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HARI DAS BAGCHI
BHOLA NATH MUKHERJI

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« NOTE ON CERTAIN EQUATIONS CONNECTED WITH GEGENBAUER FUNCTIONS »

by HARI DAS BAGCHI, M. A., Ph. D.,
Hardinge Professor of Higher Mathematics, Calcutta University
AND
BHOLA NATH MUKHERJI, M. A.,
Professor of Mathematics, Scottish Church College, Calcutta

INTRODUCTION

This short paper has, for its main objective, the investigation of the five functional equations:

$$(n+1)f_{n+1}(z) - 2(n+\lambda)zf_n(z) + (n-1+2\lambda)f_{n-1}(z) = 0\,, \quad \ldots \quad \text{(I)}$$

$$f'_{n+1}(z) - f'_{n-1}(z) = 2(n+\lambda)f_n(z)\,, \quad \ldots\ \ldots\ \ldots \quad \text{(II)}$$

$$zf'_n(z) - f'_{n-1}(z) = nf_n(z)\,, \quad \ldots\ \ldots\ \ldots \quad \text{(III)}$$

$$f'_{n+1}(z) - zf'_n(z) = (n+2\lambda)f_n(z)\,, \quad \ldots\ \ldots\ \ldots \quad \text{(IV)}$$

and

$$(z^2-1)f'_n(z) = nzf_n(z) - (n-1+2\lambda)f_{n-1}(z)\,, \quad \ldots\ \ldots\ \ldots \quad \text{(V)}$$

and of the associated differential equation

$$(1-z^2)w'' - (2\lambda+1)zw' + n(n+2\lambda)w = 0\,, \qquad (w \equiv f_n(z))\,, \quad \ldots \quad \text{(A)}$$

with special reference to their *mutual relations* and *common solutions*, (it being tacitly understood that $\lambda > -\frac{1}{2}$ and that the parameter n is an integer ≥ 1). A familiar common solution of the six equations being known to be the ultra-spherical (or, Gegenbauer) polynomial $P_n^{(\lambda)}(z)$, a second common solution, — symbolised as $Q_n^{(\lambda)}(z)$ — has been introduced in this paper, and then utilised in the establishment of a convenient formula for the *most general* type of common solution of the six equations. This new function $Q_n^{(\lambda)}(z)$ has beenn designated as Gegenbauer function *of the second kind* (1) in contrast to $P_n^{(\lambda)}(z)$, which will for this purpose be termed Gegenbauer function *of the first kind.* The concluding portion of this paper is devoted to certain integral formula involving $P_n^{(\lambda)}(z)$.

The authors believe that this paper contains on the whole some amount of *original* matter, although there are occasional references to *known* results.

ART. 1 — Out of the ${}^6C_2$ (or 15) pairs of equations, that can obviously be derived from the six original equations, *viz.* (I)-(V) and (A) by *different* modes of pairing, we shall select, for our special study, the two pairs, *viz.*,

$$\text{(I), (II)} \qquad \text{and} \qquad \text{(I), (A)}.$$

To be precise, we shall scrutinise the properties of an enumerable sequence of functions $\{f_n(z)\}$, which satisfy the two equations of each pair, *taken separately.*

Case I. — *Firstly*, assuming that $\{f_n(z)\}$ satisfies both (I) and (II), we may eliminate $f'_{n-1}(z)$ from (II) and the equation obtained from (I) by differentiation; this eliminant being readily seen to coincide with (III), the relation (IV) follows at once by taking the difference of (II) and (III). If n is now changed into $(n-1)$ in (II) and, from the resulting equation coupled with (III), $f'_{n-1}(z)$ is eliminated, the resultant easily reduces to (V) If again $f'_{n-1}(z)$ is eliminated from (III) and the equation obtained from (V) by differentiation, we arrive at (A). We thus obtain the following lemma:

LEMMA *i.* — *If $f_n(z)$ satisfies the pair of equations* (I) *and* (II), *it must satisfy the remaining four equations.*

Case II. — *Secondly*, assuming that $\{f_n(z)\}$ satisfies both (I) and (A), we may differentiate (I) twice in succession, and thus derive:

$$(n+1)f'_{n+1}(z) - 2(n+\lambda)zf'_n(z) - 2(n+\lambda)f_n(z) + (n-1+2\lambda)f'_{n-1}(z) = 0, \quad \ldots (1)$$

and

$$(n+1)f''_{n+1}(z) - 2(n+\lambda)zf''_n(z) - 4(n+\lambda)f'_n(z) + (n-1+2\lambda)f''_{n-1}(z) = 0. \quad \ldots (2)$$

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(1) It will be seen that, when $\lambda = \frac{1}{2}$, $Q_n^{(\lambda)}(z)$ reduces to $Q_n(z)$, in the same manner as $P_n^{(\lambda)}(z)$ reduces to $P_n(z)$.

If we now multiply the three equations (I), (1) and (2) respectively by

$$(n+1)(n+1+2\lambda), \quad -(2\lambda+1)z \quad \text{and} \quad (1-z^2),$$

and then add them together and further attend to the three distinct relations, — which are virtually inherent in (A), — *viz.*,

$$(1-z^2)f_\nu''(z) - (2\lambda+1)zf_\nu'(z) + \nu(\nu+2\lambda)f_\nu(z) = 0, \quad (\nu = n-1, n, n+1),$$

we get after easy reductions:

$$-2(n+\lambda).z[(n+1)(n+1+2\lambda) - n(n+2\lambda)]f_n(z)$$

$$+(n-1+2\lambda)[(n+1)(n+1+2\lambda) - (n-1)(n-1+2\lambda)]f_{n-1}(z)$$

$$-4(n+\lambda)(1-z^2)f_n'(z) + 2(n+\lambda)(2\lambda+1)zf_n(z) = 0.$$

Plainly the equation last written reduces to (V) on simplification. If, then, $(n+1)$ and $(n-1)$ be put successively for n in this equation, *viz.* (V), we have by subtraction

$$(z^2-1)\{f_{n+1}'(z) - f_{n-1}'(z)\}$$

$$= (n+1)zf_{n+1}(z) - (n+2\lambda)f_n(z) - (n-1)zf_{n-1}(z) + (n-2+2\lambda)f_{n-2}(z). \quad \ldots(3)$$

If we now write $(n-1)$ for n in (I) and then solve the resulting equation for $f_{n-2}(z)$ in terms of $f_n(z)$ and $f_{n-1}(z)$, and finally insert this value of $f_{n-2}(z)$ in the R. S. of (3), it becomes equal to

$$z[(n+1)f_{n+1}(z) + (n-1+2\lambda)f_{n-1}(z)] - 2(n+\lambda)f_n(z)$$

$$= 2(z^2-1)(n+\lambda)f_n(z), \qquad \text{by (I)}.$$

So the relation (3), when divided ont by the factor (z^2-1), simplifies to (II). Thus (II) being a consequence of the combination of (I) and (A), we apply Lemma *i* to deduce immediately the relations (III) and (IV). We are thus led to a second lemma, which reads as follows:

LEMMA *ii.* — *If $f_n(z)$ satisfies the pair of equations* (I) *and* (A), *it must satisfy the remaining four equations.*

Adopting similar methods of (elementary) analysis, the reader may deal with the remaining (15 — 2) or 13 pairs of equations, that can be formed out of the six initial equations (I)-(V) and (A).

Our next task is to reckon with the *common solutions* of the six equations.

ART. 2 — The difference equation (I) being homogeneous and of the first degree and second order, its complete solution must be expressible in the form:

$$f_n(z) = \alpha_n(z)\, g_n(z) + \beta_n(z)\, h_n(z)\,, \qquad \ldots\ \ldots\ \ldots (4)$$

where $\alpha_n(z)$ and $\beta_n(z)$ are two *linearly independent* particular solutions of (I), and $g_n(z)$ and $h_n(z)$ are two *arbitrary* functions of z, which are periodic in n *with unit period.* Restricting — as we do in the present context — the parameter n to be a positive integer, we easily see that both $g_n(z)$ and $h_n(z)$ are virtually independent of n and are as such representable simply as $g(z)$ and $h(z)$. Accordingly the *general* solution (4) of (I) takes the form:

$$f_n(z) = \alpha_n(z)\, g(z) + \beta_n(z)\, h(z)\,, \qquad \ldots\ \ldots\ \ldots (5)$$

where $g(z)$ and $h(z)$ are *arbitrary* (and hence *disposable*) functions of z.

If we now look for the *general* type of solution, common to (I) and (II), the simplest line of procedure is *firstly* to choose two (linearly independent) functions $\alpha_n(z)$ and $\beta_n(z)$, which satisfy not only (I) but also (II), and *secondly* to impose the condition that the resulting function $f_n(z)$, as defined by (5), shall satisfy (II). Actually when (5) is differentiated and the derived values of $f'_{n+1}(z)$ and $f'_{n-1}(z)$ as also the value of $f_n(z)$, as given by (5), are inserted in (II), this equation simplifies after a few steps to

$$a_n(z)\, g'(z) + b_n(z)\, h'(z) = 0\,, \qquad \ldots (6)$$

where

$$\left.\begin{aligned} a_n(z) &\equiv \alpha_{n+1}(z) - \alpha_{n-1}(z)\,, \\ \text{and} \quad b_n(z) &\equiv \beta_{n+1}(z) - \beta_{n-1}(z)\,. \end{aligned}\right\}$$

Repeating the line of argument used in the paper noted below ([2]), we eventually conclude that the relation (6) can be an *identity* in z, if and only if,

$$g'(z) = 0 \quad \text{and} \quad h'(z) = 0 \qquad \text{for } all \text{ values of } z\,.$$

([2]) See H. D. BAGCHI and P. C. CHATTERJEE: « *Note on certain equations, connected with Hermite and Weber functions* ». [Vide *Annali della Scuola Normale Superiore di Pisa* (1952), (in the press)].

This means that $g(z)$ and $h(z)$ must reduce to constants, independent of n. Hence, considering that (LEMMA *i*) *any* common solution of (I) and (II) is also a common solution of all the six equations, we may summarise our conclusions in the form of a substantive proposition:

PROP. A. — *The most general common solution of the six equations, viz.* (I)-(V) and (A), *is representable in the symbolic* form [3]:

$$f_n(z) = a\,\alpha_n(z) + b\,\beta_n(z), \qquad \ldots \ldots \ldots (7)$$

where $\alpha_n(z)$ *and* $\beta_n(z)$ *are two (linearly independent) common solutions of* (I) *and* (II) — *and therefore also of all the six equations — and* a *and* b *are two arbitrary numerical constants, independent of* n.

Should it be desired to express the common solution (7) in a more handy or tangible form, it is essential to choose particularly convenient forms for $\alpha_n(z)$ and $\beta_n(z)$. Thus for instance we may set

$$\alpha_n(z) \equiv P_n^{(\lambda)}(z)$$

and defer the selection of $\beta_n(z)$ to the succeeding article.

ART. 3 — Remembering that Gegenbauer function (of the first kind), *viz.* $P_n^{(\lambda)}(z)$ is derived from Jacobi's function of the first kind, *viz.* $P_n^{(\alpha,\beta)}(z)$ by putting

$$\alpha = \beta = \lambda - \frac{1}{2}, \qquad \left(\lambda > -\frac{1}{2}\right), \qquad \ldots \ldots \ldots (8)$$

we are entitled to define Gegenbauer function *of the second kind* $Q_n^{(\lambda)}(z)$ as being the *degenerate* form, which Jacobi's function of the second kind $Q_n^{(\alpha,\beta)}(z)$ assumes *under the same proviso*, viz. (8). That is to say, we have to set

$$Q_n^{(\lambda)}(z) \equiv Q_n^{\left(\lambda-\frac{1}{2},\,\lambda-\frac{1}{2}\right)}(z), \qquad \left(\lambda > -\frac{1}{2}\right). \qquad \ldots \ldots (9)$$

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[3] It may not be out of place to remark that if $\alpha_n(z)$ and $\beta_n(z)$ be simply supposed to be two *arbitrary* solutions of the differential equation (A), *selected at random,* they may not, and in general will not, satisfy (I) or (II), and in that case (7) may or may not represent a common solution of the six equations. Indeed the crucial point in the above context is that the two functions $\alpha_n(z)$ and $\beta_n(z)$, which occur in (7), should each be a *common* solution of (I) and (II), or, (what is the same thing) of (I) and (A).

Inasmuch as the differential equation of the second order (4), *viz.*

$$(1 - z^2)\,\omega'' + \{\beta - \alpha - (\alpha + \beta + 2)\}\, z\,\omega' + n\,(n + \alpha + \beta + 1)\,\omega = 0\,, \quad \ldots (10)$$

of which a particular solution is $Q_n^{(\alpha,\beta)}(z)$, simplifies to the form (A) under the conditions (8), it follows that the function $Q_n^{(\lambda)}(z)$, as defined by (9), is a solution of (A). To prove that $Q_n^{(\lambda)}(z)$ satisfies also (I), we proceed (5) as follows:

$$Q_n^{(\lambda)}(z) = Q_n^{\left(\lambda - \frac{1}{2},\, \lambda - \frac{1}{2}\right)}(z)\,,$$

$$= 2^{-n-1}(z^2 - 1)^{-\lambda + \frac{1}{2}} \int_{-1}^{1} (1 - t^2)^{n + \lambda - \frac{1}{2}} (z - t)^{-n-1}\, d\,t\,,$$

$$= \frac{1}{2}(z^2 - 1)^{-\lambda + \frac{1}{2}} \int_{-1}^{1} (1 - t^2)^{\lambda - \frac{1}{2}} \frac{P_n^{(\lambda)}(t)}{z - t}\, d\,t\,. \quad \ldots\ldots (11)$$

Forming the values of $Q_{n+1}^{(\lambda)}(z)$ and $Q_{n-1}^{(\lambda)}(z)$ from (11) by changing n successively into $(n + 1)$ and $(n - 1)$, we easily get

$$\left.\begin{aligned} &(n + 1)\, Q_{n+1}^{(\lambda)}(z) - 2\,(n + \lambda)\, z\, Q_n^{(\lambda)}(z) + (n - 1 + 2\,\lambda)\, Q_{n-1}^{(\lambda)}(z)\,, \\ &= \frac{1}{2}(z^2 - 1)^{-\lambda + \frac{1}{2}} \int_{-1}^{1} \frac{(1 - t^2)^{\lambda - \frac{1}{2}}\, V}{z - t}\, d\,t\,, \end{aligned}\right\} \ldots (12)$$

where

$$V \equiv (n + 1)\, P_{n+1}^{(\lambda)}(t) - 2\,(n + \lambda)\, z\, P_n^{(\lambda)}(t) + (n - 1 + 2\,\lambda)\, P_{n-1}^{(\lambda)}(t)\,. \quad \ldots (13)$$

Because $P_n^{(\lambda)}(t)$ satisfies the functional equation (I), wherein t is written for z, we have by transposition

$$(n + 1)\, P_{n+1}^{(\lambda)}(t) + (n - 1 + 2\,\lambda)\, P_{n-1}^{(\lambda)}(t) = 2\,(n + \lambda)\, t\, P_n^{(\lambda)}(t)\,.$$

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(4) See G. SZEGÖ « *Orthogonal Polynomials* » (1939). Art. 4-2 (P. 59) and Art. 4-61 (P 73). [Vide « American Mathematical Society Colloquium Publications », Vol. XXIII].

(5) See G. SZEGÖ (*loc. cit.*), Pp 73 and 74.

Hence by substitution (13) is carried over into

$$V = -2(n+\lambda)(z-t) P_n^{(\lambda)}(t),$$

so that (12) becomes

$$(n+1) Q_{n+1}^{(\lambda)}(z) - 2(n+\lambda) z Q_n^{(\lambda)}(z) + (n-1+2\lambda) Q_{n-1}^{(\lambda)}(z)$$

$$= -(n+\lambda)(z^2-1)^{-\lambda+\frac{1}{2}} \int_{-1}^{1} (1-t^2)^{\lambda-\frac{1}{2}} P_n^{(\lambda)}(t)\, dt,$$

$= 0$, in view of the *orthogonal* property of the sequence of functions $\{ P_n^{(\lambda)}(z) \}$.

Thus $Q_n^{(\lambda)}(z)$ satisfies (I).

Hence, by LEMMA *ii*, $Q_n^{(\lambda)}(z)$ must be a *common* solution of all the six equations. Furthermore $P_n^{(\lambda)}(z)$ and $Q_n^{(\lambda)}(z)$ being linearly independent of each other, we can take them as admissible substitutes for $\alpha_n(z)$ and $\beta_n(z)$ in (7).

Accordingly Prop. A assumes the special form:

PROP. B. — *The most general common solution of the six equations* (I)-(V) *and* (A) *can be thrown into the form*:

$$f_n(z) = a\, P_n^{(\lambda)}(z) + b\, Q_n^{(\lambda)}(z), \qquad \ldots\ \ldots\ \ldots (14)$$

where $P_n^{(\lambda)}(z)$ *and* $Q_n^{(\lambda)}(z)$ *are respectively the two kinds of Gegenbauer functions and* « a » *and* « b » *are two arbitrary numerical constants, independent of the positive integral parameter* « n ».

ART. 4 — We shall conclude this paper by reckoning with certain definite integrals, involving $P_n^{(\lambda)}(z)$.

In the first place let us write

$$I_{n,m}^{(\lambda)} \equiv \int_{-1}^{1} (1-z^2)^{\lambda-\frac{1}{2}} z^n P_m^{(\lambda)}(z)\, dz, \quad \ldots\ \ldots\ \ldots (15)$$

where m and n are integers ≥ 0.

Since the integrand is an *odd* function, when $n-m$ or $n+m$ is an odd integer (positive or negative), we must have

$$I_{n,m}^{(\lambda)} = 0, \qquad \text{if} \quad n \backsim m \text{ be odd.} \qquad \ldots\ \ldots\ (16)$$

To deal with the general case, we observe that if the two sets of constants $\{a_r\}_0^m$ and $\{b_r\}_0^m$ be defined as the coefficients of the right-hand side expressions of the following equalities

$$\left.\begin{aligned} z^m &= \sum_{r=0}^{r=m} a_r P_r^{(\lambda)}(z), \\ \text{and} \qquad P_m^{(\lambda)}(z) &= \sum_{r=0}^{r=m} b_r z^r, \end{aligned}\right\} \qquad \dots \dots \dots (17)$$

then we must have

$$a_m b_m = 1,$$

so that [G. SZEGÖ (*loc. cit.*), 4.7.31 (P. 84)],

$$a_m = \frac{1}{b_m} = \frac{\Gamma(\lambda)\Gamma(m+1)}{2^m . \Gamma(m+\lambda)}. \qquad \dots \dots \dots (18)$$

Further, $(1-z^2)^{\lambda-\frac{1}{2}}$ being the *weight function* associated with the orthogonal functions $\{P_m^{(\lambda)}\}(z)\}$, we have

$$\int_{-1}^{1} (1-z^2)^{\lambda-\frac{1}{2}} P_m^{(\lambda)}(z) P_r^{(\lambda)}(z)\, dz = 0, \quad \text{if} \quad m \neq r,$$

and consequently

$$\begin{aligned} I_{m,m}^{(\lambda)} &= \int_{-1}^{1} (1-z^2)^{\lambda-\frac{1}{2}} \left\{ \sum_{r=0}^{r=m} a_r P_r^{(\lambda)}(z) \right\} P_m^{(\lambda)}(z)\, dz, \\ &= a_m \int_{-1}^{1} (1-z^2)^{\lambda-\frac{1}{2}} \{P_m^{(\lambda)}(z)\}^2\, dz, \\ &= \frac{\pi}{2^{m+2\lambda-1}} \cdot \frac{\Gamma(m+2\lambda)}{\Gamma(\lambda)\Gamma(m+\lambda+1)}, \qquad \dots \dots \dots (19) \end{aligned}$$

by (18) and G. SZEGÖ (*loc. cit*), 4.7.14 (P. 81).

Now in order to calculate $I_{n,m}^{(\lambda)}$, when $n \backsim m$ is *even*, we note that, $P_n^{(\lambda)}(z)$ being a solution of (A), it is legitimate to write

$$(1-z^2)^{\lambda-\frac{1}{2}} P_m^{(\lambda)}(z) = -\frac{1}{m(m+2\lambda)} \cdot \frac{d}{dz} \{(1-z^2)^{\lambda+\frac{1}{2}} P_m'^{(\lambda)}(z)\}. \quad \dots (20)$$

Substituting (20) in the R. S. of (15) and then integrating by parts, we find after easy reductions:

$$I_{n,m}^{(\lambda)} = \frac{n\,(n-1)}{(n-m)\,(n+m+2\,\lambda)} \cdot I_{n-2,m}^{(\lambda)} . \qquad \ldots \quad \ldots (21)$$

Combining (16), (19) and (21), we conclude that

$$I_{n,m}^{(\lambda)} = 0\,, \quad \text{if} \quad n < m \quad \text{or if} \quad n - m \text{ is odd}\,,$$

and that, when $n - m$ is an even (positive) integer,

$$I_{n,m}^{(\lambda)} = I_{m,m}^{(\lambda)} \times \frac{n!}{m!} \times \frac{1}{\alpha\,\beta}\,, \qquad \ldots \quad \ldots \quad \ldots (22)$$

where

$$\alpha \equiv (n-m)\,(n-m-2)\,(n-m-4)\,\ldots\ldots\,6.4.2\,,$$

and $\beta \equiv (n+m+2\,\lambda)\,(n+m+2\,\lambda-2)\,(n+m+2\,\lambda-4)\ldots\ldots(2\,m+2\,\lambda+2).$

Let us now put

$$k \equiv \frac{n-m}{2} = \text{an integer} \qquad \text{and} \qquad p = m + \lambda + 1\,,$$

so that

$$\alpha = 2^k \times k! \qquad \text{and} \qquad \beta = 2^k \times \frac{\Gamma\,(p+k)}{\Gamma\,(p)}\,. \qquad \ldots (23)$$

Taking account of the relations (22) and (23), we readily arrive at the final formula (6):

$$I_{n,m}^{(\lambda)} = \frac{\pi}{2^{n+2\lambda-1}} \times \frac{n!}{m!\left(\dfrac{n-m}{2}\right)!} \times \frac{\Gamma(m+2\,\lambda)}{\Gamma\,(\lambda)\,\Gamma\left(\dfrac{n+m}{2}+\lambda+1\right)}\,, \qquad \ldots \quad \ldots (24)$$

it being premised as before that $(n-m)$ is an even positive integer. When, however, n and m do not conform to this condition, $I_{n,m}^{(\lambda)}$ vanishes, — a fact noticed heretofore.

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(6) In the particular case $\left(\lambda = \frac{1}{2}\right)$, $P_n^{(\lambda)}(z)$ coincides with $P_n(\lambda)$, and then the formula (24) for $I_{n,m}^{(\lambda)}$ automatically reduces to that for $I_{n,m}$, considered in Whittaker and Watson, « *Modern Analysis* » (1915), [Art. 15-211, Pp. 304-05 and Ex. 4 (P. 305)].

Another integral formula of a comparatively minor importance can be obtained as follows:

$$\int_{-1}^{1}(1-z^2)^{\lambda+\frac{1}{2}}\{P'_m(\lambda)\}^2\,dz=\int_{-1}^{1}P_m^{(\lambda)}(z)\,.\,\frac{d}{dz}\{(1-z^2)^{\lambda+\frac{1}{2}}P'_m(\lambda)\}\,dz\,,$$

$$=m\,(m+2\lambda)\int_{-1}^{1}(1-z^2)^{\lambda-\frac{1}{2}}\{P_m^{(\lambda)}(z)\}^2\,dz\,,\qquad \text{by (20)}$$

$$=2^{1-2\lambda}.\pi.\frac{\Gamma(m+2\lambda+1)}{(m+\lambda)\,\Gamma(m)\,\{\Gamma(\lambda)\}^2}\,,$$

[Cf. G. SZEGÖ, *loc. cit.*, 4.7.14 (P. 81)].