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# MIXED BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS IN THE PLANE. THE $L^p$ THEORY

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## 0. Introduction.

The main topic of this article is the mixed boundary value problem for higher order elliptic equations in a bounded planar domain  $\Omega$ . The boundary  $\partial\Omega$  of  $\Omega$  is divided by two points  $P_1, P_2$  on it:  $\partial\Omega = \partial^-\Omega \cup \partial^+\Omega$ .  $L$  is a properly elliptic operator of order  $2m$  in  $\Omega$ ;  $\{B_j^-\}, \{B_j^+\}$ ,  $1 \leq j \leq m$ , are two sets of boundary operators. The problem is to find a solution  $u$  satisfying

$$(0.1) \quad \begin{aligned} Lu &= f && \text{in } \Omega, \\ B_j^- u &= \varphi_j^- && \text{on } \partial^-\Omega, \quad B_j^+ u = \varphi_j^+ && \text{on } \partial^+\Omega, \quad i \leq j \leq m. \end{aligned}$$

$f, \varphi_j^-, \varphi_j^+$  are given functions (the data of the problem).

Introducing the map

$$Tu = (Lu|_{\Omega}, B_j^- u|_{\partial^-\Omega}, B_j^+ u|_{\partial^+\Omega})_{j=1}^m,$$

we have to study the equation  $Tu = (f, \varphi_j^-, \varphi_j^+)_{j=1}^m$  in appropriate Banach spaces. Here we take the Sobolev spaces  $W^{s+2m, p}(\Omega)$ ,  $1 < p < \infty$ ,  $s \geq 0$ . For integral values of  $s$ , this is the space of functions whose derivatives up to order  $s + 2m$  are in  $L^p(\Omega)$ . For fractional  $s$ , it is given by a certain interpolation method. The space of data is denoted by  $\Pi^{s, p}$ .

Pure boundary problems (in which one set of boundary operators is given for the whole boundary and the definition of  $T$  is accordingly modified) were widely studied in recent years [1, 2, 4, 5, 9, 11, 14, 16]. Mixed problems were studied in [15, 17, 19]. The modern approach is based on

obtaining a priori estimates of the form

$$(0.2) \quad \|u\|_{W^{2m+s, p}(\Omega)} \leq K (\|Tu\|_{H^{s, p}(\Omega)} + \|u\|_{(0)}),$$

and of the dual form (employed by Peetre [14] for  $p = 2$ )

$$(0.3) \quad \|V\|_{(H^{s, p}(\Omega))^*} \leq K (\|T^*V\|_{(W^{2m+s, p}(\Omega))^*} + \|V\|_{(0)}).$$

Here  $*$  denotes the adjoint space (or map), and the residual norm  $\|u\|_{(0)}$  ( $\|V\|_{(0)}$ ) is smaller than the one appearing on the left (and the corresponding natural imbedding is compact).

The estimate (0.2) is equivalent to: (i) the space of null solutions (of  $Tu = 0$ ) is finite dimensional and (ii) the range of  $T$  is closed. The estimate (0.3) implies: (iii) the range of  $T$  has finite codimension in  $H^{s, p}$ , ((0.3) is also implied by (i)-(iii)). If  $T$  satisfies (i)-(iii), the boundary problem is called normally solvable.

Our main result (Theorem 5.1) is the determination of the exact algebraic conditions (bearing on  $L, B_j^\pm$  and  $\Omega$ ) under which the estimates (0.2-3) are valid for mixed elliptic problems in the plane. The corresponding conditions for a pure problem (in any dimension) with a boundary set  $\{B_j\}$  are well known and is formulated as:  $\{B_j\}$  cover  $L$  at each point of  $\partial\Omega$ . Using these conditions and taking a fixed  $p$  and  $s \not\equiv 1/p, 2/p \pmod{1}$  We have (Corollary 5.1) that for every value of  $s$ , except for at most  $2m$  values of  $s \pmod{1}$ , the conditions that  $\{B_j^-\}$  cover  $L$  at each point of  $\partial^-\Omega$  and that  $\{B_j^+\}$  cover  $L$  at each point of  $\partial^+\Omega$  are necessary and sufficient for the validity of (0.2-3), hence for the normal solvability of the mixed boundary value problem in the plane. Moreover (Theorem 5.3), at the exceptional values of  $s$ , the deficiency (codimension of the range) of  $T$  has a jump. This means that the regularization theorem, which states that the solution of a pure problem is more regular provided that the data is more regular, does not carry over to the mixed case. In view of the local character of the regularization theorem the fault must be at the dividing points  $P_1, P_2$  on  $\partial\Omega$ .

The mixed estimates (0.2-3) are first obtained in a canonical situation:  $\Omega$  is a half plane,  $L, B_j^\pm$ , are homogenous operators with constant coefficient. This case is reduced to the corresponding canonical pure problem. In the reduction we use some results on the Hilbert transform on the half line  $R_+ = \{x > 0\}$ ,  $H\varphi = (2\pi i)^{-1} \int_0^\infty (\varphi(x)/(x-t)) dt, x > 0$ , considered as an operator in the space  $W^{s, p}(R_+)$ . These results will be published in [22].

Our treatment of the canonical pure problem (which is given for arbitrary dimension) is somewhat novel. We use systematically the semi-norms instead of the norms in  $W^{s,p}$ , and thus obtain directly the dual estimate (0.3) in the seminorm. This method can also be used to study the boundary problems in  $W^{s,p}$ ,  $s < 2m$ , as we shall show elsewhere (cf. also [1, 9, 11, 18]).

For  $p = 2$ , the mixed estimate (0.2) was proved and (0.3) was announced by Peetre [15], who uses  $H^{s,2}$  spaces which coincide with  $W^{s,2}$  spaces. Schechter [17] obtained (0.2) (for  $p = 2$  and integral  $s$ ) for arbitrary dimension, but under a rather complicated compatibility condition which is not an exact necessary and sufficient condition.

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NOTATIONS.  $R^n$  is the  $n$ -dimensional Euclidean space, ( $R^1 = R$ ). Points in  $R^n$  are denoted by  $P = (x_1, \dots, x_n)$  and  $|P|^2 = \sum_i x_i^2$ . We also denote  $x = (x_1, \dots, x_{n-1})$ ,  $x_n = t \cdot R_+^n (R_-^n)$  is the upper (lower) half space  $t > 0$  ( $t < 0$ ).

If  $\alpha = (\alpha_1, \dots, \alpha_n)$  is a multi-index ( $\alpha_i$  non negative integers) then  $|\alpha| = \sum_i \alpha_i$ ,  $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ , ( $D_i = \partial/\partial x_i$ ) and  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ .  $D^k$  is the generic  $k$ -order derivative,  $D_x^k$  is a pure  $k$ -order  $x$ -derivative.

Unless it is otherwise specified,  $r, l, k, j$  will denote non-negative integers,  $\sigma$  will vary in  $(0, 1)$  and  $p, p'$  are in  $(1, \infty)$  with  $1/p + 1/p' = 1$ .

$\Omega \subseteq R^n$  denotes either  $R^n, R_\pm^n$  or a bounded domain with a smooth boundary  $\partial\Omega$ .  $C^k(\Omega)$  is the set of functions with continuous derivatives up to order  $k$  in  $\Omega$  (which vanish at  $\infty$  in case  $\Omega = R^n, R_\pm^n$ ).  $C_0^k(\Omega) \subset C^k(\Omega)$  contains the functions with compact support in  $\Omega$ .

$E^*$  is the conjugate of the Banach space  $E$ . The duality between  $E^*$  and  $E$  is denoted by  $\langle e^*, e \rangle$ . Norms in a domain  $\Omega$  are denoted by  $\|u, \Omega\|$ , reference to the domain will sometimes be omitted here, as well as in integrals taken over the whole space.

Equivalence between two norms in a certain set is denoted by  $\|u\|_1 \asymp \|u\|_2$ . In this case we have  $\|u\|_1 \leq K \|u\|_2$ ,  $\|u\|_2 \leq K \|u\|_1$ ,  $K$  independent of  $u$ . The phrase ' $K$  independent of  $u$ ' will be omitted in such estimates. Mostly, the parameters on which  $K$  do depend will also not be specified.

## 1. $W^{s,p}$ spaces.

Recent expositions of the spaces  $W^{s,p}$  and their basic properties are found in [9, 10, 13]. We shall describe here several properties which we need, emphasising those which carry over to the semi-norm  $[u]_{s,p}$  in  $R^n, R_\pm^n$ .

Besides, results on compact imbeddings and multipliers in  $W^{s,p}$ , which are basic tools for studying differential problems in  $W^{s,p}$  framework, seem to have been proved explicitly only for  $p = 2$  [3] or for integral values of  $s$ .

$W^{r,p}(\Omega)$  is the completion of  $C^\infty(\bar{\Omega})$  with respect to the norm

$$\|u, \Omega\|_{r,p} = \sum_{|\alpha| \leq r} \left( \int_{\Omega} |D^\alpha u|^p dx \right)^{1/p}.$$

The semi-norm  $[u, \Omega]_{r,p}$  is obtained by summing over  $|\alpha| = r$ . For  $s = r + \sigma$  it is defined by

$$[u, \Omega]_{\sigma,p} = \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+\sigma p}} dx dy \right)^{1/p}$$

$$[u, \Omega]_{s,p} = \sum_{|\alpha| = r} [D^\alpha u, \Omega]_{\sigma,p}.$$

$W^{s,p}(\Omega)$  is now the completion of  $C^\infty(\bar{\Omega})$  with respect to the norm

$$\|u, \Omega\|_{s,p} = \|u, \Omega\|_{r,p} + [u, \Omega]_{s,p}, \quad s = r + \sigma.$$

We note that for  $\Omega = R^n, R_\pm^n$ , the semi-norm  $[u]_{s,p}$  defines a (new) norm in the set  $C^\infty(\bar{\Omega})$ , (its functions vanish at  $\infty$ , by our conventions), and the completion with respect to  $[u]_{s,p}$  is denoted by  $\mathbf{W}^{s,p}(\Omega)$ .

It was proved in [10] that for  $\Omega = R_\pm^n$  the set  $C_0^\infty(\Omega)$  is dense in  $W^{s,p}(\Omega)$  (resp.  $\mathbf{W}^{s,p}(\Omega)$ ) if and only if  $\sigma \leq 1/p$ . We denote by  $W_0^{s,p}(\Omega)$  (resp.  $\mathbf{W}_0^{s,p}(\Omega)$ ) the subspace spanned by  $C_0^\infty(\Omega)$  (i. e. its closure).

$W^{-s,p}(R^n)$  is defined as  $(W^{s,p'}(R^n))^*$ ,  $s \geq 0$ , and  $W^{-s,p}(\Omega)$  as  $(W_0^{s,p'}(\Omega))^*$ . Since  $C_0^\infty(\Omega)$  is not dense in  $W^{s,p}(\Omega)$  (for  $s > 1/p$ ) the adjoint of the latter is not a space of distributions on  $\Omega$ . However, it can be identified with  $W_{\bar{\Omega}}^{-s,p'}$ , the space of all elements in  $W^{-s,p'}(R^n)$  which are supported in  $\bar{\Omega}$ . [8, p. 34].

**THEOREM 1.1.** [10] a) *The functional  $u \rightarrow u(0)$ , defined for continuous functions is  $\mathbf{W}^{\sigma,p}(R_+)$ , is bounded in the  $[u, R_+]_{\sigma,p}$  norm for  $\sigma > 1/p$ . Thus by extension,  $u(0)$  becomes meaningful for  $u \in \mathbf{W}^{\sigma,p}(R_+)$ ,  $\sigma > 1/p$ .*

b) *For  $u \in \mathbf{W}^{\sigma,p}(R_+)$ , let  $u^*(x)$  be its extension, defined as 0 for  $x < 0$ . Then for  $\sigma \neq 1/p$*

$$u^* \in W^{\sigma,p}(R) \iff u \in \mathbf{W}_0^{\sigma,p}(R_+).$$

(For  $\sigma < 1/p$  there is no restriction; for  $\sigma > 1/p$ ,  $u(0)$  should be 0).

REMARK. The analogous results for  $W^{\sigma,p}(\mathbb{R}_+)$  are readily obtained (in fact, in this form they were proved in [10]). The results carry over to  $\mathbb{R}_+^n$   $n > 1$ , too, and also to general domains with smooth boundary. The suitable extensions for  $s \geq 1$  instead of  $\sigma$  are also clear.

$W^{s,p}(\Omega)$  are reflexive Banach spaces and  $W^{s+\varepsilon,p}(\Omega) \subset W^{s,p}(\Omega)$ ,  $\varepsilon > 0$ , the imbedding being continuous. It is probably even compact for bounded domains. (For  $p = 2$ , cf. [3]). We shall prove here the partial result:

THEOREM 1.2. *Let  $\Omega$  be bounded,  $\sigma > 0$ . The imbeddings*

$$W^{r+\sigma,p}(\Omega) \rightarrow W^{r,p}(\Omega); \quad W_{\bar{\Omega}}^{-r+\sigma,p} \rightarrow W_{\bar{\Omega}}^{-r,p}$$

are compact.

PROOF. We may assume  $0 < \sigma < 1$ .  $W^{r+\sigma,p}(\Omega)$  can be obtained by interpolation between  $W^{r,p}(\Omega)$  and  $W^{r+1,p}(\Omega)$  (Lions [7,9]). This is helpful in various proofs. In particular it yields

$$\|u\|_{r+\sigma,p} \leq K \cdot \|u\|_{r,p}^{1-\sigma} \|u\|_{r+1,p}^{\sigma}.$$

The compactness of  $W^{r+1,p}(\Omega) \rightarrow W^{r,p}(\Omega)$  being well known (the generalization of Rellich's Lemma), we obtain easily that  $W^{r+1,p}(\Omega) \rightarrow W^{r+\sigma,p}(\Omega)$  is compact. By taking adjoints we see that  $W_{\bar{\Omega}}^{-r+\theta,p} \rightarrow W_{\bar{\Omega}}^{-r,p}$  is compact,  $0 < \theta < 1$ .

Using the fact that the operator  $1 - \Delta$  ( $\Delta$  is the Laplacian) is an isomorphism of  $W_{\bar{\Omega}}^{s,p}$  onto  $W_{\bar{\Omega}}^{s-2,p}$  we can lift the indices and obtain that  $W_{\bar{\Omega}}^{\theta,p} \rightarrow W_{\bar{\Omega}}^0,p$  is compact, (and this is immediately extended to  $r + \theta$  and  $r$  instead of  $\theta, 0$ ).

Finally, for a given  $\Omega$  we take  $\bar{\Omega}_1 \supset \bar{\Omega}$ . Each  $u \in W^{\sigma,p}(\Omega)$  can be extended to  $u_1 \in W_{\bar{\Omega}_1}^{\sigma,p}$  so that

$$\|u_1, R^u\|_{\sigma,p} \leq K \|u, \Omega\|_{\sigma,p} \quad (K \text{ independent of } u).$$

The compactness of  $W^{\sigma,p}(\Omega) \rightarrow W^0,p(\Omega)$  follows now from that of

$$W_{\bar{\Omega}_1}^{\sigma,p} \rightarrow W_{\bar{\Omega}_1}^0,p.$$

COROLLARY 1.1. *Let  $\Omega \subset \mathbb{R}^n$  be a fixed compact  $s \geq 0$ . In  $W_{\bar{\Omega}}^{s,p}$  the norm  $\|u\|_{s,p}$  and the semi-norm  $[u]_{s,p}$  are equivalent.*

This corollary, the proof of which is simple and omitted, may be stated as:  $\|u\|_{s,p}$  and  $[u]_{s,p}$  are locally equivalent. As a result we obtain that  $u$  belongs to  $W^{s,p}$  if and only if  $u$  is locally in  $W^{s,p}$  and  $[u]_{s,p} < \infty$ . In

particular, differentiation in  $W^{s,p}$  can be defined as in  $W^{s,p}$  and it is readily seen that  $W^{r+\sigma,p} = \{u \mid D^r u \in W^{\sigma,p}\}$ . However, functions differing in a polynomial of degree  $< r$  are to be identified.

LEMMA 1.1 *Let  $0 < \sigma \leq 1, \varphi \in C^{r+1}(\bar{\Omega}), u \in W^{r+\sigma,p}(\Omega)$ . Then  $\varphi u \in W^{r+\sigma,p}(\Omega)$  and*

$$\|\varphi u, \Omega\|_{r+\sigma,p} \leq \left(\sup_{\bar{\Omega}} |\varphi|\right) \cdot \|u, \Omega\|_{r+\sigma,p} + K \|u, \Omega\|_{r,p}.$$

The proof for  $\sigma = 1$  follows from Leibnitz formula. For  $\sigma < 1$ , the case  $r > 0$  is reduced as usual to  $r = 0$ . The basic estimate for the semi-norm  $[\varphi u, \Omega]_{\sigma,p}$  is quite similar to the one given for  $p = 2$  in [3].

COROLLARY. *If  $L(P, D)$  is a differential operator of order  $k$  with smooth coefficients, and the high order coefficients are estimated in absolute value by  $\delta > 0$ , then*

$$\|L(P, D)u, \Omega\|_{s,p} \leq \delta \|u, \Omega\|_{s+k,p} + K \|u, \Omega\|_{r+k,p}, \quad s = r + \sigma.$$

LEMMA 1.2 *Let  $\varphi$  be sufficiently smooth and  $u \in W_{\Omega}^{-r+\sigma,p}$ . Then*

$$\|\varphi D^k u, R^n\|_{-r-k+\sigma,p} \leq \left(\sup_{\Omega} |\varphi|\right) \cdot \|u, R^n\|_{-r+\sigma,p} + K \|u, R^n\|_{-r,p}.$$

The proof is based again on lifting the indices to positive values by using the operator  $(1 - \Delta)^l$ ,  $l$  sufficiently large and then Lemma 1.1 and its corollary may be used.

THEOREM 1.3 [20, 21, 9, 13] *(the trace theorem for  $W^{s,p}$ ). Let  $s > 1/p$  and  $s \not\equiv 1/p \pmod{1}$  if  $p = 2$ . Let  $k$  be the maximal integer smaller than  $s - 1/p$ . If  $u \in W^{s,p}(\Omega)$  then the traces  $\gamma_j u = \frac{\partial^j u}{\partial \nu^j} \Big|_{\partial \Omega}$  of  $u$  and its normal derivatives up to order  $k$  can be defined,  $\gamma_j u \in W^{s-1/p-j,p}(\partial \Omega)$  and*

$$\|\gamma_j u, \partial \Omega\|_{s-1/p-j,p} \leq K \|u, \Omega\|_{s,p} \quad 0 \leq j \leq k.$$

*Conversely, given  $g_j \in W^{s-1/p-j,p}(\partial \Omega)$ ,  $0 \leq j \leq k$ , there exists a function  $u \in W^{s,p}(\Omega)$  such that  $g_j = \frac{\partial^j u}{\partial \nu^j} \Big|_{\partial \Omega}$  and*

$$\|u, \Omega\|_{s,p} \leq K \sum_{j=0}^k \|g_j, \partial \Omega\|_{s-1/p-j,p}.$$

REMARK. Theorem 1.3 remains valid for the semi-norms (in  $\Omega = \mathbb{R}_\pm^n$  and  $\partial\Omega = \mathbb{R}^{n-1}$ ). This can be shown by a simple homogeneity argument.

## 2. Pure boundary problems in a half space.

In this section we shall mainly describe Agmon-Douglis-Nirenberg results [2] for the canonical boundary value problem :

$$(2.1) \quad \begin{aligned} L(D_x, D_t) u(x, t) &= f(x, t), \quad t > 0 \\ B_j(D_x, D_t) u(x, 0) &= \varphi_j(x), \quad j = 1, \dots, m. \end{aligned}$$

We shall also obtain the additional information that the map

$$(2.2) \quad Tu = \{Lu(x, t), B_1 u(x, 0), \dots, B_m u(x, 0)\}$$

sets up an isomorphism of  $\mathbb{W}^{2m+s, p}(\mathbb{R}_+^n)$  onto the appropriate data space.

Let  $L(\xi, \tau)$  be a homogeneous polynomial of degree  $2m$ ,  $L(\xi, \tau) \neq 0$  if  $(\xi, \tau) \neq 0$ . If  $n = 2$ , we assume further that for  $|\xi| = 1$   $L(\xi, \tau)$  has  $m$  roots  $\tau_k^\pm(\xi)$ ,  $1 \leq k \leq m$ , with positive imaginary part. This is automatically satisfied for  $n > 2$ . We set

$$M(\xi, \tau) = \prod_{k=1}^m (\tau - \tau_k^\pm(\xi)), \quad |\xi| = 1.$$

$B_j(\xi, \tau)$  is a homogeneous polynomial of degree  $m_j \leq 2m - 1$ ,  $1 \leq j \leq m$ , we assume that

$\{B_j\}_{j=1}^m$  cover  $L$ : For  $|\xi| = 1$ ,  $\{B_j(\xi, \tau)\}_{j=1}^m$  are linearly independent modulo  $M(\xi, \tau)$ .

We first consider problem (2.1) in the case  $f = 0$ ,  $\varphi_j(x) \in C_0^\infty(\mathbb{R}^{n-1})$ . An explicit solution is given by :

$$(2.3) \quad u(x, t) = \sum_{j=1}^m \int K_j(x - y, t) \varphi_j(y) dy,$$

where  $K_j(x, t)$  are suitable Poisson kernels.  $K_j$  are infinitely differentiable for  $t \geq 0$  except for  $(x, t) = 0$  and satisfy

$$(2.4) \quad |D^r K_j| \leq C_r \cdot |1 + \log |P|| |P|^{m_j - n - r}.$$

If  $r \geq m_j - n$ , the kernel  $D^r K_j$  is homogeneous of degree  $m_j - n - r$  and the logarithmic term may be dropped. In particular,  $D^{m_j} K_j$  is homogeneous of degree  $-n$  and satisfies moreover  $\int_{|y|=1} D^{m_j} K_j(y, 0) dy = 0$ .



We consider now problem (2.1) in the case  $f \neq 0$ . We introduce first the fundamental solution  $\Gamma(P - P^*)$  of the equation  $Lu = 0$  with singularity at  $P = P^*$ .  $\Gamma(P)$  is of the form

$$\Gamma(p) = |P|^{2m-n-1} \psi(P/|P|) + q(P) \log |P|.$$

$q(P)$  is a polynomial of degree  $2m - n - 1$  for  $n \neq 1$  even or  $2m \geq n$ , and  $q(P) = 0$  otherwise.  $\psi(Q)$  is an analytic function on  $|Q| = 1$ .

Given  $f \in C_0^\infty(\mathbb{R}_+^n)$ , it is possible to extend  $f$  to the whole space  $\mathbb{R}^n$  so that the extension  $f_N \in C^N(\mathbb{R})$  for  $N$  sufficiently large. Having chosen some large  $N$ , we set

$$v(P) = v_N(P) = \int \Gamma(P - P^*) f_N(P^*) dP^*.$$

The function  $v$  satisfies  $Lv = f_N$  and it is easily established that

$$(2.5) \quad D^r v(P) = O(|P|^{2m-n-1-r} (1 + \log |P|)), \quad |P| \rightarrow \infty,$$

and the logarithmic term may be dropped if  $r > 2m - n - 1$ .

**THEOREM 2.1.** *If  $N$  was chosen  $\geq r + 1$ , then an explicit solution  $u$  of problem (2.1) in  $\mathbf{W}^{r+2m,p}(\mathbb{R}_+^n)$  is given by*

$$(2.6) \quad \begin{aligned} D^\alpha u &= D^\alpha v + D^\alpha u_1, \quad |\alpha| \geq 2m - 1, \\ D^\alpha u_1 &= \sum_j \int D^\alpha K_j(x - y, t) (\varphi_j(y) - \psi_j(y)) dy \end{aligned}$$

where  $\psi_j(y) = B_j v(y, 0)$ . Moreover, if  $u \in C_0^\infty(\mathbb{R}_+^n)$  and  $f = Lu$ ,  $B_j u = \varphi_j$ , then  $D^\alpha u$  has the representation (2.6).

**PROOF.** The  $D^\alpha u_1$  are the  $\alpha$  derivatives of one and the same function  $u_1$ , since they satisfy the necessary compatibility conditions. The integrals in (2.6) are convergent, due to the estimates (2.4-5) and using integration by parts, the  $j$ 'th integral can be written as

$$\pm \int D^\beta K_j D^\gamma (\varphi_j - \psi_j), \quad |\beta| = m_j, |\beta| + |\gamma| = |\alpha|.$$

The previous result about the solution of (2.1) for  $f = 0$  implies that  $Lu = f$ ,  $B_j u = \varphi_j$ ,  $1 \leq j \leq m$ . The last equations should be considered as equalities in  $\mathbf{W}^{r+2m-m_j-1/p,p}(\mathbb{R}^{n-1})$  (in fact, one can take  $r = 0$ ), that is, only  $D^\gamma B_j u = D^\gamma \varphi_j$  for  $|\gamma| \geq 2m - 1$  is assured.

The crucial point in the proof is the fact that  $u \in \mathbf{W}^{r+2m,p}(R_+^n)$ . This is clear for  $v$ . Now the  $j$ 'th term of  $D^{r+2m-1} u_1$  is  $\int D^{m_j} K_j D^{r+2m-1-m_j} (\varphi_j - \psi_j)$  and it was proved in [2, Th. 3.3 & App. 3] that

$$(2.7) \quad \left[ \int D^{m_j} K_j \cdot \chi, R_+^n \right]_{1,p} \leq O[\chi, R^{n-1}]_{1-1/p,p}$$

provided that  $\chi \in W^{1-1/p,p}(R^{n-1})$ . Hence, it suffices to show that  $D^{r+2m-1-m_j} (\varphi_j - \psi_j) \in W^{1-1/p,p}(R_{n-1})$ . Since  $\varphi_j \in C_0^\infty$ , we have to prove this for  $\psi_j$ . But it follows from (2.5) that  $D^{r+2m-1} v \in W^{1,p}(R^n)$ , hence :

$$D^{r+2m-1-m_j} B_j v(x, 0) = D^{r+2m-1-m_j} \psi_j \in W^{1-1/p,p}(R^{n-1}).$$

The assertion that  $u \in C_0^\infty(R_+^n)$  has the representation (2.6) was proved in detail in [2, Th. 4.1] for  $|\alpha| \geq 2m$ , but it is easily seen to hold for  $|\alpha| = 2m - 1$  too. This, in combination with Calderon-Zygmund results [6] for singular integrals (which are also used in proving (2.7)) yields the a-priori estimates in the seminorms

$$(2.8) \quad [u, R_+^n]_{r+2m,p} \leq K([Lu, R_+^n]_{2m,p} + \sum_j [B_j u, R^{n-1}]_{r+2m-m_j-1/p,p})$$

for  $u \in C_0^\infty(R_+^n)$ .

**THEOREM 2.2.** *The map  $T$  of (2.2) is a 1 - 1 continuous map of  $\mathbf{W}^{r+2m,p}(R_+^n)$  onto*

$$(2.9) \quad \Pi^{1,p} = \mathbf{W}^{r,p}(R_+^n) \times \Pi_j \mathbf{W}^{r+2m-m_j-1/p,p}(R^{n-1}).$$

For  $\Phi = (f, \varphi_1, \dots, \varphi_m) \in \Pi^{r,p}$ ,  $u = T^{-1} \Phi$  is given by

$$(2.10) \quad D^\alpha u = D^\alpha v + D^\alpha u_1 = D^\alpha v + \sum_j D^\alpha K_j(x-y, t) (\varphi_j(y) - \psi_j(y) dy, \\ |\alpha| \geq 2m + r.$$

**PROOF.** The estimates which show the continuity of  $T$  are

$$[Lu]_{r,p} \leq K[u]_{r+2m,p}; [B_j u, R^{n-1}]_{r+2m-m_j-1/p,p} \leq K[u, R_+^n]_{r+2m,p}.$$

They follow from Theorem 1.3 and the fact that  $k$ -order differentiation takes  $\mathbf{W}^{s+k,p}(\Omega)$  continuously into  $\mathbf{W}^{s,p}(\Omega)$ .

The estimates (2.8) can be written as

$$(2.11) \quad \|u, R_+^n\|_{r+2m, p} \leq K \|Tu\|_{\Pi^{r, p}},$$

showing that  $T$  is 1 — 1 and has a closed range. By Theorem 2.1, this range contains a dense subset of  $\Pi^{r, p}$ , hence it coincides with  $\Pi^{r, p}$ .

To prove the representation formula for  $T^{-1}\Phi$ , we note that the integrals in (2.10) for  $|\alpha| = r + 2m$  are  $D \int D^{m_j} K_j D^{r+2m-1-m_j} (\varphi_j - \psi_j)$ , and by (2.7) they belong to  $L^p(R_+^n)$  provided that  $D^{r+2m-1-m_j} (\varphi_j - \psi_j)$  are in  $\mathbf{W}^{1-1/p, p}(R)$ . But this is exactly what is required from  $\varphi_j$ , and it holds also for  $\psi_j = B_j v(x, 0)$ , since  $v \in \mathbf{W}^{2m+r, p}(R_+^n)$ . (We note however that the integrals are interpreted by extension of continuous operators into  $\mathbf{W}^{2m+r, p}$ ).

Now the formula (2.10) holds for a dense set of (compact supported)  $\Phi = (f, \varphi_1, \dots, \varphi_m) \in \Pi^{r, p}$  and by a limit process it is extended to the whole space.

**THEOREM 2.3.** *Theorem 2.2 and the estimates (2.8) remain true for  $r = \sigma$  instead of  $r$ , where  $\sigma \neq 1/p$  if  $p \neq 2$ .*

**PROOF.** By interpolating like Lions [7,9] between  $r$  and  $r + 1$  we get  $\mathbf{W}^{r+2m+\sigma, p}(R_+^n)$  on the one hand and

$$(2.12) \quad \Pi^{r+\sigma, p} = \mathbf{W}^{r+\sigma, p}(R_+^n) \times \prod_{j=1}^m \mathbf{W}^{r+\sigma+2m-m_j-1/p, p}(R^{n-1})$$

on the other hand. The results follow now from the well known property of interpolation of bounded operators. We note that by interpolating between  $L^p$  and  $\mathbf{W}^{1, p}$  we get  $\mathbf{W}^{\sigma, p}$ . This can be obtained by an homogeneity argument but in fact in Lions [7] the interpolation is proved directly for the semi-norms.

### 3. Mixed problems in a half plane.

Let now  $B^+ = \{B_j^+\}$  and  $B^- = \{B_j^-\}$  be two sets of boundary operators with constant coefficients, where  $B_j^\pm$  is homogenous of degree  $m_j^\pm$ ,  $1 \leq j \leq m$ . We assume that each set covers the elliptic operator  $L$  and that  $B_j^-$  here coincides with  $B_j$  of Section 2; in particular, the Poisson kernels  $K_j$  correspond to  $\{B_j^-\}$ .

We shall study the map  $T$  for the mixed problem in the plane :

$$(3.1) \quad Tu = \{Lu|_{R_+^2}, B^-u|_{R_-}, B^+u|_{R_+}\}.$$

Our first goal is to obtain for  $n = 2$  the mixed a-priori estimates

$$(3.2) \quad [u, R_+^n]_{s,p} \leq K ([Lu, R_+^n]_{s-2m,p} + \sum_{j,\pm} [B_j^\pm u, R_\pm^{n-1}]_{s_j^\pm,p})$$

for  $u \in \mathbf{W}^{s,p}(R_+^n)$ , where

$$(3.3) \quad s = r + \sigma + 2m; \quad s_j^\pm = r + \sigma + 2m - m_j^\pm - 1/p = l_j^\pm + \tau,$$

$$0 \leq \tau < 1, \quad 1 \leq j \leq m.$$

We assume in the following that  $\sigma \not\equiv 1/p \pmod{1}$  if  $p \neq 2$ . We also note that the first arguments are also valid for  $n > 2$ .

LEMMA 3.1. *It is sufficient to prove (3.2) for functions  $u_1$  represented in the form*

$$(3.4) \quad D^\alpha u_1 = \sum_j \int D^\alpha K_j(x-y, t) \omega_j(y) dy, \quad |\alpha| \geq 2m + r, \quad \omega_j \in \mathbf{W}^{s_j^-,p}(R^{n-1}).$$

PROOF.  $u_1 \in \mathbf{W}^{s,p}(R_+^n)$  and satisfied  $Lu_1 = 0$ ,  $B_j^- u_1(x, 0) = \omega_j(x)$ ,  $1 \leq j \leq m$ . (cf. the proofs of Theorems 2.1-2). Thus, (3.2) for  $u_1$  is

$$(3.5) \quad [u_1, R_+^n]_{s,p} \leq K \sum_{j,\pm} [B_j^\pm u_1, R_\pm^{n-1}]_{s_j^\pm,p}.$$

We have to show that (3.5) implies (3.2), and it suffices to take  $u \in C_0^\infty(R_+^n)$ . According to Section 2, such  $u$  can be represented as  $u = v + u_1$  where  $u_1$  has the form (3.4). Now

$$[u]_{s,p} \leq [u_1]_{s,p} + [v]_{s,p},$$

$$[B_j^\pm u_1, R_\pm^{n-1}]_{s_j^\pm,p} \leq [B_j^\pm u, R_\pm^{n-1}]_{s_j^\pm,p} + [B_j^\pm v, R_\pm^{n-1}]_{s_j^\pm,p},$$

$$[B_j^\pm v(x, 0), R_\pm^{n-1}]_{s_j^\pm,p} \leq [B_j^\pm v(x, 0), R_\pm^{n-1}]_{s_j^\pm,p} \leq K [v, R_+^n]_{s,p} \leq$$

$$\leq K [Lu, R_+^n]_{s-2m,p}.$$

To justify the last estimate, we observe that

$$D^{2m}v = \int D^{2m} \Gamma(P - Q) u(Q) d(Q)$$

and that  $D^{2m}I$  is a homogeneous kernel of degree  $-n$  to which Calderon-Zygmund results [6] apply.

Substituting the above estimates in (3.5) we obtain (3.2) and the lemma is established

By Theorem 1.3 and the pure a-priori estimates we have

$$[u_1, R_+^n]_{s,p} \infty \Sigma_{j,\pm} [B_j u, R_+^{n-1}]_{s_j^-, p},$$

so that (3.5) is equivalent to

$$(3.6) \quad \Sigma_j [B_j^- u_1, R_+^{n-1}]_{s_j^-, p} \leq K \Sigma_{j,\pm} [B_j^\pm u, R_\pm^{n-1}]_{s_j^\pm, p}.$$

$D^\alpha u_1$  is expressed in (3.4) in terms of  $\omega_j = B_j^- u_1(x, 0)$ . Hence it is possible to express the boundary values of derivatives of  $B_k^+ u_1$  in terms of derivatives of  $B_j^- u_1(x, 0)$ . The expressions involve derivatives of the kernels  $K_j$ .

In the two dimensional case, which will concern us from now onward, we have

$$K_j(x, t) = \frac{-1}{4\pi^2} \Sigma_\pm \int_\gamma \frac{N_j(\pm 1, \tau) (\pm x + t\tau)^{m_j^- - 1}}{M(\pm 1, \tau)} \log \frac{\pm x + t\tau}{i} d\tau, \quad m_j^- > 0,$$

$$K_j(x, t) = \frac{-1}{4\pi^2} \Sigma_\pm \int_\gamma \frac{N_j(\pm 1, \tau)}{M(\pm 1, \tau) (\pm x + t\tau)} d\tau, \quad m_j^- = 0,$$

where  $\gamma$  is a closed curve in  $Im \tau > 0$  enclosing all the zeroes of  $M(\pm 1, \tau)$  (these, we recall, are the zeroes of  $L(\pm 1, \tau)$  in  $Im \tau > 0$ ), and  $N_j(\pm 1, \tau)$  are certain polynomials in  $\tau$  satisfying  $(2\pi i)^{-1} \int_\gamma \frac{N_k B_j^-}{M^+}(\pm 1, \tau) d\tau = \delta_{jk}$ .

An easy computation shows that

$$(3.7) \quad D_x^{l_k^+} B_k^+ u_1(x, 0) = \Sigma_j (-c_{kj} (+1) \mathcal{H}^+ + c_{kj} (-1) \mathcal{H}^-) \cdot D_x^{l_j^-} B_j^- u_1(x, 0)$$

where  $\mathcal{H}^\pm$  are the upper and lower Hilbert transforms

$$\mathcal{H}^\pm f(x) = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_\infty^\infty \frac{f(t)}{x + t \pm i\epsilon} dt,$$

and  $c_{kj}(\xi)$ ,  $\xi = \pm 1$ , are the expansion coefficients of

$$B_k'^+(\xi, \tau) = B_k^+(\xi, \tau) \pmod{M(\xi, \tau)}$$

in terms of the  $B_j^-(\xi, \tau) = B_j^-(\xi, \tau) \pmod{M(\xi, \tau)}$ . Both sets  $\{B_j^-\}, \{B_j^+\}$ ,  $1 \leq j \leq m$  contain  $m$  independent polynomials of degree  $< m$ , therefore the matrices  $C(\pm 1) = [c_{kj}(\pm 1)]$  are non-singular.

REMARK 3.1. If the integrals in (3.4) are interpreted by extension of continuous operators into  $\mathbf{W}^{s,p}$  space, then the interpretation of (3.7) is likewise generalized. (by M. Riesz's theorem and interpolation,  $\mathcal{H}^\pm$  can be extended to continuous operators in  $\mathbf{W}^{s,p}(R)$  and  $W^{s,p}(R)$ ,  $s \geq 0$ ).

We introduce also the Hilbert transforms on the half line  $R^+$ :

$$(H^\pm f)(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_0^\infty \frac{f(t)}{x - t \pm i\varepsilon} dt, \quad x > 0.$$

Again,  $H^\pm$  can be extended to continuous operators in  $L^p(R^+)$ . It is convenient now to use vector notation and denote

$$\mathbf{B}^\pm = (D_x^{i_1^\pm} \cdot B_1^\pm, \dots, D_x^{i_m^\pm} \cdot B_m^\pm),$$

$$(3.8) \quad \mathcal{A} = -C(1)\mathcal{H}^+ + C(-1)\mathcal{H}^-, \quad A = C(1)H^+ + C(-1)H^-,$$

$\mathcal{A}$  and  $A$  operate on vector functions). Using these notations, (3.7) becomes

$$(3.7)' \quad \mathbf{B}^+ u_1(x, 0) = \mathcal{A} \mathbf{B}^- u_1(x, 0),$$

and the estimate (3.6) (hence (3.2) too) is reduced to the following:

$$(3.9) \quad [\varphi, R]_{\tau, p} \leq K([\varphi, R_-]_{\tau, p} + [\mathcal{A}\varphi, R_+]_{\tau, p})$$

for every vector function  $\varphi \in (C_0^\infty(R))^m$ . (It turns out, as could be expected, that the estimates (3.2) for various values of  $r$  are equivalent. They do depend however on  $\sigma$  in an essential manner, as we shall see).

At this point we need some results on the semi-infinite Hilbert transforms  $H^\pm$  and the operators  $A, \mathcal{A}$  defined in (3.8). These results will be published in [22]. Here we summarize them in

**THEOREM 3.1.** *a) For  $\sigma \neq 1/p$  the operators  $H^\pm, A$  are continuous 1-1 operators in  $(\mathbf{W}^{\sigma,p}(R_+))^m$ .  $A$  has a closed range (i. e.  $[\varphi, R_+]_{\sigma,p} \leq K[A\varphi, R_+]_{\sigma,p}$ ) if and only if the eigenvalues of the matrix  $E = C^{-1}(1)C(-1)$  are outside the ray  $\arg z = -2\pi(\sigma - 1/p)$  in the complex plane. Since  $E$  has at most  $m$  distinct eigenvalues, this condition is satisfied for every  $\sigma$  except  $m$  values at most.*

b) *If this eigenvalues-condition is satisfied for  $\sigma = \tau$ , then (3.9) is valid (for  $p = 2$ , the case  $\sigma = 1/2$  is not excluded here).*

c) *The range of  $A$  has a finite codimension  $\delta$  in  $(W^{\sigma,p}(R_+))^m$ ;  $\delta$  remains constant as  $\sigma$  increases, as long as the eigenvalues condition is satisfied. At a point  $\sigma_0$  where it is violated (a jump point),  $\delta$  gets an increment which is equal to the total multiplicity of the eigenvalues on the ray  $\arg z = -2\pi(\sigma_0 - 1/p)$ . Hence the total increment over the unit interval (of  $\sigma$ ) is  $m$ .*

By (3.3),  $\tau$  in (3.9) is congruent to  $\sigma - 1/p$ . Since the a-priori estimate (3.2) was reduced to (3.9), Theorem 3.1 (b) yields.

**THEOREM 3.2.** [19] *Suppose that  $\sigma \neq 1/p, 2/p$  if  $p \neq 2$ . The mixed a-priori estimate (3.2) is valid if and only if the eigenvalues of  $E = C^{-1}(1)C(-1)$  are outside the ray  $\arg z = -2\pi(\sigma - 2/p)$ . In particular there are at most  $m$  values of  $\sigma$  (different from  $1/p, 2/p$ ) for which (3.2) is not valid.*

**REMARK.** If  $L, B^\pm$  satisfy the condition of Theorem 3.2, they will be called  $\sigma$ -compatible. It is easy to show directly, from the definition of  $C(\pm 1)$ , that this condition is symmetric with respect to interchange of  $B^-$  and  $B^+$ . This is clear however from the symmetric form of (3.2).

We turn now to the counterpart of Theorem 2.2 for the mixed boundary problem.

**THEOREM 3.3.** *Suppose that  $\sigma \neq 1/p, 2/p$  if  $p \neq 2$  and let  $L, B^-, B^+$  be  $\sigma$ -compatible. Then the (mixed) map  $T$ , defined in (3.1), is a continuous 1-1 map of  $W^{s,p}(R_+^2)$  into*

$$H^{s-2m,p} = W^{s-2m,p}(R_+^2) \times \Pi_{j,\pm} W^{s_j^\pm,p}(R^\pm).$$

*Moreover, the range of  $T$  is closed and has a finite codimension.*

**PROOF.** As in the pure case, the continuity of  $T$  is immediate, and the a-priori estimate (3.2) means that  $T$  is 1-1 and has a closed range. It remains to show that the range has a finite codimension. For this purpose, it is sufficient to discuss the mixed problem

$$(3.10) \quad \begin{aligned} Lu &= F, \quad t > 0; \\ B^\pm u(x, 0) &= \theta^\pm(x), \quad x \in R^\pm, \end{aligned}$$

for a dense set of data. Thus we take  $F, \theta_j^\pm$  as  $C_0^\infty$  functions.

Let  $\theta^*$  be a smooth compact support extension of  $\theta^-(x)$  to the whole line  $R$ . By Theorems 2.2-3 we can find  $v \in \mathbf{W}^{s,p}(R_+^2)$  satisfying  $Lv = F$ ,  $B^-v = \theta^*$ , and by subtraction (3.10) is reduced to the problem

$$(3.11) \quad Lu = 0, \quad t > 0, \quad (u \in \mathbf{W}^{s,p}(R_+^2));$$

$$B^-u(x, 0) = 0, \quad x \in R_-, \quad B^+u(x, 0) = \theta^+ - \theta^* = g(x), \quad x \in R_+.$$

The boundary equations are equivalent to

$$(3.12) \quad \mathbf{B}_j^- u(x, 0) = 0 \text{ in } \mathbf{W}^{\tau,p}(R_-), \quad \mathbf{B}_j^+ u(x, 0) = D_j^+ g(x) = \mathbf{g}_j(x) \\ \text{in } \mathbf{W}^{\tau,p}(R_+).$$

LEMMA 3.1. *The problem (3.11) is solvable if and only if there exists a vector  $\mathbf{f}$  in  $(\mathbf{W}^{\tau,p}(R_+))^m$  satisfying  $A\mathbf{f} = \mathbf{g}$  and  $\mathbf{f}(0) = 0$  if  $\tau > 1/p$ . ( $A$  is defined in (3.8)).*

PROOF. Assume that  $\mathbf{f}$  exists, and extend it as zero for  $x < 0$ . By Theorem 1.1 (b), the extended  $\mathbf{f}$  belongs to  $(\mathbf{W}^{\tau,p}(R))^m$ , since  $\mathbf{f}(0) = 0$  if  $\tau > 1/p$ .

Let  $f_j$  be an  $l_j^-$  order primitive of  $\mathbf{f}_j$ . Then  $f_j \in \mathbf{W}^{s_j^-,p}(R)$  so that by Theorems 2.2-3 the  $m+1$  tuple  $(0, f_1, \dots, f_m)$  belongs to the range of the pure map  $T$ . Hence, there is a function  $u_1 \in \mathbf{W}^{s,p}(R_+^2)$  satisfying  $Lu_1 = 0$  for  $t > 0$  and  $B_j^-u = f_j$ , or equivalently:  $\mathbf{B}_j^-u = \mathbf{f}_j$  in  $\mathbf{W}^{\tau,p}(R)$ . Moreover  $u_1$  is represented by (2.10) (with  $v = \psi_j = 0$ ). Hence by (3.7)  $\mathbf{B}^+u_1 = \mathcal{A}\mathbf{B}^-u_1$  in  $(\mathbf{W}^{\tau,p}(R))^m$  (cf. Remark 3.1). Since  $\mathbf{B}^-u_1 = \mathbf{f} = 0$  in  $R_-$ , we obtain

$$\mathbf{B}^+u_1 = A\mathbf{B}^-u_1 = A\mathbf{f} = \mathbf{g} \text{ in } (\mathbf{W}^{\tau,p}(R_+))^m.$$

Hence  $u_1$  satisfies (3.12) and solves (3.11).

Conversely, if  $u_1$  solves (3.11), then  $\mathbf{f} = \mathbf{B}^-u_1 \in (\mathbf{W}^{\tau,p}(R))^m$  solves  $A\mathbf{f} = \mathbf{g}$  in  $(\mathbf{W}^{\tau,p}(R_+))^m$  and since  $\mathbf{f} = 0$  for  $x < 0$  we obtain  $\mathbf{f}(0) = 0$  if  $\tau > 1/p$ . The proof of the lemma is thus concluded.

To finish the proof of the theorem, we notice that  $\mathbf{g}$  is in the range of  $A$  in  $\mathbf{W}^{\tau,p}(R_+)$  if  $\mathbf{g}$  satisfies a finite number of continuous linear conditions (Theorem 3.1. c). If  $\tau > 1/p$ , the requirement  $\mathbf{f}(0) = 0$  imposes  $m$  additional conditions on  $\mathbf{g} \in (\mathbf{W}^{\tau,p}(R_+))^m$ , or equivalently on  $\mathbf{g} \in \Pi_{j=1}^m \mathbf{W}^{s_j^+,p}(R_+)$ . Since (3.10) was reduced to (3.11) by using the isomorphism of Theorems 2.2-3, we have obtained a finite number of continuous linear conditions on the data  $(F, \theta^\pm) \in \Pi^{s-2m,p}$  of the original problem, which conditions are necessary and sufficient for the solvability of (3.10).



REMARK 3.2. We noted already that the equation  $Lu = f$  with  $u \in \mathbf{W}^{r+\sigma+2m, p}$  is only a congruence modulo polynomials of degree  $< r$ . In order to obtain more adequate solutions, even for  $r > 0$ , we introduce the spaces

$$\mathcal{W}^{r+\sigma+2m, p} = \bigcap_{k < r} \mathbf{W}^{k+\sigma+2m, p} \quad \text{and} \quad \mathcal{W}^{r+\sigma, p} = \bigcap_{k < r} \mathbf{W}^{k+\sigma, p} \quad (\text{pure and mixed}).$$

It is clear that the continuity of  $T$  and the estimates  $\|u\| \leq K \|Tu\|$  carry over, under the same conditions, to the new spaces, (whose norms are, as usual, the sum of the factors norms), and so do Theorems 2.2-3, 3.3. The conditions which determine the range of  $T$  in Theorem 3.3 are now supplemented by: The primitives  $\mathbf{g}^{(-1)}, \dots, \mathbf{g}^{(-r)}$  of  $\mathbf{g}$  are also in the range of  $A$  in  $(\mathbf{W}^{\tau, p})^m$  (cf. Lemma 3.1). However, if only  $\mathbf{g}^{(-r)} \in \text{range}(A)$  is required, and  $A^{-1} \mathbf{g}^{(-r)} = h$ , then the other conditions are easily seen to be equivalent to  $h(0) = h'(0) = \dots = h^{(r-1)}(0) = 0$ . The additional condition for  $\tau > 1/p$  can be written as  $h^{(r)}(0) = 0$ . Thus we see that in going from  $r$  to  $r+1$  the codimension increases by  $m$ , the  $m$  additional conditions require continuity at  $x=0$  of one higher derivative of the boundary value vector  $B^- u$ .

It will also be interesting to consider the spaces.

$$\mathcal{W}^{r+\sigma+2m, p}(\mathbf{R}_+^2) \cap \mathcal{W}^{r+\sigma, p}(\mathbf{R}_+^2) = \{u \mid D^{2m} u \in \mathbf{W}^{r+\sigma, p}(\mathbf{R}_+^2)\}$$

and the corresponding data spaces. These spaces constitute, like  $\mathbf{W}^{s, p}$ , a monotonic decreasing scale (for  $s = r + \sigma + 2m$  varying continuously). All the results carry over provided that in the mixed case we require both  $\sigma$  and 0-compatibility. The total increment  $m$  for a unit increase in  $s$  is distributed now to the jump points, where the range is not closed, exactly as (and because) it happened for the operator  $A$  (Theorem 3.1. c). We note in particular that in general  $\sigma = 1/p, 2/p$  need not be jump points.

#### 4. The dual estimates.

The pure map  $T$  of (2.2) was shown to be an isomorphism of  $\mathbf{W}^{s, p}(\mathbf{R}_+^n)$  onto  $\mathbf{W}^{s-2m, p}$ , hence the adjoint map  $T^*$  maps  $(\mathbf{W}^{s-2m, p})^*$  isomorphically onto  $(\mathbf{W}^{s, p}(\mathbf{R}_+^n))^*$ , and in particular we have  $\|V\| \leq K \|T^* V\|$ . An element  $V \in (\mathbf{W}^{s-2m, p})^*$  has the form

$$V = \{F, \Gamma_1, \dots, \Gamma_m\} \quad \text{where} \quad F \in (\mathbf{W}^{s-2m, p}(\mathbf{R}_+^n))^*, \quad \Gamma_j \in (\mathbf{W}^{s-m_j-1/p, p}(\mathbf{R}^{n-1}))^*,$$

and  $T^*$  is defined by the relation

$$(4.1) \quad \langle T^* V, v \rangle = \langle V, Tv \rangle = \langle F, Lv \rangle + \sum_j \langle I_j, B_j v \rangle, \text{ for every } v \in \mathbf{W}^{s,p}(\mathbf{R}_+^n),$$

$\langle, \rangle$  denotes the appropriate duality in each case.

From the local equivalence of  $\|u\|_{s,p}$  and  $[u]_{s,p}$  for  $s \geq 0$  we easily obtain, by duality, that  $\|U\|_{-s,p'} \simeq \|U\|_{(\mathbf{W}^{s,p})^*}$  for all  $U$  supported in a fixed compact. Hence we obtain

**THEOREM 4.1.** *Let  $\Omega \subset \bar{\mathbf{R}}_+^n$  be a fixed compact,  $s \neq 1/p$  if  $p \neq 2$  and  $V = \{F, I_j\}_{j=1}^m \in \Pi^{-s+2m,p'} = W_{\bar{\mathbf{R}}^n}^{-s+2m,p'} \times \Pi_{j=1}^m W^{-s+m_j+1/p,p'}(\mathbf{R}^{n-1})$ . Then for all  $V$  with support  $(V) \subset \Omega$*

$$(4.2) \quad \|V\|_{\Pi^{-s+2m,p'}} = \|F\|_{-s+2m,p'} + \sum_j \|I_j\|_{-s+m_j+1/p,p'} \leq K \|T^*V\|_{-s,p'}.$$

We turn now to the mixed case. Under the conditions of Theorem 3.3 the mixed map  $T$  is continuous, 1-1 valued and has a closed range  $\Pi_1 \subset \Pi^{s-2m,p}$ , ( $s-2m = r + \sigma$ ). Moreover, the range has a finite dimensional complement  $\Pi_2$ .

The adjoint map  $T^*$  is therefore 1-1 on  $\Pi_1^*$  and annihilates  $\Pi_2^*$ , which is finite dimensional. By a well known argument (e.g. [14, section 8]) we obtain

$$V_{\Pi^*} \leq K (\|T^*V\| + \|V\|_{(0)}),$$

where  $\|V\|_{(0)}$  is any norm smaller than the norm of  $\Pi^* = (\Pi^{s-2m,p})^*$ . For functions  $V$  supported in a fixed compact we obtain as in the pure case:

**THEOREM 4.2.** *Let  $\Omega \subset \bar{\mathbf{R}}_+^2$  be a fixed compact, and let*

$$V = \{F, I_i^\pm\}_{i=1}^m \in \Pi^{-s+2m,p'} = W_{\bar{\mathbf{R}}_+^2}^{-s+2m,p'} \times \Pi_{i,\pm} W^{-s_i^\pm,p'}(\mathbf{R}_\pm).$$

*Under the conditions of Theorem 3.3, for every  $V$  such that support  $(V) \subset \Omega$*

$$(4.3) \quad \begin{aligned} \|V\|_{-s+2m,p'} &= \|F\|_{-s+2m,p'} + \sum_{j,\pm} \|I_j^\pm\|_{-s_j^\pm,p'} \\ &\leq K (\|T^*V\|_{-s,p'} + \|V\|_{(0)}), \quad (s_j^\pm = s - m_j^\pm - 1/p), \end{aligned}$$

where  $\|V\|_{(0)}$  is smaller than the norm on the left hand side. (In practice, we shall use for the components of  $V$  norms with indices which are smaller

than the corresponding indices of  $H^{-s+2m, p'}$ , and such that the compact imbedding property (Theorem 1.2) will hold. In the following we shall assume that  $\|V\|_{(0)}$  already denotes such norm).

**5. Estimates for bounded domains and normal solvability.**

The passage from the canonical case in a half space to the estimates for general elliptic problems in bounded domains is performed in a familiar technique, based on Korn's principle and a partition of unity. We shall sketch the proof only for the dual estimates in the mixed boundary value problem; the treatment of the other cases is analogous and better known.

We consider first operators which are slightly different from the canonical ones. Let

$$L = L'(D) + L''(P, D); B_j^\pm = B_j^{\prime\pm}(D) + B_j^{\prime\prime\pm}(P, D), \quad 1 \leq j \leq m.$$

All the operators are defined in  $t \geq 0$ ;  $L', B_j^{\prime\pm}$  are operators with constant coefficients, homogeneous of degrees  $2m, m_j^\pm$  respectively.  $L'', B_j^{\prime\prime\pm}$  are of the same respective orders, have  $C^\infty$  coefficients and all the high order coefficients are bounded by  $\delta > 0$ .

The maps  $T, T^*$  are defined as before, using  $L, B_j^\pm$ . The maps  $T', T''^*, T'', T''^*$  are obtained from the primed and doubly primed operators. Clearly  $T = T' + T'', T^* = T''^* + T''^*$ .

**LEMMA 5.1.** *Assume that (3.2) and (4.3) (namely the estimates for  $T', T''^*$ ) hold, and let  $\Omega \subset \bar{R}_+^2$  be a fixed compact. If  $\delta$  is sufficiently small then*

$$(5.1) \quad \|u, K_+^2\|_{s, p} \leq K (\|Tu\|_{s-2m, p} + \|u\|_{(0)}), \quad \text{support } (u) \subset \Omega;$$

$$(5.2) \quad \|V\|_{H^{-s+2m, p'}} \leq K (\|T^*V\|_{-s, p'} + \|V\|_{(0)}) \quad \text{support } (V) \subset \Omega.$$

**PROOF.** We prove (5.2). By assumption, for  $V = (E, \Gamma_j^\pm)_{j=1}^m$

$$\|V\|_{H^{-s+2m, p'}} \leq K (\|T''^*V\|_{-s, p'} + \|V\|_{(0)}).$$

Since  $\|T''^*V\| \leq \|T^*V\| + \|T''^*V\|$ , we have only to estimate  $\|T''^*V\|$ . But

$$T''^*V = L''^*V + \sum_{j, \pm} (B_j^{\prime\prime})^* \Gamma_j^\pm$$

where  $L''^*$  is the formal adjoint of  $L''$  (differentiation and multiplication are taken in the sense of distributions) and for a typical operator  $B$

$$Bu = \gamma B(P, D)u \quad B^* = B^*(P, D)\gamma^*$$

where  $\gamma$  is the trace operator, so that  $\gamma^*$  is defined by

$$\langle \gamma^* \Gamma, u(x, t) \rangle = \langle \Gamma, \gamma u(x, t) \rangle = \langle \Gamma, u(x, 0) \rangle.$$

$L''^*$  contains derivations up to order  $2m$  and the coefficients of the  $2m$ -order terms are bounded by  $\delta$ . It follows from Lemma 1.2 that

$$(5.3) \quad \|L''^* F\|_{-s, p'} \leq \delta \|F\|_{-s+2m, p'} + K \|F\|_{-r-1, p'}, \quad (2m - s = -r - \sigma),$$

and the residual norm  $\|F\|_{-r-1, p'}$  is indeed an  $\|F\|_{(0)}$  norm with respect to  $\|F\|_{-s+2m, p'}$  (Theorem 1.2).

By dualizing Theorem 1.3,

$$\|\gamma^* \Gamma\|_{-s, p'} \asymp \|\Gamma\|_{-s+1/p, p'}, \quad s > 1/p, \quad s \not\equiv 1/p \pmod{1},$$

and since  $B_j^{\pm*}(P, D)$  are of order  $m_j^{\pm}$  with high order coefficients bounded by  $\delta$ , we obtain again

$$(5.4) \quad \|B_j^{\pm*} \Gamma_j^{\pm}\|_{-s, p'} \leq \delta \|\Gamma_j^{\pm}\|_{-s_j^{\pm}, p'} + K \|\Gamma_j^{\pm}\|_{(0)}, \quad (-s_j^{\pm} = -s + m_j^{\pm} + 1/p).$$

Combining (5.3) and (5.4), we obtain

$$\|T''^* V\|_{-s, p'} \leq \delta \|V\|_{H^{-s+2m}, p'} + K \|V\|_{(0)},$$

and if  $\delta < 1/2$ , we obtain (5.2). (The proof of (5.1) is completely analogous, with Lemma 1.1 and its corollary replacing Lemma 1.2).

We can now use Lemma 5.1 for general operators with smooth coefficients where  $L', B_j^{\pm}$  denote their principal parts at  $P = 0$ . Then the estimates (5.1-2) are valid for  $u, V$ , supported in a small neighborhood of the origin. Equivalently, if  $\varphi$  is a  $C_0^\infty$  function in that neighborhood, we have

$$(5.3) \quad \|\varphi u, R_+^2\|_{s, p} \leq K (\|T \varphi u\|_{H^{s-2m}, p} + \|u\|_{(0)});$$

$$(5.4) \quad \|\varphi V\|_{-s+2m, p'} \leq K (\|T^* \varphi V\|_{-s, p'} + \|\varphi V\|_{(0)}).$$

We turn now to the main result.

**THEOREM 5.1.** *Let  $\Omega$  be a bounded planar domain with a smooth boundary  $\partial\Omega$ . Two points  $P_1, P_2$  on divide it into  $\partial\Omega = \partial^-\Omega \cup \partial^+\Omega$ .  $L(P, D)$  is a properly elliptic operator of order  $2m$  with  $C^\infty(\bar{\Omega})$  coefficients and  $\{B_j^-\}$ ,  $\{B_j^+\}$ ,  $1 \leq j \leq m$  are two sets of operators with respective orders  $m_j^\pm \leq 2m - 1$  and  $C^\infty(\bar{\Omega})$  coefficients. Suppose that at every point of  $\partial^-\Omega$  (resp.  $\partial^+\Omega$ ) the set  $\{B_j^-\}$  (resp.  $\{B_j^+\}$ ) covers  $L$ . (i. e., the covering conditions is satisfied for the principal parts of the operators at this point). Let the principal parts of  $\{B_j^-(P_1, D)\}$ ,  $\{B_j^+(P_1, D)\}$ ,  $L(P_1, D)$  be  $\sigma$ -compatible, and similarly for  $P_2$ . Finally suppose that  $s \not\equiv 1/p, 2/p \pmod{1}$  if  $p \neq 2$ . Then for  $u \in C^\infty(\bar{\Omega})$ , and  $s \geq 2m$*

$$(5.5) \quad \begin{aligned} \|u, \Omega\|_{s,p} &\leq K (\|Tu\|_{\Pi^{s-2m,p}(\Omega)} + \|u, \Omega\|_{(0)}) = \\ &= K (Lu, \Omega\|_{s-2m,p} + \sum_{j,\pm} \|B_j^\pm u, \partial^\pm \Omega\|_{s-m_j^\pm-1/p,p} + \|u, \Omega\|_{(0)}), \end{aligned}$$

and for  $V = (F, \Gamma_j^\pm) \in \Pi^{-s+m,p'}$  such that  $\text{support}(V) \subseteq \bar{\Omega}$ :

$$(5.6) \quad \|V\|_{\Pi^{-s+2m,p'}} \leq K (\|T^*V\|_{-s,p'} + \|V\|_{(0)}).$$

**PROOF.** Using diffeomorphisms which flatten the boundary locally, we obtain (5.3-4) for  $\varphi$  supported in a small  $\bar{\Omega}$  — neighborhood of  $P_1$  or  $P_2$ . For  $P \in \partial\Omega$ ,  $P \neq P_1, P_2$ , the estimates (5.3-4) are true if  $\varphi$  is supported in neighborhood  $d(P)$  which do not contain  $P_1, P_2$ . In fact, (5.3-4) amount then to the pure estimates for  $\{B_j^-\}$  (or  $\{B_j^+\}$ ) alone. (Cf. the opening paragraph of this section). For an interior point  $P$  and  $d(P) \cap \partial\Omega = \emptyset$ , (5.3-4) are again true, being in fact interior estimates which are easily established. Thus every point  $P \in \bar{\Omega}$  has a  $\bar{\Omega}$  — neighborhood  $d(P)$  such that (5.3-4) hold if  $\text{support}(\varphi)$  is in  $d(P)$ . We choose now a finite covering of  $\bar{\Omega}$  by sets  $d(P)$ , construct a corresponding position of unity, and an obvious computation leads to the desired estimates (5.5-6).

**COROLLARY 5.1.** *Let  $p$  be fixed and  $s \not\equiv 1/p, 2/p \pmod{1}$  if  $p \neq 2$ . For every value of  $s$ , except at most  $2m$  values of  $s \pmod{1}$ , the condition that  $\{B_j^-\}$  (resp.  $\{B_j^+\}$ ) cover  $L$  at each point of  $\partial^-\Omega$  (resp.  $\partial^+\Omega$ ) is necessary and sufficient for the validity of (5.5-6).*

**THEOREM 5.2.** *The estimates 5.5-6 are equivalent to the normal solvability of the mixed map  $T$  (i. e. of the boundary value problem (0.1) in  $W^{s,p}(\Omega)$ ).*

The proof of this abstract result can be found in [14, section 8].

We shall prove now that the exceptional values of  $s$ , for which the  $\sigma$ -compatibility is violated at  $P_1$  or  $P_2$ , are jump points for the codimension

of  $T$  (in Remark 3.2, this was shown for the canonical problem in half plane). It follows then that a regularization theorem for the mixed case do not exist. However, we shall prove this fact directly.

**THEOREM 5.3.** *Let  $s > 2m$  be a value for which the  $\sigma$  compatibility ( $s = \sigma \pmod{1}$ ) is violated (at  $P_1$  for instance). If  $2m \leq s_1 < s < s_2$ , there exists a function  $u$  such that  $u \in W^{s_1, p}(\Omega)$ ,  $u \notin W^{s_2, p}(\Omega)$ , and  $Tu \in \Pi^{s_2-2m, p}(\Omega)$ . In other words, there is a data-element in  $\Pi^{s_2-2m, p}(\Omega)$  for which the boundary problem is not solvable in  $W^{s_2, p}$  (as it « ought » to be) but there is a less regular solution in  $W^{s_1, p}$ .*

**PROOF.** We may assume that  $s_2 - s_1 < 1$  and that (after a suitable transformation)  $P_1$  is at the origin, the domain  $\Omega$  is locally contained in  $t > 0$  and its boundary is locally  $t = 0$ . Let  $L', B_j^{\pm}$  be the principal parts at the origin. By assumption they are not  $\sigma$ -compatible. Hence the value  $s$  is a jump point for the « tangent » boundary problem at the origin, which is defined by  $L', B_j^{\pm}$ . There exists then a function  $u$  which is locally in  $W^{s_1, p}$  but not in  $W^{s_2, p}$  and its data  $T'u = \{L'u, B_j^{\pm}u\}$  consist of  $C_0^\infty$  functions in  $R_+^2, R_\pm$ , respectively. Moreover, by the proof of Theorem 3.3 it is possible to choose a satisfying  $L'u = 0, B_j^{\pm}u = 0, x < 0$  and represented by

$$(5.7) \quad D^\alpha u = \sum_j \int_0^\infty D^\alpha K_j(x - y, t) \varphi_j(y) dy, \quad |\alpha| \geq 2m - 1,$$

where  $\varphi_j(x) = B_j^{\pm}u(x, 0)$  ( $= 0$  for  $x < 0$ ). The  $\varphi_j$  and their derivatives up to order  $l_j = [s - m_j^- - 2/p]$  are continuous (i. e. null) at  $x = 0$  while  $\varphi_j(l_j + 1)$  is discontinuous there. (Otherwise (5.7) will show that  $u \in W_{loc}^{s_2, p}$ ). The discontinuity of the corresponding derivatives of  $u$ , which contain the expressions  $\int_0^\infty (\varphi_j^{(l_j+1)})/(\pm(x - y) + t\tau) dy$ , is at most a logarithmic singularity at the origin (cfr. [12, p. 74]).

We have to show that  $Tu \in \Pi^{s_2, p}(\Omega)$ . It is clearly sufficient to prove this for a neighborhood of the origin. There,  $T = T' + T'' + T'''$  where  $T'$  is defined by  $L', B_j^{\pm}$ ,  $T''$  is defined by the lower order parts of  $L, B_j^{\pm}$  and  $T'''$  is defined by high order terms whose coefficients are zero (of the first order at least) at the origin.

By the choice of  $u$ , the components of the  $2m + 1$  tuple  $T'u$  are  $C_0^\infty$  functions. Since  $u \in W^{s_1, p}(\Omega)$ , we have

$$T'''u \in \Pi^{s_1+1-2m}(\Omega) \subset \Pi^{s_2-2m}(\Omega).$$

Finally in  $T''u$  the logarithmic singularity of the derivatives at the origin is killed by the zero of the coefficients. Thus  $T''u \in H^{s_1+1-2m}(\Omega)$  too, and the assertion is proved.

The fact that there is a jump in the codimension of range ( $T$ ) follows easily: We can choose a finite basis composed of  $C_0^\infty$  functions for the complement of the range. If there were no jump, the same basis spans this complement for both  $s_1$  and  $s_2$ . If now  $v = Tu$  is in the range for  $s_1$ , and is contained in  $H^{s_2-2m, p}(\Omega)$ , then by decomposing it in the last space we see that the component of  $u$  in the complement of range ( $T$ ) is null. Thus  $v$  belongs to the range for  $s_2$  too. This contradicts the result of the last theorem.

## BIBLIOGRAPHY

1. AGMON, S. *The  $L^p$  approach to the Dirichlet problem*, Annali di Pisa, 13 (1959), 49-92.
2. AGMON, S., DOUGLIS, A. and NIRENBERG, L. *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. Comm. Pure Appl. Math. 12 (1959), 623-727.
3. ARONSZAJN, N. and SMITH, K. T. *Theory of Bessel potentials (I)*. Annales Inst. Fourier 11 (1961), 385-475. (II-to appear).
4. BROWDER, F. E. *Estimates and existence theorems for elliptic boundary value problems* Proc. Nat. Acad. Sci. U. S. A. 45 (1959), 365-372.
5. BROWDER, F. E. *A priori estimates for solutions of elliptic boundary value problems (I) (II)*. Konink. Ned. Akad. Van Wetenschap 22 (1960), 145-159, 160-169.
6. CALDERON, A. P. and ZYGMUND, A. *On singular integrals*. Amer. J. Math. 78 (1956), 289-309.
7. LIONS, J. L. *Théorèmes de traces et d'interpolation (I)*. Annali di Pisa 13 (1959), 389-403.
8. LIONS, J. L. *Equations différentielles opérationnelles et problèmes aux limites*, Springer Verlag, Berlin, Göttingen, Heidelberg, 1961.
9. LIONS, J. L. and MAGENES, E. *Problemi ai limiti non omogenei (III)* Annali di Pisa 15 (1961), 39-101.
10. LIONS, J. L. *Problemes aux limites non homogènes (IV)* Annali di Pisa 15 (1961), 311-326.
11. LIONS, J. L. and MAGENES, E. *Problemi ai limiti non omogenei (V)*. Annali di Pisa 16 (1962), 1-44.
12. MUSKHELISHVILI, W. I. *Singular integral equation (English translation)* P. Nordhoff N. V, Groningen, 1953.
13. NIKOLSKI, M. *On the imbedding, continuation and approximation theorems for differentiable functions in many variables*. Usp. Mat. Nauk. 16, 5 (1961), 63-114.
14. PEETRE, J. *Another approach to elliptic boundary problems*. Comm. Pure Appl. Math. 14 (1961) 711-731.
15. PEETRE, J. *Mixed problems for higher order elliptic equations in two variables (I)*. Annali di Pisa 15 (1961), 337-353.
16. SCHECHTER, M. *General boundary value problems for elliptic partial differential equations*. Comm. Pure App. Math. 12 (1959), 451-485.
17. SCHECHTER, M. *Mixed problems for higher order elliptic equations*. Comm. Pure App. Math. 13 (1960), 407-425.
18. SCHECHTER, M. *Some  $L^p$  estimates for partial differential equations*. Bull. Amer. Math. Soc. 68 (1962), 470-474.
19. SHAMIR, E. *Évaluations dans  $W^{s,p}$  pour des problèmes aux limites elliptiques mixtes dans le plan*. C. P. Acad. Sci. Paris 254 (1962), 3621-3623.
20. USPENSKI, S. V. *An imbedding theorem for Sobolev classes of fractional order*. Dok Akad. Nauk S. S. S. R, 130 (1960), 992-993.
21. USPENSKI, S. V. *Properties of the classes  $W^{r,p}$  of fractional derivatives on differentiable manifolds*, Dok. Akad. Nauk S. S. S. R. 132 (1960), 60-62.
22. SHAMIR, E. *Reduced Hilbert transforms and singular integral equations*. The appear in J. D'analyse Math., (1964).