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MULTIPLE INTEGRAL PROBLEMS IN THE CALCULUS OF VARIATIONS AND RELATED TOPICS

by

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Introduction.

In this series of lectures, I shall present a greatly simplified account of some of the research concerning multiple integral problems in the calculus of variations which has been reported in detail in the papers [39], [40], [41], [42], [44], [46], and [47]. I shall speak only of problems in non-parametric form and shall therefore not describe the excellent result concerning double integrals in parametric form obtained almost concurrently by Sigalov, Danskin, and Cesari [62], [9], [5]) nor the work of L. C. Young and others on generalized surfaces. Some of my results have been extended in various ways by Cinquini [6], De Giorgi [10], Fichera [17], Nöbéling [51], Sigalov [58], [59], [60], [61], Silova [63], and Stampacchia [67], [68], [69], [70]. However, it is hoped that the results presented here will serve as an introduction to the subject.

The first part of this research reported in these lectures is an extension of Tonelli's work on single and double integral problems in which he employed the so-called direct methods of the calculus of variations ([71] through [78]). His work was stimulated, no doubt, by the succes of Hilbert, Lebesgue [31] and others in the rigorous establishment of Dirichlet's principle in certain important cases. The principle idea of these direct methods is to establish the existence of a function z minimizing an integral by showing (i) that the integral $I(z)$ is lower semicontinuous with respect to some

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kind of convergence, (ii) that $I(z) \geq d$ for the z considered and (iii) that there is a « minimizing sequence » z_n such that $I(z_n) \to d$ and $z_n \to z_0$ in the sense required.

In the case of single integral problems, where

$$(0.1) \qquad I(z) = \int_a^b f[x, z(x), z'(x)]\, dx$$

Tonelli (see, for instance [76]) was ableto carry through this program for the case that only absolutely continuous functions are admitted, the convergence is uniform, and (essentially) $f(x, z, p)$ is convex in p (if $f(x, z, p) \geq f_0(p)$ where $f_0(p)/|p| \to \omega$, it is seen from the proof of Theorem 2.4 below, that the functions in any minimizing sequence would be uniformly absolutely continuous so that a subsequence would converge uniformly to an absolutely continuous function z_0 which would thus minimize $I(z)$). Tonelli was also able to carry through the entire program for certain double integral problems using functions absolutely continuous in his sense (ACT) and uniform convergence [77], [78]. However, in general he had to assume that the integrand $f(x, y, z, p, q)$ satisfied a condition like

$$(0.2) \qquad f(x, y, z, p, q) \geq m(p^2 + q^2)^{\alpha/2} - k, \{\alpha > 2, m > 0\}.$$

If f satisfies this condition, Tonelli showed that the functions in any minimizing sequence are equicontinuous, and uniformly bounded on interior domains at least (see Lemma 4.1) and so a subsequence converges uniformly on such domains to a function still in his class. He was also able to handle the case where

$$(0.3) \qquad f(x, y, z, p, q) \geq m(p^2 + q^2) - k \text{ if } f(x, y, z, 0, 0) \equiv 0,$$

for instance by showing that any minimizing sequence can be replaced by one in which each z_n is monotone in the sense of Lebesgue (see [31] and [37], for instance) and hence equicontinuous on interior domains, etc.

However, Tonelli was not able to get a general theorem to cover the case where f satisfies (0.2) only with $1 < \alpha < 2$. Moreover, if one considers problems involving $\nu > 2$ independent variables, one soon finds that one would have to require α to be $> \nu$ in (0.2) in order to ensure that the functions in any minimizing sequence would be equicontinuous on interior domains. To see this, one needs only to notice that the functions

$$\log \log (1 + 1/r), \quad 1/r^h, 0 < r \leq 1 \; (r^2 = \sum_{\alpha=1}^{\nu} (x^\alpha)^2),$$

are limits of ACT functions in which

$$\int_{B(0,1)} | \nabla z_n |^{\nu} dx \text{ and} \int_{B(0,1)} | \nabla z_n |^{k} dx \text{ for } k < \nu/(h+1)$$

respectively, are uniformly bounded (see below for notation).

In order to carry through the program, for these more general problems, then, the writer found it expedient to allow functions which are still more general than Tonelli's ACT functions. One obtains these more general functions by merely replacing the requirement of ν-dimensional continuity in Tonelli's definition by summability, but retaining Tonelli's requirements of absolute continuity along lines parallel to the axes, summable partial derivatives, etc. But then, two such functions may differ on a set of measure zero in such a way that their partial derivatives also differ only on a set of measure zero. It is clear that such functions should be identified and this in done in forming the « spaces $\mathcal{B}_\lambda$ » discussed in Chapter I.

These more general functions have been defined in various ways and studied by various authors in various connections. Beppo Levi [32] was probably the first to use functions of this type in the special case that the function and its first derivatives are in $\mathcal{L}_2$; any function equivalent to such a function has been called strongly differentiable by Friedrichs and these functions and those of corresponding type involving higher derivatives have been used extensively in the study of partial differential equations (see [2], [3], [11], [18], [19], [20], [21], [24], [28], [30], [42], [45], [46], [47], [50], [57], [61], [66]), G. C. Evans also made use at an early date [14], [15], [16] of essentially these same functions in connection with his work on potential theory. J. W. Calkin needed them in order to apply Hilbert space theory to the study of boundary value problems for elliptic partial differential equations and collaborated with the author in setting down a number of useful theorems about these functions (see [4] and [40]). The functions have been studied in more detail since the war by some of the writers mentioned above and by Aronszajn and Smith who showed that any function in the space H_{m0} (see Professor Niremberg's lectures) can be represented as a Riesz potential of order m [1]. The writer is sure that many others have also discussed these functions and certainly does not claim that the bibliography is complete.

In Chapter I, the writer presents some of the known results concerning these more general functions. In Chapter II, these are applied to obtain theorems concerning the lower-semicontinuity and existence of minima of multiple integrals of the form

$$I(z, G) = \int_G f[x, z(x), \nabla z(x)]\, dx$$

(0.5) $$x = (x^1, \dots, x^\nu), z = (z^1, \dots, z^N), \nabla z = \{\partial z^i / \partial x^\alpha\}, dx = dx^1 \dots dx^\nu$$

$$i = 1, \dots, N, \alpha = 1, \dots, \nu$$

where the function f is assumed to be continuous in (x, z, p) for all (x, z, p) and convex in $p = \{p^i_\alpha\}$ for each (x, z). In Chapter III, the most general type of function $f(x, z, p)$ for which the integral $I(z, G)$ in (0.5) is lower-semicontinuous is discussed. In Chapter IV, the writer discusses his results concerning the differentiability of the solutions of minimum problems. In Chapter V, the writer discusses the recent application by Eells and himself of a variational method in the theory of harmonic integrals.

We consistently use the notations of (0.5). Il φ is a vector, $|\varphi|$ denotes the square root of the sum of the squares of the components. Our functions are all real-valued unless otherwise noted. If z is a vector or tensor $z_\alpha, z_{\alpha\beta}$, etc., will denote the partial derivatives $\partial z / \partial x^\alpha$, $\partial^2 z / \partial x^\alpha \partial x^\beta$, etc., or their corresponding generalized derivatives. Repeated indices are summed unless otherwise noted. If G is a domain ∂G denotes its boundary and $\bar{G} = G \cup \partial G$. $B(x_0, R)$ denotes the solid sphere with center at x_0 and radius R; we sometimes abbreviate $B(0, R)$ to B_R, $[a, b]$ denotes the closed cell $a^\alpha \leq x^\alpha \leq b^\alpha$. All integral are Lebesgue integrals. It is sometimes desirable to consider the behavior of a function (or vector) $z(x)$ with respect to a particular variable x^α; when this is done, we write $x = (x^\alpha, x'_\alpha)$ and $z(x) = z(x^\alpha, x'_\alpha)$ where x'_α stands the remaining variables; sometimes $(\nu - 1)$ dimensional integrals

$$\int_{a'_\alpha}^{b'_\alpha} f(x_0^\alpha, x'_\alpha)\, dx'_\alpha$$

appear in which case they have their obvious significance. We say that a (vector) function $z(x)$ satisfies a uniform Lipschitz condition on a set S if and only if there is a constant M such that

$$|z(x_1) - z(x_2)| \leq M \cdot |x_1 - x_2| \text{ for } x_1 \text{ and } x_2 \text{ on } S;$$

z is said to satisfy a uniform Hölder condition on S with exponent μ, $0 < \mu < 1$, if and only if there is an M such that

$$|z(x_1) - z(x_2)| \leq M \cdot |x_1 - x_2|^\mu \text{ for } x_1 \text{ and } x_2 \text{ on } S.$$

A (vector) function z is of class C^n on a domain G if and only if z and its partial derivatives of order $\leq n$ are continuous on G; z is said to be of class $C^{n+\mu}$ or C^n_μ on G if and only if z is of class C^n on G and it and all of its partial derivatives of order $\leq n$ satisfy uniform Hölder conditions with exponent μ, $0 < \mu < 1$, on G; the second notation C^n_μ is used when $\mu = 1$ (see Chapter V).

CHAPTER I

Function of class $\mathcal{B}_\lambda, \mathcal{B}'_\lambda, \mathcal{B}''_\lambda\ (\lambda \geq 1)$ and functions which are ACT.

We begin with the definitions of these classes:

DEFINITION: A function $z(x)\,(x = (x^1, \dots, x^\nu))$ *is of class* $\mathcal{B}_\lambda$ *on a domain* G if and only if z is of class $\mathcal{L}_\lambda$ on G and there are functions p_α, $\alpha = 1, \dots, \nu$, of class $\mathcal{L}_\lambda$ on G with the following property; if R is any cell with closure in G, there is a sequence z_{nR} of functions of class C' on $R \cup \partial R$ such that $z_n \to z$ and $z_{n,\alpha} \to p_\alpha$ strongly in $\mathcal{L}_\lambda$ on R.

DEFINITION: A function z *is of class* $\mathcal{B}'_\lambda$ on G if and only if

(i) z is of class $\mathcal{L}_\lambda$ on G;

(ii) if $[a, b]$ is any closed cell in G, then z is AC (absolutely continuous) in x^α on $[a^\alpha, b^\alpha]$ for almost all x'_α on $[a'_\alpha, b'_\alpha]$, $\alpha = 1, \dots, \nu$;

(iii) the partial derivatives $z_{,\alpha}$, which exist almost every-where and are measurable on account of (ii), are of class $\mathcal{L}_\lambda$ on G.

DEFINITION: A function z *is of class* $\mathcal{B}''_\lambda$ on G if and only if z is of class $\mathcal{B}_\lambda$ on G and is continuous there.

DEFINITION: A function z is *absolutely continuous in the sense of Tonelli* (ACT) on G if and only if z is of class $\mathcal{B}'_1$ and is continuous on G.

DEFINITION: Suppose z is of class $\mathcal{L}_1$ on G. We define its h average function on the set G_h by

$$z_h(x) = (2h)^{-\nu} \int_{x-h}^{x+h} z(\xi)\, d\xi, \tag{1.1}$$

G_h being the set of all x in G such that the cell $[x - h, x + h] \subset G$.

LEMMA 1.1: *Is z is of class $\mathcal{L}_\lambda$ on a domain G and z_h is its h-average function defined on G_h, then $z_h \to z$ in $\mathcal{L}_\lambda$ as $h \to 0$ on each closed cell $[a, b]$ in G and z_h is continuous on G_h.*

Proof: That z_h is continuous follows from the absolute continuity of the Lebesgue integral. Next, it is well known that $z_h(x) \to z(x)$ as $h \to 0$ for almost all x. Finally, choose $h_0 > 0$ so that $[a - h_0, b + h_0] \subset G$, keep $0 < h < h_0$, and let $\varphi(\varrho)$ be a function $\to 0$ as $\varrho \to 0$ such that $\| z \|_e \leq \varphi[m(e)]$ for $e \subset [a - h_0, b + h_0]$, where

$$\| z \|_e = \left[\int_e | z(x) |^\lambda \, dx \right]^{1/\lambda}.$$

Then the lemma follows, since

$$\| z_h - z \|_e \leq \| z_h \|_e + \| z \|_e \leq 2\varphi\,[m(e)] \text{ for } e \subset [a, b]$$

since

$$\int_e | z_h(x) |^\lambda \, dx \leq (2h)^{-\nu} \int_{-h}^{h} \left[\int_{e(\xi)} | z(x+\xi) |^\lambda \, dx \right] d\xi =$$

$$= (2h)^{-\nu} \int_{-h}^{h} \left[\int_{e(\xi)} | z(y) |^\lambda \, dy \right] d\xi \leq \left\{ \varphi\,[m(e)] \right\}^\lambda .$$

where $e(\xi)$ is the set obtained by translating e along the vector ξ.

THEOREM 1.1: *If z is of class $\mathcal{B}_\lambda$ on G, the functions p_α are uniquely determined up to null functions. If z_h is the h average of z and $p_{\alpha h}$ is that of p_α, then z_h is of class C' on G_h* and

$$(1.2) \qquad z_{h,\alpha}(x) = p_{\alpha h}(x), \qquad h > 0.$$

Proof: Let $[a, b] \subset G$, choose h_0 so $[a - h_0, b + h_0] \subset G$, and keep $0 < h < h_0$. Approximate to z and p_α by z_n and $z_{n,\alpha}$ in $\mathcal{L}_\lambda$ on $[a - h_0, b + h_0]$. Then for each h, we see that $z_{nh;\alpha} = (z_{n,\alpha})_h$ and we may obtain (1.2) by letting $n \to \infty$ on $[a, b]$. The first statement is now obvious.

DEFINITION: If z is of class $\mathcal{B}_\lambda$ on a domain G, we define its *generalizet derivative* $D_\alpha z(x)$ as the Lebesgue derivative at x of the set function $\int_e p_\alpha(x)\, dx$.

THEOREM 1.2: *If z is of class $\mathcal{B}'_\lambda$ on G, z_h is its h-average function, and $p_{\alpha h}$ is that of its partial derivative $\partial z / \partial x^\alpha$, then z_h is of class C' and* (1.2) *holds. Moreover z is of class $\mathcal{B}_\lambda$ and its corresponding partial and generalized derivatives coincide almost everywhere.*

Proof: Let $[a, b] \subset G$, choose h_0 so $[a - h_0, b + h_0] \subset G$, and keep $0 < h < h_0$. If x'_α is not in a set of measure 0 on $[a'_\alpha - h_0, b'_\alpha + h]$, then $\partial z / \partial x^\alpha \equiv p_\alpha$ is summable in x^α over $[a^\alpha - h_0, b^\alpha + h_0]$ and

$$(1.3) \qquad \int_{x_1^\alpha}^{x_2^\alpha} p_\alpha(x^\alpha, x'_\alpha)\, dx^\alpha = z(x_2^\alpha, x'_\alpha) - z(x_1^\alpha, x'_\alpha).$$

By integrating (1.3), we see that it holds for all x'_α on $[a'_\alpha, b'_\alpha]$ and all x_1^α, x_2^α on $[a^\alpha, b^\alpha]$ if z and p_α are replaced by their h-averages. Then (1.2) and the last statement follow.

THEOREM 1.3: (a) *If z_1 and z_2 are equivalent and one is of class $\mathcal{B}^\lambda$ on G, then both are and their generalized derivatives* coincide.

(b) *If z_1 and z_2 are of class $\mathcal{B}_\lambda$ on a domain G and $z_{1,\alpha}(x) = z_{2,\alpha}(x)$ almost everywhere on G, then z_1 and z_2 differ by a constant and a null function.*

These are easily proved using the h-average functions.

THEOREM 1.4: (a) *Any function z of class $\mathcal{B}_\lambda$ on G is equivalent to a function z_0 of class $\mathcal{B}'_\lambda$ on G.*

b) *z is ACT on G if and only if z is of class $\mathcal{B}''_1$ there.*

Proof: To prove (*a*), let $R = [a, b]$ be any rational cell in G and approximate to z there by functions z_n of class C' on $[a, b]$. A subsequence, still called z_n, converges to z almost everywhere and is such that

$$(1.4) \qquad \lim_{n \to \omega} \int_{a^\alpha}^{b^\alpha} | z_{n,\alpha}(x^\alpha, x'_\alpha) - z_{,\alpha}(x^\alpha, x'_\alpha) |^\lambda \, dx^\alpha = 0$$

for all x'_α not in a set $Z_{R\alpha}$ of $(\nu - 1)$-dimensional measure zero, $\alpha = 1, \ldots, \nu$. From (1.4), we see that the $z_n(x^\alpha, x'_\alpha)$ are equicontinuos in x^α and converge uniformly on $[a^\alpha, b^\alpha]$ to a function $z_{0R}(x^\alpha, x'_\alpha)$ which is AC in x^α if x'_α is not in $Z_{R\alpha}$, $\alpha = 1, \ldots, \nu$. Obviously $z_{0R} = z$ almost everywhere on R. Since the union of the $Z_{R\alpha}$ for α fixed and R running over all rational cells is still of measure zero; we see that the z_{0R} join up to form a function z_0 of class $\mathcal{B}'_\lambda$ on G.

To prove (*b*), we note first that if z is ACT on G, it is of class $\mathcal{B}''_1$ on G. Conversely, if z is of class $\mathcal{B}''_1$, we may repeat the first part of the proof taking z_n as the h_n-average of z and conclude that we may take z_{0R} always $= z$ since then z_n converges uniformly to z on R.

The following theorems are easily proved by approximations:

THEOREM 1.5: *The space $\mathcal{B}_\lambda$ of equivalence classes of functions of class $\mathcal{B}_\lambda$ is a Banach space if we define the norm by*

$$\| z \|_\lambda = \left\{ \int_G \left[| z |^2 + \sum_{\alpha=1}^{\nu} | z_{,\alpha} |^2 \right]^{\lambda/2} dx \right\}^{1/\lambda} .$$

If $\lambda = 2$, $\mathcal{B}_\lambda$ is a real Hilbert space if we define

$$(z, w) = \int_G \left(zw + \sum_{\alpha=1}^{\nu} z_{,\alpha} w_{,\alpha} \right) dx .$$

THEOREM 1.6: *If $u \in \mathfrak{B}_\lambda$ and h is of class C' and satisfies a uniform Lipschitz condition on the bounded domain G, then $h u \in \mathfrak{B}_\lambda$ on G and the generalized derivatives $(hu)_{,\alpha}$ all exist at any point x_0 where all the $u_{,\alpha}(x_0)$ exist.*

DEFINITION: A transformation $T: x = x(y)$ from a domain $\widetilde{G}$ onto G which is of class C' is said to be *regular* if and only if T is $1-1$ and T and its inverse are of class C' and satisfy uniform Lipschitz condition $(|x(y_1) - x(y_2)| \leq M \cdot |y_1 - y_2|$, etc.).

THEOREM 1.7: *If u is of class $\mathfrak{B}_\lambda$ $(\mathfrak{B}_\lambda'')$ on the bounded demain G, $x=x(y)$ is a regular transformation of class C' from the bounded domain $\widetilde{G}$ onto G and $\widetilde{u}(y) = u[x(y)]$, then $\widetilde{u}$ is of class $\mathfrak{B}_\lambda$ $(\mathfrak{B}_\lambda'')$ on $\widetilde{G}$. Moreover, if $x_0=x(y_0)$ and all the generalized derivatives $u_{,\alpha}(x_0)$ exist, then all the generalized derivatives $\widetilde{u}_{,\alpha}(y_0)$ exist and*

$$\widetilde{u}_{,\beta}(y_0) = u_{,\alpha}[x(y_0)] \cdot x^{\alpha}_{,\beta}(y_0) \tag{1.5}$$

Proof: That $\widetilde{u}$ is of class $\mathfrak{B}_\lambda$ $(\mathfrak{B}_\lambda'')$ and that we may choose the right sides of (1.5) as the «derivative functions» $\widetilde{p}_\beta$ of the definition is easily proved by approximating u on interior domains by functions of class C'. Since regular families of sets correspond under regular transformations, the last statement follows easily.

REMARKS: It is proved in [40] and [47], for instance, that if u is of class $\mathfrak{B}_\lambda$ on G, it is equivalent to a function $\bar{u}$ (namely the Lebesgue derivetive of $\int_e u\,dx$) which is of class $\mathfrak{B}_\lambda'$ and is such that any transform as in Theorem 1.7 retains this property. But the last statement of Theorems 1.7 does not hold for the partial derivatives since this would imply that z had a total differential almost everywhere contrary to an example of Sake [55]. It is clear how to define the generalized derivative in a given direction and that (Theorem 1.7) if all the $u_{,\alpha}(x_0)$ exist, then a all the *generalized* directional derivatives exist at x_0 and are given by their usual formulas there. It is now easy to prove Rademacher's famous theorem [52] that a Lipschitz function has a total differential almost everywhere: For using the result just mentioned together with Theorem 1.2 we see that if z is Lipschitz and x_0 is not in a set of measure zero, then the *partial* and generalized derivatives all exist at x_0 and the ordinary directional derivatives in a denumerable everywhere dense set of directions (independent of x_0) all exist and are given by their usual formulas; at any such point z is seen to have a total differential. Thus in Theorem 1.6, h may be Lipschitz and in Theorem 1.7, the transformation and its inverse may be Lipschitz; in this case (1.5) holds whenever all the *generalized* derivatives involved exist.

THEOREM 1.8: *The most general linear functional on the space $\mathfrak{B}_\lambda$ is of the form*

$$f(x) = \int_G (A_0 z + \sum_{\alpha=1}^{\nu} A_\alpha z_{,\alpha})\, dx \tag{1.6}$$

where the $A_\alpha\ (\alpha \geq 0) \in \mathfrak{L}_\mu$ with $\lambda^{-1} + \mu^{-1} = 1$ if $\lambda > 1$ or are bounded and measurable on G if $\lambda = 1$.

Proof: Let A_λ be the space of all vectors $\varphi = (\varphi_0, \dots, \varphi_\nu)$ with components in $\mathfrak{L}_\lambda$ and

$$\| \varphi \| = \left\{ \int_G \left[\sum_{\alpha=0}^{\nu} \varphi_\alpha^2 \right]^{\lambda/2} dx \right\}^{1/\lambda}.$$

From Theorem 1.5 it follows that the subspace of all vectors $(z, z_{,1}, \dots, z_{,\nu})$ for which $z \in \mathfrak{B}_\lambda$ on G is a closed linear manifold M in B_λ. Hence if $F(z, z_{,1}, \dots, z_{,\nu}) = f(z)$, then F can be extended to the whole space B to have same norm as f. Then F is given by (1.6).

From Theorem 1.8 we immediately obtain:

THEOREM 1.9: (a) *A necessary and sufficient condition that z_n converges weakly to z ($z_n \rightharpoonup z$) in $\mathfrak{B}_\lambda$ on G is that $z_n \rightharpoonup z$ and the $z_{n,\alpha} \rightharpoonup z_{,\alpha}$ in $\mathfrak{L}_\lambda$ on G.*

(b) *If $z_n \rightharpoonup z$ in $\mathfrak{B}_\lambda$ on G, then $z_n \rightharpoonup z$ in $\mathfrak{B}_\lambda$ on any subdomain.*

(c) *If $z_n \rightharpoonup z$ in $\mathfrak{B}_\lambda$ on G* (bounded), *$x = x(y)$ is a regular transformation of class C' from $\widetilde{G}$ onto G, $\widetilde{z}_n(y) = z_n[x(y)]$ and $\widetilde{z}(y) = z[x(y)]$, then $\widetilde{z}_n \rightharpoonup \widetilde{z}$ in $\mathfrak{B}_\lambda$ on $\widetilde{G}$.*

(d) *If $z_n \rightharpoonup z$ in $\mathfrak{B}_\lambda$ on G* (bounded) *and h is Lipschitz on G, then $hz_n \rightharpoonup hz$ in $\mathfrak{B}_\lambda$ on G.*

DEFINITION: A function z is of class $\mathfrak{B}_{\lambda 0}$ on G (bounded) if and only if it is of class $\mathfrak{B}_\lambda$ there and there existe a sequence $\{u_n\}$, each of class C' and vanishing on and near the boundary ∂G such that $z_n \to z$ (strong convergence) in $\mathfrak{B}_\lambda$ on G. The *subspace* $\mathfrak{B}_{\lambda 0}$ of $\mathfrak{B}_\lambda$ is defined correspondingly. If z and $z^* \in \mathfrak{B}_\lambda$ on G, we say thath $z = z^*$ *on ∂G in the $\mathfrak{B}_\lambda$ sense* if and only if $z - z^* \in \mathfrak{B}_{\lambda 0}$ on G·

The follawing is immediate:

THEOREM 1.10: *The subspace $\mathfrak{B}_{\lambda 0}$ is a closed linear manifold is $\mathfrak{B}_\lambda$; if $z_n \rightharpoonup z$ in $\mathfrak{B}_\lambda$ on G and each $z_n \in \mathfrak{B}_{\lambda 0}$, then $z \in \mathfrak{B}_{\lambda 0}$. If $z \in \mathfrak{B}_{\lambda 0}$ and $z_1(x) = z(x)$ for x on G and $z_1(x) = 0$ otherwise, then $z_1 \in \mathfrak{B}_{\lambda 0}$ on any $D \supset G$ and $z_{1,\alpha}(x) = 0$ for almost all x not in G.*

THEOREM 1.11 (Poincaré's inequality): *Suppose $z \in \mathfrak{B}_{\lambda 0}$ on $G \subset B(x_0, R)$.* Then

$$\int_G |z|^\lambda\, dx \leq \lambda^{-1} R^\lambda \int_G |\nabla z|^\lambda\, dx.$$

Proof: It is sufficicient to prove this for z of class C' and vanishing on $\partial B(x_0, R)$ with $G = B(x_0, R)$. Taking spherical coordinates (r, p) with $r = |x - x_0|$ and $p \in \Sigma = \partial B(0, 1)$, we obtain

$$\int_\Sigma |u(r, p)|^\lambda \, d\Sigma(p) = \int_\Sigma |u(R, p) - u(r, p)|^\lambda \, d\Sigma(p) =$$

$$= \int_\Sigma \left| \int_r^R u_r(s, p) \, ds \right|^\lambda d\Sigma \leq (R - r)^{\lambda - 1} \int_r^R \int_\Sigma |u_r(s, p)|^\lambda \, ds \, d\Sigma$$

where $u(r, p) = z(x)$. Thus

$$\int_{B(x_0, R)} |z(x)|^\lambda \, dx = \int_0^R r^{\nu - 1} \left[\int_\Sigma |u(r, p)|^\lambda \, d\Sigma(p) \right] dr$$

$$\leq \int_0^R (R - r)^{\lambda - 1} \left[\int_r^R s^{\nu - 1} \left\{ \int_\Sigma |u_r(s, p)|^\lambda \, d\Sigma(p) \right\} ds \right] dr$$

from which the result follows.

THEOREM 1.12: *Suppose $z \in \mathcal{B}_\lambda$ on G, $\Delta \subset G$, $z^* \in \mathcal{B}_\lambda$ on Δ and coincides with z on $\partial\Delta$ in the $\mathcal{B}_\lambda$ sense. Then the function Z such that $Z(x) = z^*(x)$ on Δ and $Z(x) = z(x)$ on $G - \Delta$ is of class $\mathcal{B}_\lambda$ on G and $z_{,\alpha}(x) = z^*_{,\alpha}(x)$ almost everywhere on Δ and $Z_{,\alpha}(x) = z_{,\alpha}(x)$ almost averywhere on $G - \Delta$.*

Proof: For define $Z_1(x) = z^*(x) - z(x)$ on Δ and 0 elsewhere. Then $Z(x) = z(x) + Z_1(x)$ on G and the result follows from Theorem 1.10.

LEMMA 1.2: *Suppose $z \in \mathcal{B}_\lambda$ on the cell $[a - h_0, b + h_0]$. Then*

$$\int_a^b |z_h(x) - z(x)|^\lambda \, dx \leq C_1(\nu, \lambda) \cdot h^\lambda \cdot \int_{a-h}^{b+h} |\nabla z(y)|^\lambda \, dy,$$

$$0 < h < h_0$$

where C_1 depends only on the arguments indicated.

Proof: Since we may approximate to z strongly in $\mathcal{B}_\lambda$ on $[a - h, b + h]$ by functions of class C' on that closed cell, it is sufficient to prove the lemma for such functions. Then if $x \in [a, b]$ and $|\xi^\alpha| \leq h$, we see that x and

$x + \xi$ are in $[a - h, h + b]$ so that

$$| z(x+\xi) - z(x) |^\lambda = \left| \int_0^1 \xi^\alpha z_{,\alpha}(z + t\xi)\, dt \right|^\lambda \leq | \xi |^\lambda \int_0^1 | \nabla z(x + t\xi) |^\lambda\, dt.$$

Then

$$\int_a^b | z_h(x) - z(x) |^\lambda\, dx = \int_a^b \left| (2h)^{-\nu} \int_{-h}^{h} [z(x+\xi) - z(x)]\, d\xi \right|^\lambda dx$$

$$\leq (2h)^{-\nu} \int_a^b \left[\int_{-h}^{h} | \xi |^\lambda \left\{ \int_0^1 | \nabla z(x + t\xi) |^\lambda\, dt \right\} d\xi \right] dx$$

$$= (2h)^{-\nu} \int_{-h}^{h} | \xi |^\lambda \left\{ \int_0^1 \left[\int_{a+t\xi}^{b+t\xi} | \nabla z(y) |^\lambda\, dy \right] dt \right\} d\xi$$

from which the result follows.

THEOREM 1.13: *If $z_n \rightharpoonup z_0$ in $\mathfrak{B}_{0\lambda}$ on the bounded domain G, then $z_n \to z_0$ in $\mathfrak{L}_\lambda$ on G, $\lambda \geq 1$. If $\{z_n\}$ is a sequence in $\mathfrak{B}_{0\lambda}$ with $\| z_n \|$ uniformly bounded, a subsequence converges strongly in $\mathfrak{L}_\lambda$ to some function z.*

Proof: The first statement follows from the second. For, let $\{z_p\}$ be any subsequence of $\{z_n\}$. A subsequence $\{z_q\}$ converges strongly in $\mathfrak{L}_\lambda$ to some function z which must be (equivalent to) z_0. Hence the whole sequence $z_n \to z_0$ in $\mathfrak{L}_\lambda$.

To prove the second statement, suppose $G \subset [a, b]$ and extend each z_n to be 0 outside G; then each $z_n \in \mathfrak{B}_{0\lambda}$ on $[a - 1, a + 1]$ with uniformly bounded $\mathfrak{B}_\lambda$ norm. For each h with $0 < h < 1$, we see that the z_{nh} are uniformly bounded and equicontinuous on $[a, b]$. So there is a subsequence, called $\{z_p\}$, such that z_{ph} converges uniformly to some function z_h for each h of a sequence $\to 0$. From lemma 1.2, it is easy to see first that the limiting z_h form a Cauchy sequence in $\mathfrak{L}_\lambda$ having some limit z and then that $z_p \to z$ strongly in $\mathfrak{L}_\lambda$.

In order to treat variational problems with fixed boundary values, one can, of course, practically always reduce the problem to one where the given boundary values are zero. Although one can formulate theorems about variational problems having variable boundary values on the boundary of an arbitrary bounded domain (see Chapter II), such problems become more meaningful if we restrict ourselves to domains G which are bounded and of class C' where boundary values can be defined in a more definite way as we now do.

DEFINITION: A bounded domain G is *of class* C' if and only if each point x_0 of the boundary ∂G is interior to a neighborhood $N(x_0)$ on $G \cup \partial G$ which is the image, under a regular transformation $x = x(y)$ of class C', of the half-cube Q^+: $|x^\alpha| < 1$ for $\alpha < \nu$ and $0 \leq x^\nu < 1$, where $x(0) = x_0$ and $\partial G \cap N(x_0)$ is the image of the part of Q^+ where $x^\nu = 0$. Such a neighborhood $N(x_0)$ is called a ***boundary neighborhood.***

DEFINITION: Suppose G is a domain. A finite sequence $\{h_1, \ldots, h_N\}$ of functions is said to be a ***partition of unity*** of class C' on $G \cup \partial G$ if and only if each h_i is of class C' on $G \cup \partial G$, $0 \leq h_i(x) \leq 1$ on $G \cup \partial G$ for each i, and

$$\sum_{i=1}^{N} h_i(x) \equiv i \quad \text{for } x \text{ on } G \cup \partial G .$$

The support of h_i is the closure of the set of all x on $G \cup \partial G$ for which $h_i(x) > 0$.

LEMMA 1.3: *If G is bounded domain of class C', there is a pattition of unity $\{h_1, \ldots, h_N\}$ of class C' on $G \cup \partial G$ such that the support of each h_i is either interior to a cell in G or is interior to a boundary neighborhood of $G \cup \partial G$.*

Proof. With each interior point P of G we define R_P as the largest hypercube $|x^\alpha - x_P^\alpha| < h_P$ in G and define r_P as the hypercube $|x^\alpha - x_P^\alpha| < h_P/2$. With each P on ∂G, associate a boundary neighborhood $R_P = N(P)$ which is the image under τ_P of Q^+ as in the definition; we define r_P as the part of R_P corresponding under τ_P to the part of Q^+ for which $|x^\alpha| < 1/2$, $\alpha = 1, \ldots, \nu$. There are a finite number $r_1, \ldots, r_N$ of the r_P which cover $G \cup \partial G$. Clearly each corresponding R_i is the image under a regular transformation τ_i of class C' of either the unit cube Q or the half-cube Q^+ where r_i corresponds under τ_i to the part where $|x^\alpha| < 1/2$.

Now, let $\varphi(s)$ be a fixed function of class C^∞ for all s with $\varphi(s) = 1$ for $|s| \leq 1/2$, $\varphi(s) = 0$ for $|s| \geq 3/4$, and $0 \leq \varphi(s) \leq 1$ otherwise. For each i, define $k_i(x)$ on R_i as the image under τ_i of the function $\varphi(y^1) \ldots \varphi(y^\nu)$ and define $k_i(x) = 0$ elsewhere on $G \cup \partial G$. Then the support of k_i is interior to R_i, $k_i(x) = 1$ for x on r_i, and each k_i is of class C' on $G \cup \partial G$. We then define

$$h_1(x) = k_1(x), \quad h_{i+1}(x) = k_{i+1}(x) \prod_{j=1}^{i} [1 - k_j(x)], \quad i = 1, \ldots, N-1 .$$

Then we see by induction that

$$h_1(x) + \dots + h_i(x) = 1 - \prod_{j=1}^{i} [1 - k_j(x)]$$

so that the sequence $\{h_i, \dots, h_N\}$ satisfies the desired conditions.

THEOREM 1.14: *Suppose G is bounded and of class C' and $z \in \mathfrak{B}_\lambda$ on G. Then*

(i) *there is a sequence $\{z_n\}$ of functions of class C' on $G \cup \partial G$ which converges strongly in $\mathfrak{B}_\lambda$ to z on G;*

(ii) *there is a boundary value function φ in $\mathfrak{L}_\lambda$ on ∂G (with respect to hyperarea) to which every sequance $\{z_n\}$ in (i) converges strongly in $\mathfrak{L}_\lambda$ on ∂G;*

(iii) *if $T: x = x(y)$ is a regular transformation of class C' of $\tilde{G} \cup \partial \tilde{G}$ onto $G \cup \partial G$, $\tilde{z}(y) = z[x(y)]$, and $\tilde{\varphi}(y) = \varphi[x(y)]$, then $\tilde{\varphi}$ is the boundary value function for $\tilde{z}$ on $\partial \tilde{G}$;*

(iiii) *if $\varphi(x) = 0$ for almost all x on ∂G, then $z \in \mathfrak{B}_{\lambda 0}$ on G.*

Proof: Let $\{h_1, \dots, h_N\}$ be a partition of unity on $G \cup \partial G$ of the type described in Lemma 1.3. Clearly each function $h_i z \in \mathfrak{B}_\lambda$ on G and on R_i and the thansform $w_i(y)$ under $\tau_i \in \mathfrak{B}_\lambda$ on either Q or Q^+; in the former case w_i vanishes on and near ∂Q and in the latter, w_i vanishes near $\partial Q^+ \cap \cap \partial Q$. In the latter case, w_i is equivalent to a function w_{i0} which is AC in y^α for almost all y'_α, $\alpha = 1, \dots, \nu$ on any cell where $h \le y^\nu \le 1$ (since $w_i = 0$ near $y^\nu = 1$), where $h > 0$. But since $w_{i0,\nu} \in \mathfrak{L}_\lambda$, we see that w_{i0} is AC in y^ν for $0 \le y^\nu \le 1$ for almost all y'_ν. If we extend w_{i0} to the whole of Q by setting

$$w_{i0}(y^\nu, y'_\nu) = + w_{i0}(-y^\nu, y'_\nu) \quad \text{for} \quad -1 \le y^\nu \le 0,$$

we see that $w_{i0} \in \mathfrak{B}'_\lambda$ on Q and vanishes near ∂Q. Clearly we may approximate each w_i or w_{0i} on Q strongly in $\mathfrak{B}_\lambda$ by functions w_{ni} of class C' on $\bar{Q}$ and vanishing near ∂Q. If we define z_{ni} on R_i as the transform of w_{ni} under τ_i and then define $z_n = z_{n1} + \dots + z_{nN}$, we see that z_n has the desired properties.

To prove (ii) we choose, in all cases, w_{i0} equivalent to w_i and $\mathfrak{B}'_\lambda$ on Q. Then, since w_{i0} is AC in x^ν, we see that

$$\int_{-1}^{1} | w_{i0}(y_2^\nu, y'_\nu) - w_{i0}(y_1^\nu, y'_\nu) |^\lambda \, dy'_\nu$$

(1.7)

$$\le (y_2^\nu - y_1^\nu)^{\lambda - 1} \int_{y_1^\nu}^{y_2^\nu} \int_{-1}^{1} | w_{i0,\nu}(y^\nu, y'_\nu) |^\lambda \, dy \le \varepsilon \, (y_2^\nu - y_1^\nu),$$

$$0 < y_1^\nu < y_2^\nu; \ \lim_{\varrho \to 0^+} \varepsilon(\varrho) = 0.$$

Accordingly, we see that $w_{i0}(y^\nu, y'_\nu)$ converges strongly in $\mathfrak{L}_\lambda$ in y'_ν to $w_{i0}(0, y'_\nu)$ as $y^\nu \to 0+$. If $z_n \to z$ in $\mathfrak{B}_\lambda$, z_n of class C', and we let w_{ni} be the transform of $h_i z_n$ under τ_i, then we see that (1.7) holds uniformly. Now let $\{p\}$ be any subsequence of $\{n\}$. There is a subsequence $\{q\}$ of $\{p\}$ such that (for each i) $w_{qi}(y^\nu, y'_\nu)$ converges strongly in $\mathfrak{L}_\lambda$ with respect to y'_ν on $[-1, 1]$ for almost all $y^\nu, 0 < y^\nu \leq 1$. But, on account of the uniformity in (1.7), this convergence is uniform for *all* $y^\nu, 0 \leq y^\nu \leq 1$. Hence the whole sequence $w_{i0}(0, y'_\nu)$ converges strongly to $w_{i0}(0, y'_\nu)$ in $\mathfrak{L}_\lambda$.

(iii) is now evident. To prove (iiii), $\{\tilde{z}_n\}$ be of class C' and converge strongly to z in $\mathfrak{B}_\lambda$ on G. Then $\tilde{z}_n$ and each $h_i\tilde{z}_n$ converges strongly to 0 on ∂G. If we define the w_{0i} as above, then $w_{0i}(0, y'_\nu) = 0$ for almost all y'_ν on $[+1, 1]$ if R_i is a boundary neighborhood. If we extend such w_{0i} to Q by $w_{0i}(y^\nu, y'_\nu) = -w_{0i}(-y^\nu, y'_\nu)$, $y^\nu \leq 0$ we see that w_{0i} is of class $\mathfrak{B}_\lambda$ on Q and that it and its h-average functions, for sufficiently small h vanish near ∂Q and along $y^\nu = 0$. By modifying the average function slightly for each h in a sequence $\to 0$ we may construct sequences w_{ni} tending strongly in $\mathfrak{B}_\lambda$ to $w_{0i} = 0$ such thot each $W_{ni} = 0$ near $y^\nu = 0$ as near ∂Q for those i for which R_i is a boundary neighborhood. The desired z_n, each of class C' and vanishing near ∂G can be constructed as above.

THEOREM 1.15: *If G is bounded and of class C' and if $z_n \rightharpoonup z$ in $\mathfrak{B}_\lambda$ on G, then $z_n \to z$ in $\mathfrak{L}_\lambda$ on G and $\varphi_n \to \varphi$ in $\mathfrak{L}_\lambda$ on ∂G. If $\| z_n \|$ is uniformly bounded in $\mathfrak{B}_\lambda$, and the set functions $\int_e |\nabla z_n| \, dx$ are uniformly absolutely continuous if $\lambda = 1$, there is a subsequence $\{z_p\}$ which converges weakly in $\mathfrak{B}_\lambda$ to some z on G.*

Proof: Let $\{h_1, \ldots, h_N\}$, w_{ni}, w_i, and w_{0i} have meanings as in the proof of Theorem 1.14 and let w_{n0i} be of class $\mathfrak{B}'_\lambda$ on Q (or Q^+) and be equivalent to w_{ni} and extend each w_{n0i} to Q as before. Then (1.7) holds uniformly (in case $\lambda = 1$ this is true on account of the uniform absolute continuity in that case) and $w_{ni} \to w_i$ in $\mathfrak{L}_\lambda$ on Q for each i. The argument in the proof of (ii) in Theorem 1.14 can be repeated to obtain the desired results. The last statement follows easily.

In the next section, we shall have occasion to discuss vector functions of class $\mathfrak{B}_\lambda$.

DEFINITION: A vector function $z = (z^1, \ldots, z^N)$ is of class $\mathfrak{B}_\lambda$ if and only if each of its components is; in this case

$$\| z \|_{\beta_\lambda} = \left\{ \int_G \left[\sum_{i=1}^N \left\{ (z^i)^2 + \sum_{\alpha=1}^\nu (z^i_\alpha)^2 \right\} \right]^{\lambda/2} dx \right\}^{1/\lambda}.$$

It is clear that all the theorems and lemmas of this section except Theorem 1.11 and lemma 1.2 generalize immediately to vector functions. Those two can be generalized with the help of the following well known lemma:

LEMMA 1.4: *Suppose $f_1, \dots, f_n$ are summable over the set S with respect to the maesure μ. Then $\sqrt{f_1^2 + \dots + f_n^2}$ is also and*

$$\left\{\sum_{i=1}^{n}\left[\int_S f_i(x)\,d\mu\right]^2\right\}^{1/2} \leq \int_S \left[\sum_{i=1}^{n} f_1^2(x)\right]^{1/2} dx. \tag{1.8}$$

Proof: For the left side of (1.8) equals

$$\max_{|a|=1}\sum_{i=1}^{n}\int_S a_i f_i(x)\,d\mu \leq \int_S \left[\sum_{i=1}^{n} f_i^2(x)\right]^{1/2} d\mu\,; \quad \left(|a|^2 = \sum_{i=1}^{n} a_1^2\right).$$

In addition, we need the following special case of Rellich's theorem [53]:

THEOREM 1.16: *If the vector z is of class $\mathfrak{B}_\lambda$ on the hypercube R of side h and z_R is its average over R, then*

$$\int_R |z(x) - z_R|^\lambda\,dx \leq C_2(\nu, \lambda)\cdot h^\lambda \cdot \int_R |\nabla z(x)|^\lambda\,dx$$

where C_2 depends only on the arguments indicated.

Proof: It is sufficient to prove this for vectors of class C' where R: $|x^\alpha| \leq k = h/2$. Then we have

$$\int_R\int_R \left\{\sum_{i=1}^{N}[z^i(\xi) - z^i(x)]^2\right\}^{\lambda/2} dx\,d\xi$$

$$= \int_R\int_R \left\{\sum_{i=1}^{N}\left[\int_0^1 (\xi^\alpha - x^\alpha)\,z^i_{,\alpha}[x + t(\xi - x)]\,dt\right]^2\right\}^{\lambda/2} dx\,d\xi$$

$$\leq \int_R\int_R \left[\int_0^1 |\xi - x|\cdot|\nabla z[x + t(\xi - x)]|\,dt\right]^\lambda dx\,d\xi$$

$$\leq \int_{-k}^{k}\int_{-k}^{k}\int_0^1 |\xi - x|^\lambda \cdot |\nabla z[x + t(\xi - x)]|^\lambda\,dt\,dx\,d\xi.$$

Setting $\eta^a = x^a + t(\xi^a - x^a) = (1-t)x^a + t\xi^a$, the last integral becomes

$$\int_{-k}^{k}\left\{\int_0^1 t^{-\nu-\lambda}\left[\int_{(1-t)x-tk}^{(1-t)x+tk} |\eta - x|^\lambda \cdot |\nabla z(\eta)|^\lambda d\eta\right] dt\right\} dx$$

$$= \int_{-k}^{k} |\nabla z(\eta)|^\lambda \left\{\int_0^1 t^{-\nu-\lambda}\left[\int_{R(\eta,t)} |\eta - x|^\lambda dx\right] dt\right\} d\eta$$

where $R(\eta,t)$ is the intersection of R with the hypercube $|x^a - \eta^a/(1-t)| \leq tk$. On $R(\eta, t)$ we see that

$$|\eta - x| \leq \nu^{1/2} \cdot th .$$

The result follows since $m[R(h, t)] \leq h^\nu$ and is $\leq (2th)^\nu$ for $t \leq 1/2$.

CHAPTER II

Lower-semicontinuity and existence theorems for a class of multiple integral problems.

In this chapter, we consider variational problems for integrals of the form (0,5) in which $f(x, z, p)$ is continuous in (x, z, p) for all (x, z, p) and is convex in p for each (x, z) (cf. [42], Chapter III).

DEFINITIONS: A set S in a linear space is said to be *convex* if and only if the segment $P_1 P_2$ belongs to S whenever the points P_1 and P_2 do. A function $\varphi(\xi)$ $(\xi = (\xi^1, \ldots, \xi^P))$ is said to be *convex* on the convex set S in the ξ-space if and only if

$$\varphi[(1-\lambda)\xi_1 + \lambda\xi_2] \leq (1-\lambda)\varphi(\xi_1) + \lambda\varphi(\xi_2), \qquad 0 \leq \lambda \leq 1,$$

whenever ξ_1 and $\xi_2 \in S$.

The following theorems concerning convex functions are well known and are stated without proof:

LEMMA 2.1: *Suppose $\varphi(\xi)$ is convex on the open convex set S with $|\varphi(\xi)| \leq M$ there. Then φ satisfies*

$$|\varphi(\xi_2) - \varphi(\xi_1)| \leq 2M \cdot |\xi_2 - \xi_1| / \delta$$

on any compact subset of S at a distance $\geq \delta$ from ∂S.

LEMMA 2.2: *Suppose φ and each φ_n are convex on the open convex set S and suppose $\varphi_n(\xi) \to \varphi(\xi)$ for each ξ on S. Then the convergence is uniform on any compact subset of S.*

LEMMA 2.3: *A necessary and sufficient condition that φ be convex on the open convex set S is that for each $\bar{\xi}$ in S there exists a linear function $a_p \cdot \xi^p + b$ such that*

$$(2.1) \qquad \varphi(\bar{\xi}) = a_p \bar{\xi}^p + b, \quad \varphi(\xi) \geq a_p \xi^p + b \qquad \text{for all} \qquad \xi \in S.$$

If φ is of class C' on S, this condition is equivalent to

$$E(\xi, \bar{\xi}) = \varphi(\xi) - \varphi(\bar{\xi}) - (\xi^\alpha - \bar{\xi}^\alpha)\varphi_{\alpha}(\bar{\xi}) \geq 0; \ \xi, \bar{\xi} \in S.$$

If φ is of class C'' on S, this condition is equivalent to

$$\varphi_{,\alpha\beta}(\bar{\xi})\,\eta^\alpha\,\eta^\beta \geq 0$$

for all $\bar{\xi}$ on S and all η.

DEFINITION: A linear function $a_p\,\xi^p + b$ which satisfies (2.1) for some $\bar{\xi}$ is said to *be supporting to* φ at $\bar{\xi}$.

LEMMA 2.4: *Suppose φ is convex for all ξ and satisfies*

$$\lim_{|\xi| \to +\infty} \varphi(\xi)/|\xi| = +\infty. \tag{2.2}$$

Then φ takes on its minimum. Also, if $a_1, \ldots, a_p$ are any numbers, there is a unique b such that $a_p\,\xi^p + b$ is supporting to φ for some $\bar{\xi}$. If ψ is convex and satisfies (2.2), *if $\psi(\xi) \geq \varphi(\xi)$ for each ξ, and if $a_p\,\xi^p + c$ is supporting to ψ, then $c \geq b$.*

LEMMA 2.5: *Suppose that φ_n and φ are everywhere convex and satisfy* (2.2) *and suppose that $\varphi_n(\xi) \to \varphi(\xi)$ for each ξ. Suppose $a_1, \ldots, a_p$ are any numbers and b_n and b are chosen so that $a_p\,\xi^p + b_n$ and $a_p\,\xi^p + b$ are supporting to φ_n and φ, respectively. Then $b_n \to b$. Likewise, if $a_{np} \to a_p$ for each p and b_n and b are chosen so that $a_{np}\,\xi^p + b_n$ and $a_p\,\xi^p + b$ are all supporting to f, then $b_n \to b$.*

In order to consider variational problems on arbitrary bounded domains, it is convenient to introduce the following type of weaker than weak convergence in $\mathcal{B}_1$ on such a domain.

DEFINITION: We say that $z_n \dashrightarrow z_0$ in $\mathcal{B}_1$ on the bounded domain G if and only if z_n and z_0 all $\in \mathcal{B}_1$ on G, $z_n \rightharpoonup z_0$ in $\mathcal{B}_1$ on each cell interior to G and each $z_{n,\alpha} \rightharpoonup z_{0,\alpha}$ in $\mathcal{L}_1$ on the whole of G.

THEOREM 2.1: *If G is bounded and of class C' or if all the $z_n \in \mathcal{B}_{10}$ on G and if $z_n \dashrightarrow z_0$ in $\mathcal{B}_1$ on G, then $z_n \rightharpoonup z_0$ in $\mathcal{B}_1$ on G.*

Proof: The second case can be reduced to the first by extending each z_n to be zero outside G and choosing a domain Γ of class C' such that $\Gamma \supset \bar{G}$. Thus we suppose G of class C'. If we use the notation in the proof of Theorem 1.14, we see that (1.7) holde uniformly for the w_{noi} so that an argument similar to those in the proofs of Theorems 1.14 and 1.15 and 1.13 shows that w_{noi} converge strongly in $\mathcal{L}_1$ on Q or Q^+ to something for each i. Thus z_n converges strongly in $\mathcal{L}_1$ on G to something which must be z_0.

REMARK: If G is not of class C' and the z_n are not all in $\mathcal{B}_{10}$ on G, then an example in [41] shows that $z_n \dashrightarrow z_0$ in $\mathcal{B}_1$ on G without the $\mathcal{B}_1$ norms of the z_n being uniformly bounded. If for some $\lambda > 1$,

$$\int_G |\nabla z_n|^\lambda \, dz \equiv \int_G \left[\sum_{\alpha=1}^{\nu} z^2_{n,\alpha}\right]^{\lambda/2} dy \quad (G \text{ bounded})$$

are uniformly bounded, then a subsequence $\{p\}$ of $\{n\}$ exists such that the $z_{p,\alpha} \rightharpoonup$ something in $\mathcal{L}_1$ on the whole of G.

THEOREM 2.2: *Suppose that $f(p)$ is defined of all $p = \{p^i_\alpha\}$ $(i = 1, \dots, N$ $\alpha = 1, \dots, \nu)$ and f is convex. If $z_n \dot{\rightharpoonup} z_0$ on G and*

$$I(z_0, G) = \int_G f(\nabla z_0)\, dx, \quad I(z_n, G) = \int_G f(\nabla z_n)\, dx,$$

then $I(z_0, G)$ and $I(z_n, G)$ are each finite or $+\infty$ and

$$I(z_0, G) \leq \liminf_{n \to \infty} I(z_n, G).$$

Proof: Since f is convex, there are constants a^α_i such that

$$f(p) \geq f(0) + a^\alpha_i p^i_\alpha \qquad \text{for all } p.$$

Hence

$$I(z, G) \geq f(0)\, m(G) + a^\alpha_i \int_G z^i_{,\alpha}(x)\, dx$$

with a similar inequality for $I(z_n)$. Thus the first statement follows.

If $D \subset G$, we see as above that

$$I(z_n, G) - I(z_n, D) = I(z_n, G - D) \geq f(0)[m(G - D)] + a^\alpha_i \int_{G-D} z^i_{n,\alpha}\, dx \geq$$
$$\geq \varepsilon[m(G - D)]; \quad \lim_{\varrho \to 0} \varepsilon(\varrho) = 0$$

by virtue of the uniform absolute continuity of the set functions $\int_e z^i_{n,\alpha}(x)\, dx$.

Clearly also $I(z, D) \to I(z, G)$ as D runs through an expanding sequence of domains exhausting G. Thus it is sufficient to prove the lower semicontinuity for G a hypercube of side h, say.

To do this, we define a sequence of summable functions $\varphi_q(x)$ as follows: For each q divide G into $2^{\nu q}$ hypercubes of side $h \cdot 2^{-q}$. On each of these hypercubes R, define

$$\varphi_q(x) = f(p_R) + a_i^\alpha(R, q)\,[z^i_{0,\alpha}(x) - p^i_{R\alpha}[\,, \quad x \text{ to interior to } R\,,$$

where $p^i_{R\alpha}$ is the average of $z^i_{,\alpha}$ over R and the $a_i^\alpha(R, q)$ are chosen so that $f(p_R) + a_i^\alpha \cdot (p^i_\alpha - p^i_{R\alpha})$ is supporting to f at p_R. We define the φ_{nq} similarly from z_n. Then it follows that

$$\varphi_q(x) \leq f[\nabla z_0(x)]\,, \qquad \varphi_{nq}(x) \leq f[\nabla z_n(x)]$$

(almost everywhere). On the other hand, suppose all the generalized derivatives exist at some x_0 which is not on ∂R for any hypercube R as above for any q. Let R denote the hypercube containing x_0. Then as $q \to \infty$ $p^i_{R\alpha} \to z^i_{,\alpha}(x_0)$ so that $\varphi_q(x_0) \to f[\nabla z(x_0)]$ since the a_i^α remain bounded (Lemma 2.5). Hence

$$I(z\,,\,G) = \lim_{q\to\infty} \int_G \varphi_q(x)\,dx\,. \tag{2.3}$$

Moreover, for each fixed q, $p_{nR} \to p_R$ from the weak convergence so

$$\int_G \varphi_q(x)\,dx = \sum_R f(p_R)\,m(R) = \lim_{n\to\infty} \sum_R f(p_{nR})\,m(R) =$$

$$= \lim_{n\to\infty} \int_G \varphi_{nq}(x)\;dx \leq \liminf_{n\to\infty} I(z_n\,,\,G)\,.$$

The result follows from (2.3) and (2.4).

LEMMA 2.6: *Suppose $f(x, z, p)$ is defined and satisfies a uniform Lipschitz condition with constant K for all (x, z, p), suppose $f(x, z, p)$ is convex in p for each (x, z) and suppose $f(x, z, p) \geq f_0(p)$ for all (x, z, p), where $f_0(p)$ is convex. Then, if $z_n \rightharpoonup z_0$ in $\mathcal{B}_1$ on G,*

$$I(z_0\,,\,G) \leq \liminf_{n\to\infty} I(z_n\,,\,G)\,.$$

Proof: As in the proof of Theorem 2.2, it is sufficient to prove this for a hypercube D of side d interior to G. Then $z_n \to z_0$ in $\mathcal{L}_\lambda$ on D. From the Lipschitz condition, $f(x, z, p) \leq f(0, 0, 0) + K\cdot|x| + K\cdot|z| + K\cdot|p|$ so that $I(z, D)$ and $I(z_n, D)$ are finite.

For each q, divide D into $2^{\nu \cdot q}$ hypercubes R of side $2^{-q} \cdot d$. Then, using Theorem 1.16, it follows that

$$\int_R |f[x, z(x), p(x)] - f[x_R, z_R, p(x)]| \, dx \leq K \int_R [|x - x_R| + |z(x) - z_R| \, dx \leq$$

$$\leq K \cdot 2^{-q} [2^{-1} \nu^{1/2} \cdot h^\nu + \int_R |\nabla z(x)| \, dx \qquad (h = 2^{-q})$$

$$\left| \int_D f[x, z(x), p(x)] \, dx - \sum_R \int_R f[x_R, z_R, p(x)] \, dx \right| \leq$$

$$\leq K \cdot 2^{-q} [2^{-1} \nu^{1/2} \cdot d^\nu + \int_D |\nabla z(x)| \, dx] \leq \varepsilon_q, \lim_{q \to \delta,} \varepsilon_q = 0$$

and a similar inequality holds for each z_n with ε_q independent of n on account of the weak convergence. Also

$$\sum_R \int_R |f[x_R, z_{nR}, p_n(x)] - f[x_R, z_R, p_n(x)]| \, dx \leq K \int_D |z_n(x) - z(x)| \, dx .$$

The lemma follows easily from Theorem 2.2 and the inequalities above.

THEOREM 2.3: *Suppose* $f(x, z, p)$ *is defined and continuous for all* (x, z, p), *is convex in* p *for each* (x, z) *and* $f(x, z, p) \geq f_0(p)$ *for all* (x, z, p) *where* $f_0(p)$ *is convex and* $f_0(p)/|p| \to +\infty$ *as* $p \to \infty$. *Then* $I(z, G)$ *is lower semicontinuous with respect to the convergence* $\dot{\rightarrow}$.

Proof: In order to prove this, it is sufficient to show that $f(x, z, p)$ is the limit of a non-decreasing sequence $f_n(x, z, p)$ each of which has the properties required in Lemma 2.6. In order to do this, let $b(x, z; a)$ $(a = \{a_i^\alpha\})$ be chosen so that the function $\varphi(x, z; p; a) \equiv a_i^\alpha p^i + b(x, z; a)$ is the unique supporting plane (in p) to f determined by a. By Lemmas 2.4 and 2.5 $b(x, z; a)$ is continuous in $(x, z; a)$ and $b(x, z; a) \geq b_0(a)$, the corresponding function for f_0. For each a, choose a non-decreasing sequence $b_n(x, z; a)$ of functions, each $\geq b_0(a) - 1$, each satisfyng a uniform Lipschitz condition for all (x, z), which converges to $b(x, z; a)$. We then define $\varphi_n(x, z; p; a) = a_i^\alpha p_\alpha^i + b_n(x, z; a)$ and we see that φ_n is a non decreasing sequence tending to φ for each a, each φ_n satisfying a uniform Lipschitz condition everywhere.

For each n, we define $f_n(x, z, p) = \max \varphi_n(x, z, p, a)$ for all a for which all the a_i^α are rational numbers having numerator and denominator both $\leq n$. Then it is clear that the f_n are non-decreasing and each satisfies a uniform Lipschitz condition. Now, let (x_0, z_0, p_0) and $\varepsilon > 0$ be given. Using Lemma 2.5 and the continuity of b, we see that there is a rational a such that $\varphi(z_0, p_0; \bar{a}) > f(x_0, z_0, p_0) - \varepsilon/2$. Clearly $\varphi_n(x_0, z_0, p_0; \bar{a}) > \varphi(x_0, z_0, p_0; \bar{a}) - \varepsilon/2$ for all sufficiently large n, so that $f_n(x_0, z_0, p_0) \to f(x_0, z_0, p_0)$.

We now turn to existence theorems on arbitrary domains. We begin with the following theorem (cf. [48] and [40], theorem 8.8 and [41]:

THEOREM 2.4: *Suppose $f_0(p)$ is convex in p and $f_0(p)/|p| \to +\infty$ as $p \to \infty$. Then there is a function $\varphi(\varrho) \to 0$ as $\varrho \to 0$ which depends only on f and M such that if $I(z, G) \leq M$, then*

$$\int_e |\nabla z(x)|\, dx \leq \varphi[m(e)].$$

Proof: For each integer $r \geq 1$, let E_r be the set of x in G where $r - 1 \leq |\nabla z(x)| < r$ and $\nabla z(x)$ exists and let

$$\mathcal{E}_r = \bigcup_{k=r+1}^{\infty} E_k \cup Z,\ r = 0, 1, 2, \ldots$$

where Z is the set of measure 0 where $\nabla z(x)$ does not exist. Clearly $\mathcal{E}_0 = G$ and if $r \geq 1$ and $x \in G - \mathcal{E}_r$, then $|\nabla z(x)| < r$. Let α_r be the inf. of $f_0(p)/|p|$ for $|p| \geq r - 1$. Then $\alpha_r \to +\infty$ as $r \to \infty$. Also

$$\sum_{k=r+1}^{\infty} \alpha_k \cdot (k-1) \cdot m(E_k) \leq \int_G f_0(\nabla z)\, dx \leq M$$

From this we see that

$$m(\mathcal{E}_r) \leq \frac{M}{r \cdot \alpha_{r+1}}, \quad \int_{\mathcal{E}_r} |\nabla z|\, dx \leq \frac{(r+1)M}{r \cdot \alpha_{r+1}}$$

and both $\to 0$ as $r \to \infty$. So, let e be any subset of G. Let r be the smallest integer such that $M/r\,\alpha_{r+1} \leq m(e)$. Then

$$\int_e |\nabla z|\, dx \leq \int_{e - \mathcal{E}_r} |\nabla z|\, dx + \int_{e \cap \mathcal{E}_r} |\nabla z|\, dx$$

$$\leq \frac{M}{\alpha_{r+1}} + \frac{(r+1)M}{r \cdot \alpha_{r+1}} = \varphi[m(e)]$$

and φ satisfies the conditions.

THEOREM 2.5: *Suppose $f(x, z, p)$ satisfies the hypotheses of Theorem 2.3 and G is a bounded domain. Suppose that Γ^* is a family of functions z^* in $\mathfrak{B}_1$ which is compact with respect to the convergence $\dot{\rightarrow}$ in $\mathfrak{B}_1$ on G. Suppose F is the family of all z in $\mathfrak{B}_1$ which coincide on ∂G in the $\mathfrak{B}_1$ sense with some z^* in Γ^* and suppose F contains some $\tilde{z}_1$ for which $I(\tilde{z}_1, G) < +\infty$. Then $I(z, G)$ takes on its minimum in F.*

Proof: Let $\{z_n\}$ be a minimizing sequence (i. e. $I(z_n, G) \rightarrow$ greatest lower bound for z in F); we may assume that $I(z_n, G) \leq M = I(\tilde{z}_1, G)$. Suppose $z_n = z_n^*$ on G where $z_n^* \in \Gamma^*$. A subsequence $z_q^* \dot{\rightarrow} z_0^*$ in $\mathfrak{B}_1$ on G and $z_0^* \in \Gamma^*$. By Theorem 2.4, the set functions $\int_e |\nabla z_q| \, dx$ are uniformly AC; the same is true of the set functions $\int_e |\nabla (z_q - z_q^*)| \, dx$. Since G is bounded and each $z_q - z_q^* = 0$ on ∂G, we see with the aid of Theorem 1.13 that a subsequence $z_r - z_r^* \rightarrow$ some w_0 in $\mathfrak{B}_1$ ond G and $w_0 = 0$ on ∂G. Accordingly $z_r \dot{\rightarrow} z_0 = z_0^* + w_0$ in $\mathfrak{B}_1$ on G and $z_0 \in F$. the theorem follows from the lower-semicontinuity of $I(z, G)$.

Somewhat more meaningful boundary value problems can be studied if we require G to be of class C' at least. We need the following preliminary lemma:

LEMMA 2.7: *Suppose G is bounded and of class C' and F is a family of functions of $\mathfrak{B}_\lambda$ on G such that*

$$\int_G |\nabla z|^\lambda \, dx \leq M, \quad z \in F.$$

Suppose that F satisfies one of the following additional conditions:

(i) *there is a number P and an open subset τ of G such that*

$$\int_\tau |z|^\lambda \, dx \leq P \text{ for all } z \in F; \text{ or}$$

(ii) *there is a number P and an open set σ of ∂G such that*

$$\int_\sigma |z|^\lambda \, dS \leq P \text{ for all } z \in F.$$

Then the $\mathfrak{B}_\lambda$ norms of the z in F are uniformly bounded.

Proof: We may cover $G \cup \partial G$ with a finite number of hypercubes or boundary neighborhoods $R_1, \dots, R_Q$; let φ_i map Q or Q^+ onto R_i as in the proof of Lemma 1 3. We may assume that one of the $R_i \subset \tau$ in case (i) or that $R_i \cap \partial G \subset \sigma$ in case (ii). In case (ii), we see using equation (1.7) with $y_1^\nu = 0$ that case (i) holds with $\tau = R_i$ and P replaced by P_i; here we have assumed that w_{i0} is equivalent to the transform under φ_i of the restriction of z to R_i.

Now, let $R_j \cap R_i$ be an open set τ_{ij}. For a given z, let w_{j0} be of class $\mathfrak{B}'_\lambda$ of Q or Q^+ and be equivalent to the transform under φ_j of the restriction of z to R_j. Thus there is a cell $R_{jQ} = [a, b]$ in Q or Q^+ such that case (i) holds with z replaced by w_{j0} and P by P_{j0} (independently of z in F). By using an equation like (1.7), we see in turn that case (i) holds with R_{j0} replaced $R_{j1}, R_{j2}, \dots, R_{j\nu} = Q$ or Q^+ with P replaced by $P_{j1}, \dots, P_{j\nu} = P'_j$ where R_{j2} is the cell $-1 \leq x^1 \leq 1$, $-1 \leq x^2 \leq 1$, $a^\alpha \leq x^\alpha \leq b^\alpha$ for $\alpha = 3, \dots, \nu$, etc. Thus case (i) holds with τ replaced by R_j and P by P_j. Since any R_k can be joined to the first R_i by a sequence R_n, each two adjacent members of which have an open set in common, the lemma follows.

We can now prove our second principal existence theorem:

THEOREM 2.6: *Suppose the domain G and the family F satisfy the conditions of Lemma 2.7 for some $\lambda \geq 1$ and hence for $\lambda = 1$ and suppose F contains some vector $\tilde{z}$ for which $I(\tilde{z}, G)$ is finite and suppose F is closed with respect to weak convergence in $\mathfrak{B}_1$. Suppose that $f(x, z, p)$ satisfies the conditions of Theorem 2.3 Then $I(z, G)$ takes on its minimum in F.*

Proof: Let $\{z_n\}$ be a minimizing sequence for which $I(z_n, G) \leq I(\tilde{z}, G)$. Then the set functions $\int_e z_{n,\alpha}\, dx$ are uniformly absolutely continuous on account of Theorem 2.4. Combining this with Theorem 1.15; we see that a subsequence $\{z_p\}$ can be selected which converges weakly on G in $\mathfrak{B}_1$ to some z_0 in $\mathfrak{B}_1$. Since F is closed with respect to weak convergence in $\mathfrak{B}_1$, $z_0 \in F$. The result follows from the lower semicontinuity of $I(z, G)$.

THEOREM 2.7: *Suppose G is of class C', $f(x, z, p)$ satisfies the hypotheses of Theorem 2.3, and Γ is a closed family of functions φ in $\mathfrak{L}_1$ on ∂G such that case* (ii) *of Lemma 2.7 holds. Suppose F is the family of all functions z in $\mathfrak{B}_1$ on G, each of which has boundary values in Γ and suppose F contains a function $\tilde{z}$ such that $I(\tilde{z}, G)$ is finite. Then $I(z, G)$ takes on its minimum in F.*

Proof: For the subfamily $\tilde{F}$ of z in F for which $I(z, G) \leq I(\tilde{z}, G)$ satisfies the conditions of Theorem 2.6, on account of Theorems 2.1, 2.3; and 1.15.

EXAMPLE: As an example of the use of Theorem 2.7, consider the problem of finding the surface $z = z(x)$ $(z = (z^1, z^2, z^3), x = (x^1, x^2))$ of least area of type of a disc bounded by a simple closed C consisting of a fixed arc C_1 which has only its end points on a surface S and a variable arc C_2 on S. Using theorems about conformal mapping this probleme can be reduced to that of minimizing the Dirichlet integral

$$I(z, G) = \int_G |\nabla z|^2 \, dx \left(\iint_G \sum_{i=1}^{3} (z^i_{x^1})^2 + (z^i_{x^2})^2 | \, dx^1 \, dx^2 \right)$$

among all vectors z of class $\mathfrak{B}_2''$ on G, where G is the unit circular disc, such that the restrictions of z to ∂G carry the upper semicircle of ∂G in a $1 - 1$ continuous way onto the fixed arc C_1 with (0,1) corresponding to some fixed point on C_1 and carry the lower part of ∂G in a $1 - 1$ continuous way onto the variable arc C_2. In order to apply Theorem 2.7, we let Γ consist of all strong limits in $\mathcal{L}_2$ on ∂G of the restrictions of such z to ∂G. Any vector φ in Γ is equivalent along the upper part of ∂G to a vector which carries that part of ∂G in a «monotone» way onto C_1 in which arcs of C_1 may correspond to points on ∂G; for almost all x on the lower part of ∂G, $\varphi(x) \in S$ at any rate. Since any minimizing vector z_0 certainly minimizes $I(z, G)$ among all z in $\mathfrak{B}_2$ which coincide with z_0 on ∂G in the $\mathfrak{B}_2$ sense, we see that z_0 is harmonic (see Professor Nirenberg's lectures). By arguments like those in [7] and [43], we conclude that z_0 is continuous on the upper half of ∂G and yields a conformal map of G onto the surface represented by z_0. However, an example of Courant [8] (p. 220, 221), shows that z_0 need not be continuous along the lower half of ∂G and that the limiting «curve» C_2 need not be an arc even if the surface S is regular and of class C^∞; Lewy [33] has shown that if S is analytic, the curve C_2 is analytic.

CHAPTER III

Quasi-convexity and lower-semicontinuity.

In the preceding chapter, we proved theorems concerning the lower-semicontinuity of multiple integrals $I(z, G)$ in cases where the integral function $f(x, z, p)$ is continuons and convex in p for each (x, z). This restriction on f was a natural extension to the case of several unknown functions of the ordinary requirement when $N = 1$ that the variational problem be regular or at least that Hadamard's condition

$$f_{p_\alpha p_\beta}(x, z p)\, \lambda_\alpha \lambda_\beta \geq 0 \qquad \text{for all } (x, z, p, \lambda) \tag{3.1}$$

be satisfied, f being assumed of class C''. But (3.1) holds if and only if f is convex in $(p_1, \ldots, p_\nu)$ for each $(x^1, \ldots, x^\nu, z)$.

The condition (3.1) is arrived at as follows: Suppose a function $z_0(x)$ of class C' minimizes $I(z, G)$ among all functions of z of class C' which have the same boundary values and which are near z_0 in the sense that the maximum of $| z(x) - z_0(x) | + | \nabla z(x) - \nabla z_0(x) | \leq \delta$ for some $\delta > 0$. Then it can be shown that (3.1) holds for x on G, $z = z_0(x)$, and $p = \nabla z_0(x)$. However, if this procedure is applied in the case where $N > 1$, we obtain only the condition

$$f_{p^j_\alpha p^k_\beta}(x, z, p)\, \lambda_\alpha \lambda_\beta \xi^j \xi^k \geq 0 \tag{3.2}$$

for all (x, z, p) (along the solution $z = z_0(x)$, etc.) and all $(\lambda_1, \ldots, \lambda_\nu)$ and $(\xi^1, \ldots, \xi^N)$ (see Theorem 3.3 below). This does not imply that $f(x, z, p)$ is convex in p. Moreover, it is known that integrals $I(z, G)$ which arise in parametric problems are lower semi-continuous with respect to uniform convergence; for the case of the parametric problem for surfaces in 3-space $(\nu = 2, N = 3)$, these integrands have the form

$$f(x, z, p) = F(x, z, J_1, J_2, J_3)$$

where

$$J_1 = p^2_1 p^3_2 - p^3_1 p^2_2, \quad J_2 = p^3_1 p^1_2 - p^1_1 p^3_2, \quad J_3 = p^1_1 p^2_2 - p^2_1 p^1_2$$

and F is convex in (J_1, J_2, J_3), but not in the six p^i_α.

It turns out to be rather easy to derive (see also [44]) a certain necessary and sufficient condition on f as a function of p for the lower semicontinuity of $I(z, G)$ with respect to a certain type of convergence. This question was considered for $\nu = N = 1$ by Tonelli ([72], [73], [74], [75]) and by Cesari and others for the parametric case. We begin by deriving this condition and then discuss the relation of that condition to the condition (3.2). In order not to get involved with the behavior of f at infinity we shall use the following convergence which obviously implies weak convergence in each $\mathcal{B}_\lambda$ but does not necessarily imply strong convergence in any $\mathcal{B}_\lambda$:

DEFINITION: We say that $z_n \to z$ on $G \longleftrightarrow z_n(x)$ converges uniformly to $z(x)$ on G and z and z_n each satisfy a uniform Lipschitz condition on G which is independent of n.

THEOREM 3.1: *Suppose $I(z, G)$ is lower-semicontinuous with respect to this type of convergence at any z on any G and f is continuous. Then.*

$$\int_0 f[x_0, z_0, p_0 + \nabla \zeta(x)]\, dx \geq f(x_0, z_0, p_0) \cdot m(G) \tag{3.3}$$

for any constant (x_0, z_0, p_0), any bounded domain G, and any Lipschitz vector ζ which vanishes on ∂G.

Proof: Let x_0 be any point, R be the cell $x_0^\alpha \leq x^\alpha \leq x_0^\alpha + h$, z_0 be any vector of class C' on $R \cup \partial R$, Q be the cell $0 \leq x^\alpha \leq 1$, and ζ be any vector which satisfies a uniform Lipschitz condition over the whole space and is periodic of period 1 in each x^α.

For each n, define $\zeta_n(x)$ on R by

$$\zeta_n^j(x) = n^{-1} h\, \zeta^j [nh^{-1}(x - x_0)].$$

Then the ζ_n^j tend to zero in our sense. Then, for each $n, I(z_0 + \zeta_n, R)$ can be written as a sum of integrals over the sub-hypercubes of R of side $n^{-1} h$. If r is one these the integral over it is

$$n^{-\nu} h^\nu \int_Q f[x_1 + n^{-1} h\, \xi, z_n(x_1 + n^{-1} h\, \xi), p_0(x_1 + n^{-1} h\, \xi) + \nabla \zeta(\xi)]\, d\xi,$$

where

$$r: \; x_1^\alpha \leq x^\alpha \leq x_1^\alpha + n^{-1} h, \; x_1^\alpha = x_0^\alpha + k^\alpha n^{-1} h, \; 0 \leq k^\alpha \leq n - 1$$

$$z_n(x) = z_0(x) + \zeta_n(x), \; x^\alpha = x_1^\alpha + n^{-1} h\, \xi^\alpha, \; 0 \leq \alpha \leq 1.$$

Thus we see that

$$\lim_{n\to\infty} I(z_0 + \zeta_n, R) = \int_R \left\{ \int_Q f[x, z_0(x), p_0(x) + \nabla \zeta(\xi)] \, d\xi \right\} dx \geq I(z_0, R).$$

By letting z_0 and p_0 be arbitrary constant verctors, setting $z_0(x) = z_0 + p_{0\alpha} \cdot \cdot (x^\alpha - x_0^\alpha)$, dividing by $m(R) = h^\nu$ and letting $h \to 0$, we obtain (3·3) for $G = Q$ and ζ periodic of period 1 in each x^α. But if G is any bounded domain and ζ vanishes on ∂G, we may choose a hypercube Q' containing G and extend $\zeta(x)$ to be zero in $Q' - G$. Then a simple change of variable obtains the result in general.

DEFINITION: If f is continuous in (x, z, p) for all (x, z, p) and satisfies (3.3) for all (x_0, z_0, p_0), we say that f is *quasi-convex in* p; if f depends only on p and satisfies (3.3), we say simply that f is quasi-convex.

We now prove that the condition (3.3) is sufficient for lower-semicontinuity.

LEMMA 3.1: *Suppose R is the hypercube $|x^\alpha - x_0^\alpha| \leq h$, $f(p)$ is quasi-convex, suppose p_0 is any constant tensor and suppose $\zeta_n \to 0$ in our sense or R. Then*

$$\liminf_{n\to\infty} \int_R f[p_0 + \nabla \zeta_n(x)] \, dx \geq f(p_0) \cdot m(R).$$

Proof: Suppose the ζ_n satisfy a uniform Lipschitz condition with constant M on R. We may assume that $|\zeta_n(x)| \leq M k_n h$ where each $k_n < 1/2$ and $\lim k_n = 0$. For each n, we begin by defining $\eta_n(x) = \zeta_n(x)$ on ∂R and $\eta_n(x) = 0$ for $|x^\alpha - x_0^\alpha| \leq (1 - k_n) h$; we then extend each η_n to the whole of R to satisfy a Lipschitz condition with constant $\leq M$. Then $\eta_n \to 0$, $\zeta_n - \eta_n \to 0$, $\zeta_n(x) - \eta_n(x) = 0$ on ∂R, and $\eta^j_{n,\alpha}(x) \to 0$ for each x interior to R. Henee

$$\lim_{n\to\infty} \int_R |f[p_0 + \nabla \zeta_n] - f[p_0 + \nabla (\zeta_n - \eta_n)]| \, dx = 0.$$

The result follows easily from the quasi-convexity of f.

THEOREM 3.2: *Suppose $f(x, z, p)$ is quasi-convex iu p, G is a bounded domain, and $z_n \to z_0$ on G. Then*

$$I(z_0, G) \leq \liminf_{n\to\infty} I(z_n, G).$$

Proof: Since all the arguments $[x, z_n(x), \nabla z_n(x)]$ and $[x, z_0(x), \nabla z_0(x)]$ remain in a bounded part $\mathcal{G}$ of (x, y, p)-space and since G is the union of $\mathcal{H}_0$ disjoint hypercubes, it is sufficient to prove this for the case of a hypercube R of side h. Since f is uniformly continuous on $\mathcal{G}$, there is a function $\varepsilon(\varrho)$ with $\lim_{\varrho \to 0} \varepsilon(\varrho) = 0$ such that

$$|f(x', z', p') - f(x'', z'', p'') \leq \varepsilon(\varrho) \quad \text{if} \quad |x' - x''|^2 + |z' - z''|^2 + |p' - p''|^2 \leq \varrho^2.$$

For each k, divide R up into $2^{\nu k}$ hypercubes R_{ki} of side $2^{-k} \cdot h$. Define the functions $x_k^*(x)$, $z_k^*(x)$, $p_k^*(x)$ on R to be equal on each R_{ki} to the averages over R_{ki} of x, $z_0(x)$, and $p_0(x)$ respectively, and define

$$r_k(x) = \{ |x - x_k^*(x)|^2 + |z_0(x) - z_k^*(x)|^2 + |p_0(x) - p_k^*(x)|^2 \}^{1/2}.$$

$$\zeta_n(x) = z_n(x) - z_0(x).$$

Then

$$f[x, z_n(x), \nabla z_n(x)] - f[x, z_0(x), \nabla z_0(x)] = A_n + B_{nk} - C_k + D_{nk} \tag{3.4}$$

where

$$\begin{aligned} A_n &= f[x, z_n(x), p_n(x)] - f[x, z_0(x), p_n(x)]; \quad (p_n(x) = \nabla z_n(x)) \\ B_{nk} &= f[x, z_0(x), p_0(x) + \pi_n(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x) + \pi_n(x)] \\ C_k &= f[x, z_0(x), p_0(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x)]; \quad (\pi_n(x) = \nabla \zeta_n(x)) \\ D_{nk} &= f[x_k^*(x), z_k^*(x), p_k^*(x) + \pi_n(x)] - f[x_k^*(x), z_k^*(x), p_k^*(x)]. \end{aligned} \tag{3.5}$$

We see that

$$\begin{aligned} &|A_n| \leq \varepsilon(|z_n(x) - z_0(x)|) \\ &|B_{nk}|, \quad |C_k| \leq \varepsilon[r_k(x)] \end{aligned} \tag{3.6}$$

and $I(z_n, R) - I(z_0, R) = J_n + K_{nk} + L_k + P_{nk}$, where these are the integrals of A_n, B_{nk}, C_k, and D_{kn}, respectively. Now, let $\varepsilon > 0$. We first choose a fixed k such that K_{nk} and L_k are both $< \varepsilon/2$. From (3.4), (3.5), (3.6), and Lemma 3.1, we see that

$$\lim_{n \to \infty} J_n = 0, \quad \liminf_{n \to \infty} P_{nk} \geq 0$$

since $x_k^*(x)$, $z_k^*(x)$, and $p_k^*(x)$ are each constant on each R_{ki}. Thus

$$\liminf_{n\to\infty}[I(z_n, R) - I(z_0, R)] \geq -\varepsilon .$$

Some of the theory of Chapter 2 can be carried over for the more general functions $f(x, z, p)$ which are quasi-convex in p but more has to be assumed about how f behaves as $p \to \infty$. These theorems are not of great interest and they can be found in [44].

We now investigate the concept of quasi-convexity in more detail.

LEMMA 3.2 [79], [45]: *Suppose $a_{jk}^{\alpha\beta}$ are constants and*

$$\int_G a_{jk}^{\alpha\beta} \zeta_{,\alpha}^j(x)\, \zeta_{,\beta}^k(x)\, dx \geq 0$$

for all ζ in $\mathcal{B}_{20}$ on domain G, then

$$a_{jk}^{\alpha\beta} \lambda_\alpha \lambda_\beta \xi^j \xi^k \geq 0 \text{ for all } \lambda \text{ and } \xi. \tag{3.7}$$

Proof: Let λ^1 be a unit vector with $\lambda_\alpha^1 = \lambda_\alpha$ and choose $\lambda^2, \ldots, \lambda^\nu$ so $(\lambda^1, \ldots, \lambda^\nu)$ form a normal orthogonal set. Suppose $x_0 \in G$ and let $y^\gamma = \lambda_\alpha^\gamma \cdot (x^\alpha - x_0^\alpha)$. Choose h_0 and $R > 0$ so that the set of all x for which $|y^1| \leq h_0$ and $|y_1'| \leq R$ is in G. Let ξ be an arbitrary vector and define

$$\zeta_h^j(x) = \xi^j \varphi_h(y^1) \cdot \psi(|y_1'|),$$

where

$$\varphi_h(y^1) = h - |y^1| \text{ if } |y^1| \leq h, \quad \psi(r) = R - r \text{ if } 0 \leq r \leq R$$

and φ_h and $\psi(|y_1'|) = 0$ otherwise. Then it is easy to see that

$$\lim_{h\to 0} (2h)^{-1} \cdot I(\zeta_h, G) \doteq \Gamma_{\nu-1} R^\nu a_{jk}^{\alpha\beta} \lambda_\alpha \lambda_\beta \xi^j \xi^k / \nu(\nu - 1) \geq 0$$

which proves the lemma-

We now prove the theorem mentioned in the introduction to this chapter.

THEOREM 3.3: *Suppose $f(x, z, p)$ is of class C'' for all (x, z, p) near the locus S of all points $[x, z, (x), \nabla z_0(x)]$ for x in G and suppose $z_0(x)$ is of class C' on $G \cup \partial G$ and minimizes $I(z, G)$ among all Lipschitz z which coincide with z_0 on ∂G and are such that $|z(x) - z_0(x)| + |\nabla z(x) - \nabla z_0(x)| \leq \delta$ for some $\delta > 0$. Then* (3.2) *holds for all (x, z, p) on S.*

Proof: For, let ζ be any Lipschitz function vanishing vanishing on and near ∂G. Then $z_0 + \lambda\zeta$ is sufficiently near z_0 for all sufficiently small

λ. So if $\varphi(\lambda) = I(z_0 + \lambda\zeta)$, we must have

$$\varphi''(0) = \int_G f_{p_\alpha^j p_\beta^k}[x, z_0(x), p_0(x)]\, \zeta^j_{,\alpha}\, \zeta^k_{,\beta}\, dx \geq 0$$

By selecting any point x_0 in G and proceeding as in the proof of Lemma 3.2 and then dividing by $[\Gamma_\nu R^\nu/\nu(\nu-1)]$, but letting R and h both $\to 0$ so that $h : R \to 0$, we obtain (3.2) at $[x_0, z(x_0), p(x_0)]$.

Using the result of Lemma 3.2 and the method of proof of Theorem 3.3, we conclude that if $f(p)$ is quasi-convex and of class C'', then (3.2) holds with x and z omitted. This result and the analogy with convex functions suggest the following theorem whichwe now prove.

THEOREM 3.4: *If $f(p)$ is quasi convex, then $f(p^j_\alpha + \lambda_\alpha \xi^j)$ is convex in λ for each p and ξ and convex in ξ for each p and λ.*

Proof: If f is quasi-convex, it is easy to see that its twiceiterated h-average function f_{hh} is also quasi-convex and is of class C'' as well. Then any linear function furnishes an absolute minimum to $I_{hh}(z, G)$ among all Lipschitz functions with the same boundary values. Accordingly, by Theorem 3.3 we see that f_{hh} satisfies (3.2). But then f_{hh} has the convexity properties stated in the theorem. Since f_{hh} converges uniformly to f on any bounded part of space, the theorem follows.

DEFINITION: A function $f(p)$ which satisfies the conditions in Theorem 3.4 is said to be weakly *quasi-convex.*

REMARK : The principal problem, so far unsolved, is whether or not every weakly quasi-convex function is quasi-convex.

THEOREM 3.5: *If $f(p)$ is weakly quasi-convex , it satisfies a uniform Lipschitz condition on a bounded part of space. If p is given, there are constants A^α_j such that*

$$f(p^j_\alpha + \lambda_\alpha \xi^j) \geq f(p^j_\alpha) + A^\alpha_j \lambda_\alpha \xi^j \text{ for all } \lambda, \xi. \tag{3.8}$$

If f is also of class C', then $A^\alpha_j = f_{p^j_\alpha}(p)$. If f is also of class C'' then (3.2) *holds. If f is continuous and if, for each p, constants A^α_j exist such that* (3.8) *holds, then f is weakly quasi-convex.*

Proof: If f is weakly quasi-convex, it is convex in each p^j_α separately. Hence, if $|f(p)| \leq M$ on some hypercube, any difference quotient of the form :

$$|[f(p^j_{2\alpha}) - f(p^j_{1\alpha})]/(p^j_{2\alpha} - p^j_{1\alpha})| \leq 2M/d,\ p^j_{1\alpha} < p^{j\times}_{2\alpha}$$

where d is the smaller of $b^j_\alpha - p^j_{2\alpha}$ and $p^j_\alpha - a^j_\alpha$.

Next, f_{hh} is still weakly quasi-convex and of class C'' so that (3.2) holds. Then, from the convexity in ξ for each λ, for instance, (3.8) holds with $A^{\alpha}_{hh} = f_{hhp^i_\alpha}(p)$. Since f satisfies a uniform Lipschitz condition near p, we see that the A^{α}_{hhj} are uniformly bounded as $h \to 0$ so a sequence of $h \to 0$ can be chosen so that all the A^{α}_{hhj} tend to limits. Clearly (3.8) holds in the limit. Since the unit vector in the p^j_α direction is of form $\lambda_\alpha \xi^j$, we see that $A^\alpha_j = f_{p^j_\alpha}$ if f is of class C'. The last statement follows from theorems on convex functions.

We now define a sufficient condition for f to be (strongly) quasi-convex.

THEOREM 3.6 : *A sufficient condition for f to be quasi-convex is that for each p there exist alternating forms*

$$A^{\alpha_1 \cdots \alpha_\mu}_{j_1 \cdots j_\mu} \pi^{j_1}_{\alpha_1} \cdots \pi^{j_\mu}_{\alpha_\mu}, \qquad \mu = 1, \ldots, \nu$$

(in which the coefficients are 0 unless all the $\alpha_1 \ldots \alpha_\mu$ are distinct and all the $j_1 \ldots j_\mu$ are distinct and an interchange of two α's or two j's changes the sign) *such that for all π we have*

$$f(p + \pi) \geq f(p) + \sum_{\mu=1}^{\nu} A^{\alpha_1 \cdots \alpha_\mu}_{j_1 \cdots j_\mu} \pi^{j_1}_{\alpha_1} \cdots \pi^{j_\mu}_{\alpha_\mu}. \tag{3.9}$$

Proof: For suppose p is any constant tensor, G is any bounded domain, and ζ is any Lipschitz vector which vanishes on ∂G. By extending $\zeta = 0$ outside G and approximating to it on a larger domain D with smooth boundary with functions of class C'' which vanish on and near ∂D and using Stokes' theorem we see that the integral of the *sum* on the right in (3.9) is zero. We now exhibit two interesting cases where the weak quasi-convexity of f implies its quasi-convexity.

THEOREM 3.7 : *If $f(p)$ is weakly quasi-convex and*

$$f(p) = a^{\alpha\beta}_{jk} p^j_\alpha p^k_\beta$$

then f is quasi-convex ([79], [45]).

Proof: For, if ζ is Lipschitz and vanishes on ∂G (which may as well be assumed smooth), then

$$\int_G f[p + \nabla \zeta(x)]\, dx = f(p) \cdot m(G) + \int_G a^{\alpha\beta}_{jk} \zeta^j_{,\alpha}(x) \zeta^k_{,\beta}(x)\, dx$$

If we introduce Fourier transforms (see [79])

$$Z^i(y) = (2\pi)^{-\nu/2} \int_G e^{iy^\alpha x^\alpha} \zeta^j(x)\, dx$$

we see that

$$\int_G a_{jk}^{\alpha\beta} \zeta^j_{,\alpha} \zeta^k_{,\beta} dx = \int_{-\infty}^{\infty} a_{jk}^{\alpha\beta} y^\alpha y^\beta Z^j(y) \overline{z}^k(y)\, dy \geq 0$$

since the integrand is ≥ 0 for each y.

THEOREM 3.8: *If* $N = \nu + 1$ *and*

$$f(p) = F(X_1, \dots, X_{\nu+1})$$

where F *is continuous and*

$$X_j = -\det M_j \, (j = 1, \dots, \nu), \quad X_{\nu+1} = \det M_{\nu+1}$$

$$M_{\nu+1} = \| p^1_\alpha, \dots, p^\nu_\alpha \|, \quad M_j = \| p^1_\alpha, \dots, p^{j-1}_\alpha p^{\nu+1}_\alpha, p^{j+1}_\alpha, \dots, p^\nu_\alpha \| .$$

Then f *is quasi-convex in* p *if and only if* F *is convex in* $(X_1, \dots, X_{\nu+1})$.

We omit the proof which is found in [44]; F is there required to be homogeneous of the first degree in X but this is not necessary in the proof.

Chapter IV

The differentiability of the solutions of certain variational problems with $\nu = 2$.

In this chapter we discuss the differentiability of the solutions of certain problems whose existence was proved in § 2. To save time, we shall not discuss the continuity on the boundary but shall consider only the differentiability on the interior. This work was first presented in [42], chapters 4,6, and 7 and was the culmination of a series of papers on this subject by Lichtenstein [34], [35], Hopf [27], and the writer [39]. Some of these results have recently been generalized by De Giorgi [10] and Nash [49]. Sigalov [61] announced results similar to those presented here.

We begin with the following lemma which has a proper generalization for all values of ν (see [42] and [47]):

LEMMA 4.1: *Suppose a vector $z(x) \in \mathfrak{B}_2$ on a domain G and suppose that*

$$\int_{B(x_0,r)} |\nabla z|^2 \, dx \leq L^2 (r/a)^{2\lambda} \qquad \text{for} \quad 0 \leq r \leq a, \tag{4.1}$$

whenever $B(x_0, a) \subset G$. Then

$$|z(x_2) - z(x_1)| \leq C_1(\lambda) \cdot L \cdot (|x_1 - x_2|/a)^{\lambda} \quad \text{for} \quad 0 \leq |x_1 - x_2| \leq a, \tag{4.2}$$

where

$$C_1(\lambda) = 2^{1-\lambda} \pi^{-1/2} \lambda^{-1}$$

for every pair of points (x_1, x_2) in G such that every point on the segment joining them is at a distance $\geq a$ from ∂G.

Proof: We note first that if ξ is on the segment and $s \leq a$,

$$\int_{B(\xi,s)} |\nabla z(y)| \, dy \leq \pi^{1/2} L a^{-\lambda} s^{1+\lambda},$$

using the Schwarz inequality. Next we write

$$| z(x_2) - z(x_1) | \leq | z(x) - z(x_1) | + | z(x) - z(z_2) |$$

$$| z(x) - z(x_k) | = | (x^a - x_k^a) \int_0^1 z_{,a} [x_k + t(x - x_k)] \, dt |$$

$$\leq r \int_0^1 | \nabla z [x_k + t(x - x_k)] | \, dt , \; r = | x_2 - x_1 | , \; k = 1, 2 ,$$

and then average with respect to x over $B(\bar{x}, r/2)$, $\bar{x} = (x_1 + x_2)/2$. If for a given t, $0 < t < 1$, we set $y = x_k + t(x - x_k)$, then y ranges over $B[(1 - t)x_k + t\bar{x}, rt/2]$. Then

$$\int_{B(x_0, r)} | z(x) - z(x_k) | \, dx \leq r \int_0^1 t^{-2} \left[\int | \nabla z(y) | \, dy \right] dt$$

from which the result follows.

NOTATION: If $z \in \mathcal{B}_2$ on G, we define $D(z, G) = \int_G | \nabla z |^2 \, dx$; this is called the *Dirichlet integral.*

LEMMA 4.2: *Suppose $z \in \mathcal{B}_2$ on $B(x_0, a)$ and suppose*

$$D[z, B(x_0, r)] \leq K \cdot D[Z_r, B(x_0, r)] + \psi(r), \; 0 < r \leq a \tag{4.3}$$

where

$$\int_0^a s^{-1} \psi(s) \, ds$$

converges, for every function $Z_r = z$ on $\partial B(x_0, r)$. Then

$$D[z, B(x_0 r)] \leq D[z, B(x_0, a)] (r/a)^{1/K} + K^{-1} r^{1/K} \int_0^a \varrho^{-1 - \frac{1}{K}} \psi(\varrho) \, d\varrho \tag{4.4}$$

and the right side tends to zero with r.

Proof: Let $\varphi(r) = D[z, B(x_0, r)]$. Then φ is absolutely continuous. For almost all r, $z(r, \theta)$ is AC in θ with $|z_\theta(r, \theta)|$ in $\mathcal{L}_2$. For such r, define

$$Z_r(\varrho, \theta) = \bar{z}(r) + (\varrho/r)[z(r, \theta) - \bar{z}(r)], \bar{z}(r) = \frac{1}{2\pi}\int_0^{2\pi} z(r, \theta)\, d\theta .$$

Using Fourier series, one easily sees that

$$\int_0^{2\pi} |z(r, \theta) - \bar{z}(r)|^2 d\theta \leq \int_0^{2\pi} |z_\theta(r, \theta)|^2 d\theta \leq r\varphi'(r) \tag{4.5}$$

By computing $D_2[Z_r, B(x_0, r)]$ and using (4.5) we see that

$$\varphi(r) \leq Kr\varphi'(r) + \psi(r) \tag{4.6}$$

from which (4.4) follows easily. In order to see that the right side of (4.4) tends to zero with r, we note that

$$r^{1/K}\int_r^a \varrho^{-1-1/K}\psi(\varrho)\, d\varrho \leq \int_r^{r^{1/2}} \varrho^{-1}\varphi(\varrho)\, d\varrho + r^{1/2K}\int_{r^{1/2}}^a \varrho^{-1}\varphi(\varrho)\, d\varrho .$$

THEOREM 4.1: *Suppose $f(x, z, p)$ is continuous for all (x, z, p) and is convex in p for each (x, z), and suppose there are constants m, M, and k such that*

$$m|p|^2 - k \leq f(x, z, p) \leq M|p|^2 + k, M \geq m \geq 0, \tag{4.7}$$

for all p. Suppose $I(z_0, G)$ is finite, G is a bounded domain, and z_0 minimizes $I(z, G)$ among all z in $\mathcal{B}_2$ coinciding with z_0 on ∂G. Then z_0 satisfies (4.1) *and* (4.2) *on G with*

$$\lambda = m/2M \text{ and } L^2 = D[z_0 B(x_0, a)] + 2k\pi a^2/M . \tag{4.8}$$

Thus z_0 satisfies a uniform Hölder condition on each compact subset of G.

Proof: Suppose $\overline{B(x_0, r)} \subset G$ and let Z_r be any function in $\mathcal{B}_2$ on $B(x_0, r)$ and coinciding with z_0 on $\partial B(x_0, r)$. Then, from (4.7)

$$mD[z_0, B_r] - k\pi r^2 \leq I(z_0, B_r) \leq I(Z_r, B_r) \leq MD(Z_r, B_r) + k\pi r^2$$

$$D(z_0, B_r) \leq \frac{M}{m} D(Z_r, B_r) + \frac{2k\pi}{m} r^2 \qquad (B_r = B(x_0, r)) .$$

The result follows from Lemma 4.2.

For the remainder of this section, we shall assume that $f(x, z, p)$ satisfies the following condition in addition to (4.7):

GENERAL ASSUMPTIONS: *We assume that G is a bounded domain, f satisfies the conditions of Theorem 4.1, and*

(i) *f is of class C'' for all (x, z, p)*

(ii) *there are functions $m_1(R), M_1(R)$, and $M_2(R)$ with $0 < m_1(R) \leq M_1(R)$ for all $R \geq 0$ such that*

$$m_1(R)\,|\pi|^2 \leq f_{p^j_\alpha p^k_\beta}\,\pi^j_\alpha\,\pi^k_\beta \leq M_1(R)\,|\pi|^2 \tag{4.9}$$

$$\sum_{j=1}^{N}\left\{\sum_{k=1}^{N}\left[|f_{z^j z^k}| + \sum_{\alpha=1}^{2} f^2_{p^j_\alpha z^k}\right] + \sum_{\alpha=1}^{2}\left[|f_{z^j x^\alpha} + \sum_{\beta=1}^{2} f^2_{p^j_\alpha x^\beta}\right]\right\} \leq M_2(R)\cdot|p|^2 \tag{4.10}$$

for all (x, z, p) such that $|x|^2 + |z|^2 \leq R^2$.

THEOREM 4.2: *Suppose f and G satisfy the general assumptions, z_0 satisfies the continuity conclusions of Theorem 4.1, and ζ is any Lipschitz function on G which vanishes on and near ∂G, and $\varphi(\lambda) = I(z_0 + \lambda\zeta)$. Then $\varphi'(0)$ exists and*

$$\varphi'(0) = \int_G \{f_{z^j}[x, z_0(x), p_0(x)]\,\zeta^j(x) + f_{p^j_\alpha}[x, z_0(x), p_\alpha(x)]\,\pi^j_\alpha\}\,dx \tag{4.11}$$

Proof: Let F be the compact support of ζ. Since z_0 is continuous on F, $|x|^2 + |z_0(x)|^2 \leq R^2$, for some R, for all x on F. Then, for almost all x on F,

$$f[x, z_0(x) + \lambda\zeta(x), p_0(x) + \lambda\pi(x)] = f[x, z_0(x), p_0(x)] + \lambda\,[f_{z^j}\zeta^j + f_{p^j_\alpha}\pi^j_\alpha] + \lambda^2\{A^{\alpha\beta}_{jk}(x, \lambda)\,\pi^j_\alpha\,\pi^k_\beta + 2B^\alpha_{jk}(x, \lambda)\,\pi^j_\alpha\,\zeta^k + C_{jk}(x, \lambda)\,\zeta^j\,\zeta^k\}$$

where, for instance,

$$A^{\alpha\beta}_{jk}(x, \lambda) = \int_0^1 (1 - t)\,f_{p^j_\alpha p^k_\beta}[x, z_0(x) + t\lambda\zeta(x), p_0(x) + t\lambda\,\pi(x)]\,dt.$$

Clearly all the $A^{\alpha\beta}_{jk}$, B^α_{jk}, and C_{jk} are measurable and we conclude also from the general assumptions and the Lipschitz character of ζ that

$$\varphi(\lambda) - \varphi(0) - \lambda\int_G (f_{z^j}\zeta^j + f_{p^j_\alpha}\pi^j_\alpha)\,dx = \lambda^2 K(\lambda) \tag{4.12}$$

where $K(\lambda)$ is uniformly bounded for $|\lambda| \leq 1$. The result follows.

DEFINITION: If $\varphi(0) = 0$ for every ζ as in Theorem 4.2, we say that z_0 furnishes a *stationary value* to the integral $I(z, G)$.

COROLLARY: *If f, G, and z_0 satisfy the conditions of Theorem 4.2 and if z_0 minimizes $I(z, G)$ among all sufficiently near z ($\mathfrak{B}_2$ sense) having the same boundary values, then z_0 furnishes a stationary value to $I(z, G)$.*

In order to obtain further differentiability properties of the solutions z_0, we must consider the solutions u of equations

$$(4.13)\quad \int_G [v^j_{,\alpha}(a^{\alpha\beta}_{jk} u^k_{,\beta} + b^\alpha_{jk} u^k + e^\alpha_j) + v^j(b^\alpha_{kj} u^k_{,\alpha} + c_{jk} u^k + f_j)]\, dx = 0, \quad v \in \mathfrak{B}_{20}$$

where all the coefficients are measurable and satisfy

$$(4.14)\qquad m_1 |\pi|^2 \leq a^{\alpha\beta}_{jk}(x)\, \pi^j_\alpha \pi^k_\beta \leq M_1 |\pi|^2 \qquad \text{for all } \pi,$$

$$a^{\beta\alpha}_{kj} = a^{\alpha\beta}_{jk}, \quad x \in G$$

$$(4.15)\qquad \int_{B(x_0,s)\cap G} (|b|^2 + |c| + |f|)\, dx \leq M_2^2 r^{2\lambda}, \quad e \in L_2, \quad 0 < m_1 \leq M_1.$$

We begin by considering the case where $b^\alpha_{jk} = c_{jk} = 0$ and set

$$I_0(u, v; G) = \int_G v^j_\alpha a^{\alpha\beta}_{jk} u^k_{,\beta}\, dx.$$

From our general assumptions, we see that

$$(4.16)\qquad m_1 \int_G |\nabla u|^2\, dx \leq I_0(u, u; G) \leq M_1 \int_G |\nabla u|^2\, dx.$$

From this result and the Poincare inequality (Theorem 1.11), we see that the space $\mathfrak{B}_{20}$ is a Hilbert space if we take $I_0(u, v; G)$ as an inner product and that the resulting norm is topologically equivalent to the original $\mathfrak{B}_2$ norm on $\mathfrak{B}_{20}$.

LEMMA 4.3: *If S is any set of finite measure, then*

$$\int_S |x - x_0|^{h-2}\, dx \leq 2\pi \cdot h^{-1} s^h, \quad 0 < h < 2, \quad \pi s^2 = m(S).$$

Proof: Obviously $\int_S |x - x_0|^{h-2}\, dx \leq \int_{B(x_0,s)} |x - x_0|^{h-2}\, dx$.

LEMMA 4.4: *Suppose* $u \in \mathfrak{B}_{20}$ *on* G, $f \in \mathfrak{L}_1$ *on* G, *and*

$$\int_{B(x_0,r)\cap G} |f(x)|\,dx \leq L r^{2\lambda}$$

for every circle $B(x_0, r)$. *Then* $u \cdot f \in \mathfrak{L}_1$ *on* G *and satisfies*

$$\int_{B(x_0,r)\cap G} |f(x)\cdot u(x)|\,dx \leq C_1(\lambda, \mu)\cdot L \cdot \|\nabla u\|_{L_2}\cdot g^{\mu} r^{2\lambda-\mu},\ 0<\mu<\lambda,$$

$$C_1(\lambda, \mu) = 2^{-1}\pi^{-1/2}\lambda^{1/2}\mu^{-1/2}(\lambda-\mu)^{-1/2},\ \pi g^2 = m(G);$$

u and f may be tensors.

Proof: The proof for the general vector u in $\mathfrak{B}_{20}$ will follow from the result for class C' which vanishes near ∂G. Let $x_1 \in G$ and suppose $\overline{G} \subset B(x_1, R)$ and extend u to be zero outside G. Then if we set $v^j(r, \theta) = u^j(x_1^1 + r\cos\theta, x_1^2 + r\sin\theta)$, we see that

$$\begin{aligned} u^j(x_1) = v^j(0, \theta) &= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^R v^j_{,r}(r, \theta)\,dx\,d\theta \\ &= -\frac{1}{2\pi}\int_G |\xi - x_1|^{-2}(\xi^\alpha - x_1^\alpha)\,u^j_{,\alpha}(\xi)\,d\xi. \end{aligned} \tag{4.17}$$

Hence

$$\begin{aligned} \int_{B(x_0,r)\cap G} |f(x)\cdot u(x)|\,dx &\leq \\ &\leq \frac{1}{2\pi}\int_{B(x_0,r)\cap G}\int_G |f(x)|\cdot|\xi - x|^{-1}\cdot|\nabla u(\xi)|\,d\xi\,dx. \end{aligned} \tag{4.18}$$

Applying the Schwarz inequality judiciously to (4.18), we obtain

$$\begin{aligned} \int_{B(x_0,r)\cap G} |f(x)\cdot u(x)|\,dx &\leq \frac{1}{2\pi}\left[\int_{B(x_0,r)\cap G}\int_G |\xi - x|^{2\mu-2}\cdot|f(x)|\,dx\,d\xi\right]^{1/2} \\ &\left[\int_{B(x_0,r)\cap G}\int_G |f(x)|\cdot|\xi - x|^{-2\mu}\cdot|\nabla u(\xi)|^2\,dx\,d\xi\right]^{1/2}. \end{aligned} \tag{4.19}$$

Using Lemma 4.3 we see that

$$(4.20)\qquad \int\limits_{B(x_0,r)\cap G}\int\limits_{G} |\xi - x|^{2\mu-2}\, |f(x)|\, dx\, d\xi \le \pi\mu^{-1}\, g^{2\mu}\cdot Lr^{2\lambda}.$$

Next, define,

$$\varphi_\xi(\varrho) = \int\limits_{B(\xi,\varrho)\cap B(x_0,r)\cap G} |f(x)|\, dx.$$

From our assumption on f, we see that

$$\varphi_\xi(\varrho) \le L\varrho^{2\lambda} \qquad \text{and } Lr^{2\lambda}.$$

Accordingly

$$\int\limits_{B(x_0,r)\cap G} |\xi - x|^{-2\mu}\, |f(x)|\, dx = \int\limits_0^\infty \varrho^{-2\mu}\, \varphi'_\xi(\varrho)\, d\varrho =$$

$$= \int\limits_0^\infty 2\mu\, \varrho^{-2\mu-1}\, \varphi_\xi(\varrho)\, d\varrho \le L\lambda(\lambda-\mu)^{-1}\, r^{2\lambda-2\mu}$$

(4.21)

$$\int\limits_{B(x_0,r)\cap G}\int\limits_{G} |f(x)|\cdot|\xi - x|^{-2\mu}\, |\nabla u(\xi)|^2\, dx\, d\xi \le$$

$$\le L\lambda(\lambda-\mu)^{-1}\, r^{2\lambda-2\mu}\cdot \|\nabla u\|^2_{L^2}.$$

The result follows from (4.20) and (4.21).

LEMMA 4.5: *Suppose u and f satisfy the hypotheses of Lemma 4.4. Then $fu^2 \in \mathcal{L}_1$ on G and*

$$\int\limits_{B(x_0,r)\cap G} |f(x)|\cdot|u(x)|^2\, dx \le C_2(\lambda,\mu)\cdot L\cdot\|\Delta u\|^2_{L^2}\cdot g^{\mu}\cdot r^{2\lambda-\mu};\quad 0\le\mu\le\lambda.$$

Proof: This follows from two applications of Lemma 4.4.

THEOREM 4.3: *There is an $a_0 > 0$ and depending only on m_1, M_1, M_2, and λ such that if $0 < a \le a_0$ and $B(x_0, a) \subset G$, then*

$$I[u, u; B(x_0, a)] \ge \frac{m_1}{2} D[u, B(x_0, a)] \quad \text{for all } u \in \mathfrak{B}_{20} \text{ on } B(x_0, a).$$

Proof: For

$$I[u,u;B(x_0,a)] = I_0[u,u;B(x_0,a)] + \int_{B(x_0,a)} (2b^{\alpha}_{jk}\, u^j_{\alpha}\, u^k + c_{jk}\, u^j\, u^k)\, dx$$

$$\geq D_2[u,B(x_0;a)]\cdot[m_1 - 2C_2^{1/2} M_2^{1/2} g^{\mu/2} a^{\lambda-\mu/2} - C_2 M_2 g^{\mu} a^{2\lambda-\mu}],\ 0<\mu<\lambda,$$

using Lemma 4.5 and the Schwarz inequality.

THEOREM 4.4: *If* $0 < a \leq a_0$, $B(x_0, a) \subset G$, b^{α}_{jk}, c_{jk}, *and* f *saiisfy* (4.15) *and* $e \in \mathcal{L}_2$ *on* $B(x_0, a)$, *there exists a unique* u *in* $\mathcal{B}_{20}$ *on* $B(x_0, a)$ *such that* (4.13) *holds for all* $v \in \mathcal{B}_{20}$ *on* $B(x_0, a)$. *Moreover*

$$(4.22)\qquad B[u, B(x_0, a)] \leq 2m_1^{-1}[\|e\|_L + C_1(\lambda,\mu)\cdot M_2\cdot a^{2\lambda}]^2,\ 0<\mu<\lambda.$$

Proof: From Theorem 4.3 and the Poincare inequality (Theorem 1.11), we see that the space $\mathcal{B}_{20}$ is a Hilbert space if we introduce $I(u, v)$ as inner product. Since the equation (4.13) ($G = B(x_0, a)$) can be written

$$(4.23)\qquad I(u,v) = L(v),\ L(v) = \int_{B(x_0,a)} (e^{\alpha}_j\, v^j_{,\alpha} + f_j\, v^j)\, dx$$

and since $L(v)$ is a linear functional, we see from Hilbert space theory that there is a unique u in $\mathcal{B}_{20}$ which satisfies the equation. If, now, we revert to $\{D[u, B(x_0, a)]\}^{1/2}$ as norm, we see from (4.23) and Lemma 4.4 that the norm of $L(v)$ is given by the bracket on the right in (4.22). The inequality (4.22) follows by comparing the I and D norm.

We can now prove the interior boundedness theorem:

THEOREM 4.5: *Suppose* $u \in \mathcal{L}_2$ *on* $B(x_0, a) \subset G$ *where* $0 < a \leq a_0$, $u \in \mathcal{B}_2$ *on* $B(x_0, r)$ *and* (4.13) *holds for each* $v \in \mathcal{B}_{20}$ *on* $B(x_0, r)$ *for each* r *with* $0 < r \leq a$. *Then*

$$\{D[u, B(x_0, r)]\}^{1/2} \leq C_3(m_1, M_1)\{\|e\| + C_1 L a^{2\lambda} + (a-r)^{-1}\|u\|\}$$

$$(C_1 = \min_{0<\mu<\lambda} C_1(\lambda,\mu))$$

the norm being the $\mathcal{L}_2$ *norms.*

Proof: Let h be a fixed function of class C^{∞} with $h(s) = 1$ for $s \leq 0$ and $h(s) = 0$ for $s \geq 1$ and $0 \leq h(s) \leq 1$. Choose R so $r < R < a$ and define

$$\zeta(x) = h[(|x - x_0| - r)/(R - r)],\ v^j = \zeta^2 u^j,\ U^j = \zeta u^j.$$

Then v and $U \in \mathcal{B}_{20}$ on $B(x_0, R)$. Substituting in (4.13), we obtain

$$0 = I[U, U; B(x_0, R) + \int_{B(x_0,R)} (\zeta e_j U^j_{,\alpha} + \zeta f_j U^j + \zeta\zeta_{,\alpha} e^\alpha_j u^j - a^{\alpha\beta}_{jk} \zeta_{,\alpha} \zeta_{,\beta} u^j u^k)\, dx$$

$$\geq \frac{m_1}{2} ||| U |||^2 - ||| U ||| [|| e || + C_1(\lambda, \mu) M_2 \cdot R^{2\lambda}] - h_i \cdot (R-r)^{-1} || e || \cdot || u || -$$

$$- h_1^2 M_1 \cdot (R-r)^{-2} || u ||^2$$

where $||| U |||$ is the $\mathcal{B}_{20} - D$-norm and $|| u ||$ is the $\mathcal{L}_2$ norm. Since (4.24) holds for all $R < a$, the result follows.

Lemma 4.6: *If $u \in \mathcal{B}_2$ on $B(x_0, R)$, there is a $u_1 \in \mathcal{B}_{20}$ on $B(x_0, 2R)$ such that $u_1(x) = u(x)$ on $B(x_0, R)$ and*

$$D[u_1, B(x_0, 2R)] \leq C_4 \int_{B(x_0,R)} (| \nabla u |^2 + R^{-2} | u |^2)\, dx$$

where C_4 is an absolute constant.

Proof: Define $u_2(x) = u(x)$ on $B(x_0, R)$ and extend it by reflection in the circle $B(x_0, R)$. Then $u \in \mathcal{B}_2$ on $B(x_0, 2R)$ and

$$\int_{B(x_0,2R)-B(x_0,R)} | \nabla u_2 |^2\, dx \leq \int_{B(x_0,R)} | \nabla u |^2\, dx$$

$$\int_{B(x_0 2R)-B(x_0,R)} | u_2 |^2\, dx \leq 16 \int_{B(x_0,R)} | u |^2\, dx$$

Then, define

$$u_1(x) = h[(| x - x_0 | - R)/R] \cdot u_2(x),$$

where h is function introduced in the proof of Theorem 4.5. Then u_1 is easily seen to have the desired properties.

Theorem 4.6 (Dirichlet growth theorem): *Suppose $0 < a \leq a_0$, $B(x_0, a) \subset G$, $u \in \mathcal{B}_2$ on $B(x_0, a)$, (4.13) holds for all $v \in \mathcal{B}_{20}$ on $B(x_0, a)$, and e satisfies the condition*

$$\int_{B(x_1,r)} | e |^2\, dx \leq L^2 (r/\delta)^{2\mu}, \quad 0 \leq r \leq \delta = a - | x_1 - x_0 |,$$

for some μ with $0 < \mu < \lambda/2$ and $m_1/2M$, and every circle $B(x_1, r) \subset B(x_0, a)$. Then u satisfies the condition (4.1) and (4.2) with G replaced by $B(x_0, a)$, x_0

replaced by x_1, *a by* $\delta = a - |x_1 - x_0|$, λ *replaced by* μ, *and* L *replaced by* C_5, *where* C_5 *depends only on* m_1, M_1, M_2, L, λ, μ, a, *and* $|||u|||$ *where*

$$\int_{B(x_0,a)} (|\nabla u|^2 + a^{-2}|u|^2)\,dx \equiv |||u|||^2 .$$

Thus u *satisfies a uniform Hölder condition on any* $B(x_0, R)$ *with* $R < a$ *which depends only on the quantities above and* $a - R$.

Proof: Let

$$E_j^\alpha = b_{jk}^\alpha u^k + e_j^\alpha, \quad F_j = b_{kj}^\alpha u_{,\alpha}^k + c_{jk} u^k + f_j .$$

From our hypotheses on the b's, c's, e's, and f's and from Lemmas 4.4, 4.5, and 4.6, we see that

$$\left\{\int_{B(x_1,r)} |E|^2\,dx\right\}^{1/2} \le C^{1/2}(\lambda, \mu') \cdot M_2 \cdot |||u||| \cdot a^{\mu'/2} r^{\lambda - \mu'/2} + L(r/\delta)^\mu \quad 0 < \mu' < \lambda ,$$

$$\int_{B(x_1,r)} |F|\,dx \le [M_2 r^\lambda + C_1(\lambda, \mu') \cdot M_2 a^{\mu'} r^{2\lambda - \mu'}] \cdot |||u||| + M_2^2 r^{2\lambda} .$$

Moreover u satisfies the equation

$$(4.25) \qquad I_0[u, v; B(x_1, r)] = -\int_{B(x_1,r)} (E_j^\alpha v_{,\alpha}^j + F_j v^j)\,dx ; \quad v \in \mathfrak{B}_{20} \text{ on } B(x_1, r)$$

on any $B(x_1, r) \subset B(x_0, a)$. As in the proof of Theorem 4.4, there is a unique solution U_r of (4.25) which is in $\mathfrak{B}_{20}$ on $B(x_1, r)$ and

$$D[U_r, B(x_1, r)] \le m_1^{-1}[Z_1 a^\lambda |||u||| + L]^2 (r/\delta)^{2\mu}$$

where Z_1 depends only on the quantities mentioned.

Now $V_r = u - U_r$ satisfies the homogeneous equation (4.25) and so clearly minimizes $I_0[V, V; B(x_1 r)]$ among all $V = V_r (= u)$ on $\partial B(x_1, r)$. Since $U_r \in \mathfrak{B}_{20}$ on $B(x_1, r)$, we see that

$$I_0(V_r, U_r; B_r) = 0 \text{ so } I_0(u, u; B_r) = I_0(V_r, V_r; B_r) + I_0(U_r, U_r; B_r) ,$$

where $B_r = B(x_1, r)$. Using the fact that $I_0(V_r, V_r; B_r) \le I_0(u_r, u_r; B_r)$ for any $u_r = u$ on $\partial B(x_1, r)$ and using (4.16), we see that

$$D(u, B_r) \le \frac{M_1}{m_1} D(u_r, B_r) + Z_2 (r/\delta)^{2\mu}$$

where Z_2 depends only on the quantities indicated. The results follow from Lemmas 4.2 and 4.1.

We can now resume our discussion of a solution z_0 of a variational problem of the type being discussed here.

THEOREM 4.7 : *Suppose* z_0 *gives a stationary value to* $I(z, G)$ *and satisfies the continuity conclusions of Theorem* 4.1. *Then* $z_0 \in C^{1+\mu}$ *on each domain* Γ *with* $\bar{\Gamma} \subset G$, *where* $0 < \mu < 1$, *and the derivatives* $\varepsilon \mathcal{B}_2''$ *on domains interior to* G.

Proof: Since $\varphi(0) = 0$, we see that the right side of (4.11) holds for each Lipschitz ζ with compact support in G. So, suppose $\overline{B(x_0, a)} \subset G$. Choose $A > a$ so that $\overline{B(x_0, A)} \subset G$. Then, from Theorem 4.1, we have $|x|^2 + |z_0(x)|^2 < R^2$, for some R, on $\overline{B(x_0, A)}$. Let $b = (2a + A)/3$, $c = (a + 2A/3)$, $h_0 = (A - a)/3$, let e_γ be the unit vector in the x^γ direction for $\gamma = 1,2$, let v be an arbitrary Lipschitz function having support in $B(x_0 c)$ and define

$$\zeta_h^j(x) = h^{-1}[v^j(x - he_\gamma) - v^j(x)], \; u_h^j(x) = h^{-1}[z_0^j(x + he_\gamma) - z_0^j(x)]$$

for $0 < |h| < h_0$. Then ζ_h has support in $B(x_0, A)$. Substituting ζ_h into the equation $\varphi'(0) = 0$ and using (4.11), we see that u_h satisfies equation (4.13) on $B(x_0, c)$ with coefficients $a_{hjk}^{\alpha\beta}$, etc., where

$$(4.26) \quad a_{hjk}^{\alpha\beta}(x) = \int_0^1 f_{p_\alpha^j p_\beta^k}[x + the_\gamma, (1-t) z_0(x) + tz_0(x + he_\gamma), (1-t) p_0(x) + tp_0(x + he_\gamma)] \, dt$$

for almost all x. From the general assumptions on f and from the formulas (4.26) for the coefficients, we see that the bounds (4.14) and (4.15) hold uniformly for $0 < |h| < h_0$ with

$$m_1 = m_1(R), M_1 = M_1(R), M_2 = KM_2(R), 2\lambda = m/M, G = B(x_0, c),$$

where K is a constant depending on λ and the distance of $B(x_0, A)$ from ∂G. Clearly each $u_h \in \mathcal{B}_2''$ on $B(x_0, c)$ and its L_2 norm is uniformly bounded there, and we also have

$$\int_{B(x_1, r)} |e_h|^2 \, dx \leq M_2^2 r^{2\lambda}, \quad 0 < |h| < h_0.$$

Accordingly, we see first from Theorem 4.5 that the $\mathcal{B}_2$ norms of the u_h are uniformly bounded on $B(x_0, b)$ and then from Theorem 4.6 that the u_h satisfy a uniform Hölder condition on $\overline{B(x_0, a)}$ independently of h. Thus we may let $h \to 0$ and we see that the derivatives $z_{,\gamma}^j \in \mathcal{B}_2''$ and satisfy this Hölder condition on $\overline{B(x_0, a)}$.

CHAPTER V

A variational method in the theory of harmonic integrals.

In this section, we apply our variational method to the study of armonic integrals and, more generally, use it to obtain the Kodaira decomposition theorem [29] (see Theorem 5.10 below). This approach was originally suggested by Hodge in his first paper on the subject [25]. The generality of the manifolds allowed and the methods and results obtained are closely related to those obtained by Friedrichs [20] working independently. Of course corresponding results have been obtained on smoother manifolds by a number of other authors using other methods ([12], [23], [26], [29], [38]). In this section, we shall confine ourselves to compact manifolds without boundary. The variational methods are applied to compact manifolds with boundary in [20] and [46]; boundary value problems for forms have been considered by other writers using other methods in [13], [66].

We adopt the usual definition of a compact Riemannian manifold of dimension n (instead of ν) and of class C^k or C^k_μ $(0 < \mu \leq 1)$ any two admissible coordinate systems are related by a transformation of class C^k or C^k_μ, respectively. If $0 < \mu < 1$, the class C^k_μ is the same as what we have called $C^{k+\mu}$; If $\mu = 1$, a function is of class C^k_1 if and only if its derivatives of order $\leq k$ satisfy Lipschitz conditions; transformations of class C^k_1 are defined similarly. If a coordinate system is of class C^k_μ, the induced g_{ij} are of class C^{k-1}_μ. *We shall assume that our manifold is of class at least* C^1_1.

We shall be concerned with exterior differential forms of degree r on a manifold M; we call these simply *r-forms*. In the domain of a given coordinate system such a form ω may be represented by

$$\omega = \sum_{i_1 < \dots < i_r} \omega_{i_1 \dots i_r} \, dx^{i_1} \wedge \dots \wedge dx^{i_r} \tag{5.1}$$

where $\omega_{i_1 \dots i_r}$ are the *components* of ω in that coordinate system and $\wedge$ denotes the exterior product. In order to take care of the case of non-orientable manifolds, we allow both *even* and *odd* forms. If two coordinate systems (x) and (x') overlap, the components transform according to the law

$$\begin{gathered} '\omega_{i_1 \dots i_r}(r') = \varepsilon \sum_{j_1 < \dots < j_r} \omega_{j_1 \dots j_r} [x(x(x'))] \frac{\partial (x^{j_1} \dots x^{j_r})}{\partial ('x^{i_1} \dots 'x^{i_r})}, \\ \varepsilon = \begin{cases} +1 \text{ for even forms,} \\ J/|J| \text{ form odd forms,} \end{cases} J = \frac{\partial (x^i \dots x^n)}{\partial ('x^i, \dots, 'x^n)}. \end{gathered} \tag{5.2}$$

Since the Jacobians involved in (5.2) are at least of class C_1^0 (Lipschitz), we may say that a form ω *is of class* $\mathcal{L}_2$ or $\mathcal{B}_2 \longleftrightarrow$ its components in each coordinate system are.

Given an r-form ω, we define its dual $*\omega$ by

$$*\omega = \sum_{j_1<\dots<j_{n-r}} (*\omega)_{j_1\dots j_{n-r}}\, dx^{j_1} \wedge \dots \wedge dx^{j_{n-r}}$$

(5.3)
$$({}^*\omega)_{j_1\dots j_{n-r}} = \Gamma \cdot e_{k_1\dots;k_r j_1\dots j_{n-r}} \sum_{l_1\dots l_r} g^{k_1 l_1} \dots g^{k_r l_r}\, \omega_{l_1 l_r}$$
$$= \Gamma \cdot e_{k_1\dots k_r j_1\dots j_{n-r}} \sum_{l_1<\dots<l_r} \Gamma^{(k)(l)}\, \omega_{l_1\dots l_r}$$

where $e_{p_1\dots p_n}$ is 0 if two indices p_i are the same or otherwise is ± 1 according as $p_1 \dots p_n$ is an even or odd permutation, $k_1 < .., < k_r$ are chosen so that $k_1 \dots k_r j_1 \dots j_{n-r}$ is a permutation, $\Gamma^{(k)(l)}$ is the determinant of the $g^{k_i l_j}$, and $\Gamma = \pm\sqrt{g}$ chosen so that $\Gamma\, dx^1 \wedge \dots \wedge dx^n = dS$, the positive volume element. *If two forms* ω *and* η *of the same kind* (both even or both odd) *of the same degree are in* $\mathcal{L}_2$ *on* M, we define their *inner product*

(5.4)
$$(\omega, \eta) = \int_M \omega \wedge * \eta ;$$

we form inner products only under these conditions. If ω is an r-form given in the x-system by (5.1) and if η is an s-form of the same kind with a corresponding representation, we define

(5.5)
$$\omega \wedge \eta = \sum_{(i)} \sum_{(j)} \omega_{i_1\dots i_r}\, \eta_{j_1\dots j_s}\, dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}.$$

Accordingly the inner product (ω, η) is also given by

(5.6)
$$(\omega, \eta) = \int_M (F(P; \omega, \eta)\, dS_P$$
$$F = \sum_{(i)(j)} \Gamma^{(i)(j)}\, \omega_{(i)}\, \eta_{(j)}$$

where $(i) = i_1 \dots i_r$, where $i_1 < \dots < i_r$, etc. In case P corresponds to x_0 in the x system and $g_{ij}(x_0) = \delta_{ij}$, we see that

(5.7)
$$F(P; \omega, \eta) = \sum_{(i)} \omega_{(i)}(x_0)\, \eta_{(i)}(x_0),\ dS_P\, |\, dx\, |.$$

The following theorem is well hoown and is evident.

THEOREM 5.1. *For each $r = 0, 1, \ldots, n$ the totality of r-forms of a fixed kind $\mathcal{L}_2$ on M (with equivalent forms identified) forms a real Hilbert space $\mathcal{L}_2^r$ with inner product given by* (5.4)

In order to introduce an inner preduct in $\mathcal{B}_2$ on M, we proceed as follows:

DEFINITION: Let $\mathcal{U} = (U_1, \ldots, U_Q)$ be a finite open covering of M by coordinate patches $U_q = Q_q(G_q)$, where each G_q is a Lipschitz domain in $\mathcal{E}^n$. If ω and η are in $\mathcal{B}_2$ on M we define

$$(5.8) \qquad ((\omega, \eta))_{\mathcal{U}} = (\omega, \eta) + \sum_{q=1}^{Q} \int_{G_q} \sum_{(i)} \sum_{\alpha=1}^{n} \omega^{(q)}_{(i)x^\alpha} \eta^{(q)}_{(i)x^\alpha} \, dx ,$$

where $\omega^{(q)}_{(i)}$ and $\eta^{(q)}_{(i)}$ are the components of ω and η in Q_q. Then

$$(5.9) \qquad \| \omega \|_{\mathcal{U}} = ((\omega, \omega))_u^{1/2}$$

is the expression for the norm in $\mathcal{B}_2$ on M corresponding to the inner product (5.8). It is char that convergence of ω_k to ω according to one of the norms (5.9) is equivalent to the atrong convergence in $\mathcal{B}_2$ of the compenents ω_k in *any* coordinate system to those of ω. Thus we obtain the theorem:

THEOREM 5.2: *For each coordinate cover $\mathcal{U}$ and each $r = 0, \ldots, n$* the space of r-forms in $\mathcal{B}_2$ of a given kind on M forms a real *Hilbert space* $\mathcal{B}_2^r$ *with inner product given by* (5.8). *Any two such inner product sare topologically equivalent.*

Now, if ω is an r form $\in \mathcal{B}_2$, we define $d\omega$ and $\delta\omega$ by

$$\delta\omega = (-1)^{1+n(r-1)} * d * \omega, \text{ and}$$

$$(5.10) \qquad d\omega = \sum_{(i)} \sum_{q=1} \omega_{i_1 \ldots i_r, q} \, dx^q \vee dx^{i_1} \vee \ldots \vee dx^{i_r}.$$

We note that $d\omega$ is an $(r+1)$-form (if $r \leq n-1$) and $\delta\omega$ is an $(r-1)$-form (if $r \geq 1$). Finally, we define the Dirichlet integral by

$$(5.11) \qquad D(\omega) = (d\omega, d\omega) + (\delta\omega, \delta\omega).$$

THEOREM 5.3: *d is a bounded operator from the whole of $\mathcal{B}_2^r$ into $\mathcal{L}_2^{r+1}$, and δ is a bounded operalor from the while of $\mathcal{B}_2^r$ into $\mathcal{L}_2^{r-1}$; each of these operators preserves evenness or oddness. $D(\omega)$ is a lower semi-continuous function with respect to weak convergence in $\mathcal{B}_2^r$. If ω_k tends weakly to ω_0 in $\mathcal{B}_2^r$ on M, then ω_k tends strongly to ω_0 in $\mathcal{L}_2^r$ on M.*

Proof. The first statement in clear form (5.8) since the g_{ij} are at least Lipschitz and have bounded first derivatives. New if ω_k tends weakly in $\mathcal{B}_2$ to ω, $d\omega_k$ and $\delta\omega_k$ tend weakly in $\mathcal{L}_2$ to $d\omega$ and $\delta\omega$, whence the last statement about $D(\omega)$ follws from the lower-semicontinuity of the norm in $\mathcal{L}_2$ with respect to weak convergence. The last statement is an application of Theorem 1.13.

From (5.6) and (5.7), we see that

$$(\omega, \eta) = (\eta, \omega). \tag{5.12}$$

In the coordinate system of (5.7), we see that

$$(*\omega)_{j_1 \dots j_{n-r}} = e_{i_1 \dots i_r j_1 \dots j_{n-r}} \, \omega_{i_1 \dots i_r} \qquad ((i) \text{ not summed}) \tag{5.13}$$

where $i_1 < \dots < i_r$ and $i_1 \dots i_r j_1 \dots j_{n-r}$ is a permutation. From the form (5.12), we see that

$$** \omega = (-1)^{r(n-r)} \omega . \tag{5.14}$$

From (5.5) and (5.10) it is easy to see that

$$d(\omega \vee \eta) = d\omega \vee \eta + (-1)^r \omega \vee d\eta \tag{5.15}$$

where η is any s-form (and ω is an r-form) in $\mathcal{B}_2$. From the rules of exterior multiplication and (5.5), it is easy to see that

$$\eta \vee \omega = (-1)^{rs} \omega \vee \eta . \tag{5.16}$$

From (5.4), (5.12), (5.14), and (5.16), one derives

$$(* \omega, * \eta) = (\omega, \eta) \tag{5.17}$$

If M, ω, and ζ are all smooth and ω and ζ are of the same kind and degrees r and $r-1$, respectively, we obtain

$$(\delta\omega, \zeta) = (-1)^{1+n(r-1)} (* d * \omega, \zeta) = (-1)^r (d * \omega, * \zeta)$$

$$= (-1)^r \int_M d * \omega \vee ** \zeta = (-1)^{r+(r-1)(n-r+1)} \int_M d * \omega \vee \zeta$$

$$= (-1)^{r+(r-1)(n-r+1)} \int_M [d * \omega \vee \zeta + (-1)^{n-r} * \omega \vee p\zeta$$

$$+ \int_M d\zeta \vee * \omega = (d\zeta, \omega) = (\omega, d\zeta)$$

since the first integral vanishes by Stoke's theorem for $(n-1)$-forms, the bracket being just $d[*\omega \vee \zeta]$ (see (5.15)). We emphasize the result:

$$(\delta\omega, \zeta) = (\omega, d\zeta). \tag{5.18}$$

In the case of smooth manifolds and forms, we see from (5.10) and (5.14) that

$$d(d\omega) = \delta(\delta\omega) = 0. \tag{5.19}$$

Combining this with (5.18), we see that

$$(\delta\alpha, d\beta) = 0. \tag{5.20}$$

The formulas (5.18) and (5.20) can be extended to $\mathfrak{B}_2$ forms on manifolds only of class C_1^1 by using a proper partition of unity (recall Lemma 1.3), such that if the supports of two of the h_i intersect then their union lies in one coordinate patch, to represent each form as a sum of forms whose supports have the same property. Then, for instance

$$(\delta\omega, \zeta) = \sum_{r,s} (\delta\omega_r, \zeta_s)$$

and each term may be evaluated using one coordinate patch; in that patch, the g_{ij} and the forms may be approximated by smooth forms.

In the case of a coordinate system of the type in (5.7) where we also assume that all the $\partial g_{ij}/\partial x^k = 0$ at x_0, we see from (5.10) and (5.13) that the components of $d\omega$ at x_0 are

$$\begin{aligned} (d\omega)_{i_1 \ldots i_{r+1}} &= \sum_{q=1}^{r+1} (-1)^{q-1} \omega_{i_1 \ldots i_{q-1} i_{q+1} \ldots i_{r+1}, i_q}(x_0) \\ (\delta\omega)_{i_1 \ldots i_{r-1}} &= (-1)^r \sum_{s=1}^{n-r+1} (-1)^{s-1} \omega_{i_1 \ldots i_{r-1} l_s, l_s} \end{aligned} \tag{5.21}$$

where $i_1 \ldots i_{r-1} l_1 \ldots l_{n-r+1}$ is a permutation. From (5.21), we see that Dirichlet integral $D(\omega)$ in (5.11) reduces to

$$D_0(\omega) = \int_G \sum_{(i)\alpha} \omega^2_{(i),\alpha}\, dx + 2 \sum_{(i)(j)} \sum_{\alpha\beta} \int_G [\omega_{(i),\alpha}\, \omega_{(j),\beta} - \omega_{(j),\beta}\, \omega_{(j),\alpha}]\, dx \tag{5.22}$$

for the case that ω has support in a coordinate patch having domain G and the $g_{ij} \equiv \delta_{ij}$ throughout G; the last integrals all vanish in this case.

We now prove the following important lemma, first proved for forms by Gaffney

LEMMA 5.1: *Given $\varepsilon > 0$, $0 \leq r \leq n$, and P_0 on M, there is an admissibile coordinate system mapping $B(0, \varrho)$, for some $\varrho > 0$, onto a neighborhood U of P_0, and a constant l such that*

$$D(\omega) \geq (1 - \varepsilon) \int_{B(0,\varrho)} \sum_{(i)\alpha} \omega^2_{(i),\alpha}\, dx - l(\omega, \omega) \tag{5.23}$$

for any r-form $\in \mathfrak{B}_2$ whose support is in U.

Proof: We begin by choosing a fixed coordinate system mapping some $B_R = B(0, R)$ onto a neighborhood U_R of P_0, carryng the origin into P_0, and satisfyng $g_{ij}(0) = \delta_{ij}$. From our formulas for $d\omega$ and $\delta\omega$, we see that

$$D(\omega) = \int_{B_\varrho} [a^{(i)(j)\alpha\beta} \omega_{(i),\alpha}\, \omega_{(j),\beta} + 2b^{(i)(j)\alpha} \omega_{(i),\alpha}\, \omega_{(j)} + c^{(i)(j)} \omega_{(i)}\, \omega_{(j)}]\, dx \tag{5.24}$$

where the $a's$ are combination of the g_{ij} only and so are Lipschitz and the $b's$ and $c's$ are combinations of the g_{ij} and their first derivatives and so are bounded and measurable at least. Since the $a's$ are Lipschitz and since

$$|2\alpha\beta| \leq \eta\alpha^2 + \eta^{-1}\beta^2$$

we see that we may choose ϱ so small that

$$D(\omega) \geq \left(1 - \frac{\varepsilon}{2}\right) D_0(\omega) - \frac{\varepsilon}{2} \int_{B_\varrho} \sum_{(i)\alpha} \omega^2_{(i),\alpha}\, dx - l(\omega, \omega)$$

The result follows from (5.22).

The following important theorem corresponds to Gårding's Inequality for differential equations:

THEOREM 5.4: *For each $r = 0, \ldots, n$ and coordinate covering $\mathfrak{U}$ of M, there exist constants $K_{\mathfrak{U}} > 0$ and $L_{\mathfrak{U}}$ such that*

$$D(\omega) \geq K_{\mathfrak{U}}((\omega, \omega))_{\mathfrak{U}} - L_{\mathfrak{U}}(\omega, \omega) \tag{5.25}$$

for ever $\omega \in \mathfrak{B}_2^r$.

Proof: From Theorem 5.2 it is sufficient to prove this for some particular $\mathfrak{U}$. Let $\mathfrak{U} = (U_1, \ldots, U_Q)$ be an open covering of M by coordinate patches such that each $x \in M$ is in some U_k satisfying (5.23) with $\varepsilon = \dfrac{1}{2}$, say. Let $G_1, \dots, G_Q$ be the domain in E^ν such that $U_k = Q_k(G_k)$ for all k. There exists a finite sequence $\Phi_1, \dots, \Phi_s$ of Lipschitz functions on M, each of which has support interior to some U_q, and such that

$$\sum_{s=1}^{S} \Phi_s(x) = 1 \tag{5.26}$$

for all $x \in M$.

Now if (5.25) were false for the $\mathcal{U}$ just described, there would exist a sequence $\{\omega_p\}$ of r-forms in $\mathcal{B}_2^r$ such that $D(\omega_p)$ and (ω_p, ω_p) were uniformly bounded but $\| \omega_p \|_U \to \infty$. Then, for some s, q, and some subsequence, still called ω_p, we would have

$$\int_{G_q} \sum_{(i),\alpha} (\Phi_s \, \omega_{p(1)}^{(q)})^2_{x^\alpha} \to \infty$$

where Φ_s has support in $U_{q'}$ since

$$\| \omega_p \|_{\mathcal{U}} \leq \sum_{s=1}^{s} \| \Phi_s \, \omega_p \|_{\mathcal{U}}$$

and

$$\| \Phi_s \, \omega_p \|^2_{\mathcal{U}} = (\Phi_s \, \omega_p, \Phi_s \, \omega_p) + \sum_{s=1}^{Q} \int_{G_q} \sum_{(i),\alpha} (\Phi_s \, \omega_{p(i)}^{(q)})^2_{x^\alpha} \, dx .$$

But it is easy to see that $D(\Phi_s \omega_p)$ and $(\Phi_s \omega_p, \Phi_s \omega_p)$ are uniformly bounded. From our choice of neighborhoods we have reached a contradiction with the fact that

$$D(\Phi_s \, \omega_p) \geq \frac{1}{2} \int_{G_q} \sum_{(i),\alpha} (\Phi_s \, \omega_{p(i)}^{(q)})^2_{x^\alpha} \, dx .$$

We can now present the variational method. We begin with the following lemma:

LEMMA 5.2: *Let $\mathcal{M}$ be any closed linear manifold in the space $\mathcal{L}_2^r$ of r-forms on M (of some one kind). Then either there is no form ω of $\mathcal{M}$ which is in $\mathcal{B}_2^r$ or there is a form ω_0 in $\mathcal{M}\mathcal{B}_2^r$ with $(\omega_0, \omega_0) = 1$ which minimizes $D(\omega)$ among all such forms.*

Proof: If $\mathcal{M}$ contains no form in $\mathcal{B}_2^r$, there is nothing to prove. Otherwise let $\{\omega_k\}$ be a minimizing sequence, i. e., one such that $(\omega_k, \omega_k) = 1$ and $\omega_k \in \mathcal{M} \cap \mathcal{B}_2^r$ for each $k = 1, 2, \dots$, and such that $D(\omega_k)$ opproaches its infimum for all $\omega \in \mathcal{M} \cap \mathcal{B}_2^r$. From Theorem 5.4 it follows that the

$((\omega_k, \omega_k))_{\mathcal{U}}$ are uniformly bounded. Accordingly, a subsequence, still called $\{\omega_k\}$, exists which converges weakly in $\mathcal{B}_2^r$ to some form ω_0. But from Theorem 5.3 ω_k tends strongly in $\mathcal{L}_2^r$ to ω_0 and $D(\omega)$ is lower-semicontinuous with respect to weak convergence in $\mathcal{B}_2^r$. The proof of the lemma is now complete.

DEFINITION: A *harmonic field* ω on M is a form in $\mathcal{B}_2$ on M for which $d\omega = \delta\omega = 0$ almost everywhere. We will let $\mathcal{H}^r$ denote the linear manifold of harmonic fields on M of degree r. (Strinctly speaking we have $\mathcal{H}_l^r$ and $\mathcal{H}_0^r$ for even and odd forms, respectively).

THEOREM 5.5: *For each* $r = 0, \dots, n$ $(= \dim M)$ *the linear manifold* $\mathcal{H}^r$ *is finite dimensional.*

Proof. The $\mathcal{B}_2$ forms are dense in $\mathcal{L}_2^r$, since the Lipschitz forms are. Let $M_1 = \mathcal{L}_2^r$. There is a form ω_1 in $M_1 \cap \mathcal{B}_2^r$ which minimizes $D(\omega)$ among all such forms with $(\omega, \omega) = 1$. Let M_2 be the closed linear manifold in $\mathcal{L}_2^r$ orthogonal to ω_1, and let ω_2 be the corresponding minimizing form in M_2. By continuing this process, we may determine successive minimizing forms $\omega_1, \omega_2, \omega_3, \dots$, each satisfying $(\omega_k, \omega_k) = 1$ and being orthogonal to all the preceding ones.

Now if $D(\omega_1) > 0$, there are no harmonic fields $\neq 0$ since $D(\omega_1) \leqq \leqq D(\omega_2) \leqq \dots$. On tue other hand, suppose $D(\omega_k) = 0$ for all values of K. Then by Theorem 5.4, $((\omega_k, \omega_k))_{\mathcal{U}}$ is uniformly bounded in k, whence a subsequence $\{\omega_p\}$ converges weakly in $\mathcal{B}_2^r$ and hence strongly in $\mathcal{L}_2^r$ to some form ω_0 in $\mathcal{B}_2^r$. This is impossible since the ω_k form an orthonormal system in $\mathcal{L}_2^r$.

THEOREM 5.6: *For each coordinate covering* $\mathcal{U}$ *of* M *there is a constant* λ_0 *such that*

$$D(\omega) \geqq \lambda_0 ((\omega, \omega))_{\mathcal{U}} \tag{5.27}$$

for any ω *in* $\mathcal{B}_2^r$ *wich is orthogonal to* $\mathcal{H}^r$.

Proof. For, let ω_0 be that form in $\mathcal{B}_2^r$ (there is one since each harmonic field is in $\mathcal{B}_2$) which minimizes $D(\omega)$ among all ω in $\mathcal{B}_2^r$ with $(\omega, \omega) = 1$ and ω orthogonal to $\mathcal{H}^r$. Then clearly $D(\omega_0) > 0$ and by homogeneity

$$D(\omega) \geq D(\omega_0)(\omega, \omega)$$

for all ω in $\mathcal{B}_2^r$ and orthogonal to $\mathcal{H}^r$. By Theorem 5.4 we see that

$$K_{\mathcal{U}}((\omega, \omega))_{\mathcal{U}} \leqq \{1 + L_{\mathcal{U}}/D(\omega_0)\} D(\omega),$$

from which (5.27) follows.

THEOREM 5.7: *Suppose ω_0 is any form in $\mathcal{L}_2^r$ and orthogonal to $\mathcal{H}^r$. Then there is a unique form Ω_0 in $\mathcal{B}_2^r$ and orthogonal to $\mathcal{H}^r$ such that*

$$(5.28) \qquad (d\,\Omega_0\,,\,d\zeta)+(\delta\Omega_0\,,\,\delta\zeta)=(\omega_0\,,\,\zeta)$$

for every ζ in $\mathcal{B}_2^r$. Moreover, the transformation from ω_0 to Ω_0 is a bounded linear transformation from $\mathcal{L}_2^r$ into $\mathcal{B}_2^r$.

Proof: From Theorem 5.5, we see that

$$I(\omega)\equiv D(\omega)-2(\omega\,,\,\omega_0)\geq\lambda_0\,\|\,\omega\,\|_{\mathcal{L}}^2-P\,\|\,\omega_0\,\|_{\mathcal{L}}$$

since $(\omega\,,\,\omega_0)$ is a bounded linear functional on $\mathcal{B}_2^r$; here $\|\,\omega\,\|_{\mathcal{L}}^2=((\omega,\omega))_{\mathcal{L}}$. Hence $I(\omega)$ is bounded below and is lower-semicontinuous with respect to weak convergence in $\mathcal{B}_2^r$ if ω is orthogonal ($\mathcal{L}_2$-sense) to $\mathcal{H}^r$. Accordingly there is a minimizing form Ω_0. If ζ is any form in $\mathcal{B}_2^r$ orthogonal to $\mathcal{H}^r$, we then see that

$$(5.29) \quad I(\Omega_0+\lambda\zeta)\doteq I(\Omega_0)+2\lambda\,[d\Omega_0\,,\,d\zeta)+(\delta\Omega_0\,,\,\delta\zeta)-(\omega_0\,,\,\zeta)]+\lambda^2\,D(\zeta)$$

which shows that (5.28) holds for all such ζ and Ω_0 is unique. But then (5.28) holds all ζ in $\mathcal{B}_2^r$ since any such ζ is uniquely representable in the form $\mathcal{G}=H+\zeta_0$ where $dH=\delta H=0$ and ζ_0 is in $\mathcal{B}_2^r$ and orthogonal to $\mathcal{H}^r$. Finally, if we set $\zeta=\Omega_0$ in (5.28) and use Theorem 5.7, we see that

$$\|\,\Omega_0\,\|_{\mathcal{L}}^l\leq\lambda_0^{-1}\,\|\,\Omega_0\,\|_{\mathcal{L}}\cdot\|\,\omega_0\,\|_{\mathcal{L}}$$

from which the last statement follows.

DEFINITION: The form Ω_0 of Theorem 5.7 is called the potential of ω_0.

We observe that if all forms in (5.28) and the manifold M were sufficiently smooth, the equation (5.28), together with equation (5.18) would imply that

$$(5.30) \qquad \Delta\,\Omega_0\equiv d\,\delta\Omega_0+\delta\,d\Omega_0=\omega_0\,.$$

In any coordinate system, (5.30) reduces to a system of second order equations in the components of the forms; if $r\geq 1$, these equations involve the second derivatives of the g_{ij} as well as those of the components of Ω_0. However, all the results stated so far hold for manifolds of class C_1^1 in which case the requisite second derivatives of the g_{ij} certainly do not exist.

DEFINITION: We say that ω *is of class* $\mathcal{L}_{2\lambda}$, $0 \leq \lambda < n/2$, if for each coordinate system 0 with domain B_R, there is a constant $L = L(\theta, \omega)$ such that

$$\int_{B_r} \bar{\omega}^2_{(i)}\, dx \leq L^2 r^{2\lambda},\ 0 \leq r \leq R\ (B_r = B(0, r)).$$

The class $\mathcal{B}_{2\lambda}$ is defined similarly.

The importance of the spaces $\mathcal{B}_{2\lambda}$ ariees from the fact that if $\omega \in \mathcal{B}_{2\lambda}$ with $\lambda = \mu - 1 + n/2$, $0 < \mu < 1$, then $\omega \in C^0_\mu$; this follows from the straightforwand extension of Lemma 4.1, to n dimensions. We can now state the following results concerning differentiability.

THEOREM 5.8: *Suppose that* $\omega \in \mathcal{L}^r_2 \ominus \mathcal{H}^r$ *and* Ω *is its potential.*

(i) *If* M *is of class* C^1_1, *the* Ω, $d\Omega$, *and* $\delta\Omega \in \mathcal{B}_2$.

(ii) *If* M *is of class* C^1_1, *and* $\omega \in \mathcal{L}_{2\lambda}$, *then* Ω, $d\Omega$, *and* $\delta\Omega \in \mathcal{B}_{2\lambda}$ *and hence in* C^0_μ *if* $\lambda = n/2 - 1 + \mu$, $0 < \mu < 1$.

(iii) *If* M *is of class* C^1_1 *and* $\omega \in \mathcal{B}_2$, *then* $d\Omega$ *and* $\delta\Omega$ *are the potentials of* $d\omega$ *and* $\delta\omega$, *respectively.*

(iv) *If* M *is of class* C^k_μ *and* $\omega \in C^{k-2}_\mu$, $k \geq 2$, $0 < \mu < 1$, *then* Ω, $d\Omega$ *and* $\delta\Omega \in C^{k-1}_\mu$. *If* $k \geq 3$ *and* $\omega \in C^{k-3}_\mu$, *then* $\Omega \in C^{k-1}_\mu$.

(v) *If* M *and* ω *are of class* C^∞ *or analytic, then so is* Ω. *In all case, if we set* $\alpha = d\Omega$ *and* $\beta = \delta\Omega$ *we have*

$$(5.31) \qquad \delta\alpha + d\beta = \delta(d\Omega) + d(\delta\Omega) = \omega,\ d\alpha = \delta\beta = 0.$$

THEOREM 5.9: *Suppose that* H *is a harmonic field.*

(i) *If* $M \in C^1_1$, then $H \in \mathcal{B}_{2\lambda}$ with $\lambda = n/2 - 1 + \mu$ for any μ, $0 < \mu < 1$.

(ii) *If* $M \in C^k_\mu$, $k \geq 2$, $0 < \mu < 1$, *then* $H \in C^{k-1}_\mu$.

(iii) *If* $M \in C^\infty$ *or is analytic, then so is* H.

In both Theorems 5.8 and 5.9, 0-forms have an additional degree of differentiability (except in the second part of Theorem 5.8 (iv)). It should be observed that we can form $\Delta\,\Omega$ as indicated in (5.31) even though the individual components of Ω do not have the necessary second derivatives (if $r > 0$).

Proof: Obviously H satisfies (5.28) with $\omega_0 = 0$. Then equations (5.28) are a special case of the more general equations

$$(5.32) \qquad (d\omega - \varphi, d\zeta) + (\delta\omega - \psi, \delta\zeta) = (\omega_0, \zeta)$$

Using (5.24) and (5.22) we see that equations (5.32) are equivalent to equations of the form (4.13), if ζ has snpport on some one coordinate patch, where the a's are Liptschitz, the b's and c's are bounded and measurable, and the $e's$ and $f's \in \mathcal{L}_2$. Such systems have been studied extensively by the writer in [75] and [47]. Since Professor Nirenberg's lectures are concerned with differentiability problems, the results and their proofs are omitted.

The results concerning Ω and H follow directly from the result just mentioned. To prove the differentiability of $d\Omega$ and $\delta\Omega$, we select a coordinate patch and find that we can approximate to Ω, ω, and the g_{ij} by smoot functions so That Ω is a potential of ω with respect to the altered g_{ij} at each stage. Then, if ζ has support interior to this patch, we see that (5.31), (5.18), and (5.20) imply that α and β satisfy

$$(5.33) \qquad \begin{aligned} &(d\alpha, d\zeta) + (\delta\alpha - \omega, \delta\zeta) = 0 \\ &(d\beta - \omega, d\zeta) + (\delta\beta, \delta\zeta) = 0 . \end{aligned}$$

The interior boundedness theorem (like Theorem 4.5) and an approximation theorem for such systems allow us to pass to the limit in (5.33). If $\omega \in \mathcal{B}_2$, we use (5.33) and (5.18) to see that α and β are the potentials of $d\omega$ and $\delta\omega$, respectively.

The following theorem complements the well-known orthogonal decomposition of Kodaira [29].

THEOREM 5.10: *If ω is any form in $\mathcal{L}_2$, then there exists a harmonic field H and forms α, β, and Ω in $\mathcal{B}_2$ such that*

$$(5.34) \qquad \begin{aligned} &\omega = H + \delta\alpha + d\beta , \; d\alpha = \delta\beta = 0 , \\ &\alpha = d\Omega , \; \beta = \delta\Omega , \end{aligned}$$

where Ω is the potential of $\omega - H$. If the first equation of (5.34) holds for a harmonic field H_1 and forms α_1 and β_1 in $\mathcal{B}_2$, then $H_1 = H$, $\delta\alpha_1 = \delta\alpha$, and $d\beta_1 = d\beta$.

The sets $\mathcal{C}^r$ or all forms $\delta\alpha$ for α in $\mathcal{B}_2^{r+1}$ and $\mathcal{D}^r$ of all forms $d\beta$ for β in $\mathcal{B}_2^{r-1}$ are closed linear manifolds in $\mathcal{L}_2^r$ and

$$(5.35) \qquad \mathcal{L}_2^r = \mathcal{H}^r \oplus \mathcal{C}^r \oplus \mathcal{D}^r .$$

If $M \in C_1^1$ and $\omega \in \mathcal{L}_{2\lambda}$ or $\mathcal{B}_{2\lambda}$, $0 \leq \lambda < n/2$, then $\delta\alpha$ and $d\beta$ have the same properties.

If $M \in C_\mu^k$ and $\omega \in C_\sigma^l$ with $k \geq 2$, $0 < \mu < 1$, $0 < \sigma < 1$, and either $l < k - 1$ or $l = k - 1$ and $\sigma \leq \mu$, then $\delta\alpha$ and $d\beta$ have the same differentiability properties as ω.

If M and $\omega \in C^\infty$ or are analitic, so are $\delta\alpha$ and $d\beta$.

Proof: The first statement and the differentiability results follow immediately from Theorems 5.8 and 5.9 If H, α, and β all $\in \mathcal{B}_2$ (and have properly related degrees), formulas (5.18) and (5.20) and the definition of harmonic field imply that H, $\delta\alpha$, and $d\beta$ are orthogonal in $\mathcal{L}_2$. To see that the sets $\mathcal{C}^r$ and $\mathcal{D}^r$ are closed we see, by following the construction in the first paragraph of the theorem with $\omega = \delta\alpha$ and $d\beta$ in turn, that if α and $\beta \in \mathcal{B}_2$, there are forms α_1 and β_1 in $\mathcal{B}_2$ and orthogonal to $\mathcal{H}$ such that

$$\delta\alpha_1 = \delta\alpha\,,\ d\alpha_1 = 0\,,\ \delta\beta_1 = 0\,,\ d\beta_1 = d\beta\,.$$

Then if $\delta\alpha_n \to \sigma$ in $\mathcal{L}_2$, we see that the $\alpha_{1n} \to$ some α_1 in $\mathcal{B}_2$ by Theorem 5.6. A corresponding result holds if $d\beta_n \to \tau$ in $\mathcal{L}_2$.

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