T}$, there exists a number $0<R^*<R_1$ such that the function:

$$K_j(x, y, t) = \Delta_y^{(n+j)/2}\, W_j(x, y, t) \tag{6.11}$$

is an analytic g. f. s. of L in $Z(x^0, R^*)$ for every $t\in\mathcal{T}'$.

(ii) If $0<R<R^*$, $x, y\in Z(x^0, R)$ and $t\in\mathcal{T}'$ then:

$$| D_{xy}^{i}\, D_t^{i'}\, W_j(x, y, t) | \leq \begin{cases} \text{con. } \underline{r}^{m+j-i}, & (n \text{ odd}) \\ \text{con. } \underline{r}^{m+j-i}(1 + |\log \underline{r}|), & (n \text{ even}) \end{cases} \tag{6.12}$$

where $i=0, 1, \ldots$; $i'=0, 1, \ldots, q$; $\underline{r}=|x-y|$, and the constant depends on i, i', R, $\mathcal{T}'$.

As in section 2, we shall drop out the index j of $K_j(x, y, t)$ and we shall denote by $K(x, y, t)$ any g. f. s. of the type (6.11).

In addition to the above it may be shown that:

$$K(x, y, t) = \underline{r}^{m-n} \sum_{\nu=0}^{\infty} c_\nu(y, \xi, t)\, \underline{r}^\nu + w(x, y, t) \log \underline{r}, \tag{6.13}$$

where $\xi=(x-y)/\underline{r}$. The series in (6.13) defines a function belonging to $\mathcal{A}_q(\underline{r}, y, \xi; t)$ for $(\underline{r}, y, \xi)$ in a certain neighborhood of $\underline{r}=0$, $y=x^0$ and $|\xi|=1$ and $t\in\mathcal{T}$. Therefore the coefficients $c_\nu(y, \xi, t)\in\mathcal{A}_q(y, \xi; t)$ for (y, ξ) in a corresponding neighborhood of $y=x^0$ and $|\xi|=1$ and $t\in\mathcal{T}$. The function $w(x, y, t)$ belongs to $\mathcal{A}_q(x, y; t)$ for (x, y) in a neighborhood of $x=y=x^0$ and $t\in\mathcal{T}$. For every fixed t, this function has the properties described in result I of section 2. In particular for n even and $m\geq n$ the limit:

$$\lim_{x \to y} r^{n-m} w(x, y, t) = c_0'(y, \xi, t) \tag{6.14}$$

exists and is attained uniformly with respect to ξ (for $|\xi| = 1$) and t in every compact subset of $\mathcal{J}$.

DEFINITION 7.

Let $V(x, y, t) \in \mathcal{A}_q(x, y; t)$ in a domain:

$$\{(x, y, t): x, y \in \mathcal{D}, x \neq y, t \in \mathcal{J}\}. \tag{6.15}$$

We shall say that $V(x, y, t)$ is an $\mathcal{A}_q$ g. f. s. of L in $\mathcal{D} \times \mathcal{J}$ if for every fixed $t \in \mathcal{J}$, $V(x, y, t)$ is a g. f. s. of L in $\mathcal{D}$.

By Theorem 3 and Definition 7 it is clear that results similar to III, IV, V of section 2 hold also in the case treated here. The result parallel to IV may be formulated as follows:

Let $V(x, y, t)$ [resp. $\overline{V}(x, y, t)$] be an $\mathcal{A}_q$ g. f. s. of $L(x, t, D_x)$ [resp. $\overline{L}(x, t, D_x)$] in a domain $\mathcal{D} \times \mathcal{J}$ and suppose that the coefficients of L belong to $\mathcal{A}_q(x; t)$ in $\mathcal{D} \times \mathcal{J}$. Then:

$$\overline{V}(x, y, t) = V(y, x, t) + \mu(y, x, t) \tag{6.16}$$

where $\mu(y, x, t) \in \mathcal{A}_q(x, y; t)$ in $\mathcal{D} \times \mathcal{D} \times \mathcal{J}$. If $x, y \in Z(x^0, R) \subset\subset \mathcal{D}$ then for every $t \in \mathcal{J}$ we have:

$$\mu(y, x, t) = \int_{|\xi - x^0| = R} M[\overline{V}(\xi, y, t), V(\xi, x, t)] \, dS_\xi . \tag{6.17}$$

From this result and Theorem 3 it follows that two $\mathcal{A}_q$ g. f. s. of L in $\mathcal{D} \times \mathcal{J}$ differ only by a function $w^*(x, y, t)$ belonging to $\mathcal{A}_q(x, y; t)$ in $\mathcal{D} \times \mathcal{D} \times \mathcal{J}$.

On the basis of Lemma 1 we obtain the following:

LEMMA 9.

Suppose that the operator (6.1) is uniformly elliptic in $Z(R_0) \times \mathcal{J}$. Let $u(x, t)$ be a solution of $Lu(x, t) = f(x, t)$ in $Z'(R_0)$ for every $t \in \mathcal{J}$, such that $u(x, t) \in C_{m+p, q}(x; t)$ in $Z'(R_0) \times \mathcal{J}$. Suppose that the coefficients of L belong to $C_{p+\varepsilon, q}(x; t)$ in $Z(R_0) \times \mathcal{J}$ and $f(x, t) \in C_{p+\varepsilon, q}(x; t)$ in $Z'(R_0) \times \mathcal{J}$. Then: $u(x) \in C_{m+p+\varepsilon, q}(x; t)$ in $Z'(R_0) \times \mathcal{J}$.

If in addition to the above we assume :

(6.18) $$u(x,t) = O_{0,q}(r^s \mid \log r \mid^{\sigma'}), \qquad (t \in \mathcal{J},\ r = \mid x \mid \to 0).$$

(6.19) $$f(x,t) = O_{p+\varepsilon,q}(r^{s-m} \mid \log r \mid^{\sigma'}), \qquad (t \in \mathcal{J},\ r = \mid x \mid \to 0),$$

where $0 < \sigma$ and s is any real number, then :

(6.20) $$u(x,t) = O_{m+p+\varepsilon,q}(r^s \mid \log r \mid^{\sigma'}), \qquad (t \in \mathcal{J}).$$

PROOF. In the case $q = 0$ this lemma is a direct consequence of Lemma 1. For $q > 0$ we obtain the required result by induction, using the formula :

(6.21) $$L_x D_t^{\varrho} u(x,t) = D_t^{\varrho} f(x,t) + A_{\varrho} u(x,t),$$

where $A_{\varrho} u$ is a sum of terms of the form :

(6.22) $$D_t^{\varrho'} a_{\alpha}(x,t) \cdot D_t^{\varrho''} D_x^{\alpha} u(x,t),$$

with

$$\mid \alpha \mid \leq m,\ \varrho' + \varrho'' = \varrho,\ \mid \varrho'' \mid < \mid \varrho \mid.$$

Using the results described in this section we obtain the following theorem, which is parallel to Theorem 1 :

THEOREM 4.

Let $L(x, t, D_x)$ be a uniformly elliptic operator in $\mathcal{D} \times \mathcal{J}$ whose coefficients belong to $\mathcal{A}_q(x;t)$ in this domain. Suppose that $\mathcal{D}$ contains the sphere $Z(R_0)$. Let $f(x,t)$ be a function of $C_{a,q}(x;t)$ in $Z'(R_0) \times \mathcal{J}$ (where $0 < a$ is not an integer) such that:

(6.23) $$f(x,t) = O_{a,q}(r^{-s} \mid \log r \mid^{\sigma}),$$

with $-[a] \leq s$ a real number and $0 \leq \sigma$ and integer.

Let $\mathcal{J}'$ be a compact subdomain of $\mathcal{J}$ and let R^* be the number mentioned in Theorem 3 (i). Then, if $0 < R < R^*$, there exists a solution $u(x,t)$ of $Lu(x,t) = f(x,t)$ in $Z'(R) \times \mathcal{J}'$ such that:

(*a*) $u(x,t) \in C_{m+a,q}(x;t)$ in $Z'(R) \times \mathcal{J}'$.

(*b*) If $0 < m - s$ and s' is defined as in Lemma 4, then $u(x,t) \in C_{s',q}(x;t)$ in $Z(R) \times \mathcal{J}'$.

(*c*) $$u(x,t) = O_{m+a,q}(r^{m-s} \mid \log r \mid^{\sigma'}) + P(x,t),$$

where $\sigma \leq \sigma'$ is an integer (and if n is odd and s is not an integer then $\sigma = \sigma'$) and

$$P(x, t) = \begin{cases} 0 & , \quad m \leq s, \\ \sum_{|\beta| \leq s'} (D_x^\beta u(0, t) / \beta !) x^\beta , & m > s. \end{cases}$$

This theorem is proved with the aid of three lemmas which are parallel to lemmas 3, 4, 5 by which Theorem 1 is proved. (We shall denote them by the numbers 3′, 4′, 5′ respectively). We write here in detail only the first of them:

LEMMA 3′

Let $G(x, y, t) \in \mathcal{A}_q(x, y; t)$ in the domain:

$$\{(x, y, t) : x, y \in Z(R_0), \quad x \neq y, t \in \mathcal{J}\}, \tag{6.24}$$

and suppose that it satisfies there the inequalities:

$$| D_{xy}^i D_t^{i'} G(x, y, t) | \leq \text{con} \cdot \underline{r}^{p'-n-i}(|\log \underline{r}|^{\sigma'} + 1) \qquad \begin{cases} i = 0, 1, 2, \ldots \\ i' = 0, 1, \ldots, q \end{cases} \tag{6.25}$$

where $1 < p'$ and $0 \leq \sigma'$ are integers and $\underline{r} = |x - y|$.

Let j be an integer, $0 \leq j < p' - \bar{1}$ and $p = p' - (j + 1)$. Let $f(x, t) \in C_{j+\varepsilon, q}(x; t)$ in $Z'(R_0) \times \mathcal{J}$, $0 < \varepsilon < 1$, such that:

$$f(x, t) = O_{j+\varepsilon, q}(r^{-s} |\log r|^\sigma), \quad |x| = r \to 0, t \in \mathcal{J} \tag{6.26}$$

s being a real number, $-j \leq s < n$, and $0 \leq \sigma$ an integer.

Denote

$$F(x, y, t) = D_y^\beta G(x, y, t), \qquad (|\beta| = j + 1) \tag{6.27}$$

and

$$u(x, t) = \int_{Z(R)} F(x, y, t) f(y, t)\, dy \qquad (0 < R < R_0). \tag{6.28}$$

Then $u(x, t) \in C_{p+j, q}(Z'(R))$ and if $s < p$ then $u(x, t) \in C_{s'}(Z(R))$, (s' as in Lemma 3). Moreover we have:

$$u(x, t) = O_{p+j, q}(r^{p-s} |\log r|^{\sigma''}) + P(x, t) \tag{6.29}$$

σ'' being defined as in Lemma 3 and

$$P(x, t) = \begin{cases} 0 & \text{if } s \geq p \\ \sum_{|\beta| \leq s'} (D_x^\beta u(0, t) / \beta !) x^\beta & \text{if } s < p. \end{cases}$$

The proof of this lemma is essentially the same as that of Lemma 3 since the following formula holds:

$$D_t^\varrho u(x, t) = \int_{Z(0, R)} D_t^\varrho [F(x, y, t) f(y, t)] dy, \qquad |\varrho| \leq q. \tag{6.30}$$

Lemmas 4′ and 5′ stand exactly in the same relation to Lemmas 4 and 5 as Theorem 4 to Theorem 1. Their proofs are essentially the same as the proofs of the last two lemmas. Of course, we have to replace Lemmas 2 and 3 by Lemmas 9 and 3′. In both cases the first step is the proof of the existence and continuity of all derivatives of $u(x, t)$ (respectively $u_\nu(x, t)$) mentioned in the formulation of the lemma (property (a_j)). In order to prove the other properties of $u(x, t)$ (in Lemma 4′) we may first suppose that $q = 0$ and then obtain the general result by induction on q, (see formula (6.21)). In Lemma 5′ this induction is trivial since the operator in this case is a power of the Laplacian Δ_x.

The proof of Theorem 4 is exactly parallel to that of Theorem 1.

By John's Theorem (VI section 2) and the formulas obtained in the preceding section we obtain:

THEOREM 5.

Suppose that the operator (6.1) is uniformly elliptic in $\mathcal{D} \times \mathcal{J}$ with coefficients belonging to $\mathcal{A}_q(x ; t)$ in $\mathcal{D} \times \mathcal{J}$. Suppose that $Z(R_0) \subset \mathcal{D}$. Let $u(x, t)$ be a solution of $Lu(x, t) = 0$ in $Z'(R_0) \times \mathcal{J}$, such that $u(x, t) \in C_{m, q}(x ; t)$ in this domain. Let $\mathcal{J}'$ be a compact subdomain of $\mathcal{J}$ and let $K(x, y, t)$ be an $\mathcal{A}_q$ g. f. s. of L in $Z(R^*) \times \mathcal{J}'$ (R^* being a certain number in the interval $(0, R_0)$).

If $u(x, t)$ has a finite singularity at $x = 0$, such that for $|\alpha| = m - 1$:

$$D_x^\alpha u(x, t) = o(r^{-n-j}), \qquad |x| = r \to 0 \tag{6.31}$$

for any fixed point $t \in \mathcal{J}$, and in addition to this

$$D_x^\alpha u(x, t) = O_{0,q}(r^{-n-j}), \qquad (t \in \mathcal{J}, |x| = r \to 0) \tag{6.32}$$

j being a non negative integer, then we have:

$$u(x, t) = \sum_{|\beta| \leq j} c_\beta(t) [D_y^\beta K(x, 0, t)] + w(x, t), \tag{6.33}$$

for $(x, t) \in Z(R^*) \times \mathcal{J}'$, where $w(x, t) \in \mathcal{A}_q(x ; t)$ in $Z(R^*) \times \mathcal{J}'$ and $c_\beta(t) \in C_q(\mathcal{J}')$.

In the case $0 \leq j < 2m$ the assumption (6.32) is not required. Moreover using formula (6.33) it is seen that in this case (6.32) may be obtained as a result of this theorem.

PROOF. Formula (6.33) follows by John's Theorem on the basis of (6.31). We have to prove here only the assertions concerning $c_\beta(t)$ and $w(x, t)$.

In the case $0 \leq j < m$ these assertions follow immediately from formulas (5.23) and (5.24), taking into account the properties of $K(x, y, t)$ and the fact that $u(x, t) \in C_{m,q}(x; t)$ in $Z'(R_0) \times \mathcal{T}$. The assumption (6.32) is not needed in this case.

If $m \leq j$ the required results follow by an argument similar to that described in the last part of section 5. For this argument we need also estimate (6.32).

Since $u(x, t)$ is a solution of $Lu(x, t) = 0$, and the coefficients of L belong to $\mathcal{A}_q(x; t)$ it follows that $u(x, t) \in \mathcal{A}_q(x, t)$ in the domain $Z'(R_0) \times \mathcal{T}$. This result is parallel to V section 2 and it may be proved exactly in the same way as result V, (John [7] pp. 57), using the properties of $K(x, y, t)$.

From (6.32) we obtain by integration:

$$u(x, t) = O_{0,q}(r^{m-1-n-j}), \qquad (t \in \mathcal{T}, |x| = r \to 0) \tag{6.34}$$

and, since $u(x, t) \in \mathcal{A}_q(x; t)$ in $Z'(R_0) \times \mathcal{T}$, by Lemma 9 it follows that:

$$u(x, t) = O_{\infty,q}(r^{m-1-n-j}), \qquad (t \in \mathcal{T}, |x| = r \to 0). \tag{6.35}$$

From this point on the argument is exactly parallel to that of section 5 beginning with (5.26). Of course we shall use Lemma 5′ instead of Lemma 5, and the results of the present theorem in the case $j < m$.

On the basis of theorems 4 and 5 it is possible to obtain a generalization of Theorem 2 to operators of the form (6.1) whose coefficients satisfy certain regularity conditions. We shall formulate here only a special case of this generalization, which will be needed later in the proof of some other results.

THEOREM 6.

Suppose that the operator (6.1) is uniformly elliptic in $Z(R_0) \times \mathcal{T}$ and that his coefficients belong to $C_{\varepsilon,q}(x; t)$ in this domain, q being a non-negative integer and $0 < \varepsilon < 1$.

Let $u(x, t)$ be a solution of $Lu(x, t) = f(x, t)$ in $Z'(R_0) \times \mathcal{T}$ belonging to $C_{m,q}(x; t)$ in this domain. Suppose that $u(x, t)$ has a singularity of finite order at $x = 0$ such that:

$$u(x, t) = O_{0,q}(r^{m-s}), \qquad (t \in \mathcal{I}, \ |x| = r \to 0) \tag{6.36}$$

where $n \leq s$ is a real number and $0 \leq \mu = s - [s] < \varepsilon$.

Let $L^{(0)}$ be the osculating operator:

$$L^{(0)} = \sum_{|\alpha| = 2m} a_\alpha(o, t) D_x^\alpha, \tag{6.37}$$

and let $K^{(0)}(x, y, t)$ be an $\mathcal{A}_q$ g.f.s. of $L^{(0)}$. (In this case $K^{(0)}$ will be an $\mathcal{A}_q$ g.f.s. of $L^{(0)}$ in $E_n \times \mathcal{I}$).

If $f(x, t) \in C_{\varepsilon,q}(x; t)$ in $Z'(R_0) \times \mathcal{I}$ and satisfies the condition:

$$f(x, t) = O_{\varepsilon,q}(r^{-s+\varepsilon})$$

then we have, for $|x| < R < R_0$ and $t \in \mathcal{I}$:

$$u(x, t) = \sum_{|\beta| = \nu} c_\beta(t) D_y^\beta K^{(0)}(x, 0, t) + w(x, t) + O_{m+\varepsilon,q}(r^{m-s+\varepsilon} |\log r|^{\sigma'}), \tag{6.38}$$

where $\nu = [s] - n$, $c_\beta(t) \in C_q(\mathcal{I})$, $w(x, t) \in \mathcal{A}_q(x, t)$ in $Z(R) \times \mathcal{I}$ and $0 \leq \sigma'$ is an integer.

The proof is exactly parallel to that of Theorem 2 (in the special case $j = 0$). We have only to replace Lemma 2 by Lemma 9, Theorem 1 by Theorem 4 and John's Theorem (V section 2) by Theorem 5.

From Theorem 6 and the remark to Theorem 2 it follows:

COROLLARY 6.1.

Under the assumptions of Theorem 6, if $n \leq s$ in an integer and if $u(x, t)$ satisfies the additional condition:

$$u(x, t) = o(r^{m-s}), \qquad (|x| = r \to 0) \tag{6.39}$$

for every fixed $t \in \mathcal{I}$, then:

$$u(x, t) = w(x, t) + O_{m+\varepsilon,q}(r^{m-s+\varepsilon} |\log r|^{\sigma'}) \tag{6.40}$$

where $w(x, t)$ and σ' are as in (6.38). Hence we obtain:

$$u(x, t) = O_{m+\varepsilon,q}(r^{m-s+\varepsilon} |\log r|^{\sigma'}), \qquad (t \in \mathcal{I}, \ |x| = r \to 0). \tag{6.41}$$

COROLLARY 6.2.

Suppose that n is even. Assume all the assumptions of Thorem 6 with $n \leq s < m + n/2 + 1$ and in addition suppose that:

$$D_x^\alpha u(x, t) \in L_2(Z(R_0) \times \mathcal{I}), \tag{6.42}$$

for $|\alpha| = m + n/2 - [s] \equiv \nu$. Then:

$$D_x^\alpha u(x, t) = O_{0,q}(r^{-n/2-\mu+\varepsilon} |\log r|^{\sigma'}), \quad (t \in \mathcal{I}, |x| = r \to 0) \tag{6.43}$$

for $|\alpha| = \nu$. If moreover $[s] \leq m$, the derivatives $D_t^\varrho D_x^\beta u(x, t)$, with $|\varrho| \leq q$ and $|\beta| = m - [s]$, are bounded in every compact subdomain of $Z(R_0) \times \mathcal{I}$. Also these derivatives are continuous functions of x in $Z(R_0)$ for every fixed $t \in \mathcal{I}$.

The estimate (6.43) may be proved by an argument similar to that described in the proof of Corollary 2.2. The last assertion of the present corollary follow from (6.43) by integration.

APPENDIX A.

We bring here, without proof, three lemmas of potential theory which are frequently used in this paper. Although these lemmas are well known they are usually formulated only for special cases. The essential ideas of these lemmas may be found in the paper of E. Hopf [5].

LEMMA A.1.

Let $\mathcal{D}$ and $\mathcal{I}$ be bounded domains in E_n and E_k respectively. Let $F(z, y, t)$ be a function of $2n + k$ variables which is continuos in the domain:

$$\{(x, y, t) : x, y \in \mathcal{D},\ x \neq y,\ t \in \mathcal{I}\}. \tag{A.1}$$

Suppose that $F(x, y, t)$ satisfies the following conditions:

(i) For every compact subset K of $\mathcal{D}$ there exists a constant $c_1(K)$ such that

$$|F(x, y, t)| \leq c_1(K)\, r^{-n+\mu}, \qquad (x, y \in K,\ t \in \mathcal{I}), \tag{A.2}$$

where $r = |x - y|$ and $0 < \mu < 1$.

(ii) For every compact subset K of $\mathcal{D}$ and every positive δ there exists a constant $c_2(K, \delta)$ and a function $G_K(y)$ belonging to $L_1(\mathcal{D})$, such that

$$(A.3) \qquad | F(x, y, t) | \leq c_2 (K, \delta)\, G_K(y)$$

in the set $\{(x, y, t) : x \in K,\ y \in \mathcal{D},\ |x - y| \geq \delta,\ t \in \mathcal{I}\}$.

Under these conditions the function

$$(A.4) \qquad \psi(x, t) = \int_{\mathcal{D}} F(x, y, t)\, dy$$

is continuous in $\mathcal{D} \times \mathcal{I}$.

LEMMA A.2.

Let $F(x, y, t)$ be the function mentioned in the preceding lemma. Suppose that there exists a partial derivative $DF(x,y,t)$ $\left(\text{where } D \text{ representes an operator of the form } \frac{\partial}{\partial x_i} \text{ or } \frac{\partial}{\partial t_i}\right)$ which is continuous in the domain (A.1). Suppose also that $F(x, y, t)$ and $DF(x, y, t)$ satisfy condition (i) and (ii) of Lemma A.1.

Let $\psi(x, t)$ be the function defined by (A.4). Then the partial derivative $D\psi(x, t)$ exists and is contiuuous in $\mathcal{D} \times \mathcal{I}$. This derivative is given by the formula :

$$(A.5) \qquad D\psi(x, t) = \int_{\mathcal{D}} DF(x, y, t)\, dy.$$

LEMMA A.3.

Let $G(x, y)$ be a function belonging to C_2 in the set $\{(x, y) : |x|, |y| \leq R,\ x \neq y\}$, such that:

$$(A.6) \qquad | D^i_{xy}\, G(x, y) | \leq \text{con.}\, r^{-n+\mu-i+1}\,, \qquad (i = 1, 2),$$

where $0 < \mu < 1$. Put $F(x, y) = D_{y_n} G(x, y)$.

Suppose that $f(x) \in C_\varepsilon \cap L_1$ in $Z'(R)$, where $1 - \mu < \varepsilon < 1$. Then the function

$$(A.7) \qquad \varphi(x) = \int_{Z(R)} F(x, y)\, dy$$

belongs to $C_2(Z(R)$ while the function

$$(A.8) \qquad \psi(x) = \int_{Z(R)} F(x, y) f(y)\, dy$$

belongs to $C_1(Z'(R))$, and

$$D_{x_i}\psi(x) = \int_{Z(R)} D_{x_i} F(x,y)\,[f(y) - f(x)]\,dy + f(x)\,D_{x_i}\varphi(x). \tag{A.9}$$

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