Annali della
Scuola Normale Superiore di Pisa
Classe di Scienze

Isaiah Maximoff

On continuous transformation of some functions into an ordinary derivative

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 2e série, tome 12, nº 3-4 (1947), p. 147-160
<http://www.numdam.org/item?id=ASNSP_1947_2_12_3-4_147_0>

ON CONTINUOUS TRANSFORMATION OF SOME FUNCTIONS INTO AN ORDINARY DERIVATIVE

by ISAIAH MAXIMOFF (Tcheboksary, U. S. S. R.).

1. - A function $f(x)$ is called an ordinary derivative in $[0, 1]=[0\leqq x\leqq 1]$ if there exists a continuous function $\mathcal{F}(x)$ such that $\frac{d}{dx}[\mathcal{F}(x)]=f(x)$.

This paper is intended to solve the following *problem D* which was putted to me by NICOLAS LUSIN: Let $f(x)$ be a finite function of class 1 having the property of DARBOUX in $[0, 1]$ (1). We must find a continuous and essentially increasing function $x=\varphi(t)$ $[\varphi(0)=0, \varphi(1)=1]$ such that $f[\varphi(t)]$ is an ordinary derivative in $[0\leqq t\leqq 1]$.

Let $f(x)$ be a finite function and let

$$(y) \qquad y_1, \quad y_2, \quad y_3, \ldots$$

be the sequence of all rational numbers y_n such that there are two points a_n and b_n belonging to $[0, 1]$ and satisfying the condition $f(a_n)<y_n<f(b_n)$. Denote by E_{y_n} $\{E^{y_n}\}$ the set of all points x of $[0, 1]$ satisfying the condition $f(x)<y_n$ $\{f(x)>y_n\}$. If E is a measurable set of points, then mE will denote the measure of E.

DEFINITION 1. - Let (ϱ, Δ) be any pair of positive numbers ϱ, Δ, and let E be any measurable set. We shall say that a point x_0 of E is *a density* (ϱ, Δ) *point of E* if for each interval $i_{x_0}=(x_0-\delta', x_0+\delta'')$ where $\delta=\delta'+\delta''\leqq\Delta$ there exists the inequality

$$(1) \qquad m[i_{x_0}E]\geqq\delta-\frac{1}{\varrho^3}\cdot\delta^2.$$

In our Note (2) we have introduced the following definition.

DEFINITION 2. - Let

$$\Delta_1, \quad \varrho_1, \quad \Delta_2, \quad \varrho_2, \quad \Delta_3, \quad \varrho_3, \ldots$$

---

(1) G. DARBOUX: *Sur les fonctions discontinues.* [Ann. Ec. Norm. Sup. (2) 4, pp. 109-110].

(2) *Sur les fonctions dérivées.* [Bulletin des Sciences Mathématiques, (2) 64, pp. 116-121 (1940)].

be a sequence of positive numbers such that:

(*i*) $\varrho_n \geqq |y_i|$ for all $i \leqq n$;

(*ii*) $\varrho_1 < \varrho_2 < \varrho_3 < \varrho_4 < \ldots$; $\Delta_1 \leqq \Delta_2 \leqq \Delta_3 \leqq \Delta_4 \leqq \ldots$

A finite function $f(x)$ will be called approximately (ϱ_n, Δ_n) continuous in $[0, 1]$ if there exists a system $\overline{P}$ of perfect sets,

$$(\overline{P}) \qquad \overline{P}_{y_r}^{r+s}, \quad \overline{P}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3, \ldots;\ r=1, 2, 3, \ldots)$$

such that:

$$(i) \qquad \overline{P}_{y_r}^{r+s} \subset [0, 1], \quad \overline{P}_{r+s}^{y_r} \subset [0, 1],$$

$$\overline{P}_{y_r}^{r+s} \subset \overline{P}_{y_r}^{r+s+1} \subset E_{y_r}, \quad \overline{P}_{r+s+1}^{y_r} \subset \overline{P}_{r+s}^{y_r} \subset E^{y_r};$$

$$(ii) \qquad E_{y_r} = \lim_{s=\infty} \overline{P}_{y_r}^{r+s}, \quad E^{y_r} = \lim_{s=\infty} \overline{P}_{r+s}^{y_r};$$

(*iii*) if $y_r < y_t$ and if M is the greater of the integers r, t, every point of the set

$$\overline{P}_{y_r}^{M+s} \{ \overline{P}_{M+s}^{y_t} \}, \qquad (s=0, 1, 2, 3, 4, \ldots)$$

is a density $(\varrho_{M+s}, \Delta_{M+s})$ point of the set

$$\overline{P}_{y_t}^{M+s} \{ \overline{P}_{M+s}^{y_r} \}.$$

In the same Note we have proved the following

THEOREM I. - Every approximately (ϱ_n, Δ_n) continuous in $[0, 1]$ function is an ordinary (exact) derivative in $[0, 1]$.

The proof of this theorem will rest on the following

LEMMA 1. - We suppose that for a finite summable in $[0 \leqq x \leqq 1]$ function $f(x)$ and for any interval $(x_0 - \delta_1', x_0 + \delta_1'')$ contained in $[0, 1]$, there exists a sequence of integers,

$$(n) \qquad n_1, \quad n_2, \quad n_3, \quad n_4, \ldots$$

and a sequence of measurable sets,

$$(\mathcal{E}) \qquad \mathcal{E}_1, \quad \mathcal{E}_2, \quad \mathcal{E}_3, \ldots$$

such that:

$$(i) \qquad \mathcal{E}_1 < \mathcal{E}_2 < \mathcal{E}_3 < \ldots < \mathcal{E}_n < \ldots, \qquad \lim_{n \to \infty} \mathcal{E}_n = (x_0 - \delta_1', x_0 + \delta_1'');$$

$$(ii) \qquad m(C\mathcal{E}_k) \leqq \frac{2\delta_1}{\sqrt[3]{\varrho_{n_k}}}, \qquad \delta_1 \left[1 - \frac{2\delta_1}{\sqrt[3]{\varrho_{n_1}}} \right] \leqq m\mathcal{E}_1 \leqq \delta_1,$$

where

$$\delta_1 = \delta_1' + \delta_1'' ;$$

(*iii*) if x_1 is any point of $\mathcal{E}_1$, then $|f(x_1) - f(x_0)| < \eta$, and if x_k is any point of $\mathcal{E}_k$, then $|f(x_k)| \leqq \varrho_{n_{k-1}}$.

2. - In order to solve our problem D we proceed in the following manner.

In the first place we construct for a given finite function $f(x)$ of class 1 having the property of DARBOUX *a characteristic system* P of perfect sets

$$(P) \qquad P_{y_r}^{r+s}, \quad P_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3, \ldots ; \; r=1, 2, 3, 4, \ldots) \quad (^3).$$

In the second place we pass from the characteristic system P to *a perfect system* of sets:

$$(Q) \qquad Q_{y_r}^{r+s}, \quad Q_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3, \ldots ; \; r=1, 2, 3, \ldots) \quad (^3).$$

In the third place we construct a system $\overline{Q}$ of perfect sets

$$(\overline{Q}) \qquad \overline{Q}_{y_r}^{r+s}, \quad \overline{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3, \ldots ; \; r=1, 2, 3, \ldots)$$

which enjoys the following properties:

(*i*) there exists a correspondence CS of the similitude

$$(CS) \quad \dot{Q}_{y_r}^{r+s} \longleftrightarrow \overline{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r} \longleftrightarrow \overline{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3, \ldots ; \; r=1, 2, 3, \ldots)$$

between the sets of the system $\overline{Q}$ and the sets of a perfect system $\dot{Q}$ of the sets which is obtained by making the complete corrections of the initial perfect system Q (3);

(*ii*) the set

$$\lim_{n \to \infty} \sum_{s=1}^{s=n} \overline{Q}_{y_s}^{n} \left\{ \lim_{n \to \infty} \sum_{s=1}^{s=n} \overline{Q}_{n}^{y_s} \right\}$$

is dense in the segment $[0 \leqq t \leqq 1]$;

(*iii*) if $y_r < y_t$ and if M is the greater of the integers r and t, every point of the set

$$\overline{Q}_{y_r}^{M+s} \left\{ \overline{Q}_{M+s}^{y_t} \right\}$$

is a density $(\varrho_{M+s}, \Delta_{M+s})$ point of the set

$$\overline{Q}_{y_t}^{M+s} \left\{ \overline{Q}_{M+s}^{y_r} \right\}$$

for all $s=0, 1, 2, 3, \ldots$;

---

(3) I. MAXIMOFF: *Some Theorems on the Functions of class 1 having the property of Darboux.* [Rendiconti del Circolo Matematico di Palermo].

I. MAXIMOFF: *On Continuous Transformation of Some Functions into Approximately Continuous.* [Annals of Mathematics, 1941].

(*iiii*) $$m\{C[\overline{Q}_Y^n \cdot \overline{Q}_n^y]\} \leqq a_n,$$

where a_n is a positive number such that the series $a_1+a_2+a_3+a_4+....$ is convergent and $Y\{y\}$ denote the greater $\{$the least$\}$ of the numbers $y_1, y_2, y_3,...., y_n$.

At last, we construct the continuous and essentially increasing function $x=\psi(t)$ $[\psi(0)=0, \psi(1)=1]$ which transforms the sets

$$\dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3, 4,....;\ r=1, 2, 3, 4,....)$$

respectively into the sets

$$\overline{Q}_{y_r}^{r+s}, \quad \overline{Q}_{r+s}^{y_r}.$$

If these conditions are fulfilled we shall say that

(*i*) the system $\dot{Q}$ of perfect sets

$(\dot{Q})$ $$\dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3,....;\ r=1, 2, 3,....)$$

is *the final perfect system* of sets for $f(x)$, and will be denoted par DQ;

(*ii*) the system $\overline{Q}$ of perfect sets

$(\overline{Q})$ $$\overline{Q}_{y_r}^{r+s}, \quad \overline{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3,....;\ r=1, 2, 3,....)$$

is the system of our problem D for $f(x)$.

In order to construct the sets of the system DQ and of the system $\overline{Q}$ we shall need the following lemmas.

LEMME 2. - If ϱ, Δ are positive numbers and if p is any perfect everywhere non dense set, then there exists a perfect everywhere non dense set P such that every point of p is a density (ϱ, Δ) point of P.

Proof. - We construct, first of all, a perfect everywhere non dense set P' such that every point of p is a density point of P'. Let (a, b) be any contiguous interval of P' and let

(δ) $$\delta_1, \quad \delta_2, \quad \delta_3,....$$

be any sequenee of intervals such that

(*i*) the extremities of these intervals are rational numbers;

(*ii*) the length of each of these intervals does not surpass Δ;

(*iii*) each of these intervals contains at least one of the points of p and $p \subset \sum_k \delta_k$.

Now we will consider a transformation T of the set P' into a new perfect set P, $P'=TP$. To determine this transformation T we fix one of the extremities of any contiguous interval (a, b) of the set P and we denote this extremity with $f(a, b)$. Let F be the set consisting 1) of all points $f(a, b)$, 2) of all the extremities of all the intervals δ_1, δ_2, δ_3,.... and 3) of all points of the set p. The transformation T must shorten the contiguous intervals of P without changing the points of the set F.

First of all, we shorten each contiguous interval (a, b) of P' having points in common with δ_1 in such a manner that

$$m[\delta_1 CP'] \leqq \frac{\delta_1^2}{\varrho^3}.$$

This is the first shortening which transforms the set P' into a new perfect set P''.

Without changing the points of the set F we shorten each contiguous interval of P'' having points in common with δ_2 in such a manner that

$$m[\delta_2 CP''] \leqq \frac{\delta_2^2}{\varrho^3}.$$

This is the second shortening which transforms the set P'' into a new perfect set P'''.

Without changing the points of the set F we shorten each contiguous interval of P''' having points in common with δ_3 in such a manner that

$$m[\delta_3 CP'''] \leqq \frac{\delta_3^2}{\varrho^3}$$

and so on.

This process may be continued illimitedly and as a result of it we shall obtain the perfect set P satisfying all the conditions of the lemma 2.

LEMMA 3. - If p, q are any two perfect everywhere non dense sets such that $p \cdot q = 0$, then there exists a pair of perfect everywhere non dense sets P, Q such that $P \cdot Q = 0$ and every point of the set $p\{q\}$ is a density (ϱ, Δ) point of the set $P\{Q\}$.

Proof. - First of all, we find the perfect everywhere non dense sets P', Q' such that every point of the set $p\{q\}$ is a density (ϱ, Δ) point of the set $P'\{Q'\}$. The sets p and q have no points in common, therefore the set q is contained in the sum $S = (a_1, b_1) + (a_2, b_2) + ... + (a_k, b_k)$ of the contiguous intervals of p. Let (α_r, β_r) be any interval such what the set $(a_r, b_r)q$ will be contained in (α_r, β_r) and $a_r < \alpha_r < \beta_r < b_r$. We form the perfect set

$$P = P' - \sum_r (\alpha_r, \beta_r) P'$$

and a perfect everywhere non dense set P_r contained in (α_r, β_r) and such that every point of the set $(\alpha_r, \beta_r)q$ is a density (ϱ, Δ) point of P_r. Evidently, the sum $Q = \sum_r P_r$ is a perfect everywhere non dense set. It is easily seen that P and Q are the sets satisfying all the conditions of our lemma.

3. - We now turn baek to the construction of the sets of the system $\overline{Q}$. *The procedure* $\overline{Q}$ of this construction is based on the precedent lemmas. To construct the sets

$$\overline{Q}_{y_r}^{r+s}, \quad \overline{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3, \ldots; \ r=1, 2, 3, \ldots)$$

we shall consider these sets in pairs

$$[\overline{Q}^n_{z^n_r}, \quad \overline{Q}_n^{z^n_{n+1-r}}]$$

where $z_1^n, z_2^n, z_3^n, \ldots, z_n^n$ is the increasing sequence of numbers $y_1, y_2, y_3, \ldots, y_n$ The pair $[\overline{Q}^n_{z^n_r}, \overline{Q}_n^{z^n_{n+1-r}}]$ will be called a pair of order n and of index r, $\overline{Q}^n_{z^n_r}$ is the first set and $\overline{Q}_n^{z^n_{n+1-r}}$ is the second set of this pair.

If $r=1$, this pair is said to be of class 1.

If $1<r\leqq n+1-r$, this pair is said to be of class 2.

If $r>n+1-r$, this pair is said to be of class 3.

Let c_n^s denote the number of all the pair s of class $s=1, 2, 3$. We define the order of the construction of the sets $\overline{Q}$ by the following *rule* R:

(*i*) firstly we construct the sets of the pair of index $r=1$, next the sets of the pair of index $r=2$, thereupon the sets of the pair of index $r=3$ and so on;

(*ii*) then we construct the set $\overline{Q}^n_{z^n_r}$, and after the set $\overline{Q}_n^{z^n_{n+1-r}}$ of the same pair

$$[\overline{Q}^n_{z^n_r}, \quad \overline{Q}_n^{z^n_{n+1-r}}]$$

of class 3;

(*iii*) we construct the sets of each pair of class <3 simultaneously and conjointly.

Denote by K_n^r the operation of the construction of the sets of a pair of order n and of index r if $1\leqq r\leqq n+1-r$, i. e. if this pair is of class <3.

Denote by D_n^s $(s=1, 2, 3, \ldots, 2c_n^3)$ the operation of the construction of the set belonging to the class 3 where s indicate the order of this operation defined by the rule R.

Let G_k *be one of the operations* K_n^r, D_n^s

$$(n=1, 2, 3, \ldots; \ r=1, 2, 3, \ldots; \ s=1, 2, 3, \ldots)$$

where k indicate the place of the operation G_k in the system of all these operations defined by the rule R.

Each operation Q_k must be accompanied by the passage from one perfect system Q of the sets to another perfect system which will *be denoted by* G_kQ. After each operation G_k it must be established some *correspondence of the similitude* π_k: $x \longleftrightarrow t$ between the points t of the already formed sets $\overline{Q}$ and the points x of the corresponding sets Q of the system G_kQ.

Thus, *each step of the* procedure $\bar{Q}$ consists

1) of the operation G_k, 2) of the operation construing the sets of the perfect system G_kQ and 3) of the operation establishing the correspondence π_k.

The general scheme of *the step S_k* of the procedure $\bar{Q}$ will be described in the following lines. We suppose that the correspondence $\pi_{k-1}: x \longleftrightarrow t$ is given. Denote by $X_{k-1}\{T_{k-1}\}$ the set of all points $x\{t\}$ partaking oi the correspondence π_{k-1}.

We now consider the following cases.

First case: the operation G_k brings to the simultaneous construction of the sets $\bar{Q}_1$, $\bar{Q}_2$ of one pair. Let $Q_1\{Q_2\}$ be the set of the perfect system $G_{k-1}Q$ having the same indices as those of the sets $\bar{Q}_1\{\bar{Q}_2\}$ (4). Let $q_1=X_{k-1}Q_1$, $q_2=X_{k-1}Q_2$. Evidently, we can attach to the set $q_1\{q_2\}$ a set $\bar{q}_1\{\bar{q}_2\}$ corresponding in virtue of π_{k-1} to the set $q_1\{q_2\}$. Let (a, b) $\{(A, B)\}$ be any contiguous interval of the set $q_1\{q_2\}$ and let $(\bar{a}, \bar{b})$ $\{(\bar{A}, \bar{B})\}$ be the contiguous interval of the set $\bar{q}_1\{\bar{q}_2\}$ corresponding in virtue of π_{k-1} to the interval (a, b) $\{(A, B)\}$. We will construct the perfect everywhere non dense sets $\bar{Q}_1$, $\bar{Q}_2$ in such a manner that $[\bar{a}, \bar{b}]\bar{Q}_1$, $[\bar{A}, \bar{B}]\bar{Q}_2$ are the perfect everywhere non dense sets. But it may happen that the sets $[a, b]Q_1$, $[A, B]Q_2$ are not perfect. In order to transform the sets $[a, b]Q_1$, $[A, B]Q_2$ into the perfect sets it is necessary to make *the original correction.* We must transform this original correction into the *complete correction.* This gives us the perfect system G_kQ. We now establish the correspondence of similitude $\varphi(a, b)$ $\{\varphi(A, B)\}$: $[a, b]Q_1 \longleftrightarrow [\bar{a}, \bar{b}]\bar{Q}_1$ $\{[A, B]Q_2 \longleftrightarrow [\bar{A}, \bar{B}]\bar{Q}_2\}$ where $Q_1\{Q_2\}$ is the set of the perfect system G_kQ having the same indices as those of the set $\bar{Q}_1\{\bar{Q}_2\}$. We adjoin to the correspondence π_{k-1} the correspondences $\varphi(a, b)$ $\{\varphi(A, B)\}$ for all the intervals (a, b) $\{(A, B)\}$, then we obtain the new correspondence of similitude π_k.

Second case: the operation G_k brings to the construction of the first set of a pair of class 3. In this case, we repeat the precedent reasoning, but leaving out the letters Q_2, $\bar{Q}_2$, q_2 and the intervals (A, B).

Third case: the operation G_k brings to the construction *of the second set of a pair* of class 3. In this case we repeat the precedent reasoning, but leaving out the letters Q_1, $\bar{Q}_1$, q_1 and the intervals (a, b).

It is obvious that the passage from the system $G_{k-1}Q$ to the system G_kQ runs without changing the sets Q of order $\leqq h_k-1$ if h_k is the order of the sets Q_1, Q_2. [Note. We shall say that the set $Q_{y_r}^{r+s}$ $\{Q_{r+s}^{y_r}\}$ is the set of order $r+s$]. This means that

$$G_{k-1}\, Q_{y_r}^{r+s} = G_k\, Q_{y_r}^{r+s}, \quad G_{k-1}\, Q_{r+s}^{y_r} = G_k\, Q_{r+s}^{y_r}$$

(4) We shall say that the sets Q_q^p and $\bar{Q}_q^p$ have the same indices p, q.

for $r+s \leqq h_k - 1$ where

$$G_t Q_{y_r}^{r+s}, \quad G_t Q_{r+s}^{y_r}$$

are the sets of the system $G_t Q$. In pursuance of this by continuation of the procedure $\overline{Q}$ we obtain the *final perfect system* $\dot{Q}$

$$(\dot{Q}) \qquad \dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3, \ldots ; \; r=1, 2, 3, \ldots).$$

Describing above the general scheme of the step S_k of the procedure $\overline{Q}$ we have omitted the details of the construction of the sets $\overline{Q}$. We will now complete this void space.

Here are the details.

First case: the considered pair $[\overline{Q}_{z_1^n}^n, \overline{Q}_n^{z_n^n}]$ is of class 1. Denote by ψ_{n-1} the correspondence by similitude: $x \longleftrightarrow t$ between the points t of the formed sets $\overline{Q}$ of order $n-1$ and the points x of the corresponding sets Q of the perfect system which is obtained after the precedent corrections and which will be denoted by $G_{k-1} Q$ sothat $\psi_{n-1} = \pi_{k-1}$, $Q_1 = Q_{z_1^n}^n$, $Q_2 = Q_n^{z_n^n}$ where $Q_{z_1^n}^n$, $Q_n^{z_n^n}$ are the sets of the perfect system $G_{k-1} Q$. By applying the general scheme of the step S_k we determine the sets q_1, q_2 and the contiguous intervals (a, b) $\{(A, B)\}$ of the set $q_1 \{q_2\}$. Let $\overline{q}_1 \{\overline{q}_2\}$ be the set corresponding to the set $q_1 \{q_2\}$ in virtue of π_{k-1} and let $(\overline{a}, \overline{b})$ $\{(\overline{A}, \overline{B})\}$ be the contiguous interval of $\overline{q}_1 \{\overline{q}_2\}$ corresponding to the interval (a, b) $\{(A, B)\}$ in virtue of π_{k-1}. Now we construct the perfect everywhere non dense set $P[\overline{a}, \overline{b}]$ $\{P[\overline{A}, \overline{B}]\}$ contained in the segment $[\overline{a}, \overline{b}]$ $\{[\overline{A}, \overline{B}]\}$ and containing the points $\overline{a}, \overline{b}$ $\{\overline{A}, \overline{B}\}$. After that we adjoin to the set $\overline{q}_1 \{\overline{q}_2\}$ the sets $P[\overline{a}, \overline{b}]$ $\{P[\overline{A}, \overline{B}]\}$ for all the intervals $[\overline{a}, \overline{b}]$ $\{[\overline{A}, \overline{B}]\}$, we obtain then the set

$$\overline{Q}_1 = \overline{Q}_{z_1^n}^n \; \{\overline{Q}_2 = \overline{Q}_n^{z_n^n}\}.$$

Second case: the considered pair

$$[\overline{Q}_{z_1^n}^n, \; \overline{Q}_n^{z_{n+1-r}^n}]$$

is of class 2. We assume that the sets of this pair are constructed by the operation G_k. In order to make use of the general scheme we shall put

$$Q_1 = Q_{z_r^n}^n, \quad Q_2 = Q_n^{z_{n+1-r}^n}$$

where

$$Q_{z_r^n}^n, \quad Q_n^{z_{n+1-r}^n}$$

are the sets of the perfect system $G_{k-1}Q$ and we define the sets $q_1\{q_2\}$ as it was shown in the general scheme of the step S_k. Let $\overline{q}_1\{\overline{q}_2\}$ be the set corresponding to the set $q_1\{q_2\}$ in virtue of π_{k-1}.

We form the sets

$$\overline{R}^n_{z^n_r}=\overline{q}_1+\overline{Q}^n_{z^n_{r-1}}, \quad \overline{R}^{z^n_n}_{n+1-r}=\overline{q}_2+\overline{Q}^{z^n_n}_{n+1-r}$$

It is easily seen that

$$\overline{R}^n_{z^n_r}\cdot\overline{R}^{z^n_n}_{n+1-r}=0.$$

We construct with the lemma 3 the perfect everywhere non dense set $Q^n_{z^n_r}\ \{Q^{z^n_n}_{n+1-r}\}$ such

(*i*) every point of the set $\overline{R}^n_{z^n_r}\ \{\overline{R}^{z^n_n}_{n+1-r}\}$ is a density $(\varrho_n, \varDelta_n)$ point of the set

$$\overline{Q}^n_{z^n_r}\ \{\overline{Q}^{z^n_n}_{n+1-r}\};$$

(*ii*)
$$\overline{Q}^n_{z^n_r}\cdot\overline{Q}^{z^n_n}_{n+1-r}=0.$$

Third case: the considered pair $[\overline{Q}^n_{z^n_1}, \overline{Q}^{z^n_n}_{n+1-r}]$ is of class 3. In order to make use of the general scheme we shall put

$$Q_1=Q^n_{z^n_r}\ \{Q_2=Q^{z^n_n}_{n+1-r}\ .$$

Next we determine the sets $\overline{q}_1$, $\overline{q}_2$ so as it was shown in the general scheme of the step S_k. After that we form the set

$$\overline{R}^n_{z^n_r}=\overline{q}_1+\overline{Q}^n_{z^n_{r-1}}\quad\{\overline{R}^{z^n_n}_{n+1-r}=\overline{q}_2+\overline{Q}^{z^n_n}_{n+1-r+1}\}$$

and we construct the perfect everywhere non dense set

$$\overline{Q}^n_{z^n_r}\ \{\overline{Q}^{z^n_n}_{n+1-r}\}$$

such that every point of the set $\overline{R}^n_{z^n_r}\ \{\overline{R}^{z^n_n}_{n+1-r}\}$ is a density $(\varrho_n, \varDelta_n)$ point of the set $\overline{Q}^n_{z^n_r}\ \{\overline{Q}^{z^n_n}_{n+1-r}\}$.

We shall construct all the sets $\overline{Q}$ of class 3 keeping to this model except for the set $\overline{Q}^{z^n_n}_1$ which must satisfy the following complementary condition:

the length of the greater of the contiguous intervals of the set $\overline{Q}^n_{z^n_n}+\overline{Q}^{z^n_n}_1$ contained in $[0, 1]$ is less than $\frac{1}{n^2}$.

Thus, the procedure $\overline{Q}$ of the construction of the sets

$$(\overline{Q}) \qquad \overline{Q}_{y_r}^{r+s}, \quad \overline{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3,....; \ r=1, 2, 3,....)$$

is entirely determined. The principal result of the precedent discussion may be stated in the form of the following:

THEOREM 2. - If $f(x)$ is a finite function of class 1 having the property of DARBOUX in $[0 \leqq x \leqq 1]$, then there exists a *perfect system* $\dot{Q}$ of sets

$$(\dot{Q}) \qquad \dot{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3,....; \ r=1, 2, 3,....)$$

contained in $[0 \leqq x \leqq 1]$ and a system $\overline{Q}$ of perfect everywhere non dense sets

$$(\overline{Q}) \qquad \overline{Q}_{y_r}^{r+s}, \quad \overline{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3,....; \ r=1, 2, 3,....)$$

contained in $[0 \leqq t \leqq 1]$ which enjoys the following properties:

(*i*) there exists a correspondence *CS* of similitude

$$\dot{Q}_{y_r}^{r+s} \longleftrightarrow \overline{Q}_{y_r}^{r+s}, \quad \dot{Q}_{r+s}^{y_r} \longleftrightarrow \overline{Q}_{r+s}^{y_r}, \qquad (s=0, 1, 2, 3,....; \ r=1, 2, 3,....)$$

between the sets of the system $\overline{Q}$ and the sets of the system $\dot{Q}$;

(*ii*) the set

$$\lim_{n \to \infty} \sum_{s=1}^{s=n} \overline{Q}_{y_s}^{n} \ \Big\{ \lim_{n \to \infty} \sum_{s=1}^{s=n} \overline{Q}_{n}^{y_s} \Big\}$$

is dense in the segment $[0 \leqq t \leqq 1]$;

(*iii*) if $y_r < y_t$ and if M is the greater of the integers r and t, then every point of the set

$$\overline{Q}_{y_r}^{M+s} \ \{ \overline{Q}_{M+s}^{y_t} \}$$

is a density $(\varrho_{M+s}, \Delta_{M+s})$ point of the set

$$\overline{Q}_{y_t}^{M+s} \ \{ \overline{Q}_{M+s}^{y_r} \}$$

for all $s=1, 2, 3,....$;

$$(iiii) \qquad m[C(\overline{Q}_{Y}^{n}, \overline{Q}_{n}^{y})] \leqq a_n$$

where a_n is a positive number such that the series

$$a_1 + a_2 + a_3 + ...$$

is convergent and $Y\{y\}$ is the greatest $\{$the least$\}$ of the numbers $y_1, y_2, y_3,...., y_n$.

4. - Properties of the functions $\psi_n(t)$. Let

$$x = \psi_n(t), \qquad [\psi_n(0)=0, \ \psi_n(1)=1]$$

be the continuous and essentially increasing function which transforms the sets

$$\dot{Q}^{r+s}_{y_r}, \quad \dot{Q}^{y_r}_{r+s}, \qquad (s=0, 1, 2, 3,\dots; \ r=1, 2, 3,\dots)$$

respectively into the sets

$$\bar{Q}^{r+s}_{y_r}, \quad \bar{Q}^{y_r}_{r+s}, \qquad (s=0, 1, 2, 3,\dots; \ r=1, 2, 3,\dots).$$

It is obvious that this function $x=\psi_n(t)$ express the correspondence of the similitude ψ_n. This function $\psi_n(t)$ enjoys the following properties.

First property: this function transforms the sets

$$\bar{Q}^{n}_{y_r}, \quad \bar{Q}^{y_r}_{n}, \qquad (r=1, 2, 3,\dots, n; \ n=1, 2, 3,\dots)$$

respectively into the sets

$$\dot{Q}^{n}_{y_r}, \quad \dot{Q}^{y_r}_{n}, \qquad (r=1, 2, 3,\dots; \ n=1, 2, 3,\dots).$$

Second property. Let

$$Q_m=\sum_{s=1}^{s=m}(Q^{m}_{y_s}+Q^{y_s}_{m}), \quad \bar{Q}_m=\sum_{s=1}^{s=m}(\bar{Q}^{m}_{y_s}+\bar{Q}^{y_s}_{m}).$$

Then, if t_0 is any point of Q_p and if $p<q$, we have

$$\psi_q(t_0)=\psi_p(t_0).$$

Third property: the sequence of the functions

$$\psi_1(t), \quad \psi_2(t), \quad \psi_3(t),\dots$$

is uniformly convergent. In reality, let t be any point which do not belong to the set $\bar{Q}_p$, consequently, t will belong to a contiguous interval $i_t=(J_1, J_2)$ of the set $\bar{Q}_p$. The functions $\psi_p(t)$ and $\psi_q(t)$ are continuous and essentially increasing, therefore

$$\psi_p(J_1)<\psi_p(t)<\psi_p(J_2), \quad \psi_q(J_1)<\psi_q(t)<\psi_q(J_2).$$

Since $\psi_p(J_1)=\psi_q(J_1)$, $\psi_p(J_2)=\psi_q(J_2)$, if $p<q$, we have

$$|\psi_q(t)-\psi_p(t)|<|\psi_p(J_2)-\psi_p(J_1)|.$$

But $(\psi_p(J_1), \psi_p(J_2))$ is a contiguous interval of the set Q_p corresponding in virtue of $x=\psi_p(t)$ to the contiguous interval (J_1, J_2) of the set $\bar{Q}_p$. Let l_p be the length of the greatest of the contiguous intervals of Q_p, then we have

$$(l) \qquad |\psi_q(t)-\psi_p(t)|<l_p$$

for every point t of the segment $[0\leqq t\leqq 1]$, for we have $|\psi_q(t)-\psi_p(t)|=0$

for every point t of $\bar{Q}_p$. The set $\lim_{p\to\infty} Q_p$ is dense in $[0 \leqq x \leqq 1]$, therefore $\lim_{p\to\infty} l_p = 0$. Let ε be an arbitrary, positive, small number. Then we can find a positive integer ν_ε such that $l_p < \varepsilon$ for $p > \nu_\varepsilon$. Thus,

$$|\psi_q(t) - \psi_p(t)| < \varepsilon \qquad (l)$$

in all cases when $q > p > \nu_\varepsilon$, consequently, the sequence $\psi_1(t)$, $\varphi_2(t)$, $\psi_3(t)$,.... is convergent. Let $\psi(t) = \lim_{n\to\infty} \psi_n(t)$. From the inequality (l) we deduce

$$|\psi(t) - \psi_p(t)| < l_p.$$

Since $\lim_{p\to\infty} l_p = 0$ we conclude that the sequence

$$\psi_1(t), \quad \psi_2(t), \quad \psi_3(t), \ldots.$$

is uniformly convergent, consequently, the function $\psi(t)$ is continuous in $[0 \leqq t \leqq 1]$. Since $\psi_n(0) = 0$, $\psi_n(1) = 0$, we have $\psi(0) = 0$, $\psi(1) = 1$.

Properties of the function $\psi(t)$.

The first property: if a point t_0 belong to the set $\bar{Q}_\nu$, then $\psi_\nu(t_0) = \psi(t_0)$. In reality,

$$\psi_\nu(t_0) = \psi_{\nu+1}(t_0) = \psi_{\nu+2}(t_0) = \ldots. = \psi(t_0).$$

The second property: $\psi(t)$ is an essentially increasing function in $[0 \leqq t \leqq 1]$.

Proof. - In fact, let us suppose $t_1 < t_2$. Then $\psi_n(t_1) < \psi_n(t_2)$, no matter what n is, consequently, $\lim_{n\to\infty} \psi_n(t_1) \leqq \lim_{n\to\infty} \psi_n(t_2)$, or $\psi(t_1) \leqq \psi(t_2)$. Since the set $\lim_{n\to\infty} \bar{Q}_n$ is dense in $[0 \leqq t \leqq 1]$, we can find an integer ν such that the interval (t_1, t_2) contains at least two points J_1 and J_2, $J_1 < J_2$, of the set $\bar{Q}_\nu$, consequently, $t_1 < J_1 < J_2 < t_2$. Thence we deduce $\psi_\nu(t_1) < \psi_\nu(J_1) < \psi_\nu(J_2) < \psi_\nu(t_2)$. But $\psi_\nu(J_1) = \psi(J_1)$, $\psi_\nu(J_2) = \psi(J_2)$, therefore $\psi_\nu(t_1) < \psi(J_1) < \psi(J_2) < \psi_\nu(t_2)$.

Passing to the limit we obtain

$$\psi(t_1) \leqq \psi(J_1) < \psi(J_2) \leqq \psi(t_2)$$

from which it results $\psi(t_1) < \psi(t_2)$.

Third property: the function $f[\psi(t)]$ is an ordinary derivative in $[0 \leqq t \leqq 1]$.

Proof. - At first we construct for $f(x)$ *any characteristic system* P of perfect sets

$$(P) \qquad P^n_{y_r}, \quad P^{y_r}_n, \qquad [r = 1, 2, 3, \ldots., n;\ n = 1, 2, 3, \ldots.].$$

In the second place, we pass from this system P to a perfect system Q of sets

$$(Q) \qquad Q^n_{y_r}, \quad Q^{y_r}_n, \qquad [r = 1, 2, 3, \ldots., n;\ n = 1, 2, 3, \ldots.].$$

In the third place, we construct using the theorem 2 a final perfect system $\dot{Q}$ of the sets

$(\dot{Q})$ $\qquad \dot{Q}^n_{y_r}, \quad \dot{Q}^{y_r}_n, \qquad [r=1, 2, 3,...., n; \; n=1, 2, 3,....]$

and simultaneously the system $\bar{Q}$ of perfect everywhere non dense sets

$(\bar{Q})$ $\qquad \bar{Q}^n_{y_r}, \quad \bar{Q}^{y_r}_n$

satisfying all the conditions of the theorem 2.

In the fourth place, we form the system $\dot{P}$ of the sets

$(\dot{P})$ $\qquad \dot{P}^n_{y_r}, \quad \dot{P}^{y_r}_n, \qquad [r=1, 2, 3,...., n; \; n=1, 2, 3,....],$

where

$$\dot{P}^n_{y_r}=P^n_{y_r}+\dot{Q}^n_{y_r}, \quad \dot{P}^{y_r}_n=P^{y_r}_n+\dot{Q}^{y_r}_n.$$

It is easily seen that the system $\dot{P}$ is also a characteristic system of the sets for $f(x)$.

We now assume that the function $x=\psi(t)$ transforms the sets

$(\dot{P})$ $\qquad \dot{P}^n_{y_r}, \quad \dot{P}^{y_r}_n, \qquad [r=1, 2, 3,...., n; \; n=1, 2, 3, 4,....]$

respectively into the sets

$(\bar{P})$ $\qquad \bar{P}^n_{y_r}, \quad \bar{P}^{y_r}_n.$

Let $\bar{a}$ be any point of the set $\bar{P}^n_{y_r} \{\bar{P}^{y_t}_n\}$ $(y_r<y_t; \; n\geqq r, \, n\geqq t)$ and let $a=\psi(\bar{a})$. Now we consider the following cases.

First case: the point a belong to the set

$$\dot{Q}^n_{y_r} \; \{\dot{Q}^{y_t}_n\}.$$

In this case the point $\bar{a}$ belongs to the set

$$\bar{Q}^n_{y_r} \; \{\bar{Q}^{y_t}_n\}.$$

The set $\dot{Q}^n_{y_t} \{\dot{Q}^{y_r}_n\}$ is contained in $\dot{P}^n_{y_t} \{\dot{P}^{y_r}_n\}$, therefore the set $\bar{Q}^n_{y_t} \{\bar{Q}^{y_r}_n\}$ is contained in the set $\bar{P}^n_{y_t} \{\bar{P}^{y_r}_n\}$, consequently, the point $\bar{a}$ being a density (ϱ_n, Δ_n) point of the set $\bar{Q}^n_{y_t} \{\bar{Q}^{y_r}_n\}$ is a density (ϱ_n, Δ_n) point of the set $\bar{P}^n_{y_t} \{\bar{P}^{y_r}_n\}$.

Second case: the point a does not belong to the set $\dot{Q}^n_{y_r} \{\dot{Q}^{y_t}_n\}$, consequently, the point a belongs to a contiguous interval i_a of this set. On the other hand, the point a belong to the set $\dot{P}^n_{y_r} \{\dot{P}^{y_t}_n\}$ and does not belong to the set $\dot{Q}^n_{y_r} \{\dot{Q}^{y_t}_n\}$, consequently, the point a belong to the set $P^n_{y_r} \{P^{y_t}_n\}$, for

$$\dot{P}^n_{y_r}=P^n_{y_r}+\dot{Q}^n_{y_r} \quad \{\dot{P}^{y_t}_n=P^{y_t}_n+\dot{Q}^{y_t}_n\}.$$

Finally, the point a does not belong to the set

$$Q^n_{y_r} \; \{Q^{y_t}_n\},$$

because

$$Q^n_{y_r} \subset \dot{Q}^n_{y_r} \ \{Q^{y_t}_n \subset \dot{Q}^{y_t}_n\}.$$

Since the point a belongs to the set $P^n_{y_r}$ $\{P^{y_t}_n\}$ and does not belong to the set $Q^n_{y_r}$ $\{Q^{y_t}_n\}$, this point belongs to some interval j_a contained in the set $P^n_{y_r}$ $\{P^{y_t}_n\}$, and consequently, also in the set

$$\dot{P}^n_{y_r} \ \{\dot{P}^{y_t}_n\}.$$

Then the point a belongs to the interval $i_a j_a$ contained in $\dot{P}^n_{y_r}$ $\{\dot{P}^{y_t}_n\}$ and in the interval i_a, consequently, the point $\bar{a}$ is contained in an interval contained in $\bar{P}^n_{y_r}$ $\{\bar{P}^{y_t}_n\}$ and also in $\bar{P}^n_{y_t}$ $\{\bar{P}^{y_r}_n\}$. This means that $\bar{a}$ is a density (ϱ_n, Δ_n) point of the set

$$\bar{P}^n_{y_t} \ \{\bar{P}^{y_r}_n\}.$$

Thus, every point $\bar{a}$ of the set $\bar{P}^n_{y_r}$ $\{\bar{P}^{y_t}_n\}$ is a density (ϱ_n, Δ_n) point of the set $\bar{P}^n_{y_t}$ $\{\bar{P}^{y_r}_n\}$.

Let m_n denote the number

$$m\{c[\bar{Q}^n_{z^n_n} \cdot \bar{Q}^{z^n_1}_n]\}$$

where z^n_n $\{z^n_1\}$ is the greatest $\{$the least$\}$ of the numbers $y_1, y_2, \dots, y_n$, then $m_n \leqq a_n$, consequently,

$$m\{C[\bar{E}_{z^n_n} \cdot \bar{E}^{z^n_1}]\} \leqq a_n$$

where $\bar{E}_z$ $\{\bar{E}^z\}$ is the set of all the points t of the segment $[0 \leqq t \leqq 1]$ satisfying the condition

$$f[\psi(t)] < z \ \{f[\psi(t)] > z\}.$$

From this it follows that the function $f[\psi(t)]$ is summable in the segment $[0 \leqq t \leqq 1]$.

Thus, the function $f[\psi(t)]$ is approximately (ϱ_n, Δ_n) continuous in the segment $[0 \leqq t \leqq 1]$ (see Definition 2).

From the above discussion there follows immediately the following:

THEOREM 3. - For each finite function $f(x)$ of class 1 having the property of DARBOUX in $[0 \leqq x \leqq 1]$ there exists a continuous and essentially increasing in $[0 \leqq t \leqq 1]$ function $x = \psi(t)$ $[\psi(0) = 0, \psi(1) = 1]$ such that $f[\psi(t)]$ is approximately (ϱ_n, Δ_n) continuous in $[0 \leqq t \leqq 1]$. But in virtue of the theorem 2 the function $f[\psi(t)]$ is an ordinary derivative in $[0 \leqq t \leqq 1]$.

We have thus proved the following N. LUSIN's Theorem. For each finite function $f(x)$ of class 1 having the property of DARBOUX in $[0 \leqq t \leqq 1]$ there exists a continuous and essentially increasing in $[0 \leqq t \leqq 1]$ function

$$x = \psi(t), \qquad [\psi(0) = 0, \ \psi(1) = 1]$$

such that $f[\psi(t)]$ is an ordinary derivative in $[0 \leqq t \leqq 1]$.