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SOME PROPERTIES OF HOLMGREN-RIESZ TRANSFORM (1)

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1. Introduction.

The subject of this article is closely related to the concept of a generalized derivative and integral. The history of this concept can be traced back to Leibnitz and Euler. Many authors since then have published important articles on this problem, with different approaches. The earliest paper was pubblished by Liouville [5] in which he defined the derivative of any order for a given function as a series of exponentials. Riemann [6] has treated this problem by considering the expansion of functions in a series of non-integral powers. He was thus led to the definition of the generalized derivative which involved an integral and an infinite non-integral power series.

It may be pointed out that the work was done by Riemann in 1847, but was not published until 1876 in his collected works. H. J. Holmgren [4] gave a more general and useful definition to the concept of a generalized derivative in four different forms each of which he obtained by applying a linear fractional transformation to his integral defining the derivative of any order. Grünwald [2] had defined the derivative of any order as the limit of a difference quotient and arrived at an integral form which was similar to that introduced by Holmgren.

M. Riesz [7] showed some properties of the integral of fractional order which is a generalization of the Riemann integral to more than one dimension.

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(1) This paper is a part of my doctoral dissertation which was written under the supervision of Prof. H. J. Ettlinger of the University of Texas.

The author [1] has shown the equivalence between the two definite integrals given by Holmgren and Riesz, and thus he has established one combined definition upon which the properties of Holmgren-Riesz Transform discussed in this article were developed.

2. Preliminaries and definitions:

Throughout this work the name of the transform will be referred to as « $H-R$ Transform ». m and n will be used to represent positive integers or zero. « Class $C^{(n)}$ » will be used to mean the class of functions with continuous n th derivative. « $R\alpha$ » and « $\mathcal{J}\alpha$ » will denote the real and the imaginary parts of the complex number α respectively. Other notations such as « D_x^n » or « $f^{(n)}(x)$ » and $\Gamma(x)$ are the conventional symbols representing the *nth* derivative and Gamma function.

DEFINITION 1.

If $f(x)$ is a real valued function of class $C^{(n)}$ on the interval $a \leqq x \leqq b$, and $R\alpha + n > 0$, then

$$(H) \qquad \overset{x}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha+n)} D_x^n \int_a^x (x-t)^{\alpha+n-1} f(t)\, dt \qquad (n = 0, 1, 2, 3, \ldots)$$

or

$$(R) \qquad = \sum_{i=0}^{i=n-1} \frac{f^{(i)}(a)\,(x-a)^{\alpha+i}}{\Gamma(\alpha+i+1)} + \frac{1}{\Gamma(\alpha+n)} \int_a^x f^{(n)}(t)\,(x-t)^{\alpha+n-1}\, dt, \quad (n=1,2,\ldots)$$

It is clear that the form (H) can also be written as

$$(H_1) \qquad \overset{x}{\underset{a}{I}}{}^{\alpha} f = D_x^m \overset{x}{\underset{a}{I}}{}^{\alpha+m} f$$

3. The Derivative Property and the Identity Transform:

THEOREM 1.

(i) If $f(x)$ is a function of class $C^{(m+n)}$ on $a \leqq x \leqq b$ and $R\alpha + m > 0$, then

$$(3.1) \qquad D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha} f = \overset{x}{\underset{a}{I}}{}^{\alpha-n} f$$

(ii) If $f(x)$ is a function of class $C^{(0)}$ on $a \leqq x \leqq b$, then

$$\lim_{\alpha \to 0+} \overset{x}{\underset{a}{I}}{}^{\alpha} f = \overset{x}{\underset{a}{I}}{}^{0} f = f(x)$$

for every x on $[a, b]$.

(i) Let p be a positive integer such that $p > n + 1$. Then by (H_1) we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = D_x^{m+p} \overset{x}{\underset{a}{I}}{}^{\alpha+m+p} f$$

and

$$D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha} f = D_x^{m+n+p} \overset{x}{\underset{a}{I}}{}^{\alpha+m+p} f$$

$$= \overset{x}{\underset{a}{I}}{}^{\alpha-n} f$$

(ii)[2] Let $n = 1$, $\alpha = 0$ in (H), then we have

$$\overset{x}{\underset{a}{I}}{}^{0} f = D_x \int_a^x f(t)\, dt$$

$$= f(x)^3$$

Assuming that $f(x)$ is of class $C^{(n)}$, then it is clear that as $\alpha \to 0^-$, the form (R) gives

$$\overset{x}{\underset{a}{I}}{}^{0} f = \sum_{i=0}^{i=n-1} \frac{f^{(i)}(x-a)^i}{\Gamma(i+1)} + \frac{1}{\Gamma(n)} \int_a^x f^{(n)}(t)(x-t)^{n-1}\, dt.$$

This is Taylors expansion of $f(x)$ about $x = a$ with the remainder

$$R_n = \overset{x}{\underset{a}{I}}{}^{n} f^{(n)}$$

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(2) Another proof will be shown in Art. 8.

(3) The identity transform may be extended to include the class of functions which may have an infinite discontinuity at $x = a$; in such a case we define the identity transform by:

$$\overset{x}{\underset{a}{I}}{}^{0} f = \lim_{\epsilon \to 0} \overset{x}{\underset{a+\epsilon}{I}}{}^{0} f = \lim_{\epsilon \to 0} D_x \int_{a+\epsilon}^x f(t)\, dt = f(x), \text{ for } x > a.$$

Corollary 1.1. If $f(x)$ is a function of class $C^{(n)}$ on $a \leq x \leq b$, then

$$\overset{x}{\underset{a}{I}}{}^{-n} f = D_x^n f = f^{(n)}(x). \quad (3.3)$$

[4]

This follows from Theorem 1.

Theorem 2.

(i) If $f(x)$ is of class $C^{(0)}$ on the interval $c \leq x \lesseqgtr a \leq b$, and $R\alpha > 0$, then

$$\lim_{x\to a} \overset{x}{\underset{a}{I}}{}^{\alpha} f = 0. \quad (3.4)$$

(ii) If $f(x)$ is of class $C^{(n)}$ on $a < x < b$ and $R\alpha + n > 0$, then (3.4) holds if $f^{(i)}(a) = 0$ $(i = 0, 1, 2, \dots, n-1)$, otherwise the limit does not exist.

(i) Suppose that $x < a$. Then there exists $M > 0$ such that

$$|f(t)| < M \text{ on } (x, a),$$

and

$$0 \leq \left| \int_a^x f(t)(x-t)^{\alpha-1}\, dt \right| < \frac{M}{R\alpha}(a-x)^{R\alpha} e^{-\pi \mathfrak{J}\alpha}$$

Hence

$$\lim_{x\to a^-} \overset{x}{\underset{a}{I}}{}^{\alpha} f = 0.$$

Similarly if $x > a$, we have

$$\lim_{x\to a^+} \overset{x}{\underset{a}{I}}{}^{\alpha} f = 0.$$

(ii) By (R) we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)(x-a)^{\alpha+i}}{\Gamma(\alpha+i+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha+n} f^{(n)}$$

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(4) It is noted that the transform of a negative integer index is independent of the lower limit; in other words this is also true when the function has an infinite discontinuity at the point $x = a$, provided that the derivatives of $f(x)$ exist for $x > a$. In such a case we define

$$\overset{x}{\underset{a}{I}}{}^{-n} f = \lim_{\epsilon\to 0} \overset{x}{\underset{a+\epsilon}{I}}{}^{-n} f = D_x^n f(x), \quad x > a.$$

If $f^{(i)}(a) = 0$ for $(i = 0, 1, \dots, n-1)$, then

$$\lim_{x \to a} \overset{x}{\underset{a}{I}}{}^{\alpha} f = \lim_{x \to a} \overset{x}{\underset{a}{I}}{}^{\alpha+n} f^{(n)} = 0.$$

It is clear that if $f^{(i)}(a) \neq 0$ for any i then the limit does not exist, for

$$\lim_{x \to a} (x-a)^{\alpha+i} \to \infty.$$

4. Other properties ot the Transform:

In addition to the above, $H - R$ Transform has the following properties:

(i) If $f(x)$ and $g(x)$ are two functions satisfying the condition of Definition 1, and k is an arbitrary constant then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} kf = k \overset{x}{\underset{a}{I}}{}^{\alpha} f \tag{4.1}$$

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (f \pm g) = \overset{x}{\underset{a}{I}}{}^{\alpha} f \pm \overset{x}{\underset{a}{I}}{}^{\alpha} g . \tag{4.2}$$

[5](ii) (A). If $f(x)$ is a function of class $C^{(n+1)}$ and $g(x)$ is of class $C^{(0)}$ an (a, b) and if $R_{\alpha} > 0$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} fg = \sum_{i=0}^{n} \binom{-\alpha}{i} f^{(i)}(x) \overset{x}{\underset{a}{I}}{}^{\alpha+i} g + \tag{4.3}$$

$$+ \frac{(-1)^{n+1}}{\Gamma(\alpha)} \int_a^x f^{(n+1)}(t) \left[\overset{t}{\underset{a}{I}}{}^{n+1} g(t) (x-t)^{\alpha-1} \right] dt.$$

(B). If $f(x)$ if of class $C^{(m+n+1)}$ and $g(x)$ is of class $C^{(m)}$ on (a, b) and $m - R\beta > 0$, then

$$\overset{x}{\underset{a}{I}}{}^{-\beta} fg = \sum_{p=0}^{p=n} \binom{\beta}{p} f^{(p)}(x) \overset{x}{\underset{a}{I}}{}^{-\beta+p} g + \tag{4.4}$$

$$+ \sum_{p=n+1}^{p=m+n} \sum_{i=0}^{i=n} \binom{m}{p-i} \binom{\beta-m}{i} f^{(p)}(x) \overset{x}{\underset{a}{I}}{}^{p-\beta} g +$$

$$+ \frac{(-1)^{n+1}}{\Gamma(m-\beta)} D_x^m \int_a^x f^{(n+1)}(t) \left\{ \overset{t}{\underset{a}{I}}{}^{n+1} (x-t)^{m-\beta-1} g(t) \right\} dt.$$

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[5] It is important to note that this property also holds when $a \to -\infty$. See footnote of Theorem 1.

The first two properties can be easily seen since the $H - R$. Transform has the integral form.

To establish (ii-A), we have by definition

$$\overset{\infty}{\underset{a}{I}}{}^{\alpha} fg = \frac{1}{\Gamma(\alpha)} \int_a^\infty f(t)\, g(t)\, (x-t)^{\alpha-1}\, dt. \tag{4.31}$$

Let

$$G_1(t) = \int_a^t g(z)\, (x-z)^{\alpha-1}\, dz, \tag{4.32}$$

we notice that

$$G_1'(t) = g(t)\, (x-t)^{\alpha-1} \tag{4.33}$$

$$G_1(a) = 0$$

Substituting (4.33) in (4.31) and integrating by parts, then we get

$$\overset{\infty}{\underset{a}{I}}{}^{\alpha} fg = f(x) \overset{\infty}{\underset{a}{I}}{}^{\alpha} g + \frac{1}{\Gamma(\alpha)} \int_a^\infty f'(t)\, G_1(t)\, dt. \tag{4.34}$$

Now let

$$G_2(t) = \int_a^t G_1(z)\, dz,$$

where

$$G_2'(t) = G_1(t)$$

$$G_2(a) = 0,$$

and integrating by parts the integral in (4.34), then we have

$$\overset{\infty}{\underset{a}{I}}{}^{\alpha} fg = f(x) \overset{\infty}{\underset{a}{I}}{}^{\alpha} g - \frac{1}{\Gamma(\alpha)} \Big[f'(t)\, G_2(t) \Big]_a^x + \tag{4.35}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^\infty f''(t)\, G_2(t)\, dt.$$

But we have

$$[f'(t)\, G_2(t)]_a^x = f'(x)\, G_2(x) =$$

$$= f'(x) \int_a^x d\zeta \int_a^\zeta g(z)(x-z)^{\alpha-1}\, dz =$$

$$= f'(x) \int_a^x g(z)(x-2)^{\alpha-1}\, dz \int_z^x d\zeta =$$

$$= f'(x) \int_a^x g(z)(x-z)^{\alpha}\, dz.$$

Thus (4.35) can be written as:

$$(4.36) \qquad \overset{x}{\underset{a}{I}}{}^{\alpha} fg = f(x) \overset{x}{\underset{a}{I}}{}^{\alpha} g + \binom{-\alpha}{1} f'(x) \overset{x}{\underset{a}{I}}{}^{\alpha+1} g +$$

$$+ \frac{1}{\Gamma(\alpha)} \int_a^x f''(t)\, G_2(t)\, dt.$$

Continuing this process n-times we get

$$(4.37) \qquad \overset{x}{\underset{a}{I}}{}^{\alpha} fg = \sum_{i=0}^{i=n} \binom{-\alpha}{1} f^{(i)}(x) \overset{x}{\underset{a}{I}}{}^{\alpha+1} g +$$

$$+ \frac{(-1)^{n+1}}{\Gamma(\alpha)} \int_a^x f^{(n+1)}(t)\, G_{n+1}(t)\, dt,$$

where

$$G_{n+1}(t) = \int_a^t G_n(\zeta)\, d\zeta =$$

$$= \frac{1}{\Gamma(n+1)} \int_a^t g(z)(x-z)^{\alpha-1}(t-z)^n\, dz.$$

Therefore (4.37) can be written in the form (4.3).

Now to show (ii-B), take the *mth* derivative of both sides of (4.3), and by Theorem 1 (i), we have

$$D_x^m \overset{x}{\underset{a}{I}}{}^{\alpha} fg = \overset{x}{\underset{a}{I}}{}^{\alpha-m} fg =$$

$$= \sum_{i=0}^{i=n} \binom{-\alpha}{i} \sum_{p=0}^{p=m} \binom{m}{p} f^{(i+p)}(x) \overset{x}{\underset{a}{I}}{}^{\alpha-m+p+i} g +$$

$$= \frac{(-1)^{n+1}}{\Gamma(\alpha)} D_x^m \int_a^x f^{(n+1)}(t) \left\{ \overset{t}{\underset{a}{I}}{}^{n+1} (x-t)^{\alpha-1} g(t) \right\} dt =$$

$$= \sum_{i=0}^{i=n} \binom{-\alpha}{i} \sum_{p=i}^{p=m+1} \binom{m}{p-i} f^{(p)}(x) \overset{x}{\underset{a}{I}}{}^{\alpha-m+p} g + \tag{4.41}$$

$$+ \frac{(-1)^{n+1}}{\Gamma(\alpha)} D_x^m \int_a^x f^{(n+1)}(t) \left\{ \overset{t}{\underset{a}{I}}{}^{n+1} (x-t)^{\alpha-1} g(t) \right\} dt.$$

But for $p \leqq n$

$$\sum_{i=0}^{i=n} \binom{-\alpha}{i} \binom{m}{p-i} = \binom{m-\alpha}{p}.$$

Then (4.41) can be written as

$$\overset{x}{\underset{a}{I}}{}^{\alpha-m} fg = \sum_{p=0}^{p=n} \binom{m-\alpha}{p} f^{(p)}(x) \overset{x}{\underset{a}{I}}{}^{\alpha-m+p} g + \tag{4.42}$$

$$+ \sum_{i=0}^{i=n} \sum_{p=n+1}^{p=m+n} \binom{m}{p-i} \binom{-\alpha}{i} f^{(p)}(x) \overset{x}{\underset{a}{I}}{}^{\alpha-m+p} g +$$

$$+ \frac{(-1)^{n+1}}{\Gamma(\alpha)} D_x^m \int_a^x f^{(n+1)}(t) \left\{ \overset{t}{\underset{a}{I}}{}^{n+1} (x-t)^{\alpha-1} g(t) \right\} dt.$$

Let $\alpha - m = -\beta$ in (4.42), then we get the form (4.4).

This is a generalization of Liebnitz' rule of derivatives of an integral order of the product of two functions.

If $f(x)$ is a polynominal of degree n, then we have for $R\alpha > 0$ and $R\beta > 0$,

$$\overset{x}{\underset{a}{I^{\alpha}}} fg = \sum_{i=0}^{i=n} \binom{-\alpha}{i} f^{(i)}(x) \overset{x}{\underset{a}{I^{\alpha+1}}} g \tag{4.5}$$

$$\overset{x-\beta}{\underset{a}{I}} fg = \sum_{p=0}^{p=n} \binom{\beta}{p} f^{(p)} \overset{x}{\underset{a}{I^{-\beta+p}}} g. \tag{4.6}$$

5. The Equivalence of Hadamard Integral and the $H-R$ Transform :

In his work M. Riesz [7] has called attention to the relation between the concept of finite part of the infinite integral which has been introduced by *J.* Hadamard [3] (pp. 133-158) and the extended form of his integral which is similar to the transform (R) of Definition 1. We will introduce Hadamard's method and then we will show the equivalence.

Consider the divergent integral

$$\int_a^b \frac{f(x)}{(b-x)^{q+1}}\, dx, \qquad 0 < q < 1, \tag{5.1}$$

where $f(x)$ is either of class C on $[a, b]$ or at least satisfies the Lipschitz condition. In order that this integral have a meaning Hadamard has shown that it is possible to add to it a function of the form

$$\frac{g(x)}{(b-x)^q},$$

where $g(x)$ satisfies the same condition as $f(x)$ and $f(b) = q\, g(b)$, such that

$$\lim_{x \to b} \left\{ \int_a^x \frac{f(x)\, dx}{(b-x)^{q+1}} + \frac{g(x)}{(b-x)^q} \right\} \tag{5.2}$$

exists and tends to the limit

$$\int_a^b \frac{f(x)-f(b)}{(b-x)^{q+1}}\, dx - \frac{f(b)}{q(b-a)^q} = \left|\!\overline{\int_a^b \frac{f(x)}{(b-x)^{q+1}}\, dx.}\right. \tag{5.3}$$

Such a limit is called by Hadamard The finite part of the integral (5.1).

Similarly, if $f(x)$ is of class $C^{(0)}$ on $[a, b]$, then

$$(5.4)\qquad \left|\overline{\int_a^b \frac{f(x)}{(b-x)^{q+1}}\,dx}\right. = \lim_{x\to b}\left\{\int_a^x \frac{f(x)\,dx}{(b-x)^{q+2}} - \right.$$

$$\left. - \frac{f(b)}{(q+1)(b-x)^{q+1}} - \frac{f'(b)}{q(b-x)^q}\right\} =$$

$$= \int_a^b \frac{f(x) - f(b) - (b-x)\,f'(b)}{(b-x)^{q+2}}\,dx -$$

$$- \frac{f(b)}{(q+1)(b-a)^{q+1}} - \frac{f'(b)}{q(b-a)^q}\,.$$

The equivalence. (5.3) can be written as

$$(5.5)\qquad \lim_{x\to b}\int_a^x \frac{f(x)-f(b)}{(b-x)^{q+1}}\,dx - \frac{f(b)}{q(b-a)^q}\,.$$

Assuming f is a function of class C', and integrating the integral (5.5) by parts then we have

$$(5.6)\qquad \left|\overline{\int_a^b \frac{f(x)\,dx}{(b-x)^{q+1}}}\right. = -\frac{f(a)}{q(b-a)^q} - \frac{1}{q)}\int_a^b \frac{f'(x)\,dx}{(b-x)^q}\,.$$

But by (R) we have

$$\overset{x}{\underset{a}{I}}{}^{-q} f = \frac{(x-a)^{-q} f(a)}{-q\,\Gamma(-q)} - \frac{1}{q\,\Gamma(-q)}\int_a^x (x-t)^{-q} f'(t)\,dt.$$

Therefore we find that

$$(5.7)\qquad \left|\overline{\int_a^b \frac{f(x)\,dx}{(b-x)^{q+1}}}\right. = \Gamma(-q)\,\overset{b}{\underset{a}{I}}{}^{-q} f.$$

Similary (5.4) can be written as

$$\left|\overline{\int_a^b \frac{f(x)\,dx}{(b-x)^{q+2}}} = -\frac{f(a)}{(q+1)(b-a)^{q+1}} + \frac{f'(a)}{q(q+1)(b-a)^{q}} + \right.$$

$$+\frac{1}{q(q+1)} \int_a^b \frac{f''(x)\,dx}{(b-x)^q}.$$

But we have

$$\overset{x}{\underset{a}{I}}{}^{-q-1} f = \frac{f(a)\,(x-a)^{-q-1}}{\Gamma(-q)} + \frac{f'(a)\,(x-a)^{-q}}{\Gamma(-q+1)}$$

$$+\frac{1}{\Gamma(-q+1)} \int_a^x f''(t)\,(x-t)^{-q}\,dt.$$

Therefore

$$\left|\overline{\int_a^b \frac{f(x)\,dx}{(b+x)^{q+2}}} = \Gamma(-q-1)\, \overset{b}{\underset{a}{I}}{}^{-q-1} f. \right. \tag{5.8}$$

In general, assuming that $f(x)$ is of class $C^{(n)}$ then it can be shown by similar method that

$$\left|\overline{\int_a^b \frac{f(x)\,dx}{(b-x)^{q+n}}} = \Gamma(-q-n)\, \overset{x}{\underset{a}{I}}{}^{-q-n} f. \right. \tag{5.9}$$

6. The Law of Indices.

DEFINITION 2. If $f(x)$ is of class $C^{(n)}$ for $x > a$ and if

$$\lim_{x\to a} f^{(i)}(x) = 0 \text{ or } \lim_{x\to a} f^{(i)}(x) = \pm\infty \text{ for } i \leqq n,$$

and if $\overset{x}{\underset{a}{I}}{}^{\alpha} f^{(k)}$ exists for some $k \leqq n$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f^{(n)} = D_x^{m+n}\, \overset{x}{\underset{a}{I}}{}^{\alpha+m} f. \tag{6.1}$$

LEMMA :

(A) If $x > a$ where a is a real number and $\beta \neq -m$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} (x - a)^{\alpha+\beta} \tag{6.2}$$

for all values of α and β.

(B) If $\beta = -n$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{-n} = \frac{(-1)^{n-1}}{\Gamma(n)} D_x^{m+n+1} \left\{ \frac{(x - a)^{\alpha+m+1} \ln (x - a)}{\Gamma(\alpha + m + 2)} - \frac{(x - a)^{\alpha+m+1}}{\Gamma(\alpha + m + 2)} + \frac{(x - a)^{\alpha+m+1}}{\Gamma(\alpha + m)} K(\alpha + m) \right\} \tag{6.3}$$

where $K(\alpha + m) = \int_0^1 u (1 - u)^{\alpha+m-1} \ln u \, du$ for all values of α.

PROOF :

(A) Consider the following cases :

(i) $R\,\alpha > 0$, (ii) $R \leqq 0$ $\Big\}$ $R\,\beta > -1$

(iii) $R\,\alpha > 0$, (iv) $R\,\alpha \leqq 0$ $\Big\}$ $R\,\beta \leqq -1$

Case (i). By definition we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = \frac{1}{\Gamma(\alpha)} \int_a^x (t - a)^{\beta} (x - t)^{\alpha-1} \, dt. \tag{A.1}$$

Let $u = \dfrac{x - t}{x - a}$, then we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = \frac{(x - a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 u^{\alpha-1} (1 - u)^{\beta} \, du$$

$$= \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} (x - a)^{\alpha+\beta}.$$

In this case if we let in (A.1)

$$x - t = u - a,$$

then we have the identity

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha)} \overset{x}{\underset{a}{I}}{}^{\beta+1} (x + a)^{\alpha-1}. \tag{6.4}$$

Case (ii). Suppose that $-n < R\,\alpha \leqq -n + 1$, then we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha+n} (x - a)^{\beta}.$$

And by case (i) we find that

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = D_x^n \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + n + 1)} (x - a)^{\alpha+\beta+n}$$

$$= \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} (x - a)^{\alpha+\beta}.$$

Case (iii). Let $-m - 1 < R\,\beta \leqq -m$. Suppose that $R\,\alpha > 0$, $R\,\beta > -1$, then by (6.4) we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha)} \overset{x}{\underset{a}{I}}{}^{\beta+1} (x - a)^{\alpha-1}.$$

Then we find that

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x - a)^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha)} D_x \overset{x}{\underset{a}{I}}{}^{\beta+2} (x - a)^{\alpha-1}$$

valid for $-2 < R\,\beta \leqq -1$

$$= \frac{\Gamma(\beta + 1)}{\Gamma(\alpha)} D_x^2 \overset{x}{\underset{a}{I}}{}^{\beta+3} (x - a)^{\alpha-1}$$

valid for $-3 < R\,\beta \leqq -2$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha)} D_x^m \overset{x}{\underset{a}{I}}{}^{\beta+m-1} (x-a)^{\alpha-1}$$

valid for $-m-1 < R\beta \leqq -m$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha)} D_x^m \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta+m+1)}$$

$$(x-a)^{\alpha+\beta+m}$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}.$$

Case (iv). Let $-n < R\alpha \leqq -n+1$ and $-m-1 < R\beta \leqq -m$, then by Definition 2, we have

$$\frac{\Gamma(\beta+m+1)}{\Gamma(\beta+1)} \overset{x}{\underset{a}{I}}{}^{\alpha} (x-a)^{\beta} = D_x^m \overset{x}{\underset{a}{I}}{}^{\alpha} (x-a)^{\beta+m}.$$

Then by case (ii) we get

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x-a)^{\beta+m} = \frac{\Gamma(\beta+m+1)}{\Gamma(\alpha+\beta+m+1)} (x-a)^{\alpha+\beta+m}.$$

Consequently, we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} (x-a)^{\beta} = \frac{\Gamma(\beta+1)}{(\alpha+\beta+m+1)} D_x^m (x-a)^{\alpha+\beta+m}$$

$$= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\alpha+\beta}.$$

(B) By definition 2 if $R\alpha + m > 0$, then we have

(B.1) $$\overset{x}{\underset{a}{I}}{}^{\alpha} (x-a)^{-n} = \frac{(-1)^{n-1}}{\Gamma(n)} D_x^{n+m+1} \overset{x}{\underset{a}{I}}{}^{\alpha+m} \{(x-a) \ln e^{-1} (x-a)\}$$

Let $m=0$ in (B.1), then

(B.2) $$\Gamma(\alpha) \overset{x}{\underset{a}{I}}{}^{\alpha} (x-a) \ln e^{-1} (x-a) = \int_a^x (x-t)^{\alpha-1} (t-a) \ln e^{-1} (t-a)\, dt$$

Now let in (B.2)

$$u = \frac{t - a}{x - a}.$$

Then

$$\Gamma(\alpha) \overset{x}{\underset{a}{I^\alpha}} (x - a) \ln e^{-1} (x - a) = (x - a)^{\alpha+1} \ln (x - a) \int_0^1 u (1 - u)^{\alpha-1} du$$

$$- (x - a)^{\alpha+1} \int_0^1 u (1 - u)^{\alpha-1} du$$

$$+ (x - a)^{\alpha+1} \int_0^1 u (1 - u)^{\alpha-1} \ln u \, du$$

Since $\int_0^1 u (1 - u)^{\alpha-1} du = B(2, \alpha)$, the Beta function, therefore

$$\text{(B.3)} \qquad \overset{x}{\underset{a}{I^\alpha}} (x - a) \ln e^{-1} (x - a) = \frac{(x - a)^{\alpha+1}}{\Gamma(\alpha + 2)} \ln (x - a)$$

$$- \frac{(x - a)^{\alpha+1}}{\Gamma(\alpha + 2)} + \frac{(x - a)^{\alpha+1}}{\Gamma(\alpha)} K(\alpha)$$

where

$$K(\alpha) = \int_0^1 u (1 - u)^{\alpha-1} \ln u \, du$$

Therefore (B.1) can be written as (6.3).

It may be poind out that the form (6.3) is of more general character than the one given by Riemann [6] (pp. 343-344).

THEOREM 3.

The following relation

$$\text{(6.5)} \qquad \overset{x}{\underset{a}{I^\alpha}} \overset{x}{\underset{a}{I^\beta}} f = \overset{x}{\underset{a}{I^{\alpha+\beta}}} f$$

holds if

(i) $R\,\alpha > 0$, $R\,\beta > 0$ and $f(x)$ is a function of class $C^{(0)}$ on $a \leqq x \leqq b$.

(ii) $R\,\alpha > 0$, $R\,\beta \leqq 0$ or $R\,\beta + m > 0$ such that $\beta \neq - m$ (a negative integer), and $f(x)$ is of class $C^{(m)}$ on $a \leqq x \leqq b$.

When $\beta = -m$, then

$$(6.7) \qquad \overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{-m} f = \overset{x}{\underset{a}{I}} f^{(m)} = \overset{x}{\underset{a}{I}}{}^{\alpha+m} f - \sum_{i=0}^{m-1} \frac{f^{(i)}(a)(x-a)^{\alpha-m+i}}{\Gamma(\alpha+i-m+1)}.$$

(iii) $R\alpha \leqq 0$ or $R\alpha + n > 0, R\beta > 0$ and $f(x)$ is of class $C^{(n)}$ on $a \leqq x \leqq b$.

(iv) $R\alpha \leqq 0, R\beta \leqq 0, \beta \neq -m$ and $f(x)$ is of class $C^{(m+n)}$ on $a \leqq x \leqq b$.

When $\beta = -m$ then (6.7) holds.

Proof:

Case (i). By definition we have

$$(6.51) \qquad \overset{x}{\underset{a}{I}}{}^{\alpha} \cdot \overset{x}{\underset{a}{I}}{}^{\beta} f = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x (x-t)^{\alpha-1} dt \int_a^x f(\zeta)(t-\zeta)^{\beta-1} d\zeta.$$

By changing the order of integration in (6.51) we get

$$(6.52) \qquad \overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{\beta} f = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\zeta)\, d\zeta \int_a^x (x-t)^{\alpha-1} (t-\zeta)^{\beta-1} dt.$$

Let $u = \dfrac{t-\zeta}{x-t}$, then (6.52) becomes

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{\beta} f = \frac{B(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x f(\zeta)(x-\zeta)^{\alpha+\beta-1} d\zeta$$

$$= \frac{1}{\Gamma(\alpha+\beta)} \int_a^x f(\zeta)(x-\zeta)^{\alpha+\beta-1} d\zeta$$

$$= \overset{x}{\underset{a}{I}}{}^{\alpha+\beta} f$$

since $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

Case (ii). Suppose that $-m < R\beta \leqq -m+1$, then

$$\overset{x}{\underset{a}{I}}{}^{\beta} f = \sum_{i=0}^{i-m-1} \frac{f^{(i)}(a)(x-a)^{\beta+i}}{\Gamma(\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\beta+m} f^{(m)}$$

and

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{\beta} f = \sum_{i=0}^{i=m-1} \frac{f^{(i)}(a)\,\overset{x}{\underset{a}{I}}{}^{\alpha}(x-a)^{\beta+i}}{\Gamma(\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{\beta} f^{(m)}$$

By case (i) we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{\beta+m} f^{(m)} = \overset{x}{\underset{a}{I}}{}^{\alpha+\beta+m} f^{(m)},$$

and using the Lemma (A) we get

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{\beta} f = \sum_{i=0}^{i=m-1} \frac{f^{(i)}(a)\,(x-a)^{\alpha+\beta+i}}{\Gamma(\alpha+\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha+\beta+m} f^{(m)}$$

$$= \overset{x}{\underset{a}{I}}{}^{\alpha+\beta} f$$

Now if $\beta = -m$, then by Definition 1 we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha-m} f = D_x^m\,\overset{x}{\underset{a}{I}}{}^{\alpha} f$$

$$= \sum_{i=0}^{m-1} \frac{(\alpha-m+i+1)(\alpha-m+i+2)\ldots(\alpha+i)}{\Gamma(\alpha+i+1)}$$

$$f^{(i)}(a)\,(x-a)^{\alpha-m+i} + \overset{x}{\underset{a}{I}}{}^{\alpha} f^{(m)}$$

$$= \sum_{i=0}^{m-1} \frac{f^{(i)}(a)\,(x-a)^{\alpha-m+i}}{\Gamma(\alpha+i-m+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{-m} f$$

From this last relation (6.7) follows.

Case (iii). Let $-n < R\alpha \leqq -n+1$, and $g(x) = \overset{x}{\underset{a}{I}}{}^{\beta} f$.

Then we have

$$g(x) = D_x^n\,\overset{x}{\underset{a}{I}}{}^{\beta+n} f$$

$$= \sum_{i=0}^{i=n-1} \frac{f^{(i)}(a)\,(x-a)^{\beta+i}}{\Gamma(\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\beta+n} f^{(n)}$$

and

$$\overset{x}{\underset{a}{I}}{}^{\alpha} g = \sum_{i=0}^{i=n-1} \frac{f^{(i)}(a)\,\overset{x}{\underset{a}{I}}{}^{\alpha}(x-a)^{\beta+i}}{\Gamma(\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{\beta+n} f^{(n)}.$$

By the Lemma, this may be written as

$$\overset{x}{\underset{a}{I}}{}^{\alpha} g = \sum_{i=0}^{i=n-1} \frac{f^{(i)}(a)(x-a)^{\alpha+\beta+i}}{\Gamma(\alpha+\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{\beta+n} f^{(n)} .$$

Now let

$$g_n = \overset{x}{\underset{a}{I}}{}^{\beta+n} f^{(n)}$$

then we have

$$g_n^{(n)}(x) = D_x^n \overset{x}{\underset{a}{I}}{}^{\beta+n} f^{(n)} = \overset{x}{\underset{a}{I}}{}^{\beta} f^{(n)}$$

Moreover, by Theorem 2

$$g_n^{(n)}(a) = 0$$

Therefore

$$\overset{x}{\underset{a}{I}}{}^{\alpha} g_n = \overset{x}{\underset{a}{I}}{}^{\alpha+n} g_n^{(n)}$$

or

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{\beta+n} f^{(n)} = \overset{x}{\underset{a}{I}}{}^{\alpha+n} \overset{x}{\underset{a}{I}}{}^{\beta} f^{(n)}$$

$$= \overset{x}{\underset{a}{I}}{}^{\alpha+\beta+n} f^{(n)}$$

Consequently

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{\beta} f = \overset{x}{\underset{a}{I}}{}^{\alpha+\beta} f$$

Case (iv). Let $-n < R\alpha \leqq -n+1$ and $-m < R\beta \leqq -m+1$. The we have

$$\overset{x}{\underset{a}{I}}{}^{\beta} f = \sum_{i=0}^{i=m-1} \frac{f^{(i)}(a)(x-a)^{\beta+i}}{\Gamma(\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\beta+m} f^{(m)}$$

and

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{\beta} f = \sum_{i=0}^{i=m-1} \frac{f^{(i)}(a) \overset{x}{\underset{a}{I}}{}^{\alpha} (x-a)^{\beta+1}}{\Gamma(\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{\beta+m} f^{(m)}$$

By case (iii) we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} \overset{x}{\underset{a}{I}}{}^{\beta+m} f^{(m)} = \overset{x}{\underset{a}{I}}{}^{\alpha+\beta+m} f^{(m)} .$$

And by using the Lemma, we get

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{\beta} f = \sum_{i=0}^{m-1} \frac{f^{(i)}(a)\,(x-a)^{\alpha+\beta+i}}{\Gamma(\alpha+\beta+i+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha+\beta+m} f^{(m)}$$

$$= \overset{x}{\underset{a}{I}}{}^{\alpha+\beta} f$$

If $\beta = -m$, then we have

$$\overset{x}{\underset{a}{I}}{}^{\alpha-m} f = \sum_{i=0}^{m+n-1} \frac{f^{(i)}(a)\,(x-a)^{\alpha-m+i}}{\Gamma(\alpha-m+i+1)} + \overset{x}{\underset{a}{I}}{}^{\alpha+n} f^{(m+n)}$$

Let $n = 0$ in this expansion. Then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f^{(m)} = \overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{-m} f$$

$$= \overset{x}{\underset{a}{I}}{}^{\alpha-m} f - \sum_{i=0}^{m-1} \frac{f^{(i)}(a)\,(x-a)^{\alpha-m+i}}{\Gamma(\alpha-m+i+1}$$

which is of the form (6.7).

COROLLARY 3.1.

The relation (6.5) holds for all values of α and β if $f(x)$ satisfies the conditions of Theorem 3, cases (i), (ii), (iii) and (iv), and moreover $f^{(\mu)}(a) = 0$ for ($\mu = 0, 1, 2, \ldots, n-1$), ($\mu = 0, 1, 2, \ldots, m-1$), and ($\mu = 0, 1, 2, \ldots, m + n - 1$), in the three last cases respectively.

COROLLARY 3.2.

Il $\beta \neq -m$ and $f(x)$ satisfies the conditions of the cases (i), (ii), (iii) and (iv) respectively, the commutative law

$$\overset{x}{\underset{a}{I}}{}^{\alpha}\,\overset{x}{\underset{a}{I}}{}^{\beta}\; f = \overset{x}{\underset{a}{I}}{}^{\beta}\,\overset{x}{\underset{a}{I}}{}^{\alpha}\; f \tag{6.8}$$

holds for all values of α and β. If $\beta = -m$, then (6.8) holds if $f^{(\mu)}(a) = 0$, ($\mu = 0, 1, 2, \ldots, m-1$).

7. Theorem 4.

If

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = 0, \quad \text{for every } x \text{ on } a < x \leq b$$

and

(i) $f(x)$ is a real-valued function of class $C^{(0)}$ on

$[a, b]$ and $R\,\alpha > 0$, then $f(x) \equiv 0$;

(ii) $f(x)$ is a real-valued function of class $C^{(n)}$ on

$[a, b]$ and $R\,\alpha + n > 0$, then

$$f(x) = \sum_{i=0}^{n-1} \frac{C_{i+1}\,(x - a)^{-(\alpha+i-1)}}{\Gamma(-\alpha - i)}$$

where C_i are arbitrary constants.

Case (i). The proof of this case follows from the integral property of the transform and the continuity of the function $f(x)$.

Case (ii). By definition 1

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = D_x^n \overset{x}{\underset{a}{I}}{}^{\alpha+n} f = 0.$$

Hence

$$\overset{x}{\underset{a}{I}}{}^{\alpha+n} f = \sum_{i+1}^{n} \frac{C_{n-i+1}\,(x - a)^{i-1}}{\Gamma^{(i)}}.$$

By Theorems 2 and 3, we have

$$f(x) = \sum_{i=1}^{n} \frac{C_{n-i+1} \overset{x}{\underset{a}{I}}{}^{-\alpha-n}\,(x-a)^{i-1}}{\Gamma(i)}$$

and by using the Lemma, we get

$$f(x) = \sum_{i=0}^{n-1} \frac{C_{i+1}\,(x-a)^{-\alpha-i-1}}{\Gamma(-\alpha - i)}.$$

8. Remarks:

I. It is necessary to mention that some of the results of theorems 1 and 3 have been treated by Riesz [7] pp. 10-16, with relatively different approaches.

II. It may be useful to introduce another proof for theorem 1 (ii), which may be stated as follows:

We have

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}\, dt.$$

Let $\varepsilon > 0$ such that $x - \varepsilon > a$, then

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^{x-\varepsilon} f(t)(x-t)^{\alpha-1}\, dt$$

$$+ \frac{1}{\Gamma(\alpha)} \int_{x-\varepsilon}^{x} f(t)(x-t)^{\alpha-1}\, dt \; (i).$$

On the right side of (i), since $f(t)$ is of class $C^{(0)}$ and $(x - t)^{\alpha}$ is a non-increasing function of t, then by the mean value theorem of Stieltjes integral, we can write (i) as

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = - \frac{f(x_1)}{\Gamma(\alpha+1)} \int_a^{x-\varepsilon} d\,(x-t)^{\alpha}$$

$$- \frac{f(x_2)}{\Gamma(\alpha+1)} \int_{x-\varepsilon}^{x} d\,(x-t)^{\alpha}$$

where $a \leqq x_1 \leqq x - \varepsilon$ and $x - \varepsilon \leqq x_2 \leqq x$. Thus we find that

$$\overset{x}{\underset{a}{I}}{}^{\alpha} f = - \frac{f(x_1)}{\Gamma(\alpha+1)} [\varepsilon^{\alpha} - (x-a)^{\alpha}] + f(x_2)\, \varepsilon^{\alpha}.$$

Therefore

$$\lim_{\alpha \to 0+} \overset{x}{\underset{a}{I}}{}^{\alpha} f = f(x_2).$$

Since $f(x)$ is continuous, then

$$\lim_{\alpha \to 0+} \overset{x}{\underset{a}{I}}{}^{\alpha} f = f(x).$$

III. The $H - R$ transform seems to be very useful in its application to linear differential equations. In considering the transform equations of the form [6]

$$\overset{x}{\underset{a}{I}}{}^{-w} \prod_{1}^{m+n} (x - a_i)^{\alpha_i} \overset{x}{\underset{a}{I}}{}^{-1} \prod_{1}^{m+n} (x - a_i)^{1-\alpha_i} \overset{x}{\underset{a}{I}}{}^{w-n+1}\, y(x) = 0. \tag{E}$$

Where $y(x) \in C^n$ on $[a, b]$ and $R(p - w) > 0$ $(p = 1, 2, \ldots)$, we find that for $m = 0$, equation (E) is nth order differential equation of Fuchsian type; and for positive integers m and n the equation represents a class of differential-integral equations of Fuchs-Volterra type. In particular if $m = 1$, $n = 2$, then eqnation (E) is the Riemann second order differential equation in the reduced form with singularities at a_i $(i = 1, 2, 3)$ if and only if

$$w = \sum_{i=1}^{3} \alpha_i + 1 = 0 \tag{F}$$

a condition which is satisfied by

$$w = \alpha + \beta + \gamma$$

$$\alpha_1 = \alpha + \beta + \gamma$$

$$\alpha_2 = \alpha + \beta' + \gamma$$

$$\alpha_3 = \alpha + \beta + \gamma'$$

where α, β, γ are the indices of the Riemann P-function

$$P \begin{Bmatrix} a_1 & a_2 & a_3 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

which is associated with the Riemann-Papperitz differential equation and

$$\alpha + \beta + \gamma + \alpha' + \beta' + \gamma' = 1.$$

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(6) This work has been done recently and is not included in the dissertation work presented by this paper. It is expected to be published sometime in the future.

If (F) does not hold, then the equation is a differentialintegral equation of Riemann-Volterra type. If

$$y(x) = (x - a_1)^{-\alpha} (x - a_2)^{-\beta} (x - a_3)^{-\gamma} u(x)$$

and (F) and (G) are satisfied by (E) for this case, then (E) becomes Riemann-Papperitz differential equation in $u(x)$. Equation (E) can be reduced to the Gauss's equation when for fixed i (say $i = 2$) $a_2 \to \infty$. By the operational properties some results have been obtained regarding the solutions of the transform equation.

IV. The definition of this transform has been extended to two dimensional case. The two-dimensional transform is given by the

$$\overset{u}{\underset{v}{I}}{}^{\alpha} \, \overset{v}{\underset{z}{I}}{}^{\beta} F = \frac{D_u^n}{\Gamma(\alpha + n)\Gamma(\beta + m)} \int_v^u (u - z)^{\alpha+n-1}$$

$$\left\{ D_v^m \int_z^u (v - t)^{\beta+m-1} F(z, t)\, dt \right\} dz$$

where $F(u, v) \in C^{(m+n)}$ with respect to the variables (u, v) in the region

$$T: \quad a \leqq u \leqq b; \qquad c \leqq v \leqq d$$

and

$$R(\alpha + n) > 0, \qquad R(\beta + m) > 0.$$

The relationship between this transform and that introduced by Riesz [7] in two dimensions has been established and some of its properties have been developed, which may appear later in another paper.

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