ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 15, nº 4 (1961), p. 355-370

<http://www.numdam.org/item?id=ASNSP_1961_3_15_4_355_0>

A REMARK ON THE REGULARITY AT THE BOUNDARY FOR SOLUTIONS OF ELLIPTIC EQUATIONS

by M. K. VENKATESHA MURTHY (Bombay).

§ 1. Introduction.

The object of this note is to prove the following result.

Let A be a linear elliptic operator (of order $2m$) with infinitely differentiable coefficients in a domain Ω, having a smooth boundary $\partial\Omega$, in a euclidean space and let $B_j (0 \leq j \leq 2m - 1)$ be differential operators, with infinitely differentiable coefficients on $\partial\Omega$. If $(A, \{B_j\})$ is an admissible system (see § 2), and f and g_j are functions in certain classes of infinitely differentiable functions (referred to as Friedman classes in the sequel), then any function u infinitely differentiable in $\overline{\Omega}$ and satisfying $Au = f$ in Ω, $B_j u = g_j$ in $\partial\Omega$, is itself in a Friedman class in $\overline{\Omega}$ when the coefficients of A and of B_j and the functions which define the boundary $\partial\Omega$ locally, are in certain Friedman classes.

When f, g_j, and the coefficients of A and of B_j are real analytic functions of their arguments the real analyticity of the solution u (of the system) upto the boundary has been proved, using a method of Morrey and Nirenberg [4], by Magenes and Stampacchia [3] assuming that $(A, \{B_j\})$ is an admissible system and $\partial\Omega$ is analytic. Our result includes that of Magenes and Stampacchia. In the case of the Dirichlet problem this result was proved in the case of real analytic functions by Morrey and Nirenberg [4] and in the case of functions in Friedman classes by Friedman in [2].

The notation, necessary norms, and other preliminaries are introduced in § 2. In § 3 two lemmas, which lead to L_2-estimates for u and its tangential derivatives of all orders and normal derivatives of order upto $2m$, are proved. In § 4 L_2-estimates for derivatives of all orders of u are obtained and finally an application of Sobolev's lemma yields the result.

The author wishes to thank Mr. B. V. Singbal for his valuable suggestions and help.

§ 2. Notation and Preliminaries.

Let Ω denote a bounded domain in a ν-dimensional Euclidean space and let $(x_1, \dots, x_\nu)$ be a coordinate system in Ω. First we define certain classes of infinitely differentiable (C^∞) functions on Ω. Let $\{M_n\}$ be a sequence of positive numbers satisfying the following condition: there exists a positive constant C, independent of n, such that

$$(1) \qquad \binom{n}{\lambda} M_\lambda M_{n-\lambda} \leq C M_n \qquad (\text{for } \lambda = 1, 2, \dots, n\,;\ n = 1, 2, \dots)$$

Then if p is any non-negative integer, we denote by $C\{M_{n-p}\,;\Omega\}$ the class of C^∞-functions f on Ω satisfying the following condition:

(QA) For every closed subdomain Ω_0 of Ω there exist two constants H_1 and H_2, depending on f and on Ω_0, such that, for any $x \in \Omega_0$, we have

$$\left| \frac{\partial^{|k|} f(x)}{\partial x_1^{k_1} \dots \partial x_\nu^{k_\nu}} \right| \leq H_1 H_2^{|k|} M_{|k|-p}$$

where $k = (k_1, \dots, k_\nu)$ and $|k| = k_1 + \dots + k_\nu$.

We call a class of the type $C\{M_{n-p}\,;\Omega\}$ a *Friedman class*.

Similarly we define the classes $C\{M_{n-p}\,;\overline{\Omega}\}$ when the condition (QA) is satisfied in $\overline{\Omega}$.

It is clear that (1) implies

$$(1') \qquad (n+1) M_n \leq C_1 M_{n+1}$$

with a positive constant C_1 independent of n $\left(\text{in fact we can take } C_1 = \dfrac{C}{M_1}\right)$.

Now we make some remarks on the Friedman classes $C\{M_n\,;\Omega\}$ which will be of use in the sequel.

(i) If f is a function in the class $C\{M_n\,;\Omega\}$ such that $f(x) \neq 0$ for $x \in \Omega$ then $\dfrac{1}{f}$ is itself in the class $C\{M_n\,;\Omega\}$. For, let $(x_1, \dots x_\nu)$ be a coordinate system in Ω and let d_{i_μ} denote a generic partial differentiation operator of order one. Therefore, a generic partial differentiation operator of order k can be written in the form $d_{i_1} d_{i_2} \dots d_{i_k}$. Then

$$(d_n d_{n-1} \dots d_1) \left(\frac{1}{f}\right) = \frac{1}{f^{n+1}} H_n(f)$$

where $H_n(f)$ is a homogeneous polynomial of degree n in the set of arguments $(f,\dots,f^k,\dots,d_i f,\dots,d_{i_1}\dots d_{i_r} f,\dots,d_{i_1}\dots d_{i_n} f)$ and where the degree of $d_{i_1}\dots d_{i_r} f$ is taken to be r. Any monomial of degrée k in these arguments is majorized on any closed subset Ω_0 of Ω by $(H_0 H_1)^k C^{k-1} M_k \le H_0(H_0 H_1 C)^k M_k$. Hence one can easily see that

$$|H_n(f)| \le H_0 (c H_0 H_1 C)^n M_n$$

with a suitable positive constant c independent of n.

Let $\delta = \min_{x \in \Omega_0} |f(x)|$. Taking $K_0 = \frac{H_0}{\delta}$ and $K_1 = \frac{c H_0 H_1 C}{\delta}$ we see that

$$\left| d_n d_{n-1} \dots d_1 \left(\frac{1}{f}\right) \right| \le K_0 K_1^n M_n$$

on Ω_0 which establishes (i).

(ii) If f, g are in $C\{M_n;\Omega\}$ then their product fg is itself in $C\{M_n;\Omega\}$. In fact, we have for an $\alpha = (\alpha_1,\dots,\alpha_\nu)$

$$D^\alpha(fg) = \sum_\beta \binom{\alpha}{\beta} D^\beta(f) \cdot D^{\alpha-\beta}(g)$$

by Leibniz formula, where $\beta = (\beta_1,\dots,\beta_\nu)$ and $\binom{\alpha}{\beta} = \binom{\alpha_1}{\beta_1}\dots\binom{\alpha_\nu}{\beta_\nu}$.

Then $|D^\alpha(fg)| \le \Sigma \binom{\alpha}{\beta} H_1 H_2^\beta M_\beta H_1 H_2^{\alpha-\beta} M_{\alpha-\beta} \le C' H_1^2 H_2^\alpha M_\alpha$, with a suitable constant $C' > 0$.

The remarks (i) and (ii) together imply the following:

(iii) If f, g belong to $C\{M_n;\Omega\}$ with g non-vanishing in Ω then f/g is itself in the class $C\{M_n;\Omega\}$.

Let s denote a fixed positive real number. Let H^s denote the space of all tempered distributions φ such that its Fourier transform $\widehat{\varphi}$ satisfies the condition that $(1+|\xi|^2)^{s/2}\widehat{\varphi}$ is square integrable and we define the scalar product in H^s by

$$(\varphi,\psi)_s = \int \widehat{\varphi}(\xi)\overline{\widehat{\psi}(\xi)}(1+|\xi|^2)^s d\xi \text{ for any two } \varphi,\psi \in H^s$$

and the corresponding norm

$$\|\varphi\|_s = \left[\int |\widehat{\varphi}(\xi)|^2 (1+|\xi|^2)^s d\xi\right]^{\frac{1}{2}} \text{for any } \varphi \in H^s.$$

(see [3]).

In view of the local nature of the problem it is enough to consider the solution of the problem in a hemi-sphere with the boundary conditions defined on the plane part of the boundary.

Throughout, the function u is assumed to be infinitely differentiable in the hemi-sphere together with the plane part of the boundary. This is so, for example, in the following cases:

Let the coefficients of A and B_j, and f, g_j be infinitely differentiable functions of their arguments. Then any solution u of $Au = f$, $B_j u = g_j$ is infinitely differentiable either when $(A, \{B_j\})$ is an elliptic system in the sense defined by J. Peetre [5] or when the boundary operators B_j satisfy the complementing condition of Agmon, Douglis and Nirenberg [1] with respect to the elliptic operator A.

Next we introduce the differential operators. Let ω_r denote the hemisphere $\{x_1^2 + \dots + x_\nu^2 < r^2, x_\nu > 0\}$ and $\partial_1 \omega_r$ the plane part $\{x_\nu = 0\}$ of the boundary of ω_r. Let π_0 denote the $(\nu - 1)$-dimensional subspace $\{(x_1, \dots, x_{\nu-1}, 0)\}$ and let x' denote either $(x_1, \dots, x_{\nu-1}, 0)$ or $(x_1, \dots, x_{\nu-1})$ inadvertently in the context. We adopt the following notation throughout:

If $p = (p_1, \dots, p_\nu)$ then $a_p(x)$ denotes a function $a_{p_1 \dots p_\nu}(x)$ and $D^p = \dfrac{\partial^{|p|}}{\partial x_1^{p_1} \dots \partial x_\nu^{p_\nu}}$.

Similar notation is used in π_0 also with x' in place of x and $D_{x'}^p$ in place of D^p.

All our functions are defined in ω_{R_0} together with the plane part $\partial_1 \omega_{R_0}$ of the boundary of ω_{R_0} where R_0 is a fixed positive number. Let

$$A = \sum_{|p| \leq 2m} a_p(x) D^p$$

be an elliptic linear partial differential operator of order $2m$ on ω_{R_0} with the coefficients $a_p(x)$ C^∞ in $\overline{\omega}_{R_0}$ and let

$$B_j = \sum_{|p| \leq 2m-1-j} b_p^j(x') D^p \quad (0 \leq j \leq 2m - 1)$$

be differential operators (boundary operators) where $b_p^j(x')$ are C^∞ functions on $\partial_1 \omega_{R_0}$.

Let $\varrho(t)$ be a real valued C^∞ function of the variable t $(-\infty < t < \infty)$ such that

$$\varrho(t) = 1 \text{ for } t \leq 0,$$

$$= 0 \text{ for } t \geq 1,$$

then for any pair of positive numbers r and h with $0 < r < r + h < R_0$ define

$$\varphi_{r,h}(x) = \varrho\left(\frac{|x| - r}{h}\right).$$

Then clearly we have

$$\begin{aligned} \varphi_{r,h}(x) &= 1 \text{ for } |x| \leq r \\ &= 0 \text{ for } |x| \geq r + h \end{aligned}$$

and further for any $p = (p_1, \dots, p_\nu)$ we have

$$|D^p \varphi_{r,h}(x) \leq C_2 h^{-|p|} \quad (\text{when } h < r),$$

where C_2 is a positive constant depending on ν, p and the bounds for the derivatives of ϱ.

Definition. The system $(A, \{B_j\})$ is said to be an admissible system if A and the boundary operators $\{B_j\}$ satisfy the following condition: there exists a constant C_3 such that for any C^∞ function u and for any r with $0 < r < r + h < R_0$, we have

$$(2) \qquad \sum_{q=2m} \| D^p \varphi_{r,h} u \|^2_{0,\omega_{R_0}} \leq C_3 \left\{ \| A(\varphi_{r,h} u) \|^2_{0,\omega_{R_0}} + \sum_{j=0}^{2m-1} \| B_j(\varphi_{r,h} u) \|^2_{j+\frac{1}{2},\pi_0} \right\}$$

where if $B_j(\varphi_{r,h} u)$ is considered as having its support contained in $\partial_1 \omega_{r+h}$ then $B_j(\varphi_{r,h} u)$ is extended to the whole of π_0 by taking it to be equal to zero in $\pi_0 - \partial_1 \omega_{r+h}$. Here $\|f\|_{0,\Omega}$ is defined by $\|f\|^2_{0,\Omega} = \int\limits_{\Omega} |f(x)|^2 \, dx$.

Remark. (a) When $(A, \{B_j\})$ is an admissible system the analyticity of a solution u of $Au = f$, $B_j u = g_j$ upto the boundary (the coefficients of A and of B_j, and f, g_j being real analytic functions of their arguments) was proved by Magenes and Stampacchia [3].

(b) The inequality (2) has been obtained by J. Peetre when $(A, \{B_j\})$ is an elliptic system in the sense defined in [5]. When A is elliptic and B_j satisy the complementing condition with respect to A an analogous inequality has been proved by Agmon, Douglis and Nirenberg [1].

The following is the precise statement of our theorem.

Theorem. Let $(A, \{B_j\})$ be an admissible system, with A elliptic, such that the following conditions are satisfied:

(i) the coefficients $a_p(x)$ of A are in $C\{M_n; \omega_{R_0}\}$;

and (ii) the coefficients $b_p^j(x')$ of B_j are in $C\{M_n; \partial_1 \omega_{R_0}\}$.

Then any function u, C^∞ in $\overline{\omega}_{R_0}$ and satisfying the system

$$Au = f \text{ in } \omega_{R_0},$$

$$B_j u = g_j \text{ in } \partial_1 \omega_{R_0} \quad (0 \leq j \leq 2m-1),$$

where f is in $C\{M_n ; \omega_{R_0}\}$ and g_j are in $C\{M_{n-j-1} ; \partial_1 \omega_{R_0}\}$ respectively, is a function in $C\{M_{n-2m+[\nu/2]+1} ; \omega_{R_0} \cup \partial_1 \omega_{R_0}\}$.

In the course of the proof of the theorem we need the following norms (introduced in [3]):

$$e_{k,r}(f) = \Big(\sum_{|q|=k} \| \varphi_{r,h}\, D^q_{x'} f \|^2_{0,\omega_{r+h}}\Big)^{\frac{1}{2}} \text{ with } h = \frac{R-r}{k+1}, \; k = 0, 1, 2, \ldots$$

$$e_{j,k,r}(g) = \Big(\sum_{|q|=k} \| \varphi_{r,h}\, D^q_{x'}\, g \|^2_{j+\frac{1}{2},\pi_0}\Big)^{\frac{1}{2}} \text{ with } h = \frac{R-r}{k+1}, \; k = 0, 1, 2, \ldots$$

$$(0 \leq j \leq 2m-1)$$

and
$$d_{k,r}(u) = \Big(\sum_{|q|=2m} \; \sum_{|p|=k} \| D^q D^p_{x'}\, u \|^2_{0,\omega_r}\Big)^{\frac{1}{2}} \text{ for } k = 0, 1, 2, \ldots$$

$$= \Big(\sum_{|q|=2m+k} \| D^q u \|^2_{0,\omega_r}\Big)^{\frac{1}{2}} \qquad \text{for } k = -2m, \ldots, 0.$$

We make the covention that

$$[M_k] = M_k \qquad \text{if } k \geq 0 \text{ and}$$

$$= 1 \qquad \text{if } k < 0$$

and introduce the following notation (in analogy with that introduced in [4])

$$M_{R,k}(f) = \frac{1}{M_k} \sup_{R/2 \leq r < R} (R - (r+h))^{2m+k}\, e_{k,r}(f) \qquad \text{for } k = 0, 1, 2, \ldots$$

$$M_{j,R,k}(g) = \frac{1}{M_k} \sup_{R/2 \leq r < R} (R - (r+h))^{2m+k}\, e_{j,k,r}(g)$$

$$\text{for } k = 0, 1, 2, \ldots ; \; 0 \leq j \leq 2m-1$$

and

$$N_{R,k}(u) = \frac{1}{[M_k]} \sup_{R/2 \leq r < R} (R - r)^{2m+k}\, d_{k,r}(u)$$

$$\text{for } k = -2m, -2m+1, \ldots, 0, 1, 2, \ldots$$

§ 3. In this paragraph we present two lemmas leading to the proof of the main theorem stated in the previous paragraph. In principle we obtain an L_2-estimate for the derivatives, upto order $2m$ in the transverse direction and of all orders in the tangential direction, for a function satisfying the system. To begin with we have the following result due to Magenes and Stampacchia (see [3] p. 331).

If u is any C^∞ function and if $(A, \{B_j\})$ is an admissible system then there exists a constant C_4, independent of u, r and h such that for $0 < r < r + h < R_0$, $r > h$ we have

$$(3) \qquad \sum_{|q|=2m} \| D^q u \|^2_{0,\omega_r} \leq C_4 \Big\{ \| \varphi_{r,h} A u \|^2_{0,\omega_{r+h}} + \sum_{j=0}^{2m-1} \| \varphi_{r,h} B_j u \|^2_{j+\frac{1}{2},\pi_0} + \\ + \sum_{\lambda=0}^{2m-1} \| u \|^2_{\lambda,\omega_{r+h}} h^{2\lambda-4m} \Big\}.$$

Now we observe that, for any positive integers λ, l, k, we have

$$(4) \qquad \sum_{|q|=\lambda-l} \sum_{|p|=k} | D^q D^p_{x'} u |^2 \leq \begin{cases} \sum\limits_{|q|=\lambda} \sum\limits_{|p|=k-l} D^q D^p_{x'} u |^2 & \text{for } l \leq k \\ \sum\limits_{|q|=\lambda+k-l} | D^q u |^2 & \text{always.} \end{cases}$$

It follows from this that, for $k \geq 0$, $0 \leq \lambda \leq 2m$ we have

$$(5) \qquad \sum_{|p|=k} d^2_{-\lambda,r} (D^p_{x'} u) \leq d^2_{k-\lambda,r} (u).$$

On the otherhand we also have

$$(6) \qquad \sum_{|p|=k} e^2_{0,r} (D^p_{x'} f) = e^2_{k,r} (f)$$

$$\sum_{|p|=k} e^2_{j,0,r} (D^p_{x'} g) = e^2_{j,k,r} (g).$$

Taking $h = \dfrac{R - r}{k + 1}$ $(R/2 \leq r < R)$, (3) can now be written in the form

$$(7) \qquad d^2_{0,r} (u) \leq C_4 \Big\{ e^2_{0,r} (Au) + \sum_{j=0}^{2m-1} e^2_{j,0,r} (B_j u) + \sum_{\lambda=1}^{2m} d^2_{-\lambda,r+h} (u) \cdot h^{-2h} \Big\}.$$

Lemma 3.1. If u is any C^∞ function and if $(A, \{B_j\})$ is an admissible system then there exists a positive constant C_5 indendent of u, R and of

k, such that for any $R < R_1$, $k > 0$ the following inequality holds:

$$(8) \qquad N_{R,k}(u) \leq C_5 \left\{ M_{R,k}(Au) + \sum_{j=0}^{2m-1} M_{j,R,k}(B_j u) + \sum_{\lambda=1}^{2m} N_{R,k-\lambda}(u) + \right.$$

$$\left. + \sum_{\tau=1}^{2m+k} (H_2 R)^{\tau} N_{R,k-\tau}(u) \right\}.$$

Proof. Consider any one of the tangential derivatives $D^q_{x'} u$ of u with $|q| = k$ and apply (3) in the form (7) taking $R/2 \leq r < R$ and $h = \dfrac{R-k}{k+1}$. We obtain

$$(9) \qquad d^2_{0.r}(D^q_{x'} u) \leq C_4 \left\{ e^2_{0,r}(A(D^q_{x'} u)) + \sum_{j=0}^{2m-1} e^2_{j,0,r}(B_j(D^q_{x'} u)) + \right.$$

$$\left. + \sum_{\lambda=0}^{2m} d^2_{-\lambda,r+h}(D^q_{x'} u) \cdot h^{-2\lambda} \right\}.$$

Using Leibniz formula for the derivation of a product of two functions and the fact that

$$\binom{q_1}{s_1}\binom{q_2}{s_2}\cdots\binom{q_\nu}{s_\nu} \leq \binom{k}{\mu}$$

where q_i and s_i are non-negative integers such that $q_1 + \ldots + q_\nu = k$ and $s_1 + \ldots + s_\nu = \mu$ we have the inequalities

$$\sum_{|q|=k} |A(D^q_{x'} u)| \leq \sum_{|q|=k} |D^q_{x'}(Au)| + \sum_{|p| \leq 2m} \sum_{\mu=1}^{k} \binom{k}{\mu} \sum_{|s|=\mu} |D^s_{x'} a_p| \sum_{|t|=k-\mu} |D^t_{x'} D^p u|$$

and

$$\sum_{|q|=k} |B_j(D^q_{x'} u)| \leq \sum_{|q|=k} |(D^q_{x'}(B_j u)| + \sum_{|p| \leq 2m-1-j} \sum_{\mu=1}^{k} \binom{k}{\mu} \sum_{|s|=\mu} |D^s_{x'} b^j_p|$$

$$\cdot \sum_{|t|=k-\mu} |D^t_{x'} D^p u|.$$

Summing over all q with $|q| = k$ in (9), using the following majorizations

$$\left(\sum_{|q|=\lambda}^{k} |D^q a_p(x)|^2 \right)^{\frac{1}{2}}, \quad \left(\sum_{|q|=\lambda} |D^q_{x'} b^j_p(x')|^2 \right)^{\frac{1}{2}} \leq H_1 H_2^{\lambda} M_{\lambda}$$

(with the constants H_1, H_2 suitably changed) and making use of (5), (6) we obtain

$$d_{k.r}(u) \leq C_6 \left\{ e_{k,r}(Au) + \sum_{j=0}^{2m-1} e_{j,k,r}(B_j u) + \sum_{\lambda=1}^{2m} d_{k-\lambda,r+h}(u)\, h^{-\lambda} + \right.$$

$$+ \sum_{\mu=1}^{k} \binom{k}{\mu} H_1 H_2^{\mu} M_{\mu} \left(\sum_{|p| \leq 2m} \cdot \sum_{|t|=k-\mu} \| D_{x'}^{t} D^{p} u \|_{0,\omega_{r+h}}^{2} \right)^{\frac{1}{2}} +$$

$$\left. + \sum_{j=0}^{2m-1} \sum_{\mu=1}^{k} \binom{k}{\mu} H_1 H_2^{\mu} M_{\mu} \left(\sum_{|p| \leq 2m-1-j} \sum_{|t|=k-\mu} \| \varphi_{r,h} D_{x'}^{t} D^{p} u \|_{j+\frac{1}{2},\pi_0}^{2} \right)^{\frac{1}{2}} \right\}$$

where C_6 is a positive constant independent of u, r, h, k, R. Moreover we have $\| \varphi_{r,h} v \|_{j+\frac{1}{2},\pi_0} \leq C_7 \| v \|_{j+1,\omega_{r+h}}$ (see [3]) with C_7 independent of v. From this remark it is clear that the last term of the second member of the above inequality can be majorized by the last but one term. Hence

$$(10) \qquad d_{k,r}(u) \leq C_6 \left\{ e_{k,r}(Au) + \sum_{j=0}^{2m-1} e_{j,k,r}(B_j u) + \sum_{\lambda=1}^{2m-1} d_{k-\lambda,r+h}(u) \cdot h^{-\lambda} + \right.$$

$$\left. + C_8 \sum_{\mu=1}^{k} \binom{k}{\mu} H_2^{\mu} M_{\mu} \left(\sum_{|p| \leq 2m} \sum_{|t|=k-\mu} \| D_{x'}^{t} D^{p} u \|_{0,\omega_{r+h}}^{2} \right)^{\frac{1}{2}} \right\}$$

where C_8 is a positive constant independent of u, r, h and k. Applying the inequality (4) to the last term of the second member of (10) we obtain

$$(11) \qquad d_{k,r}(u) \leq C_6 \left\{ e_{k,r}(A u) + \sum_{j=0}^{2m-1} e_{j,k,r}(B_j u) + \sum_{\lambda=1}^{2m} d_{k-\lambda,r+h}(u) \cdot h^{-\lambda} + \right.$$

$$\left. + C_8 \sum_{\mu=1}^{k} \binom{k}{\mu} H_2^{\mu} M_{\mu} \sum_{l=0}^{2m} d_{k-\mu-l,r+\mu}(u) \right\}.$$

Multiplying both sides of (11) by $\dfrac{1}{M_k}(R-r)^{2m+k}$ we have the following estimates:

$$\frac{1}{M_k}(R-r)^{2m+k} e_{k,r}(Au) = \frac{1}{M_k} \left[\frac{R-r}{R-(r+h)} \right]^{2m+k} (R-(r+h))^{2m+k} e_{k,r}(Au) \leq$$

$$\leq \left(1 + \frac{1}{k}\right)^{2m+k} M_{R,k}(Au)$$

$$\frac{1}{M_k}(R-r)^{2m+k} e_{j,k,r}(B_j u) \leq \left(1 + \frac{1}{k}\right)^{2m+k} M_{j,R,k}(B_j u) \quad \text{for} \quad 0 \leq j \leq 2m-1.$$

Further since $h = \dfrac{R - r}{k + 1}$

$$\frac{1}{M_k}(R - r)^{2m+k} d_{k-\lambda, r+h}(u)\, h^{-\lambda} \leq \frac{1}{M_k}(k + 1)^{+\lambda}\left(1 + \frac{1}{k}\right)^{2m+k-\lambda} [M_{k-\lambda}]\, N_{R,k-\lambda}(u) \leq$$

$$\leq \frac{[(k - \lambda)\,!]}{k\,!} C_1^\lambda (k + 1)^\lambda \left(1 + \frac{1}{k}\right)^{2m+k-\lambda} N_{R,k-\lambda}(u)$$

because $[M_{k-\lambda}] \leq \dfrac{(k - \lambda)\,!}{k\,!} C_1^\lambda M_k$. Similarly we have

$$\binom{k}{\mu} \frac{M_\mu}{M_k}(R - r)^{2m+k} d_{k-\mu-l, r+h}(u) \leq \binom{k}{\mu} \frac{M_\mu\,[M_{k-\mu-l}]}{M_k}\left(1 + \frac{1}{k}\right)^{2m+k-\mu-l}$$

$$\cdot\, R^{\mu+l} N_{R,k-\mu-l}(u).$$

But by (1) $M_\mu\,[M_{k-\mu-l}] \leq \left[\binom{k - l}{\mu}\right]^{-1} C\, M_{k-l}$ and by (1′) it follows that $M_{k-1} \leq$ $\leq C_1^l \dfrac{[(k - l)\,!]}{k\,!} M_k$. Hence we have:

$$\binom{k}{\mu} \frac{M_\mu}{M_k}(R - r)^{2m+k} d_{k-\mu-l, r+h}(u) \leq C\, C_1^1 \frac{[(k - \mu - l)\,!]}{(k - \mu)\,!}\left(1 + \frac{1}{k}\right)^{2m+k-\mu-l}$$

$$\cdot\, R^{\mu+1} N_{R,k-\mu-l}(u).$$

Then the inequality (11) becomes

$$\frac{1}{M_k}(R - r)^{2m+k} d_{k,r}(u) \leq C_6\left(1 + \frac{1}{k}\right)^{2m+k}\Bigg\{M_{R,k}(Au) + \sum_{j=0}^{2m-1} M_{j,R,k}(B_j u) +$$

$$+ \sum_{\lambda=1}^{2m} \frac{[(k - \lambda)\,!]}{k\,!}(k + 1)^\lambda C_1^\lambda \left(1 + \frac{1}{k}\right)^{-\lambda} N_{R,k-\lambda}(u) +$$

$$+ C_8 \sum_{\mu=1}^{k} \sum_{l=0}^{2m} H_2^\mu R^{\mu+l}\, C\, C_1^l \frac{[(k - \mu - l)\,!]}{(k - \mu)\,!}\left(1 + \frac{1}{k}\right)^{-(\mu+l)} \cdot N_{R,k-\mu-l}(u)\Bigg\}.$$

Since $k^\lambda \dfrac{[(k - \lambda)\,!]}{k\,!} \leq \lambda^\lambda \leq 2m^{2m}$ and $\dfrac{[(k - \mu - l)\,!]}{(k - \mu)\,!} \leq 1$ it follows that there exists a constant C_5 such that

$$\frac{1}{M_k}(R-r)^{2m+k}\, d_{k,r}(u) \le C_5 \Big\{ M_{R,k}(Au) + \sum_{j=0}^{2m-1} M_{j,R,k}(B_j u) + $$
$$+ \sum_{\lambda=1}^{2m} N_{R,k-\lambda}(u) + \sum_{\mu=1}^{k} \sum_{l=0}^{2m} (H_2 R)^{\mu+l}\, N_{R,k-\mu-l}(u) \Big\}.$$

Taking $\mu + l = \tau$ in the last term of the second member and the supremum for $R/2 \le r < R$ of the first member we obtain (8) and this completes the proof of the lemma.

Lemma 3.2. Let $(A, \{B_j\})$ be an admissible system and u be any C^∞ function satisfying the system

$$Au = f \quad \text{in} \quad \omega_{R_0},$$

$$B_j u = g_j \quad \text{in} \quad \partial_1 \omega_{R_0} \quad (0 \le j \le 2m-1),$$

with f and g_j respectively in the classes $C\{M_n\,;\,\omega_{R_0}\}$ and $C\{M_{n-j-1}\,;\,\partial_1\omega_{R_0}\}$. Then there exist two positive constants M and λ such that

$$N_{R,k}(u) \le M\,\lambda^k \quad \text{for} \quad k = -2m,\ -2m+1, \dots \tag{12}$$

Proof. We can suppose, if necessary after some modification that the constants H_1, H_2 and R_1 are the same as before and are such that

$$\Big(\sum_{|q|=k} |D^q f(x)|^2\Big)^{\frac{1}{2}} \le H_1\, H_2^k\, M_k \quad \text{for} \quad x \in \overline{\omega}_{R_1}, \quad k = 0, 1, 2, \dots;$$

and

$$\Big(\sum_{|q|=k} |D^q_{x'}\, g_j(x')|^2\Big)^{\frac{1}{2}} \le H_1 H_2^k\, M_k \quad \text{for} \quad x' \in \partial_1\omega_{R_1}, \quad k = 0,1,2,\dots;\ 0 \le j \le 2m-1.$$

Let β_ν^2 denote the volume of the unit ball in the ν-dimensional Euclidean space. Then for $R < R_1$ we have

$$M_{R,k}(f) \le \frac{1}{M_k}\, R^{2m+k} \Big(\sum_{|q|=k} \int_{\omega_R} |D^q_{x'} f|^2\, dx \Big)^{\frac{1}{2}} \le H_1\, R^{2m+k}\, H_2^k\, R^{\nu/2}\, \beta_\nu.$$

Similarly using $\| \varphi_{r,h} W \|_{j+\frac{1}{2},\pi_0} \le \widetilde{C}\, \| W \|_{j+1,\omega_{r+h}}$, with a positive costant $\widetilde{C}$

independent of W, we obtain

$$M_{j,R,k}(g_j) \leq \tilde{C} \frac{1}{M_k} R^{2m+k} \left(\sum_{l=0}^{j+1} \sum_{|q|=k+l} \int_{\partial_1 \omega_R} | D^q_{x'} g_j |^2 \, dx' \right)^{\frac{1}{2}} \leq$$

$$\leq C_9 R^{2m+k} H_1 H_2^k R^{\frac{\nu-1}{2}} \beta_{\nu-1} \left(\sum_{l=0}^{j+1} \left(\frac{M_{k+l-j-1}}{M_k} \right)^2 \right)^{\frac{1}{2}} \leq$$

$$\leq C_{10} R^{2m+k} H_1 H_2^k R^{\frac{\nu-1}{2}} \beta_{\nu-1}$$

using (1′), where C_9, C_{10} are positive constants independent of R and k. Then the inequality (8) becomes, for any k and $R < R_1$,

$$N_{R,k}(u) \leq C_5 \left\{ C_{11} (H_2 R)^k + \sum_{\lambda=1}^{2m} N_{R,k-\lambda}(u) + \sum_{\tau=1}^{2m+k} (H_2 R)^\tau N_{R,k-\tau}(u) \right\}.$$

Now proceeding, as in the proof of Magenes and Stampacchia, with the constants $M \geq 3 C_5 C_{11}$ and $\lambda = (3C_5 + 1)(H_2 R_1 + 1)$ we obtain

$$N_{R,k}(u) \leq M \lambda^k \quad \text{for} \quad k = -2m, \; -2m+1, \ldots$$

after using an induction argument on k. This completes the proof of lemma 3.2.

§ 4. We complete the proof of the main theorem (see § 2) in this paragraph. For this purpose it is necessary to obtain estimates of the type (12) for all derivatives, tangential as well as transversal, of u. To obtain such estimates we follow a procedure used by Morrey and Nirenberg in [4]. We introduce the following norms analogous to those in § 2.

For $p \geq 0$, $q \geq -2m$ define

$$(13) \quad N_{R,p,q}(u) = \frac{1}{[M_{p+q}]} \sup_{R/2 \leq r < R} (R - r)^{2m+p+q} \left(\sum_{|\lambda|=p} \int_{\omega_r} | D_y^{2m+q} D^\lambda_{x'} u |^2 \, dx \right)^{\frac{1}{2}}.$$

Analogous to (4) we have

$$\sum_{|\lambda|=p} | D_y^{2m+q} D^\lambda_{x'} u | \leq \sum_{|\mu|=q} \sum_{|\lambda|=p} | D^{2m+\mu} D^\lambda_{x'} u | \leq$$

$$\leq \begin{cases} \displaystyle\sum_{|\mu|=2m} \sum_{|\lambda|=p+q} | D^{2m} D^{\mu+\lambda}_{x'} u | & \text{if} \quad p \geq 0, \; q \leq 0 \\ \displaystyle\sum_{|\mu|=p+q+2m} | D^\mu u | & \text{in all cases.} \end{cases}$$

This implies that

$$N_{R,p,q}(u) \leq N_{R,p+q}(u) \quad \text{if} \quad p \geq 0, \quad q \leq 0. \tag{14}$$

We now prove the following extension of the estimation (12): if R is smaller than or equal to a fixed number depending only on the given differential equation, then

$$N_{R,p,q}(u) \leq \bar{M}\,\bar{\lambda}^{p+q}\,\theta^{p}\,;(p \geq 0, q \geq -2m) \tag{15}$$

with $\bar{M}, \bar{\lambda} \geq 1$ and $\theta \leq \frac{1}{2}$ fixed constants, $\bar{\lambda}$ and θ depending only on the equation.

The following is a sketch of the derivation of the estimate (15). Let us denote x_ν by y for convenience. By assumption $y = 0$ is not a charecteristic surface for the given equation $Au = f$. Hence one can solve for the normal derivative $D_y^{2m} u$ of u in terms of the derivatives involving normal derivatives of u of orders less than $2m$:

$$D_y^{2m} u = g + \sum_{t=1}^{2m} b_t\, D_y^{2m-t} D_{x'}^{t} u \tag{16}$$

where in view of the remarks on the classes $C\{M_n\,;\Omega\}$, made in § 2, g and b_t are functions belonging to the class $C\{M_n\,;\omega_{R_0}\}$. This implies that both

$$\sum_{|\lambda|=p} |D_y^q D_{x'}^{\lambda}\, g(x)|, \sum_{|\lambda|=p} |D_y^q D_{x'}^{\lambda}\, b_t(x)| \leq H_1 H_2^{p+q} M_{p+q} \tag{17}$$

for suitable constants H_1, H_2 and $R_0 \leq 1$. We can assume these constants to be the same as before by suitable choice. Then we have from (16)

$$\sum_{|\lambda|=p} |D_y^{2m+q} D_{x'}^{\lambda} u| \leq \sum_{|\lambda|=p} |D_y^q D_{x'}^{\lambda} g| + \sum_{|\lambda|=p} \sum_{t=1}^{2m} |D_y^q D_{x'}^{\lambda} (b_t D_y^{2m-t} D_{x'}^{t} u)|.$$

Hence

$$\sum_{|\lambda|=p} |D_y^{2m+q} D_{x'}^{\lambda} u| \leq \tag{18}$$

$$\leq H_1 H_2^{p+q} M_{p+q} + \sum_{t=1}^{2m} \sum_{|\lambda|=p} \sum_{\alpha} \sum_{\beta=0}^{q} \binom{\lambda}{\alpha} \binom{q}{\beta} H_1 H_2^{|\alpha|+\beta} M_{|\alpha|+\beta}\, |D_y^{2m+q-\beta-t} D_{x'}^{\lambda-\alpha+t} u|.$$

It is clear from (12) and (14) that (15) follows for $-2m \leq q \leq 0$ and all $p \geq 0$ provided that $R < R_1 < R_0$ (R_1 chosen suitably) and

$$\bar{M}\,\bar{\lambda}^{p+q}\,\theta^{p} \geq M\lambda^{p+q} \quad \text{for} \quad -2m \leq q \leq 0, \ p \geq 0. \tag{19}$$

We prove (15) for $q \geq 0$, $p \geq 0$ by induction on q. Let us assume that (15) holds for all values of q less than a certain positive integer which we again denote by q. Squaring both sides of (18) and integrating over ω_r we obtain

$$\left(\sum_{|\lambda|=p} \int_{\omega_r} | D_y^{2m+q} D_{x'}^{\lambda} u |^2 \, dx \right)^{\frac{1}{2}}$$

$$\leq H_1 H_2^{p+q} M_{p+q} \beta_\nu r^{\nu/2} + \sum_{t=1}^{2m} \sum_{|\alpha|=0}^{p} \sum_{\beta=0}^{q} \binom{p}{|\alpha|} \binom{q}{\beta} H_1 H_2^{|\alpha|+\beta} M_{|\alpha|+\beta} \cdot$$

$$\cdot \left(\sum_{|\lambda|=p} \int_{\omega_r} | D_y^{2m+q-\beta-t} D_{x'}^{\lambda-\alpha+t} u |^2 \, dx \right)^{\frac{1}{2}}$$

Multiplying both sides of this inequality by

$$\frac{(R - r)^{2m+p+q}}{[M_{p+q}]} \overline{M}^{-1} \overline{\lambda}^{-(p+q)} \theta^{-p}$$

for $R < R_1$, taking the supremum over all r with $R/2 \leq r < R$ and using the induction assumption we obtain

$$\overline{M}^{-1} \overline{\lambda}^{-(p+q)} \theta^{-p} N_{R,p,q} \leq \frac{K H_1}{\overline{M}} \left(\frac{H_2 R}{\overline{\lambda}} \right)^{p+q} \theta^{-p} + \tag{20}$$

$$+ \sum_{t=1}^{2m} \theta^t \sum_{|\alpha|=0}^{p} \sum_{\beta=0}^{q} \binom{p}{|\alpha|} \binom{q}{\beta} M_{|\alpha|+\beta} \left(\frac{(H_2 R)}{\theta \overline{\lambda}} \right)^{|\alpha|} \left(\frac{(H_2 R)}{\overline{\lambda}} \right)^{\beta} \frac{[M_{p+q-|\alpha|-\beta}]}{[M_{p+q}]}$$

where K is a suitable constant.

But by (1) we have the inequality

$$[M_{p+q-|\alpha|-\beta}] M_{|\alpha|+\beta} \leq C \left[\binom{p+q}{|\alpha|+\beta} \right]^{-1} [M_{p+q}].$$

Then the inequality (20) becomes

$$\overline{M}^{-1} \overline{\lambda}^{-(p+q)} \theta^{-p} N_{R,p,q}(u) \leq \frac{K H_1}{\overline{M}} \left(\frac{H_2 R}{\overline{\lambda}} \right)^{p+q} \theta^{-p} +$$

$$+ H_1 C \sum_{t=0}^{2m} \theta^t \sum_{|\alpha|=0}^{p} \sum_{\beta=0}^{q} \binom{p}{|\alpha|} \binom{q}{\beta} \left[\binom{p+q}{|\alpha|+\beta} \right]^{-1} \left(\frac{H_2 R}{\theta \overline{\lambda}} \right)^{|\alpha|} \left(\frac{H_2 R}{\overline{\lambda}} \right)^{\beta}$$

Here all the terms in the summation over α, β are less than unity. Taking $\frac{H_2 R}{\theta \bar{\lambda}} \leq \frac{1}{2}$ and $\theta \leq \frac{1}{2}$ the second member does not exceed $\frac{KH_1}{\bar{M}} + 8\, CH_1 \theta$ which is again less than unity if $\bar{M} \geq 2KH_1$ and $\theta \leq \frac{1}{16\, CH_1}$. Thus we have proved that

$$N_{R,p,q}(u) \leq \bar{M}\, \bar{\lambda}^{p+q}\, \theta^p$$

holds for all q with $q \geq -2m$ and for $R < R_1$ if we take $\theta = \frac{1}{16 CH_1}$, $\bar{\lambda} = \max\left(\frac{2H_2R_1}{\theta}, \frac{\lambda}{\theta}\right)$ and $\bar{M} \geq 2KH_1$.

As we have already said in the introduction the result is deduced by applying Sobolev's lemma to the L_2-norms of the derivatives of u. For this we need estimates for the square integrals of the type

$$\tilde{d}_p^2(u, \omega_r) = \sum_{|q|=p} \int_{\omega_r} |D^q u|^2\, dx.$$

These are easily obtained from (15) as follows:

$$\tilde{d}_p^2(u, \omega_r) = \sum_{t=0}^{p} \sum_{|q|=p-t} \int_{\omega_r} |D_y^t D_{x'}^q u|^2\, dx$$

$$\leq \sum_{t=0}^{p} \left[\frac{\bar{M}\, \bar{\lambda}^{p-2m}\, \theta^{p-t}\, M_{p-2m}}{(R-r)^p}\right]^2.$$

Hence

$$\tilde{d}_p(u, \omega_r) \leq \frac{\bar{M}\, \bar{\lambda}^p}{(R-r)^p}\, M_{p-2m} \left(\sum_{t=0}^{p} \theta^{2t}\right)^{\frac{1}{2}}$$

Thus we obtain

$$\tilde{d}_p(u, \omega_r) \leq \frac{2\,\bar{M}\,(\bar{\lambda})^p\, M_{p-2m}}{(R-r)^p}. \tag{21}$$

Now we apply Sobolev's lemma in the form used in [4], namely, for $x \in \omega_r \cup \partial_1 \omega_r$

$$|D^p u(x)| \leq C' \left[\sum_{l=0}^{[\nu/2]+1} r^{2l-\nu}\, \tilde{d}_l^2(D^p u, \omega_r)\right]^{\frac{1}{2}}$$

$$\leq C' \left[\sum_{l=0}^{[\nu/2]+1} r^{2l-\nu} \left\{\frac{2\bar{M}\,(\bar{\lambda})^{p+l}}{(R-r)^{p+l}}\, M_{p+l-2m}\right\}^2\right]^{\frac{1}{2}}.$$

Since $R - r \leq r$ and $R \leq 1$ we obtain the following inequality

$$| D^p u(x) | \leq \frac{K' \bar{\lambda}^{[\nu/2]+p+1}}{(R - r)^{[\nu/2]+p+1}} M_{p+[\nu/2]+1-2m}$$

after using (1′) (K' being a positive constant independent of p). This proves the fact that $u \in C \{ M_{n-2m+[\nu/2]+1} ; \omega_{R_0} \cup \partial_1 \omega_{R_0} \}$ thus completing the proof of the theorem.

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