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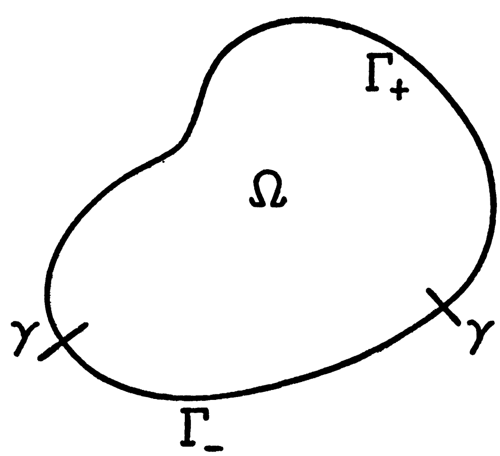
MIXED PROBLEMS FOR HIGHER ORDER ELLIPTIC EQUATIONS IN TWO VARIABLES, I (*)

JAAK PEETRE (New York)

Introduction.

In the theory of elliptic partial differential equations one is in the first place concerned with boundary problems where the boundary conditions are the same along the entire boundary. But in many physical applications boundary problems occur where the boundary conditions are different on different portions of the boundary. These are the mixed problems. A typical example is provided by the mixed Neumann-Dirichlet problem for Laplace's operator: To find u such that

$$
(1) \qquad \begin{cases} \Delta u = f \text{ in } \Omega \\ \partial u / \partial \vec{n} = g^{+} \text{ on } \Gamma_{+} \\ u = g^{-} \text{ on } \Gamma_{-} \end{cases}
$$



(*) This paper was written while the author was a temporary member of the Institute of Mathematical Sciences, New York University.

when f, g^+, g^- are given. Here Ω is a bounded domain in R^n with C^∞ boundary Γ of dimension $n-1$, and Γ_+ and Γ_- are two disjoint open portions of Γ such that $\gamma = \bar{\Gamma}_+ \cap \bar{\Gamma}_-$ is C^∞ of dimension $n-2$; further $\vec{n}$ denotes the interior normal of Γ.

This problem as well as its generalization to general second order operators has been extensively studied in the literature, specially by the Italian school (Fichera, Miranda, Magenes, Stampacchia; cf. [4] for references). The study of mixed problems for higher order operators was initiated by Schechter [8]. Let there be given a differential operator A on $\bar{\Omega}$,

$$A = A(p, D) = \sum_{|\alpha| \leq m} a_\alpha(p) D_\alpha, \tag{2}$$

and l differential operators $B_1^+, \ldots, B_l^+$ on $\bar{\Gamma}_+$,

$$B_j^+ = B_j^+(p, D) = \sum_{|\alpha| \leq m_j^+} b_{j\alpha}^+(p) D_\alpha \; (j = 1, 2, \ldots, l), \tag{3}$$

and l differential operators $B_1^-, \ldots, B_l^-$ on $\bar{\Gamma}_-$,

$$B_j^- = B_j^-(p, D) = \sum_{|\alpha| \leq m_j^-} b_{j\alpha}^-(p) D_\alpha \; (j = 1, 2, \ldots, l). \tag{4}$$

(The notation is that of [6], in particular

$$D_\alpha = D_{\alpha_1} \ldots D_{\alpha_k} = (-i\,\partial/\partial x_{\alpha_1}) \ldots (-i\,\partial/\partial x_{\alpha_k}).)$$

All coefficients are assumed to be C^∞ in their sets of definition indicated. It is also assumed that A is elliptic and moreover that $B_1^+, \ldots, B_l^+$ «cover» A over $\bar{\Gamma}_+$ and that $B_1^-, \ldots, B_l^-$ «cover» A over $\bar{\Gamma}_-$. It follows that in particular A will be «properly» elliptic and $m = 2l$. The generalization of (1) then consists of finding u such that

$$\begin{cases} Au = f \text{ in } \Omega \\ B_j^+ u = g_j^+ \text{ on } \Gamma_+ \; (j = 1, 2, \ldots, l) \\ B_j^- u = g_j^- \text{ on } \Gamma_- \; (j = 1, 2, \ldots, l) \end{cases} \tag{5}$$

when f, $g_1^+, \ldots, g_l^+$, g_1^+, $g_1^-, \ldots, g_l^-$ are given. Fundamental in the whole theory is now the following inequality

$$(6) \qquad \| u, \Omega \|_s \leq C (\| Au, \Omega \|_{s-m} + \sum_{j,\pm} \| B_j^{\pm} u, \Gamma_{\pm} \|_{s-m_j^{\pm}-1/2} +$$

$$+ \| u, \Omega \|_{s-1}), \ u \varepsilon H^s (\Omega), \ s > s_0 ,$$

where

$$s_0 = \sup \mu_j^{\pm} + 1/2,$$

$\mu_j^{\pm}$ being the « normal » order of $B_j^{\pm}$. (The definition of the norms and the corresponding spaces will be given in Section 1 and Section 4). The inequality (6) was proven ([1]) by Schechter [8] for $s = m$ under the auxiliary assumption that a certain « compatibility condition » (of algebraic nature) is fulfilled. In this paper we wish to prove (6) for general s and without the compatibility condition, which turns out to be superfluous. However, if $n > 2$ considerable technical complications arise and we have not been able to settle this case entirely. Therefore we consider here only the case $n = 2$, hoping to be able to return to the case $n > 2$ on another occasion. Now our results prove to be almost complete: *The inequality* (6) *holds for all* $s > s_0$ *except when* $s = \sigma_v \bmod 1$ $(v = 1, \dots, q)$ *where* $\sigma_1, \dots, \sigma_q$ *are certain well-defined real numbers and* $q \leq 2l$. It follows that for the problem (5) no regularity can hold, i. e. even if $f, g_1^+, \dots, g_l^+, g_2^-, \dots, g_l^-$ are all smooth, u may not be smooth; in order to get a smooth solution u one must pose auxiliary conditions on $f, g_1^+, \dots, g_l^+, g_1^-, \dots, g_l^-$, the number of which augments as the required smoothness of u is augmented. In the case of the problem (1), this phenomenon was observed by Fichera [1] (cf. also [2]).

A few words about the proof. It is course sufficient to prove (6) in the case when A, B_j^+, B_j^- have constant coefficients and $\Omega = R_+^n$, $\Gamma_+ = R_+^{n-1}$, $\Gamma_- = R_-^{n-1}$ (see Section 1). The main step of the proof is to convert (6) into the corresponding inequality for what we call a Wiener-Hopf type problem:

$$\| h, R^{n-1} \| \leq C (\| K^+ h, R_+^{n-1} \| + \| K^- h, R_-^{n-1} \|),$$

where h is a vector whose entries are functions and K^+ and K^- matrixes whose entries are convolution operators. If $n = 2$, then K^+ and K^- can

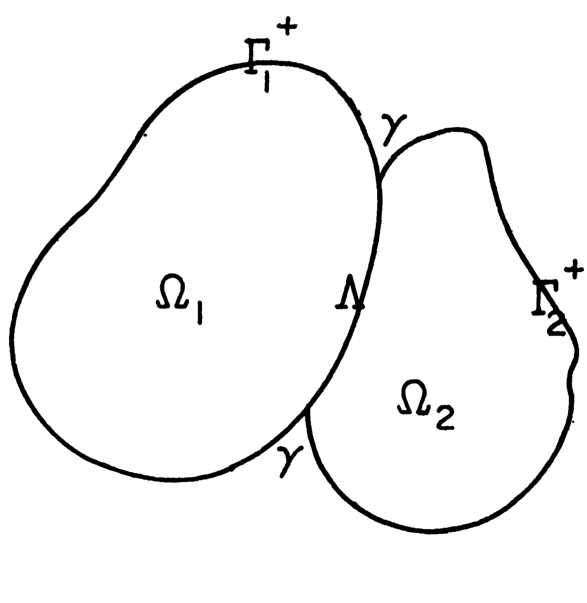
([1]) Tho tell the truth, Schechter uses stronger norms for the boundary terms; consequently he has to limit himself to functions vanishing in a neighborhood of γ. It is however not very difficult, as was also pointed out by Schechter himeslf, te change the original ideas of [8] so as to obtain (6) with the « right » norms.

be expressed easily in terms of the Hilbert transform. Using now a result of J. Schwartz [11] on the spectral properties of the « reduced » Hilbert transform, the above result easily follows. Our method is closely related to the classical Wiener-Hopf technique (cf. [12]) which has been long employed by the physicists in connection with the mixed and other related (cf. below) problems, e. g. in electromagnetic theory. Whenever the Wiener-Hopf technique is applicable, estimates of the type (6) can be obtained. Our method, on the other hand, works also when the Wiener-Hopf techinque fails — and, as we will see, it fails in general. In this context we will also exhibit the rôle of the compatibility condition used by Schechter [8].

Related to the mixed problem is the so-called transmission problem. A typical example, that often occurs in Physics, is the problem of finding u_1 and u_2 such that

$$(7)\quad \begin{cases} \Delta u_1 = f_1 \text{ in } \Omega_1, \ \Delta u_2 = f_2 \text{ in } \Omega_2 \\ u_1 = g_1^+ \text{ on } \Gamma_1^+, \ u_2 = g_2^+ \text{ on } \Gamma_2^+ \\ \partial u_1/\partial \vec{n}_1 + \partial u_2/\partial \vec{n}_2 = g_1^- \text{ on } \Lambda \\ u_1 - u_2 = g_2^- \text{ on } \Lambda \end{cases}$$

when $f_1, f_2, g_1^+, g_2^+, g_1^-, g_2^-$ are given. Here Ω_1 and Ω_2 are two bounded domains with C^∞ boundaries Γ_1 and Γ_2 of dimension $n-1$, $\Lambda = \Gamma_1^- = \Gamma_2^-$



is the interior of the common boundary $\Gamma_1 \cap \Gamma_2$, Γ_1^+ and Γ_2^+ are the interiors of the remaining part of Γ_1 and Γ_2, $\gamma = \Gamma_1^+ \cap \Gamma_1^- = \Gamma_2^+ \cap \Gamma_2^-$ is supposed to be C^∞ of dimension $n - 2$.

The generalization of (7) to higher order operators was considered by Schechter [9] and, in the much simpler case when Λ is compact, by the present author (unpublished). As in [8], the methods of this paper are also applicable to the study of this generalization of the transmission problem. We then obtain a corresponding extension of the results of [9].

Mixed problems of a different nature (over-determined problems) were considered in Schechter [10].

It is assumed throughout that the reader is acquainted with the ideas and methods of our previous papers [6], [7].

The plan of the paper is the following. In Section 1 we give some definitions and we restate the problem in the case of constant coefficients and a half-space. In Section 2 the reduction to the Wiener-Hopf situation is carried out. In Section 3 this situation is studied in the case $n - 1 = 1$. In Section 4 we give the routine extension to the case of variable coefficients and a bounded domain. Finally, in Section 5 we show that no regularity can be expected for the mixed problem.

I would like to thank Prof. Schechter for several stimulating discussions in connection with the work presented in this paper. I am further indebted to Prof. J. Schwartz for letting me know some unpublished results of his, which, as we saw above, are all important for the success of our approach.

1. Statement of the problem.

Let R^n be the n dimensional real Cartesian space; the elements of R^n are thus sequences $(x_1, \dots, x_n)$ of real numbers. Let R^n_+ (R^n_-) be the subset of R^n defined by the relation $x_n > 0$ $(x_n < 0)$. Let R^{n-1} be the subset of R^n defined by the relation $x_n = 0$; the elements of R^{n-1} are thus sequences $(x_1, \dots, x_{n-1}, 0)$ or simply $(x_1, \dots, x_{n-1})$. Let R^{n-1}_+ (R^{n-1}_-) be the subset of R^{n-1} defined by the relation $x_{n-1} > 0$ $(x_{n-1} < 0)$.

We are going to use the following norms:

$$\| u, R^n \|_s = \int (1 + |\xi_1|^2 + \dots + |\xi_n|^2)^s \, | u(\xi_1, \dots, \xi_n) |^2 \, d\xi \dots d\xi_n,$$

where $u(\xi_1, \dots, \xi_n)$ is the Fourier transform (cf. footnote (2) below) of $u = u(x_1, \dots, x_n)$,

$$u(\xi_1, \dots, \xi_n) = \int e^{-i(x_1\xi_1 + \dots + x_n\xi_n)} u(x_1, \dots, x_n)\, dx_1 \dots dx_n ,$$

and

$$\| u, R^n_\pm \|_s = \inf \| u_1, R^n \|_s$$

where inf is taken over all u_1 in R^n whose restriction to $R^n_\pm$ is u. We denote by $H^s(R^n)$ and $H^s(R^n_\pm)$ the corresponding spaces. In a similar fashion $\| u, R^{n-1} \|_s$, $\| u, R^{n-1}_\pm \|_s$, $H^s(R^{n-1})$, $H^s(R^{n-1}_\pm)$ are defined.

Let there be given in R^n $2l+1$ differential operators A, B_j^+, B_j^- with constant coefficients:

$$\begin{cases} A = A(D) = \sum_{|\alpha| \leq m} a_\alpha D_\alpha , \\ B_j^+ = B_j^+(D) = \sum_{|\alpha| \leq m_j^+} b_{j\alpha}^+ D_\alpha \quad (j = 1, 2, \dots, l), \\ B_j^- = B_j^-(D) = \sum_{|\alpha| \leq m_j^-} b_{j\alpha}^- D_\alpha \quad (j = 1, 2, \dots, l). \end{cases}$$

We make the following

HYPOTHESIS: A is elliptic. Both B_j^+ and B_j^- cover A (in the sense of Schechter, cf. e. g. [8]).

It follows that A will be properly elliptic and $m = 2l$.

We consider the following inequality:

$$(\mathcal{J}) \qquad \| u, R^n_+ \|_s \leq C (\| Au, R^n_+ \|_{s-m} + \sum_{j,\pm} \| B_j^\pm u, R^{n-1}_\pm \|_{s - m_j^\pm - 1/2} + \| u, R^n_+ \|_{s-1}), \quad u \in H^s(R^n_+) ;$$

and we will consider the following

(2) Here, and in the sequel, we use the following convention, which turns out to be convenient: If $u = u(x_1, \dots, x_n)$ is a function of $(x_1, \dots, x_n)$ then its Fourier transform with respect to $(\xi_1, \dots, \xi_v)$ will be denoted by $u(\xi_1, \dots, \xi_v, x_{v+1}, \dots, x_n)$,

$$u(\xi_1, \dots, \xi_v, x_{v+1}, \dots, x_n) = \int e^{-i(x_1\xi_1 + \dots + x_v\xi_v)} u(x_1, \dots, x_n)\, dx_1 \dots dx_n .$$

Thus the same letter u serves to denote several different functions.

PROBLEM: For which values of s does $(\mathcal{I})$ hold true?

This problem for the case $n=2$ will be completely solved in what follows. In the next Section we will however as yet make no restriction on the dimension.

REMARK. It is clear that s must satisfy the inequality $s > s_0$, where s_0 is as in the Introduction.

2. Reduction to a Wiener-Hopf type problem.

We will carry out the reduction in several steps.

1^0 It is clear that it is sufficient to consider the case when A, B_j^+, B_j^- are homogeneous.

2^0 Let us show that $(\mathcal{I})$ is equivalent to

$$(\mathcal{I}') \qquad \| u, R_+^n \|_s \leq C \big(\sum_{j,\pm} \| B_j^\pm u, R_\pm^{n-1} \|_{s-m_j^\pm-1/2} + \| u, R_+^n \|_{s-1} \big),$$

$$Au = 0, \quad u \,\varepsilon\, H^s(R_+^n).$$

It is clear that $(\mathcal{I})$ implies $(\mathcal{I}')$. Suppose thus that $(\mathcal{I}')$ is fulfilled and let us prove $(\mathcal{I})$. We observe that, in view of the « partial regularity » (cf. [6], in particular the proof of Theorem 1, for details), it is sufficient to consider a such that $u(\xi_1, \dots, \xi_{n-1}, x_n)$ (the Fourier transform [2]) with respect to $(\xi_1, \dots, \xi_{n-1})$) vanishes for $\xi_1^2 + \dots + \xi^2{}_{n-1} \leq 1$. But for such u one can always find a u_1 such that $Au_1 = Au$ and

$$(8) \qquad \| u_1, R_+^n \|_s \leq C \| Au, R_+^n \|_{s-m}.$$

On the other hand, according to our assumption, $u_0 = u - u_1$ satisfies the inequality

$$(9) \qquad \| u_0, R_+^n \|_s \leq C \big(\sum_{j,\pm} \| B_j^\pm u_0, R_\pm^{n-1} \|_{s-m_j^\pm-1/2} + \| u_0, R_+^n \|_{s-1} \big).$$

Putting together (8) and (9), $(\mathcal{I}')$ easily follows.

3^0) Obviously $(\mathcal{I}')$ is equivalent to

$$(\mathcal{I}'') \qquad \sum_j \| D_n^{j-1} u, R^{n-1} \|_{s-(j-1)-1/2} \leq$$

$$\leq C \big(\sum_{j,\pm} \| B_j^\pm u, R_\pm^{n-1} \|_{s-m_j^\pm-1/2} + \| u, R_+^n \|_{s-1} \big),$$

$$Au = 0, \quad u \,\varepsilon\, H^s(R_+^n);$$

in view of well-known properties of the Dirichlet problem.

4⁰ In view of [6], formula (10)(3), ($\mathcal{G}''$) can be written as

$$\sum_j \| (1 + \Delta_{n-1})^{(s-(j-1)-1/2)/2} D_n^{j-1} u, R^{n-1} \| \leq \tag{10}$$

$$\leq C \left(\sum_{j,\pm} \| Y_\pm (D_{n-1} \pm i (1 + \Delta_{n-2})^{1/2})^{s-m_j^\pm - 1/2} B_j^\pm u, R^{n-1} \| + \| u, R_+^n \|_{s-1} \right).$$

(Δ_v is the Laplacian with respect to $(x_1, \dots, x_v)$, $\Delta_v = D_1^2 + \dots + D_v^2$; $Y_\pm^{n-1}$ is the characteristic function of R^{n-1}). Now replace u by $u_\varepsilon(x_1, \dots, x_n) = u(x_1/\varepsilon, \dots, x_n/\varepsilon)$ where $\varepsilon > 0$. Letting ε tend to 0 we then obtain

$$\sum_j \| \Delta_{n-1}^{(s-(j-1)-1/2)/2} D_n^{j-1} u, R^{n-1} \| \leq \tag{11}$$

$$\leq C \sum_{j,\pm} \| Y_\pm (D_{n-1} \pm i \Delta_{n-2}^{1/2})^{s-m_j^\pm - 1/2} B_j^\pm u, R^{n-1} \| .$$

(No remainder term!) Conversely, if (11) holds for all $u \varepsilon H^s(R_+^n)$ with $Au = 0$, (10) follows so that (11) is equivalent to (10) and hence to ($\mathcal{G}''$).

5⁰ Put now

$$h_j = \Delta_{n-1}^{(s-(j-1)-1/2)/2} D_n^{j-1} u. \tag{12}$$

Then we get

$$\sum_j \| h_j, R^{n-1} \| \leq C \sum_{j,\pm} \| Y_\pm \sum K_{jk}^\pm h_k, R^{n-1} \| \tag{13}$$

where $K_{jk}^\pm$ are certain convolution operators which are easily computed in terms of the *characteristic forms* of B_j^+ and B_j^- respectively (cf. [5], p. 85). In fact, since u is a solution of $Au = 0$, all boundary data of u can be

(3) This formula states:

$$\| u, R_\pm^v \|_s = \| Y_+ (D_v \pm i (1 + \Delta_{v-1})^{1/2})^s u_1, R^v \|$$

where $u \varepsilon H^s(R_\pm^v)$ is the restriction of $u_1 \varepsilon H^s = H^s(R^v)$ to $R_\pm^v$, and follows easily from the fact that the mapping

$$P : u_1 \to (D_v \pm i (1 + \Delta_{v-1})^{1/2})^{-s} Y_\pm (D_v \pm i (1 + \Delta_{v-1})^{1/2})^s u_1$$

is the projection operator of the subspace of H^s orthogonal to $H^s_{R_\pm^v}$ and observing that (formally) $H^s(R_\pm^v) = H^s / H^s_{R_\pm^v}$.

expressed in terms of the Dirichlet data or, what is the same, in terms of $(h_1, \ldots, h_l)$. Since A, B_j^+, B_j^- are homogeneous (cf. 1^0), $K_{jk}^{\pm}$ will be homogeneous of degree 0 (i. e. their Fourier transforms $K_{jk}^{\pm}(\xi_1, \ldots, \xi_{n-1})$ will be homogeneous functions of degree 0). Put now

$$h = (h_1, \ldots, h_l) \; \| h \| = \Sigma \| h_j, R^{n-1} \|,$$

$$K^+ = \{K_{jk}^+\}, \quad K^- = \{K_{jk}^-\}.$$

Then (13) can be written « in matrix form » as

$$\| h \| \leq C \sum_{\pm} \| Y_{\pm} K^{\pm} h \|. \tag{14}$$

Now the « vectors » h, as defined by (12), form a dense set of

$$H = \prod_j H^0(R^{n-1}),$$

for the problem

$$\Delta_{n-1}^{(s-(j-1)-1/2)/2} D_n^{j-1} u = h_j, \; Au = 0, \; u \,\varepsilon\, H^s(R_+^n),$$

can be solved for all h such that $h(\xi_1, \ldots, \xi_{n-1})$ vanishes for $|\xi_1|^2 + \ldots + |\xi_{n-1}|^2 \leq \varepsilon$ where $\varepsilon > 0$ is arbitrary, and such h form *a* dense set in H. Hence (14) holds for all $h \,\varepsilon\, H$. Conversely, if (14) holds for all $h \,\varepsilon\, H$, then (11) and consequently ($\mathcal{J}''$) follow.

6^0 Put $\Phi = K^- h$, $M = K^+(K^-)^{-1}$. Then (14) reads

$$\| \Phi \| \leq C(\| Y_+ M\Phi \| + \| Y_- \Phi \|), \quad \Phi \,\varepsilon\, H. \tag{15}$$

(Note that K^+, K^-, M are invertible operators!) Let $H_+(H_-)$ be the subspace of H of those elements Φ in H for which $Y_- \Phi = 0$ $(Y_+ \Phi = 0)$. We claim that (15) is equivalent to

$$\| \Phi \| \leq C \| Y_+ M \Phi \|, \quad \Phi \,\varepsilon\, H_+. \tag{16}$$

In fact, it is clear that (15) implies (16). Conversely, if (16) is fulfilled, let us write $\Phi = \Phi_+ + \Phi_-$ where $\Phi_+ \,\varepsilon\, H_+$ and $\Phi_- \,\varepsilon\, H_-$; we then obtain

$$\| \Phi_+ \| \leq C \| Y_+ M \Phi_+ \| \leq C(\| Y_+ M \Phi \| + \| Y_- \Phi \|),$$

by the triangle inequality, and consequently

$$\| \Phi \| \leq \| \Phi_+ \| + \| \Phi_- \| \leq C(\| Y_+ M \Phi \| + \| Y_- \Phi \|),$$

which proves (15).

To sum up, we have now established the following result:

The Problem is equivalent to finding when (16) *holds true, M being any invertible matrix of convolution operators that are homogeneous of degree* 0.

REMARKS. 1. If $M = 1$ (or, more generally, $M = a =$ constant), then (16) trivially holds true. This may be considered to be, in a sense, the case when the equation $Y_+ K_+ h = Y_+ k$, $Y_- K_- h = Y_- k$, $h \varepsilon H$, $k \varepsilon H$, can be solved by the Wiener-Hopf technique (cf. [12]). It follows thus that Wiener-Hopf technique can in general not be used to get a priori estimates of the type ($\mathcal{G}$).

2. If $Re\,(M\,\Phi\,,\ \Phi) \geq c\,(\Phi\,,\ \Phi)\,,\quad \Phi\ \varepsilon\ H\,,\quad c > 0$ (or, more generally, $Re\,(aM\,\Phi,\,\Phi) \geq c\,(\Phi,\,\Phi),\ \Phi\ \varepsilon\ H,\ c > 0$), then (16) also holds true for we obtain easily

$$c\,\|\,\Phi\,\|^2 = c\,(\Phi,\,\Phi) \leq Re\,(M\,\Phi,\,\Phi) \leq \|\,Y_+ M\,\Phi\,\|\quad \|\,\Phi\,\|$$

for $\Phi\ \varepsilon\ H_+$. This again is, in a sense, the case considered by Schechter [8].

3. Estimates for the Wiener-Hopf type problem in the case $n - 1 = 1$.

We take now $n - 1 = 1$ or $n = 2$, and we will write $x = x_1$, $t = x_2$, $\xi = \xi_1$, $t = \xi_2$. Since M is homogeneous of degree 0, the Fourier transform is of the form

$$M\,(\xi) = \begin{cases} m_+ & \text{if}\quad \xi > 0 \\ m_- & \text{if}\quad \xi < 0 \end{cases}$$

where m_+ and m_- are constant invertible matrixes. Hence we may write

$$M = m_+ + (m_- - m_+)\,\mathcal{H} = m_+\,(1 + \gamma\,\mathcal{H})$$

so that

$$Y_+ M\,\Phi = m_+\,(Y_+\,\Phi + \gamma\,Y_+\,\mathcal{H}\,\Phi) = m_+\,(\Phi + \gamma\,\mathcal{H}_+\,\Phi),\quad \Phi\ \varepsilon\ H_+\,,$$

where

$$\gamma = m_+^{-1}\,m_- - 1$$

and $\mathcal{H}\,(\mathcal{H}_+)$ is the Hilbert transform (the Hilbert transform reduced to R_+) (4).

---

(4) We recall that the Hilbert transform is the convolution operator whose Fourier transform is 0 if $\xi > 0$ and 1 if $\xi < 0$. Its kernel will be $\dfrac{1}{2\pi i}\,\dfrac{1}{x + i0}$. By the reduced Hilbert transform we mean the operator obtained by restricting the domain and the range to H_+, $\mathcal{H}_+ = Y_+\,\mathcal{H}$.

Hence (16) is equivalent to

$$\| \Phi \| \leq C \| \Phi + \gamma \mathcal{H}_+ \Phi \|, \quad \Phi \in H_+ .$$

Now it is known (J. Schwartz [11]) that *a*) $\mathcal{H}_+$ is self-adjoint, *b*) its spectrum consists of the interval $0 \leq \lambda \leq 1$ and is of finite multiplicity, *c*) the spectral measure is (equivalent to) $d\lambda$. By the spectral theorem we then obtain

$$\| \tilde{\Phi} \| \leq C \| (1 + \lambda\gamma) \tilde{\Phi} \|, \tilde{\Phi} \in \tilde{H}_+ ,$$

where $\Phi \to \tilde{\Phi}$ is a unitary mapping from H_+ onto $\tilde{H}_+ = L^2(0,1; d\lambda)$ that diagonalizes $\mathcal{H}_+$ (5). Hence we obtain

$$\| e \| \leq C \| (1 + \lambda\gamma) e \|$$

for every numerical vector e and almost every λ in the interval $0 \leq \lambda \leq 1$. By property *c*) this will then be true for all λ. Put now

$$c = 1 + \gamma .$$

(Obviously $c = m_+^{-1} m_-$!) Then we have

$$\| e \| \leq C \| (1 - \lambda) e + \lambda c e \| \tag{17}$$

for all λ with $0 \leq \lambda \leq 1$. It is readily verified that the significance of (17) is precisely that c has no negative eigenvalues. Hence we have proven

The inequality (16) *holds if and only if the matrix c has no negative eigenvalues.*

Let us now return to our original inequality ($\mathcal{I}$). If we go from s to, say, $s + \theta$, it is easily seen that the matrix c goes over into $e^{2\pi i \theta} c$. Hence we have the following.

THEOREM. *There is a finite set of real numbers* $\sigma_1, \dots, \sigma_q$ $(q \leq 1)$ *such that* ($\mathcal{I}$) *holds for all s with* $s > s_0$ *and* $s \neq \sigma_v$ mod 1, $v = 1, 2, \dots, q$.

In the sequel $\sigma_1, \dots, \sigma_q$ will be called the *exceptional values* of s.

EXAMPLES. 1 Take $l = 1$, $A = \Delta_2$, $B^+ = d/d\varrho^+$, $B^- = d/d\varrho^-$, where ϱ^+ and ϱ^- are two non-zero real vectors, $\varrho^+ = (\varrho_1^+, \varrho_2^+)$, $\varrho^- = (\varrho_1^-, \varrho_2^-)$. Let

(5) In [11] J. SCHWARTZ gives an explicit formula for the diagonalization, from which, by the way, the properties *b*), *c*) follow (the property *a*) is trivial!).

$u \in H^s(R^2_+)$ be a solution of $Au = 0$. Then

$$u(\xi, t) = e^{-t|\xi|} [u(\xi, t)]_{t=0}$$

and hence

$$[B^\pm u(\xi, t)]_{t=0} = (-\varrho_2^\pm |\xi| + i\varrho_1^\pm \xi) [u(\xi, t)]_{t=0} .$$

It follows that

$$K^\pm(\xi) = (\xi \pm i0)^{s-3/2} |\xi|^{-s+1/2} (-\varrho_2^\pm |\xi| + i\varrho_1^\pm \xi)$$

and consequently

$$M(\xi) = \begin{cases} w \text{ if } \xi > 0 \\ e^{2\pi i(s-3/2)} \bar{w} \text{ if } \xi < 0 \end{cases}$$

where

$$w = \frac{-\varrho_2^+ + i\varrho_1^+}{-\varrho_2^- + i\varrho_1^-} .$$

Hence

$$c = e^{2\pi i(s-3/2)} \bar{w}/w .$$

Put $w = e^{\pi i \theta}$. Then

$$c = e^{2\pi i(s-3/2-\theta)}$$

and the exceptional value of s is

$$\sigma \equiv \vartheta \mod 1.$$

REMARK to example 1. If $\varrho^+ = \varrho^-$ then $\sigma \equiv 0$. In this case, as is easily seen, the problem is essentially equivalent to the following one: Given a function $u \in H^r(R_+)$, $r \geq 0$, under which conditions can u be continued by 0 outside R_+ to be a function in $H^r(R)$. We thus have found that this is always possible if $r \neq 1/2 \mod 1$ provided u satisfies the obvious requirements:

$$u(0) = Du(0) = \ldots = D^\omega u = 0 \; (D = -i d/dx),$$

ω being the largest integer ≥ 0 such that $r - \omega - 1/2 > 0$. On the other hand, if $0 < r < 1$ (obvious extension to the general case), it follows from e. g. [5], Proposition 5, p. 29, that a necessary and sufficient condition is given by

$$\int |u(x)|^2 / |x|^{1+2r} \, dx < \infty. \tag{18}$$

Using an inequality of Sobolev type, it is easily verified directly that (18) is always fulfilled for $r < 1/2$ and that (18) implies $u(0) = 0$ for $r > 1/2$, whilst no particular conclusion is possible for $r = 1/2$.

2. Take $l = 2$, $A = \Delta_2^2$, $B_1^+ = \partial^2/\partial t^2$, $B_2^+ = \partial^3/\partial t^3$, $B_1^- = 1$, $B_2^- = \partial/\partial t$ (cf. [8]). Let $u \in H^s(R_+^2)$ be a solution of $Au = 0$. Then

$$u(\xi, t) = (1 + |\xi| t) e^{-t|\xi|} [u(\xi, t)]_{t=0} + t e^{-t|\xi|} [(\partial/\partial t) u(\xi, t)]_{t=0}$$

It follows, after some calculations, that

$$K^+(\xi) = \begin{cases} \begin{Bmatrix} -1 & -2i \\ 2 & 3i \end{Bmatrix}, \ \xi > 0, \\ e^{\pi i(s-1/2)} \begin{Bmatrix} -1 & -2i \\ -2 & -3i \end{Bmatrix}, \ \xi < 0, \end{cases}$$

$$K^-(\xi) = \begin{cases} \begin{Bmatrix} 1 & 0 \\ 0 & i \end{Bmatrix}, \ \xi > 0, \\ e^{-\pi i(s-1/2)} \begin{Bmatrix} 1 & 0 \\ 0 & -i \end{Bmatrix}, \ \xi < 0, \end{cases}$$

and consequently

$$M(\xi) = \begin{cases} \begin{Bmatrix} -1 & -2 \\ 2 & 3 \end{Bmatrix}, \ \xi > 0, \\ e^{2\pi i(s-1/2)} \begin{Bmatrix} -1 & 2 \\ -2 & 3 \end{Bmatrix}, \ \xi < 0, \end{cases}$$

Hence

$$c = e^{2\pi i(s-1/2)} \begin{Bmatrix} -7 & 12 \\ 4 & -7 \end{Bmatrix}.$$

Clearly the eigenvalues of $c\ e^{-2\pi i(s-1/2)}$ are $\lambda = -7 + 4\sqrt{3}$ and $\lambda = -7 - 4\sqrt{3}$. Since $4\sqrt{3} < 7$, it follows that the only exceptional value is

$$\sigma \equiv 1/2 \mod 1.$$

4. Extension to variable coefficients.

We outline now very briefly how to extend the results hitherto obtained to the variable coefficient case. In view of the results on multiplication in H^s established in [6], [7], it is clear that $(\mathcal{G})$ holds also for operators

A, B_j^+, B_j^- with variable coefficients of the form

$$(19)\quad \begin{cases} A = A(x, D) = A'(D) + \sum_{|\alpha| \le m} c_\alpha(p) D_\alpha, \\ B_j^+ = B_j^+(x, D) = B_j^{+'}(D) + \sum_{|\alpha| \le m_j^+} d_{j\alpha}^+(p) D_\alpha, \\ B_j^- = B_j^-(x, D) = B_j^{-1}(D) + \sum_{|\alpha| \le m_j^-} d_{j\alpha}^-(p) D_\alpha, \end{cases}$$

$c_\alpha, d_{j\alpha}^+, d_{j\alpha}^-$ being C^∞ functions with compact support, if it holds for A', $B_j^{+'}, B_j^{-'}$, provided only the quantity

$$\delta = \Sigma \sup |c_\alpha(p)| + \Sigma \sup | d_{j\alpha}^+(p) + \Sigma \sup | d_{j\alpha}^-(p)|$$

is sufficiently small. Conversely, if ($\mathcal{I}$) holds for some operators A, B_j^+, B_j^- of the form (19) with δ sufficiently small, it holds also for $A', B_j^{+'}, B_j^{-'}$. Now every system of operators A, B_j^+, B_j^- such that $A(0, D)$, $B_j^+(0, D), B_j^-(0, D)$ satisfy the Hypothesis of Section 1 can be written near 0 in the form (19), in fact with $A'(D) = A(0, D), B_j^{+'}(D) = B_j^+(0, D), B_j^{-'}(D) = B_j^-(0, D)$. Hence (19) holds such A, B_j^+, B_j^- if it holds for $A(0, D), B_j^+(0, D), B_j^-(0, D)$, provided u is restricted to functions with the support contained in a sufficiently small neighborhood of 0. Conversely, if ($\mathcal{I}$) holds for A, B_j^+, B_j^- it also holds for $A(0, D) B_j^+(0, D)$, $B_j^-(0, D)$. Now it is quite easy, utilizing a finite partition of unity and again the results of multiplication in [6], [7], to establish the analogue of ($\mathcal{I}$) in the case of variable coefficients and a bounded domain. This is entirely a routine question so we just state the final result (given already in the Introduction).

Let $\Omega, \Gamma, \Gamma_+, \Gamma_-, \gamma$ be as in the Introduction. We assume also that γ consists of exactly two points p' and p''. (This assumption if of course no essential restriction but is added just for convenience). Let there further be given $2l + 1$ differential operators A, B_j^+, B_j^- as in the Introduction formular (2), (3), (4). We make the following.

HYPOTHESIS. A is elliptic in $\bar{\Omega}$. $B_1^+, \ldots, B_l^+$ cover A on $\bar{\Gamma}_+$ and $B_1^-, \ldots, B_l^-$ cover A on $\bar{\Gamma}_-$.

Then we have the following

THEOREM. *There is a finite number of real numbers* $\sigma_1, \ldots, \sigma_q$ $(q \le 2l)$ *such that* (6) *holds for all* s *with* $s > s_0$ *and* $s \not\equiv \sigma_v$ *mod* 1, $v = 1, 2, \ldots, q$.

Here

$$\| u, \Omega \|_s = \inf \| u_1, R^n \|_s$$

and $H^s(\Omega)$ is the corresponding space. In a similar fashion $\| u, \Gamma_{\pm} \|_s$ and $H^s(\Gamma_{\pm})$ are introduced. (To be precise, one has to cover Γ with a set of local coordinates and define $\| u, \Gamma_{\pm} \|_s$ as the sum of the semi-norms defined in terms of these local coordinates. A more intrinsic definition can be given by utilizing the Riemannian structure of Γ; cf. [5], p. 29).

It follows also that:

The exceptional values of A, B_j^+, B_j^- *are made up by the union of the exceptional values of* $A\ (p', D)$, $B_j^+\ (p', D)$, $B_j^-\ (p', D)$ *and the exceptional values of* $A\ (p'', D)$, $B_j^+\ (p'', D)$, $B_j^-\ (p'', D)$.

5. Various observations.

Put (cf. [6])

$$K^s(\Omega) = H^{s-m}(\Omega) \times \Pi\, H^{s-m_j^{\pm}-1/2}(\Gamma_+)$$

and consider the mapping

$$T : u \to (Au, B_1^+ u, \dots, B_1^+ u, B_1^- u, \dots, B_1^- u)$$

of $H^s(\Omega)$ into $K^s(\Omega)$ where $s > s_0$. It follows (cf. [6] for details) from (6) in view of the Hahn-Banach principle that: 1) *The nullspace N of T is of finite dimension,* and that 2) *The range R of T is closed.* It is also possible to prove that: 3) *R is of finite codimension.* The proof, which combines the methods of this paper with the general ideas of [6], will be published in the second part of this paper.

REMARK. If $n > 2$, then 3) is certainly not true in general, as simple examples show.

Let us mention here a consequence of 3). Let $\varrho = \varrho(s)$ be the codimension of $R = R^s$ and let $\nu = \nu(s)$ be the dimension of $N = N^s$. Clearly ϱ is increasing and ν is decreasing for s increasing. It follows that ν is constant for large s. We claim that the following statement holds true:

The exceptional values $\sigma_1, \dots, \sigma_q$ mod 1 *correspond to jumps of the function* ϱ, *for s large.*

Let σ be an exceptional value. Suppose that there exist s' and s'' such that $s' < \sigma < s''$ and $\varrho(s') = \varrho(s'')$. Split up $H^s(\Omega)$, $s = s', s''$, in a direct sum $N^s + M^s$; this can obviously be done in such a manner that $M^{s''} = M^{s'} \cap H^{s''}(\Omega)$; moreover we have $N^{s''} = N^{s'}$. Similarly split up $K_s(\Omega)$ in a direct sum $R^s + Q^s$ which can be done in such a manner that $Q^{s''} = Q^{s'}$; moreover we have $R^{s''} = R^{s'} \cap K^{s''}(\Omega)$. Then the equation $Tu = v$ has a unique solution $u \varepsilon M^s$ for every $v \varepsilon R^s$, $s = s', s''$. Define Sv as u if $v \varepsilon R^s$ and as 0 if $v \varepsilon Q^s$ and extend by linearity to the whole of $K^s(\Omega)$. Then S will be a continuous linear mapping of $K^s(\Omega)$ into $H^s(\Omega)$, $s = s', s''$. By general results on interpolation in Hilbert space (cf. Lions [3]), this is then true for all s with $s' \leq s \leq s''$, thus in particular for $s = \sigma$. But this obviously gives a contradiction so that σ must actually be a jump.

REMARK. It follows that no regularity can hold in general. If $m = 2$, this is, essentially, the result of Fichera [1] (cf. also [2]).

The phenomenon that no regularity can hold for mixed probtems was, in the case of second order operators, also observed by LIENARD [13] (cf. MAGENES and STAMPACCHIA [14], p. 325).

In the case of second order operators and arbitrary n, the Hölder continuity of the solutions was proven by STAMPACCHIA [15], [16].

The results of J. SCHWARTZ [11] about the spectral properties of the reduced Hilbert transform, that are utilized in the paper, were previously obtained by KOPPELMAN and PINCUS [17] (cf. also WIDOM [18] where the Lp case in considered).

The above observations were communicated to me by Prof. Magenes and Prof. Koppelman.

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