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The L_p approach to the Dirichlet problem. Part I : regularity theorems

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THE L_p APPROACH TO THE DIRICHLET PROBLEM (*)

by SHMUEL AGMON

PART I
REGULARITY THEOREMS

1. Introduction.

In this paper we present a L_p approach to the Dirichlet problem and to related regularity problems for higher order elliptic equations. Although this approach is not as simple as the well known Hilbert space approach developed by Vishik [32] Gårding [14], Browder [6 ; 7], Friedrichs [12], Morrey [22], Nirenberg [23], Lions [18] and others, it has the advantage of a greater generality. Thus, for example, we shall be able to treat the non-homogeneous Dirichlet problem in a much more general situation not restricted to solutions having a finite Dirichlet integral (in this connection see Magenes-Stampacchia [19 , § 9] and the recent paper of Miranda [20]). The method is also applicable to elliptic operators which are not necessarily strongly elliptic. We remark further that the same method could be used to solve a general class of boundary value problems. This will be done in a subsequent paper where we shall also derive L_p integral inequalities for a system of differential operators acting on functions satisfying general boundary conditions, simular to the « coercive » L_2 inequalities derived by Aronszajn [4] Agmon [2] and Schechter [25].

Recently Schechter [26 ; 27] presented a Hilbert space approach to general boundary value problems including the Dirichlet problem for non-strongly elliptic equations. His method is based on the L_2 estimates of Agmon-

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Douglis-Nirenberg [3] (see also [2; 25]) and on known L_2 regularity theorems. Our L_p method which utilizes new regularity theorems is quite different and the results we obtain are stronger in various respects [1]. Other existence results utilizing the continuity method were given by Agmon-Douglis-Nirenberg [3].

The L_p approach to the Dirichlet problem is based on a L_p regularity theory for very weak solutions of the Dirichlet problem. To obtain such a regularity theory we use some of the ideas of a method originally devised by Nirenberg [23] with the following essential modification: instead of using Gårding's inequality we use the explicit solution of the Dirichlet problem for elliptic operators with constant coefficients in a half-space, and the L_p estimates for such solutions derived in [3].

The paper is divided into two parts. In Part I we give the basic regularity theory, both in the interior and at the boundary. This part has an independent interest and entails most of the work. We remark that when we consider the simpler problem of interior regularity we consider also weak solutions of overdetermined elliptic systems and derive L_p estimates for such solutions. In Part II we shall combine the regularity theory with some general results on Banach spaces (using in particular a result of Fichera [10]) to develop the L_p existence theory for the Dirichlet problem.

2. Notations and definitions.

Throughout the paper we denote by G a bounded domain in n dimensional space with boundary ∂G and closure $\bar{G}$. We denote by $x = (x_1, \ldots, x_n)$ the generic point in the space and put $|x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}}$. We say that G is of class C^j if with every point $x^0 = (x_1^0, \ldots, x_n^0) \in \partial G$ one can associate a sphere S having its center in x^0 such that $\partial G \cap S$ admits a representation of the form:

$$x_k = g\,(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n) \tag{2.1}$$

for a suitable k; g being a function defined in some neighborhood U of $(x_1^0, \ldots, x_{k-1}^0, x_{k+1}^0, \ldots, x_n^0)$ possessing there continuous derivatives up to the order j.

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(1) For instance, Schechter's method is applicable only to such problems for which the solution of the adjoint problem is unique, whereas we get the alternative in the general case without any uniqueness assumption

Similarly, G is said to be of classe $C^{0,1}$ if around each of its points the boundary ∂G admits a local representation of the form (2.1) with a function g satisfying a Lipschitz Condition in the neighborhood U.

Finally, G is said to possess the cone property if every point in $\overline{G}$ is a vertex of a closed right spherical cone of fixed opening and height which belongs to $\overline{G}$. It is readily seen that if G is of classe $C^{0,1}$ then it also has the cone property.

We shall denote by $C^k(G)$ (resp. $C^k(\overline{G})$) the class of complex valued functions possessing continuous derivatives up to ther order k $(0 \leq k \leq \infty)$ in G (resp. $\overline{G}$). The class of infinitely differntiable functions with compact support in G will be denoted by $C_0^\infty(G)$.

Let, now, j be a non-negative integer and p a real number ≥ 1. For a function $u(x)$ belonging to $C^j(\overline{G})$ define the norm:

$$\| u \|_{j,L_p(G)} = \left(\int_G \sum_{|\alpha| \leq j} | D^\alpha u |^p \, dx \right)^{1/p}, \tag{2.2}$$

where here and in the following α stands for the multi-index $(\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and D^α is the partial derivative:

$$D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}.$$

We also put:

$$D_i = \frac{\partial}{\partial x_i} \quad (i = 1, \ldots, n) \quad \text{and} \quad D = (D_1, \ldots, D_n).$$

The linear space $C^j(\overline{G})$ is clearly not complete under the norm (2.2). Completing it we obtain a Banach space which we denote by $H_{j,L_p}(G)$. We retain the notation $\| \ \|_{j,L_p(G)}$ for the norm in $H_{j,L_p}(G)$. The space $H_{0,L_p}(G)$ is simply the space $L_p(G)$ and we shall usually write $\| \ \|_{L_p(G)}$ for the norm in this space.

The classes of functions $H_{j,L_p}(G)$ were investigated by many authors (Sobolev [30], Morrey [21], Friedrichs [11], Stampacchia [31], Deny and Lions [9], Gagliardo [13] and others). Some of the properties of these classes will be described in the next section. Here we limit ourselves to some remarks.

By the identification mapping we can consider $H_{j,L_p}(G)$ as a linear subset of $L_p(G)$. A function $u \in H_{j,L_p}(G)$ will possess generalized derivatives up to the order j which we term strong L_p derivatives. To define these let

$\{u_i\}_{i=1}^{\infty}$ be a sequence of functions in, $C^j(\overline{G})$ such that

$$\lim_{i\to\infty} \| u_i - u \|_{j,L_p(G)} = 0 .$$

Then, there exist functions $u^\alpha \in L_p(G)$ $(0 \leq |\alpha| \leq j)$ such that

$$\lim_{i\to\infty} \| D^\alpha u_i - u^\alpha \|_{L_p(G)} = 0 .$$

The functions u^α are by definition the strong L_p derivatives $D^\alpha u$ of u in G. They are uniquely defined.

We shall say that a function u belong locally to H_{j,L_p} in G — writing $u \in H_{j,L_p}^{\text{loc.}}(G)$ — if for every $x \in G$ there exists a sphere $S \subset G$ with center at x such that $u \in H_{j,L_p}(S)$. It is readily seen (using a partition of unity) that if $u \in H_{j,L_p}^{\text{loc.}}(G)$ then $u \in H_{j,L_p}(G_1)$ for every domain G_1 such that $\overline{G}_1 \subset G$.

In connection with the Dirichlet problem we shall have to consider the subclass of functions in H_{j,L_p} which together with some of their derivatives, vanish at the boundary in a generalized sense. To make this more precise suppose that G is of class $C^{0,1}$. Let $u \in H_{1,L_p}(G)$. Then, as it is well known, one can define for such u its trace $\gamma(u)$ on the boundary. For instance, one can use the following procedure. For $u \in C^1(\overline{G})$ $\gamma(u)$ is simply the restriction of u on ∂G. In this case it is easily established that

$$(2.3) \qquad \left(\int_{\partial G} |\gamma(u)|^p \, d\sigma \right)^{1/p} \leq \text{Const.} \, \| u \|_{1,L_p}(G)$$

with a constant which is independent of u. Hence, $u \to \gamma(u)$ is a bounded linear transformation from $C^1(\overline{G})$ (considered as a subset in $H_{1,L_p}(G)$) into $L_p(\partial G)$. Since $C^1(\overline{G})$ is dense in $H_{1,L_p}(G)$ one can extend the transformation in a unique manner by continuity to the whole of $H_{1,L_p}(G)$. This defines the trace on the boundary of a function $u \in H_{1,L_p}(G)$ as an element of $L_p(\partial G)$.

Let, now, m, j be positive integers such that $m \leq j$. We denote by $H_{j,L_p}(G; \{D^\alpha\}_{|\alpha| \leq m-1})$ the class of functions $u \in H_{j,L_p}(G)$ which satisfy the boundary conditions

$$(2.4) \qquad D^\alpha u = 0 \quad \text{on} \quad \partial G \quad \text{for} \quad 0 \leq |\alpha| \leq m - 1 ,$$

where (2.4) is taken in the sense that

$$\gamma(D^\alpha u) = 0 \quad \text{as an element of } L_p(\partial G).$$

We observe that $H_{j,L_p}(G\,;\,\{D^\alpha\}_{|\alpha|\leq m-1})$ is a closed subspace of $H_{j,L_p}(G)$, and that a function u belonging to $H_{j,L_p}(G\,;\,\{D^\alpha\}_{|\alpha|\leq m-1}) \cap C^{m-1}(\bar{G})$ satisfies the boundary conditions (2.4) pointwise in the ordinary sense.

3. Calculus and properties of the classes H_{j,L_p}.

We have remarked already that a function belonging to $H_{j,L_p}(G)$ possesses strong L_p derivates up to the order j in G. Considering such a function u as a distribution in G (Schwartz [28]), it is readily seen that the strong L_p derivatives are also the distribution derivatives of u which are thus functions belonging to $L_p(G)$. It is very convenient that under general conditions on the domain G one can reverse this statement. We have:

THEOREM 3.1. *Suppose that G is of class $C^{0,1}$. Let $u \in L_p(G)$ $(p \geq 1)$ and assume that the distribution derivatives of u of order $\leq j$ are functions belonging to $L_p(G)$. I. e., assume that there exist functions $u^\alpha(x) \in L_p(G)$, $0 < |\alpha| \leq j$ (weak derivatives in the terminology of Friedrichs) such that*

$$\int_G u D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_G u^\alpha \varphi\, dx, \tag{3.1}$$

for all $\varphi \in C_0^\infty(G)$. Then, $u \in H_{j,L_p}(G)$ and its distribution derivatives u^α coincide with its strong L_p derivatives $D^\alpha u$ ($|\alpha| \leq j$).

The weaker conclusion that $u \in H_{j,L_p}^{\text{loc.}}(G)$ is well known and was established by various authors (Friedrichs [11], Sobolev [30], Deny-Lions [9]). The theorem as stated is due to Gagliardo (2) [13]. For more regular domains it was established by Babich [5].

The following remarks are obvious. If $u \in H_{j,L_p}(G)$ and $a \in C^j(\bar{G})$, then $v = au$ belongs to $H_{j,L_p}(G)$ and the strong derivatives of v are obtained by the standard Leibniz rule. If, moreover, G is of class $C^{0,1}$ then the boundary values $\gamma(D^\alpha v)$ ($|\alpha| \leq j-1$) are obtained by the same rules. The classes H_{j,L_p} are preserved by homeomorphism of class C^j. That is, let $x^* \to x(x^*)$ be a one to one mapping of $\bar{G}^*$ onto $\bar{G}$ such that the mapping and its inverse possess continuous derivatives up to the order j in the corresponding closed domains. Then the mapping $u \to u^*$, $u^*(x^*) = u(x(x^*))$, is a homeo-

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(2) It should be pointed out that Gagliardo is not using the notion of a weak derivative but a different notion which is, however, equivalent to it. Also, the proof of the main approximation theorem [13; p 112] could be repeated word by word for functions possessing weak derivatives in the sense of (3.1).

morphism between $H_{j,L_p}(G)$ and $H_{j,L_p}(G^*)$. Also, to compute the strong derivatives of u^* one applies the usual chain rule. The same remark applies to the trace at the boundary of derivatives of order $\leq j-1$ when G is of class $C^{0,1}$.

Most of the following lemmas are the L_p modified versions of the calculus L_2 lemmas given in Nirenberg [23]. Unless otherwise stated we shall assume in these lemmas that G is of class $C^{0,1}$.

LEMMA 3.1. *Let $u \in L_p(G)$, $p > 1$. Suppose that u is a weak limit in L_p of a sequence of functions $\{u_k\}$ which belong to $H_{j,L_p}(G)$ and possess uniformly bounded norms $\| u_k \|_{j,L_p(G)}$. Then, $u \in H_{j,L_p}(G)$ and its derivatives of order $\leq j$ are the weak L_p limits of the corresponding derivatives of the functions u_k.*

Proof: From the weak compactess of the unit sphere in L_p $(p > 1)$ it follows that there there exists a subsequence $u_{k'}$ such that $D^\alpha u_{k'}$ converges weakly in L_p to a function $u^\alpha \in L_p(G)$, $|\alpha| \leq j$. Hence, for every function $\varphi \in C_0^\infty(G)$:

$$\int_G \varphi\, u^\alpha\, dx = \lim_{k' \to \infty} \int_G \varphi\, D^\alpha u_{k'}\, dx$$

$$= \lim_{k' \to \infty} (-1)^{|\alpha|} \int_G D^\alpha \varphi \cdot u_{k'}\, dx = (-1)^{|\alpha|} \int_G D^\alpha \varphi \cdot u\, dx .$$

Thus, u^α is the distribution (weak) derivative $D^\alpha u$. But then, since $u^\alpha \in L_p(G)$, it follows from Theorem 3.1 that $u \in H_{j,L_p}(G)$ and that u^α coincides with the strong L_p derivative $D^\alpha u$. Moreover, from the uniqueness of the derivatives it follows that the whole sequence $D^\alpha u_k$ converges weakly to $D^\alpha u$ and not only a subsequence.

Using Theorem 3.1 one also obtains readily the following

LEMMA 3.2. *Suppose that u belongs to $H_{j,L_p}(G)$ and that its j' th order derivatives belong to $H_{k,L_p}(G)$, then $u \in H_{j+k,L_p}(G)$.*

NOTATION: Let $h = (h_1, \dots, h_n)$ be a real non-vanishing vector. We shall use the symbol δ_h to denote the difference quotient operator:

$$\delta_h u = \frac{u(x+h) - u(x)}{|h|} . \tag{3.2}$$

LEMMA 3.3. *Let $u \in H_{j,L_p}(G)$ $(j \geq 0, p > 1)$. Suppose that there exists a constant C such that for every subdomain G_1, $\overline{G}_1 \subset G$:*

$$\| \delta_h u \|_{j,L_p(G_1)} \leq C \tag{3.3}$$

for all sufficently small vectors h. Then $u \in H_{j+1,L_p}(G)$ and

$$\| D_i u \|_{j,L_p(G)} \leq C , \quad i = 1 , \dots , n . \tag{3.4}$$

Proof: Consider first the case $j = 0$. From (3.3) and the weak compactness of the unit sphere in L_p it follows that there exists a sequence of vectors $\{h^m\}_{m=1}^{\infty}$ in the direction of the x_i axis, $h^m \to 0$, such that the sequence $\delta_{h^m} u$ (m sufficiently large) tends weakly in $L_p(G_1)$ to a function u_i; and this in every fixed subdomain G_1, $\overline{G}_1 \subset G$. Since $\| u_i \|_{L_p(G_1)} \leq C$ for all such subdomains, it follows further that $u_i \in L_p(G)$.

Now, from the definition of weak convergence we find that for all functions $\varphi \in C_0^\infty(G)$:

$$\int_G \varphi\, u_i \, dx = \lim_{m\to\infty} \int_G \varphi \cdot \delta_{h^m} u \, dx$$

$$= \lim_{m\to\infty} \int \delta_{-h^m} \varphi \cdot u \, dx = - \int_G D_i \varphi \cdot u \, dx.$$

This shows that u_i is the distribution derivative $D_i u$ in G. Since $D_i u \in L_p(G)$ ($i = 1 , \dots , n$) we conclude from Theorem 3.1 that $u \in H_{1,L_p}(G)$. Clearly, we also have

$$\| D_i u \|_{L_p(G)} \leq C .$$

Next, assume that $j \geq 1$. Let again $\{h^m\}$ be a sequence of vectors in the direction of x_i tending to zero. It is easily seen that $\delta_{h^m} u$ converges to $D_i u$ in $L_p(G_1)$. Assuming without loss of generality that G_1 is of class $C^{0,1}$ and applying Lemma 3.1 to the sequence $\{\delta_{h^m} u\}$, it follows that $D_i u \in H_{j,L_p}(G_1)$ and that

$$\| D_i u \|_{j,L_p(G_1)} \leq C \quad (i = 1 , \dots , n) .$$

From this and from Lemma 3.2 we conclude that $u \in H_{j+1,L_p}(G_1)$ for any subdomain G_1 of class $C^{0,1}$ (and consequently for any subdomain G_1, $\overline{G}_1 \subset G$). Since all the distribution derivatives of u of order $\leq j+1$ are functions belonging to $L_p(G)$ it follows from Theorem 3.1 that $u \in H_{j+1,L_p}(G)$. That (3·4) holds is obvious.

By the same argument used to prove Lemma 3.3 for $j = 0$ one obtains

LEMMA 3.3′. *Denote by* Σ_R *the hemisphere* $| x | < R$, $x_n > 0$. *Let* u *be a function belonging to* $L_p(\Sigma_R)$, $p > 1$. *Suppose that there exists a constant* C *such that for every* $R' < R$:

$$\| \delta^h u \|_{L_p(\Sigma_{R'})} \leq C ,$$

for all sufficiently small vectors h of the form $h = (h_1, \dots, h_{n-1}, 0)$. Then the distribution derivatives $D_i u$ for $i = 1, \dots, n-1$ are functions belonging to $L_p(\Sigma_R)$ with $\| D_i u \|_{L_p(\Sigma_R)} \leq C$.

The following known lemma will be useful.

LEMMA 3.4. *Suppose that G has the cone property. Then, for all functions $u \in H_{j,L_p}(G)$ $(j \geq 1)$ and every $\varepsilon > 0$ the following inequality holds:*

$$\| u \|_{j-1,L_p(G)} \leq \varepsilon \sum_{|\alpha|=j} \| D^\alpha u \|_{L_p(G)} + C \| u \|_{L_p(G)} \tag{3.5}$$

where C is a constant depending only on ε, j, p and G.

Lemma 3.4 for somewhat more regular domains was established by Nirenberg [24] ([3]). The inequality for domains which have the cone property was proved by Gagliardo [13].

Finally, we conclude this section with the well known integral inequalities of Sobolev [30].

THEOREM 3.2. *Suppose that G has the cone property. Then the functions u belonging to $H_{j,L_p}(G)$ $(p > 1)$ satisfy the following relations.*

(i) *If $p < \frac{n}{j}$ then $u \in L_q(G)$ where q is defined by $\frac{1}{q} = \frac{1}{p} - \frac{j}{n}$. Also,*

$$\| u \|_{L_q(G)} \leq \text{Const.} \| u \|_{j,L_p(G)} \tag{3.6}$$

with a constant depending only on n, j, p and G.

(ii) *If $p = \frac{n}{j}$ then $u \in L_q(G)$ for every $1 < q < \infty$ and* (3.6) *holds.*

(iii) *If $p > \frac{n}{j}$ then u is a continuous function (after correction on a null set) such that*

$$\sup_G |u| \leq \text{Const.} \| u \|_{j,L_p(G)}, \tag{3.6$'$}$$

with the same constant dependence as above.

REMARK: If the boundary of the domain is somewhat more regular, e. g. if G is of class $C^{0,1}$, one can assert in case (iii) of the theorem that u satisfies a Hölder condition in $\overline{G}$.

4. Some lemmas related to elliptic operators with constant coefficients.

Let $A(x, D)$ be a linear differential operator with complex coefficients operating on functions $u(x)$ defined in a domain of E_n. Denote by A' the

([3]) The analogous one dimensional case is due to Halperin and Pitt.

leading part of A, i. e. the part of highest order terms. A is said to be elliptic in the domain if for every point x in the domain the characteristic form $A'(x, \xi) \neq 0$ for all real vectors $\xi = (\xi_1, \ldots, \xi_n) \neq 0$. It is well known that if $n \geq 3$ and A is elliptic then its order is even. This is not necessarily true for $n = 2$.

In this section we shall consider an elliptic operator A of even order $2m$ with constant coefficients and with no lower order terms:

$$A(D) = \sum_{|\alpha|=2m} a_\alpha D^\alpha . \tag{4.1}$$

A being elliptic there exists a constant $\lambda \geq 1$ such that

$$\frac{1}{\lambda} |\xi|^{2m} \leq |A(\xi)| \leq \lambda |\xi|^{2m} \tag{4.1)'$$

for all real vectors ξ. We term λ the ellipticity constant of A.

We denote by $x' = (x_1, \ldots, x_{n-1})$ the generic point in E_{n-1} and whenever convenient write x in the form (x', x_n). We also put $D_{x'} = (D_1, \ldots, D_{n-1})$ and $D = (D_{x'}, D_n)$.

Write the operator (4.1) in the form $A(D_{x'}, D_n)$. For a fixed real vector $\xi' = (\xi_1, \ldots, \xi_{n-1}) \neq 0$ consider the roots (in ξ_n) of the polynomial $A(\xi', \xi_n)$. If $n \geq 3$ the ellipticity of A implies the exactly half the roots possess a positive imaginary part (see [3]). This is not necessarily true for $n = 2$ if the coefficients are not real. In general we shall say that A satisfies the « roots condition » if for every fixed real vector $\xi' \neq 0$ the polynomial $A(\xi', \xi_n)$ has exactly m roots with a positive imaginary part.

The following two lemmas are basic for the proof of regularity in L_p of weak solutions of elliptic equations. The first rather known lemma will be used to establish interior regularity (and L_p estimates) of weak solutions of elliptic equations and overdetermined elliptic systems. The second lemma will be used to establish regularity at the boundary of weak solutions of the Dirichlet problem. In both lemmas A will stand for the elliptic operator (4.1) and p will denote a number > 1. In Lemma 4.2 we shall assume in addition, if $n = 2$, that A satisfies the « roots condition » introduced above. We shall denote by S_R the sphere $|x| < R$ and by Σ_R the half sphere $|x| < R$, $x_n > 0$.

LEMMA 4.1. *Given a function* $f \in C_0^\infty(S_R)$ *there exists a function* $v \in C^\infty(\overline{S}_R)$ *such that*

$$Av = f \quad in \quad \overline{S}_R \tag{4.2}$$

and

$$\| v \|_{2m, L_p(S_R)} \leq C \| f \|_{L_p(S_R)}, \tag{4.2}'$$

where C is some constant depending only on n, m, p, R and λ (but not on f or v).

LEMMA 4.2. *Given a function $f \in C_0^\infty(\Sigma_R)$ there exists a function $v \in C^\infty(\bar{\Sigma}_R)$ such that*

$$\begin{cases} Av = f \quad in \quad \bar{\Sigma}_R \\ D_n^j v = 0 \quad for \quad x_n = 0 \quad (|x| \leq R), \quad j = 0, \ldots, m-1, \end{cases} \tag{4.3}$$

and

$$\| v \|_{2m, L_p(\Sigma_R)} \leq C \| f \|_{L_p(\Sigma_R)}, \tag{4.3}'$$

where C is some constant depindig only on n, m, p, R and λ.

To establish Lemma 4.1 we simply define

$$v(x) = \int_{S_R} f(y) F(x-y)\, dy \tag{4.4}$$

where $F(x-y)$ is a suitable chosen fundamental solution of A with pole at $x = y$. It is well known (e. g. F. John [16]) that there exists a fundamental solution having the form:

$$F(x) = |x|^{2m-n} \psi\left(\frac{x}{|x|}\right) + P(x) \log |x|. \tag{4.5}$$

where $P(x)$ is a polynomial of degree $2m-n$ if n is even, $2m \geq n$, and $P(x)$ is zero otherwise; $\psi(y)$ is an analytic function defined on $|y| = 1$. From (4.5) it follows that

$$| D^\alpha F(x) | \leq \text{Const.} \ | x |^{2m-n-|\alpha|} \tag{4.6}$$

for (i) $|\alpha| \geq 0$, in case n is odd or n is even and greater than $2m$; (ii) $|\alpha| > 2m-n$ if n is even and not greater than $2m$. If n is even and $0 \leq |\alpha| \leq 2m-n$, then

$$|D^\alpha F(x)| \leq \text{Const.} \ |x|^{2m-n-|\alpha|} (1 + |\log |x||). \tag{4.6}'$$

Inspection of the explicit formulas for the fundamental solution (in [16]) shows that the constants in (4.6) and (4.6)′ depend only on m, n, λ and

$|\alpha|$. Furthermore, it is easily established that $D^\alpha F(x)$ for $|\alpha| = 2m$ is a homogeneous function of degree $-n$ with a zero mean on the sphere $|x| = 1$.

Choosing a proper normalization of F, the function v defined by (4.4) is infinitely differentiable and satisfies (4.2). Furthermore, from the properties of the fundamental solution mentioned above and from the well known theorem of Calderon and Zygmund [8] on convolution transforms with singular kernels, it follows readily that

$$\| v \|_{2m, L_p(S_R)} \leq C \| f \|_{L_p(S_R)}$$

where C is a constant dependig only n, m, p, R and λ. Hence, the function v defined by (4.4) answers all the requirements of Lemma 4.1.

The proof Lemma 4.2 is more involved and depends on the solution of the Dirichlet problem for A in *a* half space and related L_p estimates.

We shall denote by E_n^+ the half space $x_n > 0$. In its simplest form the Dirichlet problem for A in E_n^+ is the following

PROBLEM: *Given functions $\varphi_1(x'), \dots, \varphi_m(x')$, infinitely differentiable and of compact support in E_{n-1}, find an infinitely differentiable function $u(x', x_n)$ in $\overline{E}_n^+$ such that*

$$\tag{4.7} \begin{cases} Au = 0 \quad in \quad \overline{E}_n^+, \\ D_n^{j-1} u(x', 0) = \varphi_j(x') \quad for \quad j = 1, \dots, m. \end{cases}$$

This problem (a special case among a whole class of boundary value problems) was solved in [3] (4), where it was shown that there exist kernels $K_j(x', x_n)$ $(j = 1, \dots, m)$, defined and infinitely differentiable in $\overline{E}_n^+$ except for the origin, such that a solution of (4.7) is given by the formula:

$$\tag{4.8} u(x', x_n) = \sum_{j=1}^{m} \int_{E_{n-1}} \varphi_j(y') K_j(x' - y', x_n)\, dy'.$$

We mention the following properties of the kernels K_j also established [3]. Let q be a non-negative integer having the same parity as $n - 1$. The kernel K_j admits a representation of the form

$$\tag{4.9} K_j(x', x_n) = \Delta_{x'}^{\frac{1}{2}(n-1+q)} K_{j,q}(x', x_n)$$

(4) For $n = 2$ and A real the solution was given in [1].

where $\Delta_{x'}$ is the Laplacean $\sum_{i=1}^{n-1} D_i^2$, and $K_{j,q}$ are certain kernels which are infinitely differentiable in $\overline{E}_n^+$ except for the origin which, moreover, satisfy the following inequalities in E_n^+:

$$(4.10) \qquad | D^\alpha K_{j,q}(x) | \leq \text{Const.} \, | x |^{j-1+q-|\alpha|} (1 + | \log | x||)$$

for $| \alpha | \leq j - 1 + q$, and

$$(4.10)' \qquad | D^\alpha K_{j,q}(x) \leq \text{Const.} \, | x |^{j-1+q-|\alpha|}$$

for $| \alpha | \geq j + q$, where the constants in (4.10) and (4.10)' depend only on $n, m, q, | \alpha |$ and the ellipticity constant λ.

Let, now, $w(x', x_n)$ be an infinitely differentiable function with compact support in $\overline{E}_n^+$. By the preceding a solution $u \in C^\infty(\overline{E}_n^+)$ of the Dirichlet problem

$$(4.11) \qquad Au = 0 \quad \text{in} \quad \overline{E}_n^+,$$

$$D_n^{j-1} u(x', 0) = D_n^{j-1} w(x', 0) \quad \text{for} \quad j = 1, \dots, m,$$

is given by the formula

$$(4.11)' \qquad u(x', x_n) = \sum_{j=1}^{m} \int_{E_{n-1}} D_n^{j-1} w(y', 0) \cdot K_j(x' - y', x_n) \, dy'.$$

Moreover, we have

LEMMA 4.3. *The solution u satisfies the following inequality in $L_p, p > 1$:*

$$(4.12) \qquad \sum_{|\alpha|=2m} \| D^\alpha u \|_{L_p(E_n^+)} \leq c_0 \sum_{|\alpha|=2m} \| D^\alpha w \|_{L_p(E_n^+)},$$

where c_0 is a constant depending only on m, n, p and λ. If, in addition, the support of w is contained in the halfsphere Σ_R then

$$(4.12)' \qquad \| u \|_{2m, L_p(\Sigma_R)} \leq C_0 \| w \|_{2m, L_p(\Sigma_R)}$$

where C_0 is a constant depending only on n, m, p, λ and R.

Lemma 4.3. was proved (essentially) in [3] (compare also Koselev [17] for the L_p estimates involved). For the sake of completeness we shall present a somewhat simplified version of the proof later on. It is with the aid of this lemma that we shall now give the

Proof of Lemma 4.2. Extend the function $f(x)\,(f \in C_0^\infty(\Sigma_R))$ as zero outside Σ_R. Denote by ζ_R some fixed infinitely differentiable function such that $\zeta_R \equiv 1$ for $|x| \leq R$, $\zeta_R \equiv 0$ for $|x| \geq 2R$. Define:

$$w(x) = \zeta_R(x) \int_{E_n} f(y)\, F(x-y)\, dy\,, \tag{4.13}$$

where F is the fundamental solution of A introduced before. Clearly, w is infinitely differentiable, $w \equiv 0$ for $|x| \geq 2R$, and

$$\begin{cases} Aw = f \quad \text{for} \quad |x| \leq R\,, \\ \| w \|_{2m, L_p(\Sigma_{2R})} \leq C_1 \| f \|_{L_p(\Sigma_R)}\,, \end{cases} \tag{4.14}$$

where C_1 is a constant depending only on n, m, p, λ and R. Let, now, u be the solution of the Dirichlet problem (4.11) given by (4.11)′ with w defined by (4.13). Put:

$$v = w - u\,.$$

Then, v has all the properties required by Lemma 4.2. Indeed, $v \in C^\infty(\overline{E}_n^+)$. By (4.14) and (4.11):

$$\begin{cases} Av = Aw = f \quad \text{for} \quad x \in \overline{\Sigma}_R\,, \\ D_n^{j-1} v = D_n^{j-1} w - D_n^{j-1} u = 0 \quad \text{for} \quad x_n = 0\,, j = 1, \dots, m\,. \end{cases}$$

Finally, using the estimate (4.12)′ of Lemma 4.3 and (4.14), we get

$$\| v \|_{2m, L_p(\Sigma_R)} \leq \| w \|_{2m, L_p(\Sigma_R)} + \| u \|_{2m, L_p(\Sigma_R)}$$

$$\leq \| w \|_{2m, L_p(\Sigma_R)} + C_0 \| w \|_{2m, L_p(\Sigma_{2R})} \leq C \| f \|_{L_p(\Sigma_R)}\,.$$

where C_0, C are constants depending only on n, m, p, λ and R. This establishes the lemma.

We shall conclude the section with a proof of Lemma 4.3 based on the properties of the kernels K_j mentioned before. We shall need first the following

SUBLEMMA: *Let $G(x) = G(x', x_n)$ be a kernel, defined and measurable in the half space E_n^+ such that*

$$|G(x)| \leq C |x|^{-n}\,, \tag{4.15}$$

for some constant C. For $v \in L_p(E_n^+)$, $p > 1$, consider the transform

$$u(x', x_n) = \iint_{E_n^+} v(y', y_n)\, G(x' - y', x_n + y_n)\, dy'\, dy_n . \tag{4.16}$$

Then, $u \in L_p(E_n^+)$ and

$$\| u \|_{L_p(E_n^+)} \leq \gamma\, C \,\| v \|_{L_p(E_n^+)} , \tag{4.16'}$$

where γ is a constant depending only on n and p.

Proof: Set

$$\begin{cases} M(x) = \quad |x|^{-n} & \text{for} \quad x_n > 0 , \\ M(x) = -|x|^{-n} & \text{for} \quad x_n < 0 . \end{cases}$$

Extend v as zero for $x_n \leq 0$. Then, for $x_n > 0$:

$$|u(x)| \leq C \iint_{E_n^+} |v(y', y_n)|\, M(x' - y', x_n + y_n)\, dy'\, dy_n \tag{4.17}$$

$$= C \iint_{E_n} |v(y', -y_n)|\, M(x - y)\, dy'\, dy_n .$$

Now, $M(x)$ is an odd homogeneous kernel of degree $-n$ bounded on $|x| = 1$. Hence, we are in a position to apply the Calderon-Zygmund theorem [8] to the last integral (4.17), from which it follows readily that

$$\| u \|_{L_p(E_n^+)} \leq \gamma\, C \,\| v \|_{L_p(E_n)} = \gamma\, C \,\| v \|_{L_p(E_n^+)} ,$$

γ depending only on n and p. This proves the sublemma.

To prove Lemma 4.3 we shall first transform formula (4.11)'. To this end note that (integrating by parts with respect to y_n)

$$\int_{E_{n-1}} D_n^{j-1} w(y', 0) \cdot K_j(x' - y', x_n)\, dy' \tag{4.18}$$

$$= - \iint_{E_n^+} D_n^{j} w(y', y_n) \cdot K_j(x' - y', x_n + y_n)\, dy'\, dy_n -$$

$$- \iint\limits_{E_n^+} D_n^{j-1} w(y', y_n) \cdot D_n K_j(x' - y', x_n + y_n)\, dy'\, dy_n ,$$

where here and in the following all differential operators under the integral sign act on the y variable unless otherwise indicated by a subscript. Summing (4.18) over $j = 1, \dots, m$ we obtain for the solution u of (4.11) the representation:

$$u(x', x_n) = \sum_{j=0}^{m} \iint\limits_{E_n^+} D_n^j w(y', y_n) \cdot \widetilde{K}_j(x' - y', x_n + y_n)\, dy'\, dy_n , \tag{4.19}$$

where

$$\begin{cases} \widetilde{K}_j = - K_j - D_n K_{j+1} , \quad j = 1, \dots, m-1 , \\ \widetilde{K}_0 = - D_n K_1 , \ \ \widetilde{K}_m = K_m . \end{cases} \tag{4.20}$$

Using (4.19) and (4.9) we observe that if q is a non-negative integer having the same parity as $n - 1$ then

$$\widetilde{K}_j(x', x_n) = \Delta_{x'}^{\frac{1}{2}(n-1+q)} \widetilde{K}_{j,q}(x', x_n), \tag{4.21}$$

where $\widetilde{K}_{j,q}$ are kernels given by

$$\begin{cases} \widetilde{K}_{j,q} = - K_{j,q} - D_n K_{j+1,q} , \quad j = 1, \dots, m-1 , \\ \widetilde{K}_{0,q} = - D_n K_{1,q} , \quad \widetilde{K}_{m,q} = K_{m,q} . \end{cases} \tag{4.20)'$$

From (4.20)′ it is readily seen that the inequalities (4.10)-(4.10)′, satisfied by the kernels $K_{j,q}$, are also satisfied by the kernels $\widetilde{K}_{j,q}$.

Put:

$$u_j(x) = \iint\limits_{E_n^+} D_n^j w(y) \cdot \widetilde{K}_j(x' - y', x_n + y_n)\, dy \quad (x_n > 0) , \tag{4.22}$$

so that by (4.19) $u = \sum_0^m u_j$. To establish the lemma it will suffice to show that the inequalities (4.12) and (4.12)′ hold for u_j. We shall prove this for j odd. The proof for j even is similar.

Choose $q = 2m + n + 1$. From (4.22) and (4.21) we obtain after obvious integration by parts with respect to y':

$$(4.23)\qquad u_j(x) = \iint_{E_n^+} D_n^j w(y) \cdot \Delta_{y'}^{\frac{1}{2}(n-1+q)} \widetilde{K}_{j,q}(x'-y', x_n+y_n)\, dy$$

$$= \iint_{E_n^+} D_n^j \Delta_{y'}^{m-\frac{1}{2}(j+1)} w(y) \cdot \Delta_{y'}^{n+\frac{1}{2}(j+1)} \widetilde{K}_{j,q}(x'-y', x_n+y_n)\, dy$$

$$= -\sum_{i=1}^{n-1} \iint_{E_n^+} D_i D_n^j \Delta_{y'}^{m-\frac{1}{2}(j+1)} w \cdot D_i \Delta_{y'}^{n+\frac{1}{2}(j-1)} \widetilde{K}_{j,q}(x'-y', x_n+y_n)\, dy .$$

Differentiating (4.23) we thus obtain:

$$(4.23)'\qquad -D^\alpha u_j(x) =$$

$$= \sum_{i=1}^{n-1} \iint_{E_n^+} D_i D_n^j \Delta_{y'}^{m-\frac{1}{2}(j+1)} w \cdot D_i \Delta_{y'}^{n+\frac{1}{2}(j-1)} D_x^\alpha \widetilde{K}_{j,q}(x'-y', x_n+y_n)\, dy .$$

Suppose, first, that $|\alpha| = 2m$. Using the estimates (4.10)-(4.10)$'$ which, as was pointed out before, are also satisfied by the kernels $\widetilde{K}_{j,q}$ $(q=2m+n+1)$, we find that

$$(4.24)\qquad | D_i \Delta_{x'}^{n+\frac{1}{2}(j-1)} D^\alpha \widetilde{K}_{j,q}(x', x_n) | \leq c\, |x|^{-n}$$

with a constant c depending only on n, m and λ. Thus, applying the Sublemma to a typical integral of (4.23)$'$ it follows readily that

$$\| D^\alpha u_j \|_{L_p(E_n^+)} \leq \gamma_1\, c \sum_{|\beta|=2m} \| D^\beta w \|_{L_p(E_n^+)}$$

where γ_1 depends only on n and p. This yields (4.12).

Suppose, now, that $0 \leq |\alpha| \leq 2m-1$. From (4.10)-(4.10)$'$ one finds readily that in this case

$$(4.25)\qquad | D_i \Delta_{x'}^{n+\frac{1}{2}(j-1)} D^\alpha \widetilde{K}_{j,q}(x', x_n) | \leq \text{Const.}\, (|x|^{-(n-1)} + |x|^{2m})$$

with a constsnt depending only on n, m and λ. If, furthermore, the support of w is contained in $\overline{\Sigma}_R$ it follows easily from (4.23) and (4.25) that for $|\alpha| < 2m$:

$$\| D^\alpha u_j \|_{L_p(\Sigma_R)} \leq C_1 \sum_{|\beta|=2m} \| D^\beta w \|_{L_p(\Sigma_R)} \tag{4.26}$$

where C_1 is a constant depending only on n, m, λ, p and R. Since by the preceding (4.26) (with a suitable constant) holds also for $|\alpha| = 2m$ we conclude that the functions u_j (and consequently u) satisfy (4.12)$'$. This completes the proof of Lemma 4.3.

5. Preliminary regularity lemmas.

In this section we begin with the discussion of the regularity problems of weak solutions of elliptic equations in the framework of the L_p theory. We shall discuss both the problem of interior regularity (also for weak solutions of overdetermined elliptic systems), and the problem of regularity at the boundary for weak solutions of the Dirichlet problem,

We consider a linear elliptic differential operator A of order $2m$ (variable complex coefficients) defined in $\overline{G}$:

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha . \tag{5.1}$$

We denote by A' the leading part of A and by λ some constant ≥ 1 (ellipticity constant) such that

$$\frac{1}{\lambda} |\xi|^{2m} \leq |A'(x, \xi)| \leq \lambda |\xi|^{2m} \tag{5.1$'$}$$

for all real vectors ξ and $x \in \overline{G}$. We introduce the following

DEFINITION 5.1. *The coefficients of A will be said to satisfy* Condition $\{j; K\}$ *(in $\overline{G}$), j being a positive integer and $K > 0$, if*

(i) $$a_\alpha \in C^{|\alpha|+j-2m}(\overline{G}) \quad \text{for} \quad |\alpha| > 2m - j ,$$

whereas the remaining coefficients are measurable bounded functions in G.
(ii) *The following inequalities hold in G:*

$$|D^\beta a_\alpha| \leq K \quad \textit{for} \quad |\alpha| > 2m - j , \quad |\beta| \leq |\alpha| + j - 2m ,$$

and

$$|a_\alpha| \leq K \quad \textit{for} \quad |\alpha| \leq 2m - j .$$

We also introduce the scalar product notation $(f, g)_G$ to be used from now on throughout the paper:

$$(5.2) \qquad (f, g)_G = \int_G f \bar{g} \, dx .$$

Here f and g are two functions defined in G such that the integral (5.2) makes sense.

In this and in the following section the domains of definition G will be either the sphere S_R $(|x| < R)$ or the hemisphere Σ_R $(|x| < R, x_n > 0)$, We shall denote by $\partial_1 \Sigma_R$ the part of the boundary $\partial \Sigma_R$ situated on the hyperplane $x_n = 0$, and by $\partial_2 \Sigma_R$ the part of $\partial \Sigma_R$ situated on $|x| = R$ $(\partial \Sigma_R = \partial_1 \Sigma_R \cup \partial_2 \Sigma_R)$. We also recall that by δ_h we denote the difference quotient operator:

$$\delta_h u = \frac{u(x+h) - u(x)}{|h|},$$

$h = (h_1, \dots, h_n)$ being a real vector $\neq 0$.

We shall now state two basic regularity lemmas.

LEMMA 5.1. *Let A be an elliptic differential operator of order $2m$ defined in $\bar{S}_R$, with coefficients satisfying* Condition $\{1\ ; K\}$. *Let, further, u be a function belonging to $L_p(S_R)$, $p > 1$, such that for all functions $\varphi \in C_0^\infty(S_R)$ the following inequality holds:*

$$(5.3) \qquad |(u, A\varphi)_{S_R}| \leq C \|\varphi\|_{2m-1, L_{p'}(S_R)},$$

where p' is the exponent conjugate to p: $\dfrac{1}{p} + \dfrac{1}{p'} = 1$, and C is a constant. Then, there exists a positive number $r_0 < R$ and a constant c_0 such that

$$(5.4) \qquad \|\delta_h u\|_{L_p(S_{r_0})} \leq c_0 (C + \|u\|_{L_p(S_R)}),$$

for all sufficiently small vectors h. Both r_0 and *c_0 depend only on n, m, p, R, K and the ellipticity constant λ.*

LEMMA 5.2. *Let $A(x, D)$ be an elliptic differential operator of order $2m$ defined in $\bar{\Sigma}_R$, with coefficients satisfying* Condition $\{1; K\}$. *If $n = 2$ assume also that $A'(0, D)$ satisfies the «roots condition» introduced in* § 4. *Let, further, u be a function belonging to $L_p(\Sigma_R)$, $p > 1$, such that*

$$(5.5) \qquad |(u, A\varphi)_{\Sigma_R}| \leq C \|\varphi\|_{2m-1, L_{p'}(\Sigma_R)}$$

for all functions $\varphi \in C^\infty(\overline{\Sigma}_R)$ *satisfying the boundary conditions :*

$$(5.6)\qquad \begin{cases} D_n^j \varphi = 0 \quad on \quad \partial_1 \Sigma_R, \qquad j = 0, \dots, m-1. \\ \varphi \equiv 0 \quad in\ a\ neighborhood\ of\ \partial_2 \Sigma_R. \end{cases}$$

Then, there exist a positive nomber $r_0 < R$ *and a constant* c_0, *having the same dependence as in Lemma* 5.1, *such that*

$$(5.7)\qquad \| \delta_h u \|_{L_p(\Sigma_{r_0})} \leq c_0 (C + \| u \|_{L_p(\Sigma_R)}),$$

for all sufficiently small vectors h parallel to the hyperplane $x_n = 0$.

Proof of Lemma 5.1 *and Lemma* 5.2: We shall prove both lemmas at the same time. In the sequel σ_r will denote the sphere S_r in the case of Lemma 5.1, and the hemisphre Σ_r in the case of Lemma 5.2.

By our assumption the coefficients of A are measurable functions bounded by K in σ_R. Moreover, the highest order coefficients possess first order derivatives also bounded by K. Without loss of generality we may assume in the following that A contains no lower order terms. Indeed in the general case let $A = A' + A''$ where A' is the leading part and A'' contains only terms of order $\leq 2m - 1$. By Hölder's inequality

$$| (u, A'' \varphi)_{\sigma_R} | \leq K \| u \|_{L_p(\sigma_R)} \cdot \| \varphi \|_{2m-1, L_{p'}(\sigma_R)}.$$

Consequently the operator A could be replaced by A' which satisfies the conditions of the lemmas with C replaced by

$$C' = C + K \| u \|_{L_p(\sigma_R)}.$$

Let r be a positive number $\leq R/6$ to be fixed later on, and let ζ_r be a real C^∞ function such that $\zeta_r \equiv 1$ for $|x| \leq r/3$, $\zeta \equiv 0$ for $|x| \geq \frac{2}{3} r$. Set:

$$(5.8)\qquad u_r = \zeta_r u, \qquad x \in \overline{\sigma}_R,$$

and extend u_r as zero outside $\overline{\sigma}_R$. Let, further, v be and arbitrary function belonging to $C^\infty(\overline{\sigma}_r)$ which in the case of Lemma 5.2 ($\sigma_r = \Sigma_r$) also satisfies the boundary conditions

$$(5.9)\qquad D_n^j v = 0 \quad \text{for} \quad x_n = 0, \qquad j = 0, \dots, m-1.$$

We have:

$$(5.10)\qquad (u, A(\zeta_r v))_{\sigma_r} = (u, \zeta_r A v)_{\sigma_r} + (u, B v)_{\sigma_r} = (u_r, A v)_{\sigma_r} + (u, B v)_{\sigma_r},$$

where B is a linear differential operator of order $2m-1$ with coefficients which are linear combinations in $a_\alpha D^\beta \zeta_r$. Hence, making use of Hölder's inequality, we obtain

$$(5.10)' \qquad |(u, A(\zeta_r v))_{\sigma_r} - (u_r, Av)_{\sigma_r}| \leq c_2 \| u \|_{L_p(\sigma_r)} \| v \|_{2m-1, L_{p'}(\sigma_r)},$$

where here and in the following $c_2, c_3, \ldots,$ denote constants depending only on $n, m, p, R, K, \lambda,$ and r.

Consider now the function $\varphi = \zeta_r v$ extended as zero outside $\bar{\sigma}_r$. $\varphi \in C_0^\infty(S_R)$ in the case of Lemma 5.1, while $\varphi \in C^\infty(\bar{\Sigma}_R)$ and satisfies the boundary conditions (5.6) in the case of Lemma 5.2. Hence, applying the inequality (5.3) in the first case, and the inequality (5.5) in the second, we conclude that

$$(5.11) \qquad |(u, A(\zeta_r v))_{\sigma_r}| \leq c_3 C \| v \|_{2m-1, L_{p'}(\sigma_r)}.$$

Combining (5.11) with (5.10)' we thus get:

$$(5.12) \qquad |(u_r, Av)_{\sigma_r}| \leq c_4 \| v \|_{2m-1, L_{p'}(\sigma_r)} (C + \| u \|_{L_p(\sigma_r)})$$

for all functions $v \in C^\infty(\bar{\sigma}_r)$ which in the case of Lemma 5.2 also satisfy the boundary conditions (5.9).

Next, let h be an arbitrary vector such that $0 < |h| < r/6$. In the case of Lemma 5.2 h is restricted, in addition, to be of the form $h = (h_1, \ldots, h_{n-1}, 0)$. Define the function

$$(5.13) \qquad f_h(x) = |\delta_h u_r|^{p-1} \operatorname{sign}(\delta_h u_r) \text{ (5)}.$$

Then, $f_h \in L_{p'}(\sigma_R)$,

$$(5.13)' \qquad \| f_h \|_{L_{p'}(\sigma_R)} = \| \delta_h u_r \|^{p-1}_{L_p(\sigma_R)},$$

and the support of f_h is contained in $\bar{\sigma}_{R/3}$. Let, further, $\tilde{f}_h$ be a C^∞ function with support in $\sigma_{R/3}$ such that

$$(5.14) \qquad \| \tilde{f}_h - f_h \|_{L_{p'}(\sigma_R)} \leq \frac{1}{3} \| f_h \|_{L_{p'}(\sigma_R)}.$$

---

(5) As usual: sign $z = \dfrac{z}{|z|}$ if $z \neq 0$, sign $0 = 0$.

We now make use of the lammas established in § 4. Thus, in the case of Lemma 5.1 apply Lemma 4.1 to the elliptic operator $A^0 = A(0, D)$, function $f = \widetilde{f}_h$ and exponent p'. There exists by the lemma a function $v_h \in C^\infty(\overline{S}_R)$ such that

$$A^0 v_h = \widetilde{f}_h \quad \text{in} \quad \overline{S}_R, \tag{5.15}$$

and

$$\| v_h \|_{2m, L_{p'}(S_R)} \leq \gamma \, \| \widetilde{f}_h \|_{L_{p'}(S_R)}, \tag{5.15$'$}$$

where γ is a constant depending only on n, m, p, R and λ.

Similarly, in the case of Lemma 5.2, applying Lemma 4.2 it follows that there exists a function $v_h \in C^\infty(\overline{\Sigma}_R)$ such that

$$\begin{cases} A^0 v_h = \widetilde{f}_h & \text{in} \quad \Sigma_R \\ D_n^j v_h = 0 & \text{on} \quad \partial_1 \Sigma_R \ j = 0, \dots, m-1, \end{cases} \tag{5.16}$$

and

$$\| v_h \|_{2m, L_{p'}(\Sigma_R)} \leq \gamma \, \| \widetilde{f}_h \|_{L_{p'}(\Sigma_R)}, \tag{5.16$'$}$$

where γ is a constant having the same dependence as above.

Using (5.13)$'$ and (5.14) we observe that in both cases

$$\| v_h \|_{2m, L_{p'}(\sigma_R)} \leq 2\gamma \, \| \delta_h u_r \|^{p-1}_{L_p(\sigma_R)}. \tag{5.17}$$

Consider the function $\delta_{-h} v_h$. It is a well defined infinitely differentiable function in $\overline{\sigma}_r$. Moreover, in the case of Lemma 5.2 it also satisfies the boundary conditions (5.9). Hence, applying (5.12) to $v = \delta_{-h} v_h$ we have:

$$| (u_r, A(\delta_{-h} v_h))_{\sigma_r} | \leq c_4 \, \| \delta_{-h} v_h \|_{2m-1, L_{p'}(\sigma_r)} (C + \| u \|_{L_p(\sigma_r)}). \tag{5.18}$$

Also, one checks easily that

$$\| \delta_{-h} v_h \|_{2m-1, L_{p'}(\sigma_r)} \leq N \, \| v_h \|_{2m, L_{p'}(\sigma_R)}. \tag{5.19}$$

where N is a constant depending only on n (one can actually take $N = n$). Combining (5.18), (5.19) and (5.17) we thus get

$$| (u_r, A(\delta_{-h} v_h))_{\sigma_r} | \leq c_5 \, \| \delta_h u_r \|^{p-1}_{L_p(\sigma_r)} (C + \| u \|_{L_p(\sigma_r)}). \tag{5.20}$$

Put:

$$A_h = \frac{A(x+h, D) - A(x, D)}{|h|} = \sum_{|\alpha|=2m} \frac{a_\alpha(x+h) - a_\alpha(x)}{|h|} D^\alpha .$$

Using the relation $f\,\delta_{-h}\, g = \delta_{-h}(f g) - \delta_{-h} f . g(x-h)$, and noting that the support of u_r is contained in $\bar{\sigma}_{2r/3}$ while $|h| < r/6$, we readily obtain:

$$(5.21) \qquad (u_r, A(\delta_{-h} v_h))_{\sigma_r} = (u_r, \delta_{-h}(A v_h))_{\sigma_r} - (u_r, A_{-h} v_h(x-h))_{\sigma_r} =$$

$$= (\delta_h u_r, A v_h)_{\sigma_r} - (u_r, A_{-h} v(x-h))_{\sigma_r} .$$

Since the coefficients of A_h are bounded by $n K$ we have (using 5.17):

$$(5.22) \qquad |(u_r, A_{-h} v_h(x-h))_{\sigma_r}| \leq c_6 \| u_r \|_{L_p(\sigma_r)} \| v_h \|_{2m, L_{p'}(\sigma_R)} \leq$$

$$\leq c_7 \| u_r \|_{L_p(\sigma_r)} \| \delta_h u_r \|^{p-1}_{L_p(\sigma_r)} .$$

Thus, combining (5.20), (5.21) and (5.22), we get

$$(5.23) \qquad |(\delta_h u_r, A v_h)_{\sigma_r}| \leq |(u_r, A(\delta_{-h} v_h))_{\sigma_r}| + |(u_r, A_{-h} v_h(x-h))_{\sigma_r}| \leq$$

$$\leq c_8 \| \delta_h u_r \|^{p-1}_{L_p(\sigma_r)} (C + \| u \|_{L_p(\sigma_r)}) .$$

Write

$$(5.24) \qquad (\delta_h u_r, A v_h)_{\sigma_r} = (\delta_h u_r, A^0 v_h)_{\sigma_r} + (\delta_h u_r, (A - A^0) v_h)_{\sigma_r} .$$

Using (5.15) (resp. (5.16)) we have:

$$(5.25) \qquad (\delta_h u_r, A^0 v_h)_{\sigma_r} = (\delta_h u_r, \tilde{f}_h)_{\sigma_r} = (\delta_h u_r, f_h)_{\sigma_r} + (\delta_h u_r, \tilde{f}_h - f_h)_{\sigma_r} .$$

By (5.13):

$$(5.25)' \qquad (\delta_h u_r, f_h)_{\sigma_r} = \| \delta_h u_r \|^p_{L_p(\sigma_r)} .$$

Also, from Hölder's inequality and (5.14) we get

$$(5.25)'' \qquad |(\delta_h u_r, \tilde{f}_h - f_h)_{\sigma_r}| \leq \| \delta_h u_r \|_{L_p(\sigma_r)} \| \tilde{f}_h - f_h \|_{L_{p'}(\sigma_r)} \leq$$

$$\leq \frac{1}{3} \| \delta_h u_r \|^p_{L_p(\sigma_r)} .$$

Thus, combining (5.25), (5.25)′ and (5.25)″ we obtain

$$(5.26) \qquad |(\delta_h u_r, A^0 v_h)_{\sigma_r}| \geq$$

$$\geq (\delta_h u_r, f_h)_{\sigma_r} - |(\delta_h u_r, \tilde{f}_h - f_h)_{\sigma_r}| \geq \frac{2}{3} \| \delta_h u_r \|^p_{L_p(\sigma_r)} .$$

Now, it follows from our assumption that the coefficients of $A - A^0$ are bounded by $n K r$ in σ_r. From this and from (5.17) we get, using once more Hölder's inequality:

$$(5.27) \qquad |(\delta_h u_r, (A - A^0) v_h)_{\sigma_r}| \leq \| \delta_h u_r \|_{L_p(\sigma_r)} \| (A - A^0) v_h \|_{L_{p'}(\sigma_r)} \leq$$

$$\leq r K n^{2m+1} \| \delta_h u_r \|_{L_p(\sigma_r)} \| v_h \|_{2m, L_{p'}(\sigma_r)} \leq 2 \gamma r K n^{2m+1} \| \delta_h u_r \|^p_{L_p(\sigma_r)} .$$

We shall fix now r choosing

$$(5.28) \qquad r = \min \left(\frac{R}{6}, \frac{1}{6 \gamma K n^{2m+1}} \right) .$$

With this choice of r we obtain from (5.27) and (5.26):

$$(5.29) \qquad |(\delta_h u_r, A v_h)_{\sigma_r}| \geq |(\delta_h u_r, A^0 v_h)_{\sigma_r}| -$$

$$- |(\delta_h u_r, (A - A^0) v_h)_{\sigma_r}| \geq \frac{1}{3} \| \delta_h u_r \|^p_{L_p(\sigma_r)} .$$

Finally, from (5.23) and (5.29) we get

$$\frac{1}{3} \| \delta_h u_r \|^p_{L_p(\sigma_r)} \leq c_8 \| \delta_h u_r \|^{p-1}_{L_p(\sigma_r)} (C + \| u \|_{L_p(\sigma_r)}) ,$$

or,

$$(5.30) \qquad \| \delta_h u_r \|_{L_p(\sigma_r)} \leq 3 c_8 (C + \| u \|_{L_p(\sigma_r)}) .$$

If we now choose $r_0 = \dfrac{r}{6}$, $c_0 = 3 c_8$, and note that $\delta_h u_r = \delta_h u$ for $|x| \leq r_0$, $|h| \leq r_0$, we obtain from (5.30):

$$\| \delta_h u \|_{L_p(\sigma_{r_0})} \leq \| \delta_h u_r \|_{L_p(\sigma_r)} \leq c_0 (C + \| u \|_{L_p(\sigma_R)}) .$$

for all h sufficiently small (h parallel to $x_n = 0$ in the case of Lemma 5.2). This establishes the lemmas.

Lemma 5.1 and Lemma 5.2 yield (respectively) the following corollaries.

COROLLARY 5.1. *Suppose that the conditions of Lemma* 5.1 *hold. Then,* $u \in H^{\text{loc.}}_{1,L_p}(S_R)$. *Moreover, for every $R' < R$ the following inequality holds:*

$$\| u \|_{1,L_p(S_{R'})} \leq c_1 (C + \| u \|_{L_p(S_R)}) \tag{5.31}$$

where c_1 is a constant depending only on m, n, p, λ, K, R and R'.

COROLLARY 5.2. *Suppose that the conditions of Lemma* 5.2. *hold. Then, for every $R' < R$ the distribution derivatives $D_i u$, for $i = 1, \ldots, n-1$, are functions belonging to $L_p(\Sigma_{R'})$ such that*

$$\sum_{i=1}^{n-1} \| D_i u \|_{L_p(\Sigma_{R'})} \leq c_1 (C + \| u \|_{L_p(\Sigma_R)}), \tag{5.32}$$

where c_1 is a constant having the same dependence as above.

To prove Corollary 5.1 let $d = R - R'$ and denote by $S_{x^0,r}$ the sphere $|x - x^0| < r$. Apply Lemma 5.1 to u in $\bar{S}_{x^0,d}$ (after obvious translation of variable), $x^0 \in \bar{S}_{R'}$. From the lemma, combined with Lemma 3.3, it follows that there exist positive constants $r_0 < d$ and c_0, both r_0 and c_0 depend only on n, m, p, λ, K, R and d, such that $u \in H_{1,L_p}(S_{x^0,r_0})$ and

$$\| D_i u \|_{L_p(S_{x^0,r_0})} \leq c_0 (C + \| u \|_{L_p(S_R)}), \qquad i = 1, \ldots, n. \tag{5.33}$$

Covering $\bar{S}_{R'}$ by a finite number of spheres S_{x^0,r_0} $(x^0 \in \bar{S}_{R'})$, using (5.33), we conclude that $u \in H_{1,L_p}(S_{R'})$ and that, furthermore, the inequality (5.31) holds.

Corollary 5.2 follows similarly from Lemma 5.2, Lemma 3.3′ and Corollary 5.1. (6).

6. The basic regularity theorems.

We pass to the main regularity results in the framework of the L_p theory. The first theorem dealing with interior regularity is the following

THEOREM 6.1. *Let A be an elliptic operator of order $2m$* (7) *defined in $\bar{S}_R$, with coefficients satisfying Condition $\{j; K\}$, j being an integer such that*

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(6) It should be observed here that in the exceptional case $n = 2$ the operator $A'(x^0, D)$ satisfies the «roots condition» for every $x^0 \in \bar{\Sigma}_R$. This follows by a simple continuity argument from our assumption that this is true for $x^0 = 0$.

(7) The theorem also holds for the elliptic operators in two variables of odd order.

$1 \leq j \leq 2m$. *Let, further, u be a function such that* $u \in L_q^{\text{loc.}}(S_R)$ *for some* $q > 1$ *and such that for all functions* $\varphi \in C_0^\infty(S_R)$ *the following inequality holds :*

$$(6.1) \qquad |(u, A\varphi)_{S_R}| \leq C \|\varphi\|_{2m-j, L_{p'}(S_R)},$$

where p' *is some fixed number* > 1 *and* C *is a constant.*

Denote by p *the exponent conjugate to* p' : $\frac{1}{p} + \frac{1}{p'} = 1$. *Then,* $u \in H_{j,L_p}^{\text{loc.}}(S_R)$. *Moreover, if* $0 < R' < R$ *and* $R_1 = (R + R')/2$, *then*

$$(6.2) \qquad \|u\|_{j, L_p(S_{R'})} \leq c_1 (C + \|u\|_{L_p(S_{R_1})}).$$

where c_1 *is a constant depending only on* n, m, p, λ, K, R *and* R'.

Proof: Assume first that $j = 1$ and that $q \geq p$, so that we also have $u \in L_p^{\text{loc.}}(S_R)$. In this case the theorem follows from Corollary 5.1 applied to u in S_{R_1}.

Next, let $j = 1$ but $1 < q < p$. To prove the theorem in this case it will suffice to show that actually $u \in L_p^{\text{loc.}}(S_R)$, thus reducing the proof to the case just established.

Now, let q' be the exponent conjugate to q. Since $q' > p'$ it follows from (6.1) that we also have

$$|(u, A\varphi)_{S_R}| \leq \text{Const.} \|\varphi\|_{2m-1, L_{q'}(S_R)},$$

for all functions $\varphi \in C_0^\infty(S_R)$. Hence, by the result just established (p replaced by q) we conclude that $u \in H_{1,L_q}^{\text{loc.}}(S_R)$. Invoking Sobolev's inequalities (Theorem 3.2) it follows that $u \in L_p^{\text{loc.}}(S_R)$ if either $q \geq n$ or $q < n$ but $q_1 = q\,n/(n - q) \geq p$. On the other hand if $q < n$ and $q_1 < p$ Sobolev's inequalities give only that $u \in L_{q_1}^{\text{loc.}}(S_R)$. In this case (noting that $q_1 > q$) we repeat the same argument with q replaced by q_1; either arriving at the desired result $u \in L_p^{\text{loc.}}(S_R)$, or at least proving that $u \in L_{q_2}^{\text{loc.}}(S_R)$ with $q_2 > q_1$. Carrying on in this manner we obtain after a finite number of steps that $u \in L_p^{\text{loc.}}(S_R)$. This yields the theorem for $j = 1$.

To prove the theorem for $j \geq 2$ we use induction — supposing the theorem is true for $j - 1$ $(1 \leq j - 1 < 2m)$ we shall prove it for j.

We first observe that without loss of generality we may assume that A contains no terms of order $\leq 2m - j$:

$$A = \sum_{2m-j < |\alpha| \leq 2m} a_\alpha D^\alpha.$$

(In the general case one can omit from A all terms of order $\leq 2m - j$, the resulting operator will still satisfy an inequality of the form (6.1) in S_{R_1} with C replaced by $C + \| u \|_{L_p(S_{R_1})})$. Put $b_{\alpha,i} = D_i a_\alpha$ and define

$$B_i = \sum_{2m-j<|\alpha|\leq 2m} b_{\alpha,i} D^\alpha .$$

From the induction assumption it follows that $u \in H^{\text{loc.}}_{j-1,L_p}(S_R)$, and in particular that $D_i u \in L_p^{\text{loc.}}(S_R)$. Set $R_0 = (R' + R_1)/2$. For $\varphi \in C_0^\infty(S_{R_0})$ we have:

$$(6.3) \qquad (D_i u , A \varphi)_{S_{R_0}} = (u , D_i A \varphi)_{S_{R_0}} = (u , A D_i \varphi)_{S_{R_0}} + (u , B_i \varphi)_{S_{R_0}} .$$

Now, from (6.1) we get

$$(6.4) \qquad |(u , A D_i \varphi)_{S_{R_0}}| \leq C \| D_i \varphi \|_{2m-j,L_{p'}(S_{R_0})} \leq C \| \varphi \|_{2m-(j-1),L_{p'}(S_{R_0})} .$$

Also, since

$$(u , B_i \varphi)_{S_{R_0}} = \sum_{2m-j<|\alpha|\leq 2m} (u , b_{\alpha,i} D^\alpha \varphi)_{S_{R_0}} = (-1)^{j-1} \Sigma (D^\beta (\bar{b}_{\alpha,i} u) , D^\gamma \varphi)_{S_{R_0}} ,$$

where $D^\beta D^\gamma = D^\alpha$ with $|\beta| = j-1$, $|\gamma| \leq 2m - (j-1)$, we find readily that

$$(6.5) \qquad |(u , B_i \varphi)_{S_{R_0}}| \leq \mu K \| u \|_{j-1,L_p(S_{R_0})} \| \varphi \|_{2m-(j-1),L_{p'}(S_{R_0})}$$

where μ is a constant depending only on n and m.

Combining (6.3), (6.4) and (6.5) we obtain the inequality

$$(6.6) \qquad |(D_i u , A \varphi)_{S_{R_0}}| \leq C_1 \| \varphi \|_{2m-(j-1),L_{p'}(S_{R_0})}$$

which holds for all functions $\varphi \in C_0^\infty(S_{R_0})$, $i = 1, \dots, n$, and where

$$(6.6') \qquad C_1 = C + \mu K \| u \|_{j-1,L_p(S_{R_0})} .$$

The inequality (6.6) shows that the derivatives $D_i u$ satisfy the conditions of the theorem in S_{R_0} with j replaced by $j - 1$. Hence, using the induction assumption, we conclude that $D_i u \in H^{\text{loc.}}_{j-1,L_p}(S_{R_0})$ (and consequently $D_i u \in H^{\text{loc.}}_{j-1,L_p}(S_R)$). Also, denoting by c_k constants which depend only on n, m, p, λ, K, R and R', we have:

$$(6.7) \qquad \| D_i u \|_{j-1,L_p(S_{R'})} \leq c_2 (C + \| u \|_{j-1,L_p(S_{R_0})}) .$$

Thus (using Lemma 3.2) we infer that $u \in H^{\text{loc.}}_{j,L_p}(S_R)$ and

$$\| u \|_{j,L_p(S_{R'})} \leq c_3 (C + \| u \|_{j-1,L_p(S_{R_0})}) . \tag{6.8}$$

Finally, noting that by the induction assumption we also have:

$$\| u \|_{j-1,L_p(S_{R_0})} \leq c_4 (C + \| u \|_{L_p(S_{R_1})}) , \tag{6.8$'$}$$

we derive from (6.8) and (6.8)$'$ the desired inequality (6.2). This establishes the theorem.

We now pass to the more refined result yielding regularity at the boundary. We shall first deal with functions defined in the hemisphere Σ_R, and with the regularity of such functions near the flat part of the boundary.

THEOREM 6.2. *Let $A(x, D)$ be an elliptic operator of order $2m$ defined in $\overline{\Sigma}_R$, with coefficients satisfying Condition $\{j ; K\}$, j being an integer such that $1 \leq j \leq 2m$. If $n = 2$ assume in addition that $A'(x^0, D)$ satisfies the «roots condition» for every fixed $x^0 \in \overline{\Sigma}_R$.*

Let, further, u be a function belonging to $L_q(\Sigma_R)$ for some $q > 1$ such that

$$| (u , A \varphi)_{\Sigma_R} | \leq C \| \varphi \|_{2m-j, L_{p'}(\Sigma_R)} , \tag{6.9}$$

for all functions $\varphi \in C^\infty(\overline{\Sigma}_R)$ satisfying the boundary conditions:

$$\begin{cases} D_n^k \varphi = \quad \text{on} \quad \partial_1 \Sigma_R \quad \text{for} \quad k = 0 , \ldots , m-1 , \\ \varphi \equiv 0 \quad \text{in some neighborhood of } \partial_2 \Sigma_R , \end{cases} \tag{6.9$'$}$$

where p' is some fixed number > 1 and C is a constant.

Denote by p the exponent conjugate to p'. Then, $u \in H_{j,L_p}(\Sigma_{R'})$ for every $R' < R$. Moreover, setting $R_1 = (R + R')/2$, we have:

$$\| u \|_{j, L_p(\Sigma_{R'})} \leq c_1 (C + \| u \|_{L_p(\Sigma_{R_1})}) \tag{6.10}$$

where c_1 is a constant depending only on n, m, p, λ, K, R and *R'.*

The proof of the theorem depends on Corollary 5.2 and the following lemma.

LEMMA 6.1. *Let $u \in L_p(\Sigma_R)$, $p > 1$. Suppose that the distribution derivatives $D_i u$ for $i = 1, \ldots, n-1$ are functions belonging to $L_p(\Sigma_R)$. Suppose, moreover, that there exist an integer $l > 0$ and a constant C_1 such that*

$$| (u , D_n^l \varphi)_{\Sigma_R} | \leq C_1 \| \varphi \|_{l-1, L_{p'}(\Sigma_R)} \tag{6.11}$$

for all functions $\varphi \in C_0^\infty(\Sigma_R)$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$. *Then, the (distribution) derivative* $D_n u$ *is also a function belonging to* $L_p(\Sigma_{R'})$ *for every* $R' < R$, *and*

$$(6.12) \qquad \| D_n u \|_{L_p(\Sigma_{R'})} \le c \left(C_1 + \sum_{i=1}^{n-1} \| D_i u \|_{L_p(\Sigma_R)} + \| u \|_{L_p(\Sigma_R)} \right)$$

where c *is a constant depending only on* n, l, p, R and R'.

The lemma in the special case $p=2$ is due to Lions (see [19]). For general p (and also its analogue for Hölder classes of functions) the lemma is given (essentially) in Agmon-Douglis-Nirenberg [3] where, however, instead of (6.11) it is assumed that

$$(6.13) \qquad D_n^l u = \sum_{|\alpha| \le l-1} D^\alpha f_\alpha ,$$

where f_α are certain functions belonging to $L_p(\Sigma_R)$ and (6.13) is understood in the weak (distribution) sense. Clearly, (6.13) implies an inequality of the form (6.11). The converse implication is also true (see [19]). For the sake of completeness we shall furnish in the sequel a variant proof of Lemma 6.1 which is not using the representation formula (6.13).

We shall now give the

Proof of Theorem 6.2. By the interior regularity result of Theorem 6.1 we know already that $u \in H^{\text{loc.}}_{j,L_p}(\Sigma_R)$. In the following we shall assume that the operator A contains no terms of order $\le 2m - j$. This (as in the proof of Theorem 6.1) entails no loss of generality. Given $0 < R' < R$, we set $R_1 = (R' + R)/2$ and $R_0 = (R' + R_1)/2$. We also denote by c_k constants which depended only on n, m, p, λ, K, R and R'.

To prove the theorem suppose first that $j = 1$ and that $q \ge p$ so that, in particular, $u \in L_p(\Sigma_R)$. Since in this case u satisfies the conditions of Lemma 5.2 in S_{R_1}, applying Corollary 5.2, we conclude that

$$D_i u \in L_p(\Sigma_{R_0}) \qquad \text{for} \qquad i = 1, \dots, n-1 ,$$

and that

$$(6.14) \qquad \sum_{i=1}^{n-1} \| D_i u \|_{L_p(\Sigma_{R_0})} \le c_2 (C + \| u \|_{L_p(\Sigma_{R_1})}) .$$

To complete the proof in this case we need only to show that $D_n u \in L_p(\Sigma_{R'})$ and that

$$(6.14)' \qquad \| D_n u \|_{L_p(\Sigma_{R'})} \le c_3 (C + \| u \|_{L_p(\Sigma_{R_1})}) .$$

To this end write A in the form:

$$A = aD_n^{2m} + \sum_{i=1}^{n-1} \sum_{|\alpha|=2m-1} a_{\alpha,i} D_i D^\alpha , \tag{6.15}$$

where $a_{\alpha,i}$ is either a coefficient in A or zero. Let $\varphi \in C_0^\infty(\Sigma_{R_0})$. We have:

$$\begin{aligned}(\bar{a}\, u , D_n^{2m} \varphi)_{\Sigma_{R_0}} &= (u , a D_n^{2m} \varphi)_{\Sigma_{R_0}} \\ &= (u , A \varphi)_{\Sigma_{R_0}} + \sum_{i=1}^{n-1} \sum_{|\alpha|=2m-1} (D_i (\bar{a}_{\alpha,i}\, u) , D^\alpha \varphi)_{\Sigma_{R_0}} .\end{aligned} \tag{6.16}$$

Combining (6.16), (6.14) and (6.9) we infer

$$| (\bar{a}\, u , D_n^{2m} \varphi)_{\Sigma_{R_0}} | \leq c_4 (C + \| u \|_{L_p(\Sigma_{R_1})}) \| \varphi \|_{2m-1, L_{p'}(\Sigma_{R_0})} \tag{6.17}$$

for all functions $\varphi \in C_0^\infty(\Sigma_{R_0})$.

Applying now Lemma 6.1 to the function $\bar{a}\, u$ in Σ_{R_0} we conclude that $D_n (\bar{a}\, u) \in L_p(\Sigma_{R'})$ and that

$$\| D_n (\bar{a}\, u) \|_{L_p(\Sigma_{R'})} \leq c_5 (C + \| u \|_{L_p(\Sigma_{R_0})} + \sum_{i=1}^{n-1} \| D_i (\bar{a}\, u) \|_{L_p(\Sigma_{R_0})}) . \tag{6.18}$$

Combining (6.18) and (6.14) (using $\lambda^{-1} \leq |a| \leq \lambda$, $|D_i a| \leq K$) we conclude that $D_n u \in L_p(\Sigma_{R'})$ and that (6.14)$'$ holds. This yields the theorem in the case considered.

Next, suppose that $j = 1$ but that $1 < q < p$. We shall reduce this case to the preceding one by showing that actually $u \in L_p(\Sigma_{R'})$ for every $R' < R$. We proceed as in the proof of Theorem 6.1. By the above argument (p replaced by q) we have $u \in H_{1,L_q}(\Sigma_{R'})$. Hence, using Sobolev's inequalities we conclude that $u \in L_p(\Sigma_{R'})$ if $q \geq n$, or if $q < n$ but $q_1 = q\, n/(n-q) \geq p$. On the other hand, if $q < n$ and $q < q_1 < p$ we conclude that $u \in L_{q_1}(\Sigma_R)$. Repeating in the last case the same argument with q replaced by q_1 etc., we conclude after a finite number of steps that $u \in L_p(\Sigma_{R'})$ for every $R' < R$. This completes the proof of the theorem for $j = 1$.

Finally, for $j \geq 2$ we use induction — assuming the theorem is true for $j-1$ $(1 \leq j-1 < 2m)$ we establish it for j. From the induction assumption we note that $u \in H_{j-1,L_p}(\Sigma_{R'})$ so that in particular $D_i u \in L_p(\Sigma_{R'})$ for every $R' < R$.

Consider a derivative $D_i u$ with $i \neq n$. Let φ be a function belonging to $C^\infty(\bar{\Sigma}_R)$ with support in $\bar{\Sigma}_{R_0}$, such that

$$(6.19) \qquad D_n^k \varphi = 0 \quad \text{for} \quad x_n = 0\,, \qquad k = 0, \ldots, m-1\,.$$

We now proceed to estimate $(D_i u, A\varphi)_{\Sigma_{R_0}}$ in exactly the same manner as in the proof of Theorem 6.1. Using the fact that $D_i \varphi$, $i \neq n$, satisfies (6.9)$'$, wc need only to rewrite relations (6.3) to (6.6), replacing everywhere the sphere S_{R_0} by the hemisphere Σ_{R_0}. Rewriting the final inequality (6.6) we thus conclude that

$$(6.20) \qquad |(D_i u, A\varphi)_{\Sigma_{R_0}}| \leq C_1 \|\varphi\|_{2m-(j-1), L_{p'}(\Sigma_{R_0})}$$

for $i = , \ldots, n-1$ and all functions $\varphi \in C^\infty(\bar{\Sigma}_R)$ with support in $\bar{\Sigma}_{R_0}$, satifying the boundary conditions (6.19). Here C_1 is the constant (6.6)$'$.

Hence the derivatives $D_i u\,(i \neq n)$ satisfy the conditions of Theorem 6.2 in Σ_{R_0} with j replaced by $j-1$. Applying the induction, setting $R'' = (R' + R_0)/2$, we infer that

$$(6.21) \qquad \sum_{i=1}^{n-1} \|D_i u\|_{j-1, L_p(\Sigma_{R''})} \leq c_6 (C + \|u\|_{j-1, L_p(\Sigma_{R_0})}) \leq c_7 (C + \|u\|_{L_p(\Sigma_{R_1})})\,.$$

From (6.21) and from the validity of the theorem for $j-1$ it follows that all derivatives $D^\alpha u$ such that $0 \leq |\alpha| \leq j$, $\alpha \neq (0, \ldots, 0, j)$ belong to $L_p(\Sigma_{R'})$ and satisfy

$$(6.22) \qquad \|D^\alpha u\|_{L_p(\Sigma_{R'})} \leq c_8 (C + \|u\|_{L_p(\Sigma_{R_1})})\,.$$

To complete the proof of the theorem we need only to show that $D_n^j u \in L_p(\Sigma_{R'})$ and satisfies (6.22). This we do again with the aid of Lemma 6.1. Write A in the form:

$$(6.23) \qquad A = a D_n^{2m} + \sum_{i=1}^{n-1} \sum_{|\alpha| = 2m-1} a_{\alpha,i} D_i D^\alpha + \sum_{2m-j \leq |\alpha| \leq 2m-1} a_\alpha D^\alpha.$$

Let $\varphi \in C_0^\infty(\Sigma_{R''})$. Using integration by parts, we have:

$$(6.24) \qquad (-1)^{j-1} (D_n^{j-1}(\bar{a} u), D_n^{2m-j+1} \varphi)_{\Sigma_{R''}} = (u, a D_n^{2m} \varphi)_{\Sigma_{R''}}$$
$$= (u, A\varphi)_{\Sigma_{R''}} + \sum_{i=1}^{n-1} \sum_{|\alpha| = 2m-1} (D_i(\bar{a}_{\alpha,i} u), D^\alpha \varphi)_{\Sigma_{R''}} -$$
$$- \sum_{2m-j \leq |\alpha| \leq 2m-1} (\bar{a}_\alpha u, D^\alpha \varphi)_{\Sigma_{R''}}\,.$$

Consider a typical term in the first sum on the right of (6.24). Writing $D^\alpha = D^\beta D^\gamma$, with $|\beta| = j-1$ and $|\gamma| = 2m-j$, we find readily that

$$(6.24)' \qquad |(D_i(\bar{a}_{\alpha,i}u), D^\alpha\varphi)_{\Sigma_{R''}}| = |(D^\beta D_i(a_{\alpha,i}u), D^\gamma\varphi)_{\Sigma_{R''}}|$$
$$\leq c_9(\|D_i u\|_{j-1,L_p(\Sigma_{R''})} + \|u\|_{L_p(\Sigma_{R''})})\|\varphi\|_{2m-j,L_{p'}(\Sigma_{R''})}.$$

Similarly, for a typical term in the last sum on the right of (6.24) we find (integrating $|\alpha| - (2m-j)$ times by parts):

$$(6.24)'' \qquad |(\bar{a}_\alpha u, D^\alpha\varphi)_{\Sigma_{R''}}| \leq c_{10}\|u\|_{j-1,L_p(\Sigma_{R''})}\|\varphi\|_{2m-j,L_{p'}(\Sigma_{R''})}.$$

Combining (6.24), (6.24)′, (6.24)″ and (6.21) we thus get for all functions $\varphi \in C_0^\infty(\Sigma_{R''})$:

$$(6.25) \qquad |D_n^{j-1}(\bar{a}u), D_n^{2m-j+1}\varphi)_{\Sigma_{R''}}|$$
$$\leq c_{11}\left(C + \sum_{i=1}^{n-1}\|D_i u\|_{j-1,L_p(\Sigma_{R''})} + \|u\|_{j-1,L_p(\Sigma_{R''})}\right)\|\varphi\|_{2m-1,L_{p'}(\Sigma_{R''})} \leq$$
$$\leq c_{12}(C + \|u\|_{L_p(\Sigma_{R_1})})\|\varphi\|_{2m-j,L_{p'}(\Sigma_{R''})}.$$

Applying, now, Lemma 6.1 to the function $D_n^{j-1}(\bar{a}u)$ in $\Sigma_{R''}$ with $l = 2m-j+1$, we conclude that $D_n^j(\bar{a}u) \in L_p(\Sigma_{R'})$ and that

$$(6.26) \qquad \|D_n^j(\bar{a}u)\|_{L_p(\Sigma_{R'})} \leq c_{13}(C + \|u\|_{L_p(\Sigma_{R_1})}).$$

Finally, from (6.26) and (6.22) it follows that $D_n^j u \in L_p(\Sigma_{R'})$ and that $D_n^j u$ satisfies an inequality of the form (6.22). This completes the proof of the theorem.

We shall conclude this section presenting a

Proof of Lemma 6.1. We first note that by an obvious approximation argument the inequality

$$(6.27) \qquad |(u, D_n^l\varphi)_{\Sigma_R}| \leq C_1\|\varphi\|_{l-1,L_{p'}(\Sigma_R)}$$

holds not only for $\varphi \in C_0^\infty(\Sigma_R)$, but also for all functions $\varphi \in C^l(\Sigma_R)$ with compact support in Σ_R. Moreover, we claim that (6.27) holds for all functions $\varphi \in C^l(\bar{\Sigma}_R)$ satisying the boundary conditions:

$$(6.27)' \qquad \begin{cases} D_n^j\varphi = 0 & \text{on} \quad \partial_1\Sigma_R, \quad j = 0, \dots, l, \\ \varphi \equiv 0 & \text{in a neighborhood of } \partial_2\Sigma_R. \end{cases}$$

Indeed, if φ is such a function and $\varepsilon > 0$, define:

$$\begin{cases} \varphi_\varepsilon(x) = \varphi(x-\varepsilon) & \text{for} \quad x \in \Sigma_R, \quad x \geq \varepsilon, \\ \varphi_\varepsilon(x) \equiv 0 & \text{for} \quad x \in \Sigma_R, \quad x < \varepsilon. \end{cases}$$

If ε is sufficiently small then φ_ε will be a C^l function with compact support in Σ_R. Hence, by the previous remark, (6.27) holds for φ_ε. Letting $\varepsilon \to 0$ we establish the same for φ.

Now, define u as zero in $x_n > 0$, $|x| \geq R$. Then, extend u into the half-space $x_n < 0$, putting

$$u(x', x_n) = \sum_{j=1}^{2l+1} \lambda_j u\left(x', -\frac{x_n}{j}\right) \quad \text{for} \quad x_n < 0, \tag{6.28}$$

$x' = (x_1, \dots, x_{n-1})$, where the constants λ_j are chosen so that

$$\sum_{j=1}^{2l+1} \lambda_j \left(-\frac{1}{j}\right)^k = 1 \quad \text{for} \quad k = -1, 0, 1, \dots, 2l-1. \tag{6.28)'$$

Clearly $u \in L_p(S_R)$ and

$$\| u \|_{L_p(S_R)} \leq \gamma_1 \| u \|_{L_p(\Sigma_R)}. \tag{6.29}$$

where here and in the following $\gamma_1, \dots, \gamma_5$, denote constants depending only on l. Also, the distribution derivatives $D_i u$, for $i \neq n$ are functions belon ging to $L_p(S_R)$ such that

$$\| D_i u \|_{L_p(S_R)} \leq \gamma_2 \| D_i u \|_{L_p(\Sigma_R)}. \tag{6.29)'$$

Let, now, χ be an arbitrary function of $C_0^\infty(S_R)$, extended as zero outside S_R. Write

$$(u, D_n^{2l} \chi)_{S_R} = \int_{x_n>0} u \overline{D_n^{2l} \chi}\, dx + \int_{x_n<0} u \overline{D_n^{2l} \chi}\, dx. \tag{6.30}$$

Using (6.28) to transform the last integral in (6.30), we find that

$$(u, D_n^{2l} \chi)_{S_R} = \int_{x_n>0} u \overline{D_n^{2l} \chi^*}\, dx, \tag{6.30)'$$

where

$$\chi^*(x', x_n) = \chi(x', x_n) + \sum_{j=1}^{2l+1} \lambda_j \left(\frac{1}{j}\right)^{2l-1} \chi(x', -jx_n). \tag{6.31}$$

It is readily seen from (6.31) and (6.28)′ that χ^* is a C^∞ function with support in S_R such that

$$D_n^j \chi^* = 0 \quad \text{for} \quad x_n = 0, \quad j = 0, \dots, 2l. \tag{6.31'}$$

Putting $\varphi = D_n^l \chi^*$, we write (6.30′) in the form:

$$(u, D_n^{2l} \chi)_{S_R} = (u, D_n^l \varphi)_{\Sigma_R}. \tag{6.32}$$

Since, now, $\varphi \in C^\infty(\bar{\Sigma}_R)$ and satisfies the boundary conditions (6.27)′ it follows from the preceding that it satisfies (6.27). In terms of χ (using (6.31), (6.32)) we have

$$|(u, D_n^{2l} \chi)_{S_R}| \leq \gamma_3 C_1 \| \chi \|_{2l-1, L_{p'}(\Sigma_R)}. \tag{6.33}$$

Next, for $i \neq n$, we have:

$$(u, D_i^{2l} \chi)_{S_R} = -(D_i u, D_i^{2l-1} \chi)_{S_R},$$

from which, using (6.29)′, we obtain that

$$|(u, D_i^{2l} \chi)_{S_R}| \leq \gamma_4 \| D_i u \|_{L_p(\Sigma_R)} \| \chi \|_{2l-1, L_{p'}(S_R)}. \tag{6.33'}$$

Let $A = D_1^{2l} + \dots + D_n^{2l}$. Combining (6.33) and (6.33)′ we conclude that

$$|(u, A\chi)_{S_R}| \leq \gamma_5 \left(C_1 + \sum_{i=1}^{n-1} \| D_i u \|_{L_p(\Sigma_R)}\right) \| \chi \|_{2l-1, L_{p'}(\Sigma_R)} \tag{6.34}$$

for all functions $\chi \in C_0^\infty(S_R)$.

Since A is elliptic, the inequality (6.34) allows us to apply Theorem 6.1 (Corollary 5.1) to u in S_R. We conclude that $u \in H_{1,L_p}^{\text{loc.}}(S_R)$ and that for every $R' < R$ (using (6.29))

$$\| D_n u \|_{L_p(S_{R'})} \leq c \left(C + \sum_{i=1}^{n-1} \| D_i u \|_{L_p(\Sigma_R)} + \| u \|_{L_p(\Sigma_R)}\right),$$

where c is a constant depending only on n, l, p, R and R'. This establishes the lemma.

7. The interior regularity in L_p of weak solutions of elliptic equations and overdetermined systems.

Let $\{A_i(x, D)\}_{i=1}^N$ be a system of N linear differential operators of respective order m_i:

$$A_i(x, D) = \sum_{|\alpha| \leq m_i} a_\alpha^i(x) D^\alpha \qquad (i = 1, \dots, N), \tag{7.1}$$

defined in a closed bounded domain $\overline{G}$. We shall say that the system $\{A_i\}$ is elliptic in $\overline{G}$ if there exists no real vector $\xi \neq 0$ and a point $x \in \overline{G}$, such that

$$\sum_{|\alpha| = m_i} a_\alpha^i(x) \xi^\alpha = 0 \quad \text{for} \quad i = 1, \dots, N \quad (\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}). \tag{7.2}$$

If the leading coefficients of A_i are continuous in $\overline{G}$, ellipticity implies that there exists a constant $\lambda \geq 1$ such that

$$\frac{1}{\lambda^2} \leq \sum_{i=1}^N \Big| \sum_{|\alpha| = m_i} a_\alpha^i(x) \xi^\alpha \Big|^2 \leq \lambda^2, \tag{7.2$'$}$$

for all real unit-vectors ξ and $x \in \overline{G}$. We term such a constant an ellipticity constant of the system.

For an overdetermined system of operators having the same order the above definition of ellipticity coincides with given by Schwartz [29] (see also Hörmander [15]). We point out, however, that we are not imposing the restriction that the operators A_i be of the same order.

In the following $\{A_i\}_{i=1}^N$ will denote either an elliptic operator ($N = 1$) or an elliptic overdetermined system ($N \geq 2$) defined in $\overline{G}$ and given by (7.1). The formally adjoint A_i^* of A_i is the operator

$$A_i^*(x, D) u = \sum_{|\alpha| \leq m_i} (-1)^{|\alpha|} D^\alpha \big(\overline{a_\alpha^i(x)} u\big). \tag{7.3}$$

It is a differential operator in the ordinary sense if $a_\alpha^i \in C^{|\alpha|}(\overline{G})$. Clearly the system $\{A_i^*\}$ will also be elliptic.

We shall consider a weak solution u of the adjoint system

$$A_i^* u = f_i, \quad i = 1, \dots, N, \tag{7.4}$$

in the sense that

$$(u, A_i \varphi)_G = (f_i, \varphi)_G, \; i = 1, \dots, N, \tag{7.5}$$

for all functions $\varphi \in C_0^\infty(G)$. Note that (7.5) has a sense when the coefficients of A_i are merely measurable bounded functions.

The main interior L_p regularity result for such weak solutions is the following

THEOREM 7.1. *Let u be a function belonging to $L_q^{\text{loc.}}(G)$ for some $q > 1$. Suppose that u satisfies (7.5) where f_i $(i = 1, \dots, N)$ are given functions belonging to $L_p^{\text{loc.}}(G)$, $p > 1$, and where $\{A_i\}_{i=1}^N$ $(N \geq 1)$ is the elliptic system introduced above. Assume also that the coefficients of A_i satisfy Condition $\{l ; K\}$ in $\bar{G}$, l being some positive integer* [8], *and put $j = \min(l, m_1, \dots, m_N)$. Then, $u \in H_{j,L_p}^{\text{loc.}}(G)$. Moreover, if G_0, G_1 are any two subdomains G_0, G_1, such that $\bar{G}_0 \subset G_1 \subset \bar{G}_1 \subset G$, then*

$$(7.6) \qquad \| u \|_{j,L_p(G_0)} \leq c \left(\sum_{i=1}^N \| f_i \|_{L_p(G_1)} + \| u \|_{L_p(G_1)} \right),$$

where c is a constant depending only on n, $\max m_i$, p, N, K, the ellipticity constant λ and the domains.

Proof: Put $m_0 = \min m_i$, $m = \max m_i$, and let d be the distance between ∂G_0 and ∂G_1. Denote by $\bar{A}_i$ the differential operator with coefficients complex conjugate to those of A_i. Given a point $x^0 \in \bar{G}_0$, define:

$$A_{x^0}(x, D) = \sum_{i=1}^N A_i(x, D) \bar{A}_i(x^0, D) \Delta^{m-m_i}$$

where Δ is the Laplacean. A_{x^0} is a linear differential operator of order $2m$ with coefficients satisfying Condition $\{l ; c_0 K\}$ in $\bar{G}$, c_0 being some constant depending only on n, m, and N. Also, A_{x^0} is elliptic at x^0 and consequently, by continuity, is elliptic in some neighborhood of x^0. More precisely, since the coefficients of the leading part A'_{x^0} possess first derivatives bounded by $c_0 K$, it is readily seen that there exists a positive number $\varrho \leq d$, ϱ depending only on n, m, N, K, λ and d, such that

$$\frac{1}{2\lambda^2} | \xi |^{2m} \leq | A'_{x^0}(x, \xi) | \leq 2\lambda^2 | \xi |^{2m}$$

for $| x - x^0 | \leq \varrho$ and all real vectors ξ. Thus, denoting by $S_{x^0,r}$ the sphere $| x - x^0 | < r$, A_{x^0} is elliptic in $\bar{S}_{x^0,\varrho}$ and $2\lambda^2$ can serve as its ellipticity constant.

---

[8] Condition $\{l ; K\}$ for the coefficients of A_i is defined as in § 5 (Def. 5.1) except that $2m$ should be replaced by the order m_i of A_i.

Now, let $\varphi \in C_0^\infty(S_{x^0,\varrho})$. By (7.5) we have:

$$(u\,,\,A_i(x\,,D)\,\bar{A}_i(x^0,D)\,\Delta^{m-m_i}\,\varphi)_{S_{x^0,\varrho}} = (f_i\,,\,\bar{A}_i(x^0,D)\,\Delta^{m-m_i}\,\varphi)_{S_{x^0,\varrho}},$$

which after summation yields:

$$(u\,,\,A_{x^0}\,\varphi)_{S_{x^0,\varrho}} = \sum_{i=1}^{N}(f_i\,,\,\bar{A}_i(x^0,D)\,\Delta^{m-m_i}\,\varphi)_{S_{x^0,\varrho}} \tag{7.7}$$

It follows from (7.7) that

$$|\,(u\,,\,A_{x^0}\,\varphi)_{S_{x^0\,\varrho}}\,| \leq C\sum_{i=1}^{N}\|\,f_i\,\|_{L_p(S_{x^0,\varrho})}\,\|\,\varphi\,\|_{2m-m_0,L_{p'}(S_{x^0,\varrho})} \tag{7.8}$$

where C is a constant depending only on $n\,,m\,,N$, and K.

The conclusion of the theorem follows now immediately from (7.8) and Theorem 6.1 applied to u in $S_{x^0,\varrho}$ (elliptic operator A_{x^0}), using a finite covering of $\bar{G}_0$ by spheres $S_{x^i,\varrho/2}$ $(x^i \in \bar{G}_0)$.

The following is an easy consequence, and at the same time a generalization, of Theorem 7.1.

THEOREM 7.1′. *Suppose that the conditions of Theorem* 7.1 *hold and that in addition* $f_i \in H^{\text{loc.}}_{k_i,L_p}(G)$, $k_i \geq 0$. *Set* $k = \min(l\,,k_1+m_1\,,\ldots,k_N+m_N)$. *Then* $u \in H^{\text{loc.}}_{k,L_p}(G)$, *and for any two subdomains* $G_0\,,G_1$ *such that* $\bar{G}_0 \subset G_1 \subset \subset \bar{G}_1 \subset G$ *the following inequality holds*:

$$\|\,u\,\|_{k,L_p(G_0)} \leq c\left(\sum_{i=1}^{N}\|\,f_i\,\|_{k_i,L_p(G_1)} + \|\,u\,\|_{L_p(G_1)}\right), \tag{7.9}$$

where c is a constant depending only on n, $\max(m_i+k_i)$, $N\,,p\,,K\,,\lambda$ *and the domains.*

Proof: The special case $k_1 = \ldots = k_N = 0$ is Theorem 7.1. In the general case put:

$$A_{i,\alpha}(x\,,D) = A_i(x\,,D)\,D^\alpha \quad \text{and} \quad f_{i,\alpha} = (-1)^{k_i}\,D^\alpha f_i\,,$$

for $|\,\alpha\,| = k_i$, $i = 1\,,\ldots,N$. Integrating by parts we deduce from (7.5) that

$$(u\,,\,A_{i,\alpha}\,\varphi)_G = (f_{i,\alpha}\,,\,\varphi)_G \tag{7.10}$$

for $\varphi \in C_0^\infty(G)$, $|\,\alpha\,| = k_i$, $i = 1\,,\ldots,N$.

The conclusion of the theorem follows now from (7.10) and Theorem 7.1 applied to the function u, elliptic system $\{A_{i,\alpha}\}$ and the corresponding system of functions $\{f_{i,\alpha}\}$.

Suppose now that the conditions of Theorem 7.1′ hold with $k_i = m - m_i$ and $l = m$. It follows from the theorem that $u \in H^{\text{loc.}}_{m,L_p}(G)$. Using integration by parts it follows in a standard way from (7.5) that u is a strong solution (in $H^{\text{loc.}}_{m,L_p}$) of the adjoint system (7.4). If, moreover, the conditions of Theorem 7.1′ hold with $k_i = m - m_i + j$, $k = m + j$, where $j > n/p$, then it follows from Sobolev's inequalities that $u \in C^m(G)$, $f_i \in C(G)$ (after correction on a null set) and that u satisfies (7.4) in the classical sense. Finally, if the coefficients of the system and the f_i are infinitely differentiable one obtains the well known result that u is also infinitely differentiable (for overdetermined elliptic systems see, for instance, Schwartz [29]).

With the aid of Theorem 7.1 we establish now the following a priori estimates for a system of differential operators.

THEOREM 7.2. *Let $\{A_i\}_{i=1}^N$ be an elliptic system of differential operators of respective order m_i defined in $\bar{G}$. Set $m_0 = \min m_i$, and suppose that the coefficients of A_i satisfy Condition $\{m_i ; K\}$ in $\bar{G}$. Let G_0 be a subdomain such that $\bar{G}_0 \subset G$. Then, for all functions $u \in C_0^\infty(G_0)$:*

$$(7.11) \qquad \| u \|_{m_0, L_p(G_0)} \leq c \left(\sum_{i=1}^N \| A_i u \|_{L_p(G_0)} + \| u \|_{L_p(G_0)} \right)$$

where c is a constant independent of u.

Proof: Put $A_i u = f_i$. Then, for every function $\varphi \in C_0^\infty(G)$:

$$(7.12) \qquad (u, A_i^* \varphi)_G = (f_i, \varphi)_G, \quad i = 1, \dots, N,$$

where $\{A_i^*\}$ is the formally adjoint system. The inequality (7.11) follows now from (7.12) and from Theorem 7.1 applied to u in G, system $\{A_i^*\}$ and $l = j = m_0$.

The estimate (7.11) for a single elliptic operator was established by various authors (see, for instance, Nirenberg [24]). For $p = 2$ and $m_1 = m_2 = \dots = m_N$ estimate follows from the more general Gårding's inequality [14]. For general p the estimate (7.11) was (essentially) established in Agmon-Douglis-Nirenberg [3; Th. 15.1″] by a different method.

In the special case of an elliptic system of operators having the same order the smoothness assumptions imposed on the coefficients of A_i in Theorem 7.2 could be relaxed considerably, namely, we have

THEOREM 7.2′. *Let $\{A_i(x, D)\}_{i=1}^N$ be an elliptic system of operators in $\bar{G}$, having the same order m. Suppose that the coefficients of highest order terms in A_i are continuous, whereas the remaining coefficients are measurable and* *bounded in $\overline{G}$. Then, for all functions $u \in C_0^\infty(G)$ we have :*

$$\| u \|_{m,L_p(G)} \leq c \left(\sum_{i=1}^{N} \| A_i u \|_{L_p(G)} + \| u \|_{L_p(G)} \right), \tag{7.13}$$

where c is a constant independent of u.

We sketch the proof. Using Lemma 3.4 we may assume without loss of generality that $A_i(x, D)$ contains no terms of order $< m$. Let x^0 be an arbitrary point of $\overline{G}$ and put $A_i^0 = A_i(x^0, D)$. By Theorem 7.2 the inequality (7.13) holds for the elliptic system with constant coefficients $\{A_i^0\}$. Hence, there exists a constant $c_0 > 0$ such that for all $u \in C_0^\infty(G)$ we can write

$$\| u \|_{m,L_p(G)} \leq c_0 \left(\sum_{i=1}^{N} \| A_i^0 u \|_{L_p(G)} + \| u \|_{L_p(G)} \right) \tag{7.14}$$

$$\leq c_0 \left(\sum_{i=1}^{N} \| A_i u \|_{L_p(G)} + \| u \|_{L_p(G)} \right) + c_0 \sum_{i=1}^{N} \| (A_i^0 - A_i) u \|_{L_p(G)}.$$

Using the continuity of the coefficients of A_i it is readily seen that there exists a number $\varrho > 0$ (independent of x^0) such that if the support of u is contained in the sphere $| x - x^0 | < \varrho$, then the last term on the right of (7.14) is less than $\frac{1}{2} \| u \|_{m,L_p(G)}$. From this and (7.14) it follows that there exists a number $\delta > 0$ such that (7.13) holds for all functions $u \in C_0^\infty(G)$ which in addition possess support of diameter $< \delta$. Finally, one drops the restriction on the support of u in a standard way by using a suitable partition of unity and using once more Lemma 3.4.

8. Regularity at the boundary.

We pass to the problem of regularity at the boundary in L_p of weak solutions of the Dirichlet problem. We consider an elliptic operator A of order $2m$ defined in $\overline{G}$:

$$A(x, D) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha. \tag{8.1}$$

If $n = 2$ we assume in addition that A satisfies the roots condition in $\overline{G}$ (i. e. for every $x^0 \in \overline{G}$ the principal part $A'(x^0, D)$ satisfies the condition on the roots introduced in § 4). We denotè by $C^l(\overline{G}; \{D^\alpha\}_{|\alpha| \leq m-1})$ $(m \leq l)$ the subclass of function $v \in C^l(\overline{G})$ satisfying the boundary conditions:

$$D^\alpha v = 0 \quad \text{on} \quad \partial G \quad \text{for} \quad 0 \leq |\alpha| \leq m-1. \tag{8.2}$$

We also recall that $H_{l,L_p}(\overline{G}\,;\{D^\alpha\}_{|\alpha|\le m-1})$ denotes the subclass of functions $v \in H_{l,L_p}(G)$ satisying (8.2) in the generalized (trace) sense (see § 2).

We now state the basic

THEOREM 8.1. *Let u be a function belonging to $L_q(G)$ for some $q > 1$. Suppose that for all functions $v \in C^{2m}(\overline{G}\,;\{D^\alpha\}_{|\alpha|\le m-1})$ the following inequality holds:*

$$(8.3) \qquad |(u\,,Av)_G| \le C\,\|v\|_{2m-j,L_{p'}(G)},$$

where A is the elliptic operator (8.1), j is a positive integer $\le 2m$, $p' > 1$ and C a constant. Suppose also that the coefficients of A satisfy condition $\{j\,,K\}$ in $\overline{G}$ and that G is of class C^{2m}. Then, $u \in H_{j,L_p}(G)\left(\frac{1}{p}+\frac{1}{p'}=1\right)$ and

$$(8.4) \qquad \|u\|_{j,L_p(G)} \le c_1\,(C + \|u\|_{L_p(G)}),$$

where c_1 is a constant depending only on $n\,,m\,,p\,,K\,,\lambda$ (the ellipticity constant), and the domain.

Proof: By an obvious covering argument it suffices to show that for every $x^0 \in \overline{G}$ there exists a neighborhood Ω^0 in the relative topology of $\overline{G}$ such that $u \in H_{j,L_p}(\Omega^0)$, and such that $\|u\|_{j,L_p(\Omega^0)}$ is majorized by the right side of (8.4) with a constant c_1 depending in addition on Ω^0. For a point x^0 in the interior this follows from Theorem 7.1, taking for Ω^0 a sufficiently small sphere with center at x^0. Suppose that $x^0 \in \partial G$. In this case there exists a sufficiently small neighborhood Ω of x^0 in $\overline{G}$, and a measure preserving homeomorphism (⁹) of class C^{2m}: $x \to \tilde{x}$ which takes $\overline{\Omega}$ onto the hemisphere $\overline{\Sigma}_1$: $|\tilde{x}| \le 1$, $\tilde{x}_n \ge 0$. Let $\tilde{A}$ be the transformed elliptic operator under the mapping and put $\tilde{u}(\tilde{x}) = u(x(\tilde{x}))$ ($\tilde{A}$ and $\tilde{u}$ defined in $\overline{\Sigma}_1$). Let, further, $\tilde{v}$ be an arbitrary function belonging to $C^{2m}(\overline{\Sigma}_1\,;\{D^\alpha\}_{|\alpha|\le m-1})$ and vanishing in some neighborhood of $\partial_2\Sigma_1$ (the curved part of $\partial\Sigma_1$). Put $v(x) = \tilde{v}(\tilde{x}(x))$ and extend v as zero in $\overline{G} - \overline{\Omega}$. It is readily seen that $v \in C^{2m}(\overline{G}\,;\{D^\alpha\}_{|\alpha|\le m-1})$. Using (8.3) we have:

$$(8.5) \qquad |(\tilde{u}\,,\tilde{A}\tilde{v})_{\Sigma_1}| = |(u\,,A\,v)_G|$$
$$\le C\,\|v\|_{2m-j,L_{p'}(G)} \le c_0\,C\,\|\tilde{v}\|_{2m-j,L_{p'}(\Sigma_1)}$$

(⁹) One can take a mapping of the form: $\tilde{x}_1 = x_1\,,\dots,$

$$\tilde{x}_{k-1} = x_{k-1}\,,\quad \tilde{x}_k = x_{k+1}\,,\dots,\quad \tilde{x}_{n-1} = x_n\,,$$
$$\tilde{x}_n = x_k - f(x_1\,,\dots,x_{k-1}\,,x_{k+1}\,,\dots,x_n)\,.$$

where c_0 depends only on the mapping. Applying now Theorem 6.2 to the function $\tilde{u}$ in Σ_1 we conclude that $\tilde{u} \in H_{j,L_p}(\Sigma_r)$ for every $r < 1$ and consequently that $u \in H_{j,L_p}(\Omega^0)$, $\bar{\Omega}^0$ being the image of $\bar{\Sigma}_r$ under the mapping. We also obtain by the same theorem the desired estimate. This establishes Theorem 8.1.

From Theorem 8.1 one deduces easily the regularity up to the boundary of weak solutions of the Dirichlet problem:

$$\text{(8.6)} \qquad \begin{cases} A^* u = f \quad \text{in} \quad G , \\ D^\alpha u = 0 \quad \text{on} \quad \partial G, \quad 0 \leq |\alpha| \leq m-1 . \end{cases}$$

THEOREM 8.2. *Let u be a function belonging to $L_q(G)$ for some $q > 1$. Suppose that u is a weak solution of* (8.6) *in the sense that*

$$\text{(8.7)} \qquad (u, A v)_G = (f, v)_G$$

for all functions $v \in C^{2m}(\bar{G}; \{D^\alpha\}_{|\alpha| \leq m-1})$, where A is the elliptic operator (8.1) *and f is a function belonging to $L_p(G)$, $p > 1$. Suppose, moreover, that the coefficients of A verify condition $\{j; K\}$, $1 \leq j \leq 2m$, and that G is of class C^{2m}. Then, $u \in H_{j,L_p}(G)$ and*

$$\text{(8.8)} \qquad \| u \|_{j,L_p(G)} \leq c\,(\| f \|_{L_p(G)} + \| u \|_{L_p(G)}),$$

where c is a constant depending only on n, m, p, K, λ and the domain.

Proof: From (8.7) we obtain the inequality

$$\text{(8.9)} \qquad |(u, A v)_G| \leq \| f \|_{L_p(G)} \| v \|_{L_{p'}(G)}$$

for all functions $v \in C^{2m}(\bar{G}; \{D^\alpha\}_{|\alpha| \leq m-1})$, and the result follows by Theorem 8.1.

A case of special interest is

THEOREM 8.2'. *If the conditions of Theorem 8.2 hold with $j = 2m$ (i. e. if $a_\alpha \in C^{|\alpha|}(G)$ for $|\alpha| > 0$, $a_{(0,\dots,0)}$ being bounded) then $u \in H_{2m,L_p}(G; \{D^\alpha\}_{|\alpha| \leq m-1})$ and satisfies* (8.6) *in the strong L_p sense.*

Proof: We know already that $u \in H_{2m,L_p}(G)$ and consequently that $A^* u = f$ in the strong L_p sense. To complete the proof we need only show that the trace of $D^\alpha u$ on the boundary (considered as an element of $L_1(\partial G)$) is zero for $0 \leq |\alpha| \leq m-1$. For functions u which are sufficiently smooth this follows in a well known manner from (8.7). With some precautions the proof for functions of class $H_{2m,L_p}(G)$ is similar. For the sake of completeness we present a formal proof.

It suffices to show that given $x^0 \in \partial G$ there exists a neighborhood π of x^0 on ∂G such that the traces $\gamma(D^\alpha u)$ ($|\alpha| \leq m-1$) are zero when restricted to π. Since the last property remains invariant under a domain homeomorphism of class C^{2m} we may assume without loss of generality that π is the $n-1$ dimensional sphere $x_n = 0$, $|x'| < r$ ($x' = (x_1, \dots, x_{n-1})$). We may also assume that the cylinder: $x' \in \pi$, $0 < x_n < \delta$, for some $\delta > 0$, belongs to G. Furthermore, noting that the trace $\gamma(D_n^j u) \in H_{2m-1-j\, L_p}$ on π (this follows from the estimate (2.3)), and that

$$\gamma(D_{x'}^{\alpha} D_n^j u) = D_{x'}^{\alpha} \gamma(D_n^j u) \quad \text{on} \quad \pi \tag{8.10}$$

for all derivatives $D_{x'}^{\alpha}$ of order $|\alpha| \leq 2m-1-j$ not involving x_n, we conclude that it will suffice to show that $\gamma(D_n^j u)$ is a null-function on π ($j = 0, \dots, m-1$).

Let $0 \leq j \leq m-1$ and assume that $\gamma(D_n^k u)$ is a nullfunction on π for every $k \leq j-1$ (there is no assumption for $j=0$). We shall show that $\gamma(D_n^j u)$ is also a nullfunction on π and the result will follow by induction. Let $\varphi(x') \in C_0^\infty(\pi)$ and let $\zeta(x_n)$ be a C^∞ function on $x_n \geq 0$ such that $\zeta \equiv 0$ for $x_n \geq \delta$, $\zeta(x_n) = (-x_n)^{2m-1-j}/(2m-1-j)!$ for $0 \leq x_n \leq \delta/2$. Put:

$$w(x', x_n) = \varphi(x')\, \zeta(x_n).$$

Since $u \in H_{2m, L_p}(G)$ we can integrate $(u, A\,w)_G$ by parts to obtain the usual Green's formula with boundary values taken in the generalized trace sense. A simple calculation shows that

$$(u, A\,w)_G = (A^* u, w)_G + \int_\pi \gamma(D_n^j(a\,u))\, \varphi(x')\, dx' \tag{8.11}$$

where a is the coefficient of D_n^{2m} in A. Since $A^* u = f$ and $(u, A\,w)_G = (f, w)_G$; we conclude from (8.11) that

$$\int_\pi \gamma(D_n^j(a\,u))\, \varphi(x')\, dx' = 0$$

for all $\varphi \in C_0^\infty(\pi)$. This implies that $\gamma(D_n^j(a\,u))$ and consequently $\gamma(D_n^j u)$ are null functions on π (since $a \neq 0$), and completes the proof.

Suppose that the function f in Theorem 8.2′ belongs to $L_\infty(G)$. Then, by the theorem, $u \in H_{2m, L_p}(G)$ for every p so that, using Sobolev's inequalities, $u \in C^{2m-1}(\bar{G}\,;\ \{D^\alpha\}_{|\alpha|\le m-1})$ (10). If, moreover, $f \in C^1(G)$ and $a_\alpha \in C^{|\alpha|+1}(G)$ then, by Theorem 7.1′, u also belongs to $C^{2m}(G)$ (11). Thus, in this case u is an ordinary solution of (8.6).

As a side application of Theorem 8.2′ we mention

THEOREM 8.3. *Let u be a function belonging to $C^{2m}(G) \cap C^{m-1}(\bar{G})$, such that:*

$$
\begin{cases} A\,u = 0 & in \quad G\,, \\ D^\alpha u = 0 & on \quad \partial G\,, \quad |\alpha| \le m-1\,, \end{cases} \tag{8.12}
$$

where A is the elliptic operator (8.1). *If G is of class C^{2m} and the coefficients $a_\alpha \in C^{|\alpha|}(G)$, then $u \in H_{2m,L_p}(G)$ for every p.*

To prove Theorem 8.3 it suffices to show that

$$(u\,,\,A^* v)_G = 0$$

for all functions $v \in C^{2m}(G\,;\ \{D^\alpha\}_{|\alpha|\le m-1})$, the result will then follow from Theorem 8.2′. This, however, follows from (8.12) by Green's formula applied to u and v in G (using a suitable approximation procedure).

Theorem 8.3 is useful in connection with uniqueness theorems for the Dirichlet problem (for strongly elliptic equations) where it is necessary to assume in general that $u \in H_{m,L_2}(G)$. The theorem shows that this extra condition is really superfluous (12).

The main applications of the regularity theorems will be given in Part II. In conclusion we add only the following

REMARK: Combining Theorem 8.2 with the a priori L_p estimates up to the boundary given in Agmon-Douglis-Nirenberg [3] one can show (with suitable regularity assuptions on the domain and the coefficients of A) that if in Theorem 8.2 $f \in H_{k,L_p}(G)$, then $u \in H_{2m+k,L_p}(G)$. Similarly, using the Schauder estimates in integral form of [3] one can show that if f belongs to the Hölder class $C^{k+\mu}(\bar{G})$ (k a non-negative integer and $0 < \mu < 1$), then $u \in C^{2m+k+\mu}(\bar{G})$.

---

(10) More precisely, u belongs to the Hölder class $C^{2m-1+\mu}(\bar{G})$ for every $\mu < 1$.

(11) We note that the same conclusion holds with the following weaker assumptions: f and $a_{(0,\ldots,0)}$ are continuous functions satisfying locally a Hölder condition $a_\alpha \in C^{|\alpha|}(G)$ for $|\alpha| > 0$. For a proof see Agmon-Douglis-Nirenberg [3; Appendix 5].

(12) In this connection see also Miranda [20] Lemma 11.1.

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