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THE HIGHER TOPOLOGICAL FORM OF PLATEAU'S PROBLEM

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§ 1.

The object of the following exposition is to provide, in a more concentrated and perspicuous form than hitherto, an outline of the methods and results of the author's recent work on the general topological form of the PLATEAU problem. In order that the essential features of our theory may stand out more clearly, all proofs and similar details have been omitted; for these, we refer to the papers listed at the end (1), particularly, [2, 3, 4, 5].

The guiding theme is a comparison between the two main procedures that we have followed: the first based on the direct study of GREEN's function for a general RIEMANN surface, the second on θ-functions (2). These two modes of treatment will be confronted under the respective headings of *real harmonic* and *complex analytic*. Indeed, an exact one-one correspondence can be pointed out between the basic formulas of the two methods — thus, the formulas (10.2) and (12.7) for $A(\mathrm{g}, R)$ correspond, the formula (12.6) for $\mathbf{F}'^2(w)$ to (10.4) for $\frac{\partial \mathbf{H}(Q)}{\partial \xi} \cdot \frac{\partial \mathbf{H}(Q)}{\partial \eta}$, and the identity (15.2) in θ-functions to the variational formula (10.5) for GREEN's function.

As formulated by the author a number of years ago (3), the precise statement of our problem is the following.

Given:

1) *k contours $(\Gamma)=(\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$ in the form of Jordan curves in n-dimensional euclidean space, each of assigned form, position, and sense of description*;

2) *a prescribed genus h or topological characteristic* (4) *r;*

3) *either character of orientability, i.e., two-sided or one-sided.*

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(1) References to these will be made by numbers in square brackets.

(2) Chronologically, these two methods were published in the reverse order.

(3) Bulletin Amer. Math. Soc., v. 36 (1930), p. 50.

(4) The definition of r is the maximum number of circuits, no linear combination of which separates the surface. For a two-sided surface, $r=2h$. For a one-sided surface, r may be odd or even; examples: Möbius surface with h handles, $r=2h+1$; Klein surface with h handles, $r=2h+2$. See HILBERT and COHN-VOSSEN: *Anschauliche Geometrie*, 1932.

To determine a minimal surface M corresponding to these data; i.e., M shall have the k boundaries (Γ) and no others, shall have the genus h or characteristic r, and shall be two- or one-sided, as prescribed.

This problem was solved completely by the author in a number of papers published in recent years — first, particular cases: $k=1$, $h=0$, [7, 8]; $k=2$, $h=0$, [9]; $k=1$, $r=1$ (Möbius strip), [10]; and then the general case [1-6].

Subsequent to the author's work, an alternative method of treating the problem, with details for the particular case of k arbitrary, $h=0$, was given by R. COURANT [11-13]. Independently, the same method was presented for the simplest case, $k=1$, $h=0$, by L. TONELLI [14]. The method of these authors is based essentially on permitting the vector $\mathbf{x}(u, v)$ in the multiple DIRICHLET functional $D(\mathbf{x})$ (see (6.2)) to be arbitrary, provided sufficiently regular in respect of continuity and derivatives. The author's method, not fundamentally different (5), restricts $\mathbf{x}$, in the main, to be a harmonic vector $\mathbf{H}(u, v)$. The respective problems

$$D(\mathbf{x})=\min., \qquad D(\mathbf{H})=\min.,$$

are exactly equivalent in virtue of the relation: $D(\mathbf{H})\leqq D(\mathbf{x})$ whenever $\mathbf{H}$ and $\mathbf{x}$ have the same boundary values.

A still more general form of the PLATEAU problem has been formulated and solved by the author [5, 6], where an infinite number of boundaries and infinite connectivity of the required minimal surface M are permitted. In other words, M may have the topological structure of the RIEMANN surface associated with a perfectly general real analytic curve or function, $\mathfrak{C}$. The case — which alone will be considered here — of finite values of k and h corresponds to an algebraic curve or function.

§ 2.

The minimal surface M whose existence is to be established will be obtained as conformal image of a RIEMANN surface R having the topological form prescribed for M: k boundaries and genus h.

R may always be considered as one of the conjugate halves of the complete RIEMANN surface $\mathfrak{R}$ of a real algebraic curve $\mathfrak{C}$: $P(x, y)=0$ (real coefficients); i.e., R is the abstract geometric manifold which results by identification of conjugate complex points $(x+iy,\ x-iy)$ of $\mathfrak{C}$. $\mathfrak{R}$ is a closed surface — i.e., without boundaries — of genus

(2.1) $$p=2h+k-1=r+k-1.$$

(5) As remarked by TONELLI, loc. cit., p. 333.

R, on the other hand, has k boundaries — namely, the real branches C of $\mathfrak{C}$ — and is of genus h; i. e., R admits h and no more non-mutually-intersecting circuits which do not separate it. In relation to $\mathfrak{R}$, the riemannian manifold R is often called a *semi Riemann surface.*

The interchange of conjugate complex points of $\mathfrak{R}$ constitutes an inversely conformal transformation T of $\mathfrak{R}$ into itself, which is involutory: $T^2=1$. A RIEMANN surface $\mathfrak{R}$ with such a related transformation T is called, following KLEIN (6), *symmetric*, and T-equivalent points w, $\overline{w}$ of $\mathfrak{R}$ are termed *symmetric* or also *conjugate.* The points fixed under T form what are called the *transition curves* of $\mathfrak{R}$; these correspond exactly to the real branches of the algebraic curve $\mathfrak{C}$, since any real point is its own conjugate complex.

Two cases as to $\mathfrak{R}$ may present themselves, termed respectively *orthosymmetric* or *diasymmetric*: either C may separate $\mathfrak{R}$ or not. In the former case, the semi RIEMANN surface R may be identified with either of the two conjugate halves of $\mathfrak{R}$ bounded by C. R is then a two-sided manifold. In the latter case, R is a one-sided manifold; for the conjugate points w, $\overline{w}$ represent two antipodal points between which it is possible to pass by a continuous path without crossing the boundary C — a circumstance which typifies one-sidedness.

To fix the ideas, the wording of the sequel will be arranged with the two-sided case in mind, but is easily adjusted to the one-sided case. Indeed, the latter may be referred to the former by means of the standard device of a two-sided covering surface in two-one point correspondence with the one-sided surface (see [3], § 2, arts. 2, 3).

One form of the RIEMANN surface of $\mathfrak{C}$ is the two-dimensional locus $\mathfrak{S}$ in the four-dimensional space $(x=x_1+ix_2, y=y_1+iy_2)$ of the equation $P(x_1+ix_2, y_1+iy_2)=0$ of $\mathfrak{C}$. Another form is the many-sheeted surface $\mathfrak{R}_x$ over the complex x-plane, or $\mathfrak{R}_y$ over the complex y-plane. $\mathfrak{R}_x$, $\mathfrak{R}_y$ are exactly the orthogonal projections of $\mathfrak{S}$ on the planes mentioned, and the correspondence between $\mathfrak{S}$, $\mathfrak{R}_x$, and $\mathfrak{S}$, $\mathfrak{R}_y$ thereby set up is conformal.

In all respects, conformally equivalent RIEMANN manifolds are identical for our purposes. For this reason, the algebraic curve $\mathfrak{C}$ may, without affecting anything essential, be replaced by any algebraic curve $\mathfrak{C}'$ equivalent to $\mathfrak{C}$ by a real birational transformation, where « real » means: respecting conjugate points. This is because the RIEMANN surfaces $\mathfrak{R}$ and $\mathfrak{R}'$ of any two such birationally equivalent curves are conformally equivalent, so that the corresponding semi-surfaces R, R' are precisely conformally equivalent.

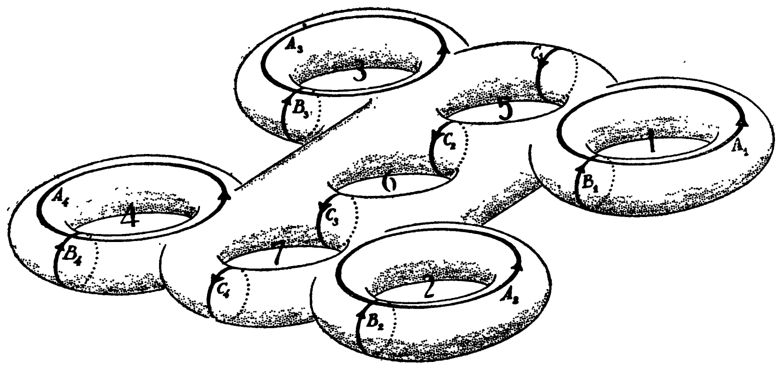
(6) F. KLEIN: *Über Riemanns Theorie der algebraischen Functionen und ihrer Integrale*, 1882.

§ 3.

The closed RIEMANN surface $\mathfrak{R}$ of genus p has p circuits A_j of the first system and p circuits B_j of the second system. The inverse conformal transformation T associates the circuits of each system, or their indices, in pairs j, j' so that we have

$$TA_j = -A_{j'}, \qquad TB_j = B_{j'}. \tag{3.1}$$

This is illustrated by the following figure, where, for the semi RIEMANN surface R, $k=4$, $h=2$: while, for the complete surface $\mathfrak{R}$, $p=7$. The inverse conformal automorphism T of $\mathfrak{R}$ is the reflection in the plane containing C_1,



C_2, C_3, C_4, the boundaries of R. The indices j, j' are paired according to the substitution

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 1 & 2 & 5 & 6 & 7 \end{pmatrix}, \qquad T^2 = 1. \tag{3.2}$$

We term corresponding indices j, j' *symmetric*, and call the index j *self-symmetric* or *alter-symmetric* according as $j'=j$ or $j' \neq j$. The essence of the case $h=0$ is that then, as is evident from a figure, all indices are self-symmetric. On the other hand, the index corresponding to any handle of R is alter-symmetric, since there is a distinct image handle on the conjugate semi-surface R'. We shall denote alter-symmetric indices by Greek letters: α, β, λ, μ, and the respective symmetric indices belonging to R' by α', β', λ', μ'.

On $\mathfrak{R}$ there exist exactly p linearly distinct normal abelian integrals of the first kind, $v_j(w)$. Their characteristic property is to be continuable indefinitely on $\mathfrak{R}$ as multiform functions without any singular point. « Normal » means that the period of $v_j(w)$ with respect to A_k is δ_{jk} ($=1$ if $j=k$, $=0$ if $j \neq k$). The period of $v_j(w)$ as to B_k is denoted by τ_{jk}, and we have the well-known relation of symmetry:

$$\tau_{jk} = \tau_{kj}. \tag{3.3}$$

It is readily shown that, due to the symmetry of $\mathfrak{R}$, we have the relations

$$v_{j'}(w) = -\overline{v_j(\overline{w})}, \tag{3.4}$$

$$\tau_{j'k'} = -\overline{\tau}_{jk}, \tag{3.5}$$

where the bar denotes the conjugate complex quantity.

The following quantities play an important role in our formulas:

$$t_{\alpha\beta} = \frac{1}{2\pi i}(\tau_{\alpha\beta} - \tau_{\alpha'\beta} - \tau_{\alpha\beta'} + \tau_{\alpha'\beta'}), \tag{3.6}$$

where the range of the alter-symmetric indices α, β is

$$\alpha,\ \beta = 1,\ 2, \ldots,\ h, \tag{3.7}$$

while α', β' are the respective symmetric indices:

$$\alpha',\ \beta' = h+1,\ h+2, \ldots,\ 2h. \tag{3.8}$$

The quantities $t_{\alpha\beta}$ are real, as can be seen by (3.5), in fact,

$$t_{\alpha\beta} = \mathfrak{R}\,\frac{1}{\pi i}(\tau_{\alpha\beta} - \tau_{\alpha'\beta}). \tag{3.9}$$

They are also symmetric, as follows from (3.3): $t_{\alpha\beta} = t_{\beta\alpha}$.

It can be proved quite easily that the determinant $T = |t_{\alpha\beta}|$ does not vanish. Hence, we can construct the reciprocal matrix

$$T_{\alpha\beta} = \text{cofactor of } t_{\alpha\beta} \text{ in } T \div T, \tag{3.10}$$

where we have the usual relation between reciprocal matrices:

$$T_{\alpha\lambda} t_{\lambda\beta} = \delta_{\alpha\beta} \tag{3.11}$$

(summation convention for $\lambda = 1, 2, \ldots, h$).

§ 4.

The θ-functions on $\mathfrak{R}$ play a fundamental role in our theory.

For the genus $p=1$ of $\mathfrak{R}$, we have the elliptic θ-function of JACOBI:

$$\theta(u, \tau) = \sum_{n=-\infty}^{+\infty} e^{2\pi i u\left(n+\frac{1}{2}\right) + i\pi\tau\left(n+\frac{1}{2}\right)^2 + \pi i\left(n+\frac{1}{2}\right)}, \tag{4.1}$$

with a single summation index n and a single period τ. This is not essentially different from the elliptic function $\sigma(u)$ of WEIERSTRASS with the periods 1, τ, the relation between θ and σ being

$$\sigma(u) = A e^{Bu^2} \theta(u), \tag{4.2}$$

where A, B are certain constants as to u, being functions only of τ. From σ, WEIERSTRASS derived the other two elliptic functions

(4.3) $$\zeta(u)=\frac{d\log\sigma(u)}{du},\qquad \wp(u)=-\frac{d^2\log\sigma(u)}{du^2},$$

fundamental to his theory.

RIEMANN generalized the one-index θ-series of JACOBI to a p-index θ-series, function of the p primary variables u_j and of a matrix of periods τ_{jk}, namely:

(4.4) $$\theta(u;\tau)=\sum\exp\left[2\pi i u_j\left(n_j+\frac{1}{2}\varrho_j\right)+\right.$$
$$\left.+i\pi\tau_{jk}\left(n_j+\frac{1}{2}\varrho_j\right)\left(n_k+\frac{1}{2}\sigma_k\right)+i\pi\sigma_k\left(n_k+\frac{1}{2}\varrho_k\right)\right],$$

where the summation $\sum$ is with regard to the indices $n_1, n_2, \ldots, n_p$, which vary independently over all integer values from $-\infty$ to $+\infty$. The summation convention as to repeated indices applies to $j, k=1, 2, \ldots, p$.

The half-integers $\frac{1}{2}\varrho_k$, $\frac{1}{2}\sigma_k$ constitute the *characteristic* of the θ-function (7); usually their values are 0 or $\frac{1}{2}$. We presume an odd characteristic, i. e.,

(4.5) $$\varrho\sigma=\varrho_1\sigma_1+\varrho_2\sigma_2+\ldots+\varrho_p\sigma_p=\text{odd integer}.$$

This implies that θ is an odd function of its arguments u:

(4.6) $$\theta(-u)=-\theta(u).$$

We also suppose

(4.7) $$\varrho_{j'}=\varrho_j,\qquad \sigma_{j'}=\sigma_j$$

(satisfied automatically in the self-symmetric case $j'=j$), which, together with (3.5) $\tau_{j'k'}=-\bar{\tau}_{jk}$, implies the relation of conjugacy

(4.8) $$\theta(-\bar{u}_{j'})=e^{\pi i\varrho\sigma}\overline{\theta(u_j)}=-\overline{\theta(u_j)}.$$

The θ-function has important period properties, expressed by the formulas

(4.9) $$\theta(u_j+\delta_{jk})=e^{\pi i\varrho_k}\theta(u_j),$$

(4.10) $$\theta(u_j+\tau_{jk})=e^{-2\pi i\left(u_k+\frac{1}{2}\tau_{kk}+\frac{1}{2}\sigma_k\right)}\theta(u_j).$$

Finally, we note the partial differential equation obeyed by the θ-function:

(4.11) $$\frac{\partial^2\theta}{\partial u_j\partial u_k}=4\pi i\frac{\partial\theta}{\partial\tau_{jk}}.$$

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(7) The introduction of the characteristic in the RIEMANN θ-series is due to HERMITE.

Following RIEMANN, let us substitute for each argument u_j in $\theta(u)$ the corresponding abelian integral $v_j(z, w)=v_j(z)-v_j(w)$. The resulting function of z, complex variable on the RIEMANN surface $\mathfrak{R}$, namely $\theta(v(z, w))$ is called a θ-function on this RIEMANN surface. Since the periods of $v_j(z)$ as to A_k, B_k are respectively δ_{jk}, τ_{jk}, it follows by (4.9, 10) that when z performs a circuit of A_k, $\theta(v(z, w))$ receives the factor $e^{\pi i \varrho_k}$, and, for a circuit of B_k, the factor indicated in (4.10), where $u_k=v_k(z, w)$. The multiform function $\theta(v(z, w))$ has no singular point on $\mathfrak{R}$.

Writing $\tau_{jk}=\tau'_{jk}+i\tau''_{jk}$, we know that for any RIEMANN surface $\mathfrak{R}$, the quadratic form $\tau''_{jk}x_jx_k$ is positive definite ([8]). This implies, referring to (4.4), that the θ-series and all its derived series are absolutely and uniformly convergent for all bounded values of the variables u, τ; accordingly, it is permissible to differentiate this series term-wise to any order — this is the way in which the partial differential equation (4.11) is obtained.

Generalizing the definitions (4.3) of ζ, $\wp$ in the theory of elliptic functions $(p=1)$, we have for a general value of the genus p the systems of functions ζ_j, $\wp_{jk}$ $(j, k=1, 2,\ldots, p)$, namely:

$$\zeta_j(u;\tau)=\frac{\partial \log \theta(u;\tau)}{\partial u_j}, \qquad \wp_{jk}(u;\tau)=-\frac{\partial^2 \log \theta(u;\tau)}{\partial u_j \partial u_k}. \tag{4.12}$$

The period properties of $\theta(u)$ imply, by logarithmic differentiation of (4.9, 10), the following period properties of ζ_j:

$$\zeta_j(u_k+\delta_{km})=\zeta_j(u_k), \tag{4.13}$$

$$\zeta_j(u_k+\tau_{km})=\zeta_j(u_k)-2\pi i\delta_{jm}. \tag{4.14}$$

From these, it follows, by differentiation as to u_k, that $\wp_{jk}(u)$ is $2p$-fold periodic in the exact sense, remaining unchanged in value when each argument u_l is replaced by $u_l+\delta_{lm}$ or $u_l+\tau_{lm}$, for $m=1, 2,\ldots, p$.

We define also, following RIEMANN, the normal integral of the second kind on $\mathfrak{R}$:

$$t(z, w)=-\frac{\partial \log \theta(v(z, w))}{\partial w}=\zeta_j(v(z, w))v_j'(w) \tag{4.15}$$

(summation convention for j). The only singularity of $t(z, w)$ is a pole of the first order at $z=w$ with residue equal to unity. By the period properties of $\theta(v(z, w))$, the periods of $t(z, w)$ as to the circuits A_k all vanish, while the period of $t(z, w)$ as to B_k is $-2\pi i v_k'(w)$.

([8]) E. PICARD: *Traité d'Analyse,* v. 2, 1925, p. 483.

§ 5.

A minimal surface M is, by definition, a *harmonic* and *conformal* image of a RIEMANN surface R. Vectorially written, the formulas for M are

$$\mathbf{x}=\mathbf{H}(Q), \tag{5.1}$$

$$\frac{\partial \mathbf{H}(Q)}{\partial \xi}\cdot\frac{\partial \mathbf{H}(Q)}{\partial \eta}=0. \tag{5.2}$$

Here Q denotes an arbitrary point of R; ξ, η are any two perpendicular directions on R at Q; $\mathbf{x}$, $\mathbf{H}$ are vectors in euclidean space of any number n of dimensions; and the dot denotes the scalar product of vectors. (5.1) expresses the harmonic character of M, and (5.2) its conformal representation on R, for, according to (5.2), perpendicular directions on R correspond to perpendicular directions on M (9).

Since any harmonic function is the real part of an analytic function of a complex variable, we may represent M in an alternative form as follows:

$$x_i=\mathfrak{R}F_i(w), \qquad (i=1, 2,\ldots, n) \tag{5.3}$$

$$\sum_{i=1}^{n} F_i'^2(w)=(E-G)-2iF=0, \tag{5.4}$$

or vectorially,

$$\mathbf{x}=\mathfrak{R}\mathbf{F}(w), \tag{5.5}$$

$$\mathbf{F}'^2(w)=0. \tag{5.6}$$

In (5.4), E, F, G are the coefficients in the ds^2 of M, and (5.4) or (5.6), i. e., $E=G$, $F=0$, express the conformality between M and R.

The parallelism between (5.1, 2), on the one hand, and (5.5, 6), on the other, will be continued systematically throughout our theory. According as the one or other pair of formulas is applied, we have two modes of treatment of the problem: the *real harmonic* and the *complex analytic*. Beginning a little later, we shall present these two methods succesively in a dualistic way.

§ 6.

Classically, going back to LAGRANGE (10), the PLATEAU problem presented itself as one of least area:

$$\mathfrak{A}(S)\equiv\iint\sqrt{EG-F^2}\,du\,dv=\text{min.} \tag{6.1}$$

among all surfaces S having a given boundary Γ.

(9) Which is sufficient to secure conformality, according to standard theorems on maps (TISSOT's theorem).

(10) J. L. LAGRANGE: *Miscellanea Taurinensia* (1760-1761); also *Oeuvres*, v. 1, p. 335.

The essential contribution made by the author was to employ, instead of (6.1), the minimum principle

$$D(\mathbf{x}) \equiv \iint \frac{1}{2}(E+G)dudv \equiv \frac{1}{2}\iint (\mathbf{x}_u^2+\mathbf{x}_v^2)dudv \tag{6.2}$$

$$\equiv \frac{1}{2}\iint \sum_{i=1}^{n} \left[\left(\frac{\partial x_i}{\partial u}\right)^2 + \left(\frac{\partial x_i}{\partial v}\right)^2\right] dudv = \min.$$

Indeed, the defining formulas of a minimal surface given in the preceding section represent exactly the vanishing of the first variation of $D(\mathbf{x})$.

The functional $D(\mathbf{x})$ depends on the parametric representation $\mathbf{x}=\mathbf{x}(u, v)$ of S as well as on this surface itself. The minimum condition (6.2), besides distinguishing S as a minimal surface M, determines its parametric representation as conformal.

In the general topological form of our problem, the assigned boundary (Γ) of M consists of k JORDAN curves, and the topological type of M, as well as that of all the surfaces S with which it is compared, is prescribed.

S is represented parametrically on a RIEMANN surface R of the assigned topological form (genus h and k boundaries), and is given by an equation $\mathbf{x}=\mathbf{x}(u, v)$ which converts the bounding curves $(C)=(C_1, C_2, \ldots, C_k)$ of R into the given contours (Γ) in a one-one continuous way. We shall denote by $\mathbf{g}$, abbreviation for $\mathbf{x}=\mathbf{g}(z)$, this one-one continuous or parametric representation of (Γ) on (C).

Then, for a given RIEMANN surface R and given parametric representation $\mathbf{g}$ of (Γ), it is a classic result that the minimum value of $D(\mathbf{x})$ is attained for the harmonic function $\mathbf{H}(u, v)$ on R determined by the boundary values $\mathbf{g}$ on (C); i. e.,

$$D(\mathbf{H}) \leqq D(\mathbf{x}) \tag{6.3}$$

if $\mathbf{x}(u, v)$ is any (piece-wise continuously differentiable) vector function on R with the same boundary values $\mathbf{g}$ as $\mathbf{H}(u, v)$.

$D(\mathbf{H})$ is completely determined by $\mathbf{g}$ and R; it is a functional of these arguments, which we denote by $A(\mathbf{g}, R)$:

$$A(\mathbf{g}, R) \equiv D(\mathbf{H}) \equiv \frac{1}{2}\iint_R (\mathbf{H}_u^2+\mathbf{H}_v^2)dudv. \tag{6.4}$$

According to (6.3), the minimum principle (6.2) is completely equivalent to

$$A(\mathbf{g}, R) = \min. \tag{6.5}$$

We may say that $A(\mathbf{g}, R)$ is the « minimizing core » of $D(\mathbf{x})$. The minimum in (6.5) subsists simultaneously with regard to all RIEMANN surfaces R of the prescribed topological type and all parametric representations $\mathbf{g}$ of (Γ).

Our method of solution of the PLATEAU problem consists of two parts, namely:

1°). The proof of the *attainment* of the minimum of $A(\mathrm{g}, R)$.

2°). The discussion of the variational condition (EULER-LAGRANGE equation),

$$\delta A(\mathrm{g}, R) = 0, \tag{6.6}$$

to prove that it expresses the defining conditions (5.1, 2) or (5.5, 6) for a minimal surface.

§ 7.

As a preliminary to the proof of the attainment of the minimum of $A(\mathrm{g}, R)$, we enlarge the set $[R]$ of all RIEMANN surfaces of the given topological type so as to secure *closure*; i. e., we adjoin those RIEMANN surfaces R' which, while not belonging to $[R]$, can be expressed as the limit of a sequence of surfaces of $[R]$. R' either consists of a number of separate parts or is of lower topological type than the surfaces R.

Also, we enlarge the set [g] of one-one continuous correspondences between (C) and (Γ) so as to include the case where a whole arc of a contour Γ_j corresponds to a single point of C_j, or *vice versa*; as an extreme case (degenerate representation), all of Γ_j may correspond to a single point of C_j, while all of C_j corresponds to a single point of Γ_j.

The set $[\mathrm{g}, R]$ of all representations of the given contours (Γ), thus enlarged by the adjunction of *improper representations*, is, by construction, closed. The functional $A(\mathrm{g}, R)$ can be shown to be lower semi-continuous — this results from the positive nature of the integrand in (6.4) (or in (10.2), that follows).

Therefore, following a standard pattern of WEIERSTRASS-FRÉCHET, the minimum of $A(\mathrm{g}, R)$ is attained for some representation (g^*, R^*), which, as far as we know at first, may be proper or improper. The proof is logically the same as that for the minimum of a continuous function of a real variable on a closed interval (11).

It remains to exclude the eventuality of improper character of the minimizing representation (g^*, R^*).

To obviate the possibility of a singular nature for R^*, we need the following hypothesis:

$$d(\Gamma, h) < d(\Gamma, h'), \tag{7.1}$$

where the notation is

$$d(\Gamma, h) = \min A(\mathrm{g}, R), \qquad d(\Gamma, h') = \min A(\mathrm{g}, R'), \tag{7.2}$$

« min » being used in the sense of « lower bound » without prejudice of the

(11) See [8], p. 231.

question of attainment. R ranges over all RIEMANN surfaces which are *properly* of the assigned topological form, and $\mathfrak{g}$ over all parametric representations of (Γ). R' denotes all RIEMANN surfaces which are *improperly* of the assigned topological form in the sense explained in the first paragraph of this section. Thus, $d(\Gamma, h')$ is the least of the quantities

$$d(\Gamma_1, h_1)+d(\Gamma_2, h_2)+ \dots +d(\Gamma_m, h_m), \tag{7.3}$$

where

$$(\Gamma)=(\Gamma)_1+(\Gamma)_2+ \dots +(\Gamma)_m, \tag{7.4}$$

$$h_1+h_2+ \dots +h_m \leqq h, \tag{7.5}$$

and where we must have either $m>1$ or the $<$ sign prevailing in (7.5).

As a matter of fact, it suffices to consider only *primary reductions* of R to R'; i. e.,

$$d(\Gamma, h')=\text{least } [d(\Gamma_1, h_1)+d(\Gamma_2, h_2),\ d(\Gamma, h-1)] \tag{7.6}$$

for

$$(\Gamma)=(\Gamma)_1+(\Gamma)_2, \qquad h=h_1+h_2, \tag{7.7}$$

where an actual partition of the contours must take place, i. e., at least one of the sets $(\Gamma)_1$, $(\Gamma)_2$ is not empty.

We may observe that *in all cases* the relation $\leqq$ holds in (7.1).

The possibility that the parametric representation $\mathfrak{g}^*$ be improper is also easily avoided. The type of improper $\mathfrak{g}$ first described is avoided because then $A(\mathfrak{g}, R) = +\infty$, whereas *we assume in the main part of theory that* $d(\Gamma, h)=\min A(\mathfrak{g}, R)$ *is finite*:

$$d(\Gamma, h)< +\infty. \tag{7.8}$$

The second type of improper $\mathfrak{g}$ can be shown to be inconsistent with (5.2) or (5.6), while the degenerate type, as can be proved, contradicts (7.1).

A more concrete interpretation of $d(\Gamma, h)$ is as a lower bound of areas:

$$d(\Gamma, h)=a(\Gamma, h)=\min \mathfrak{A}(S), \tag{7.9}$$

where S ranges over all surfaces of genus h bounded by the k contours (Γ). The equality of d and a has been proved in the author's previous papers [12] on the basis of the conformal mapping of polyhedra approaching to S.

With the analogous definition for $a(\Gamma, h')$, referring to surfaces S reduced in their topological type, the sufficient conditions (7.8, 1) for the existence of the required minimal surface M can be put in the form

$$a(\Gamma, h)< +\infty, \qquad a(\Gamma, h)<a(\Gamma, h'). \tag{7.10}$$

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[12] See [3], § 19.

It can also be easily proved that, for the area of M, we have

$$\mathfrak{A}(M) = a(\Gamma, h) \leqq \mathfrak{A}(S), \tag{7.11}$$

where S may be any surface of the topological type of M bounded by (Γ). Accordingly, M also solves the least area problem which constitutes the original form of the problem of PLATEAU.

In summary, our main result is [13]:

If $d(\Gamma, h)$ is finite, and in the relation $d(\Gamma, h) \leqq d(\Gamma, h')$, applying to all data (Γ, h), we actually have for our particular data

$$d(\Gamma, h) < d(\Gamma, h'),$$

then there exists a minimal surface M of genus h bounded by the k given contours (Γ).

The minimum value $d(\Gamma, h)$ of the multiple Dirichlet integral $D(\mathbf{x})$ is equal to the minimum area $a(\Gamma, h)$ of all surfaces S of genus h bounded by (Γ). Accordingly, $a(\Gamma, h)$ may replace $d(\Gamma, h)$ in the preceding sufficient conditions, giving them a more concrete form.

Finally, the minimal surface M solves the least area problem for the data (Γ, h):

$$\mathfrak{A}(M) = a(\Gamma, h) \leqq \mathfrak{A}(S)$$

for every surface S of genus h bounded by (Γ).

§ 8.

If the given contours are perfectly general JORDAN curves, then the finiteness condition (7.8) is generally not verified, but rather

$$d(\Gamma, h) = a(\Gamma, h) = +\infty. \tag{8.1}$$

Indeed, the necessary and sufficient condition for $d(\Gamma, h)$ to be finite is that each contour be capable of bounding some simply-connected surface of finite area, i. e.,

$$d(\Gamma_j) = a(\Gamma_j) < +\infty \quad \text{for} \quad j = 1, 2, \ldots, k. \tag{8.2}$$

In turn, a sufficient (but not necessary) condition for this is the rectifiability of each contour.

Relatively simple examples have been given by the author of contours, all of the surfaces bounded by which have infinite area, e. g., the spiral defined in spherical polar coordinates by the equations

$$\varrho = \cos\varphi, \quad \theta = \tan^5\varphi \quad (\varphi = \text{latitude}, \ \theta = \text{longitude}) \quad [14].$$

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(13) First given in [1]. See also [3], Theorems, I, II.

(14) J. DOUGLAS: *An analytic closed space curve that bounds no orientable surface of finite area*. Proc. Nat. Acad. Sci., U. S. A., v. 19 (1933), pp. 448-451. See also: J. DOUGLAS: *A Jordan space curve having the infinite area property at each of its points*. Ibid., v. 24 (1938), pp. 490-495.

If one or more such contours belong to the set (Γ), then we have the general situation (8.1).

To give also in this case a sufficient condition for the existence of M, to replace (7.1), we define, first in the case $d(\Gamma, h)$ finite, the essentially non-negative quantity

$$e(\Gamma, h) = d(\Gamma, h') - d(\Gamma, h) \geqq 0, \tag{8.3}$$

and then, regardless of the finite or infinite value of $d(\Gamma, h)$,

$$\bar{e}(\Gamma, h) = \max \limsup e(\Gamma_m, h), \tag{8.4}$$

also essentially non-negative. Here $(\Gamma)_m$ denotes any sequence, $m = 1, 2, 3, \ldots$, of *finite-area-bounding* contour-systems, k in number, which tend to (Γ) as $m \to \infty$. The « lim sup » is with respect to m, and the « max » with respect to all possible sequences $(\Gamma)_m$.

With these definitions, *a sufficient condition for the existence of the minimal surface M is that*

$$\bar{e}(\Gamma, h) > 0 \tag{8.5}$$

(*not merely* $\geqq 0$, *which is always the case*).

Since in the present case we have $\mathfrak{A}(S) \equiv +\infty$ for all surfaces S bounded by (Γ), the least area property of M now loses its meaning. But every completely interior portion M_1 of M has a finite area which is a minimum for its own topological structure (genus h_1) and boundaries $(\Gamma)_1$ (15).

§ 9.

The attainment of the minimum of $A(\mathfrak{g}, R)$ being established, our two alternative modes of procedure are distinguished by their treatment of the variational condition

$$\delta A(\mathfrak{g}, R) = 0. \tag{9.1}$$

Accordingly, we divide the rest of this paper into: (I) Real Harmonic Method, (II) Complex Analytic Method.

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(15) [3], § 17.

I. - Real harmonic method.

§ 10.

PLAYING the principal role in all our formulas is the GREEN's function $G(P, Q)$ of the RIEMANN surface R, as determined uniquely by the following properties:

(i) as function of the point P, $G(P, Q)$ is uniform and harmonic on R;

(ii) $G(P, Q)$ has a logarithmic singularity at Q:

$$G(P, Q) = \log \frac{1}{PQ} + (\text{function of } P \text{ regular at } Q);$$

(iii) for P on the boundary C of R, $G(P, Q) = 0$.

In terms of GREEN's function, the solution of the DIRICHLET problem for R can be represented explicitly. Let $\mathbf{H}(Q)$ denote the uniform, regular, harmonic vector function on R with the boundary values $\mathbf{g}(P)$ on C; then

$$\mathbf{H}(Q) = \frac{1}{2\pi} \int_C \mathbf{g}(P) \frac{\partial G(P, Q)}{\partial n} ds, \tag{10.1}$$

where $\frac{\partial}{\partial n}$ denotes differentiation in the direction of the interior normal to C at the point P.

By substituting this formula in (6.4), we obtain — after certain transformations involving principally GREEN's formula for converting a regional integral into a contour integral — the following explicit formula for $A(\mathbf{g}, R)$:

$$A(\mathbf{g}, R) = \frac{1}{8\pi} \int_C \int_C [g(P_1) - g(P_2)]^2 \frac{\partial^2 G(P_1, P_2)}{\partial n_1 \partial n_2} ds_1 ds_2. \tag{10.2}$$

Here, written vectorially,

$$[\mathbf{g}(P_1) - \mathbf{g}(P_2)]^2 = \sum_{i=1}^{n} [g_i(P_1) - g_i(P_2)]^2 \tag{10.3}$$

is the square of the distance between two arbitrary points on the given system of contours (Γ). The second normal derivative of GREEN's function is an essentially positive quantity.

Also, referring back to the condition (5.2) for a minimal surface, we obtain for the expression figuring there the formula

$$\frac{\partial \mathbf{H}(Q)}{\partial \xi} \cdot \frac{\partial \mathbf{H}(Q)}{\partial \eta} = -\frac{1}{16\pi^2} \int_C \int_C [\mathbf{g}(P_1) - \mathbf{g}(P_2)]^2 \cdot \frac{\partial^2}{\partial n_1 \partial n_2} \left\{ \frac{\partial G(P_1, Q)}{\partial \xi} \frac{\partial G(P_2, Q)}{\partial \eta} + \frac{\partial G(P_1, Q)}{\partial \eta} \frac{\partial G(P_2, Q)}{\partial \xi} \right\} ds_1 ds_2. \tag{10.4}$$

Only a few simple transformations are needed to give this formula exactly the form indicated.

The fundamental formulas (10.2), (10.4) being established, the principal feature of the proof is the demonstration, by means of an explicit construction, that an arbitrary representation ($\mathbf{g}$, R) of the given contours (Γ) can be varied so that the variation of GREEN's function is precisely:

$$\delta G(P_1, P_2) = \frac{\partial G(P_1, Q)}{\partial \xi} \frac{\partial G(P_2, Q)}{\partial \eta} + \frac{\partial G(P_1, Q)}{\partial \eta} \frac{\partial G(P_2, Q)}{\partial \xi}, \tag{10.5}$$

where Q denotes an arbitrarily chosen point of R, and ξ, η any preassigned pair of perpendicular directions through Q on R.

Combining the formulas (10.2, 4, 5), we have — under our particular variation of ($\mathbf{g}$, R) —

$$\delta A(\mathbf{g}, R) = -2\pi \frac{\partial \mathbf{H}(Q)}{\partial \xi} \cdot \frac{\partial \mathbf{H}(Q)}{\partial \eta}. \tag{10.6}$$

Hence for the minimizing representation ($\mathbf{g}^*$, R^*) of $A(\mathbf{g}, R)$, we have

$$\frac{\partial \mathbf{H}^*(Q)}{\partial \xi} \cdot \frac{\partial \mathbf{H}^*(Q)}{\partial \eta} = 0, \tag{10.7}$$

where $\mathbf{H}^*(Q)$ is the harmonic function on R^* determined by the boundary values $\mathbf{g}^*$ on C^*, in accordance with (10.1). Consequently, the surface

$$\mathbf{x} = \mathbf{H}^*(Q) \tag{10.8}$$

is a minimal surface M, and solves the PLATEAU problem for the data (Γ, h) — for R, and therefore M, is of genus h, and the values $\mathbf{x} = \mathbf{g}^*(P)$ of $\mathbf{H}^*(Q)$ on the k boundaries (C^*) of R^* form some parametric representation of the given contours (Γ).

It remains only to describe precisely the special variation of the semi RIEMANN surface R of a real algebraic curve which has the effect (10.5) on the corresponding GREEN's function.

§ 11.

We imagine that the variation of R to R_ε takes place by simultaneous variation of the individual points: P_1 to $P_1(\varepsilon)$, P_2 to $P_2(\varepsilon)$, etc. Then GREEN's function $G(P_1, P_2)$, which depends on the form of the RIEMANN surface R_ε and the position of the points $P_1(\varepsilon)$, $P_2(\varepsilon)$, becomes, for given points P_1, P_2 on R, a function of ε:

$$G_\varepsilon(P_1(\varepsilon), P_2(\varepsilon)) = G(\varepsilon; P_1, P_2). \tag{11.1}$$

We define the variation of $G(P_1, P_2)$ by the usual formula

$$\delta G(P_1, P_2) = \frac{\partial}{\partial \varepsilon} G(\varepsilon; P_1, P_2)\Big|_{\varepsilon=0}, \tag{11.2}$$

where it may be remarked that δ denotes a derivative instead of, as is more customary, a differential.

We are now in a position to state our

Variational theorem concerning Green's function.

THEOREM. - *Let $\mathfrak{A}$ denote any real algebraic curve, on whose Riemann surface, $\mathfrak{S}$ or $\mathfrak{R}$, the points Q, $\bar{Q}$ are any two conjugate imaginary points.*

Let the tangent lines t, $\bar{t}$ to $\mathfrak{A}$ at Q, $\bar{Q}$ intersect in the real point O. Choose a reference triangle with one vertex at O; then in homogeneous line coordinates u, v, w, the equation of $\mathfrak{A}$ will evidently have the form

$$\mathfrak{A}: \qquad (au^2+buv+cv^2)K(u, v)+wL(u, v, w)=0,$$

where

$$t, \bar{t}: \qquad au^2+buv+cv^2=0$$

represents the conjugate imaginary tangents t, $\bar{t}$. K and L are homogeneous polynomials.

Construct now the family of curves with parameter ε,

$$\mathfrak{A}_\varepsilon: \qquad [(a+a'\varepsilon)u^2+(b+b'\varepsilon)uv+(c+c'\varepsilon)v^2]K(u, v)+wL(u, v, w)=0,$$

where a', b', c' are fixed but arbitrary real coefficients.

Then by rectilinear projection from O, the Riemann surfaces S, S_ε of A, A_ε are set into one-one continuous and conformal correspondence, P to $P(\varepsilon)$, except in the immediate vicinity of Q, $\bar{Q}$. In particular, the real branches C, C_ε of $\mathfrak{A}$, $\mathfrak{A}_\varepsilon$ are thereby set into one-one continuous correspondence. This depends on the circumstance that all the real tangents from O to $\mathfrak{A}_\varepsilon$ — which are defined by the real factors of the equation $K(u, v)=0$ — remain invariant, since this equation is independent of ε.

In fact, for the same reason, all the tangents, real and imaginary, from O to $\mathfrak{A}_\varepsilon$, except t_ε, $\bar{t}_\varepsilon$, remain fixed. These, however, vary with ε; and t_ε, for instance, intersects the Riemann surface $\mathfrak{S}$ of $\mathfrak{A}$ in two points near to Q [16]*, which, as ε passes through the value zero from positive to negative, always enter Q from two opposite directions α', α'' and leave in the perpendicular opposite directions β', β''. Let the angle-bisectors of these α, β directions be the perpendicular directions ξ, η.*

Denote by $G(P_1, P_2)$ the Green's function of either conjugate semi-

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[16] We suppose that the contact of the tangents t, $\bar{t}$ to $\mathfrak{A}$ at Q, $\bar{Q}$ respectively is ordinary two-point contact. Higher contact can always be avoided by means of a preliminary birational transformation.

surface R of the symmetric Riemann surface $\mathfrak{S}$ ([17]). *Then, under the precedingly described variation of the form of* $\mathfrak{S}$ *and the position of the points* P_1, P_2, *we have for Green's function the variational formula*

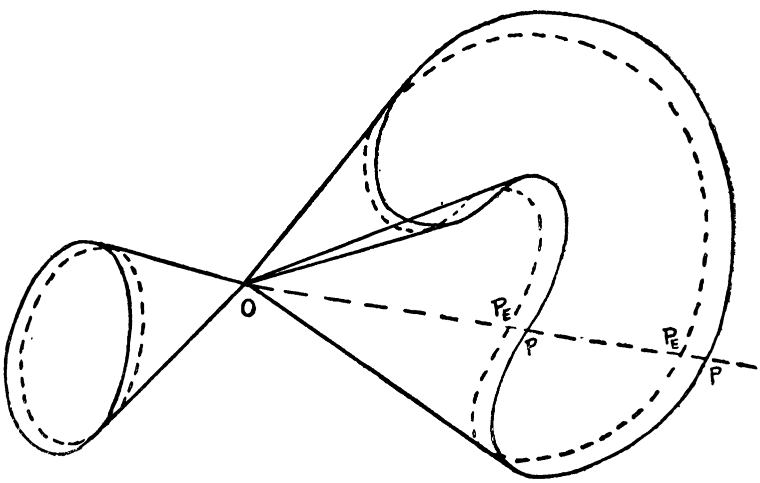
$$\delta G(P_1, P_2) = \frac{\partial G(P_1, Q)}{\partial \xi} \frac{\partial G(P_2, Q)}{\partial \eta} + \frac{\partial G(P_1, Q)}{\partial \eta} \frac{\partial G(P_2, Q)}{\partial \xi}$$

(*apart from an inessential numerical factor*).

Finally, the directions ξ, η *can be made to coincide with any preassigned perpendicular directions at Q by proper choice of* a', b', c'.

This theorem seems of interest not only for its direct application to the PLATEAU problem, but also for its interplay of fundamental analytic and geometric entities. A detailed proof of the theorem is given in [4], § 5.

The figure illustrates the real branches of the curve $\mathfrak{C}$ drawn full, and of the varied curve $\mathfrak{C}_\varepsilon$ drawn dotted. The real tangents from O ore indicated by



full drawn lines. One other projecting line from O is drawn dotted, and upon it corresponding points P, P_ε are indicated.

It is evident how the fact that $\mathfrak{C}_\varepsilon$ remains always tangent to the same real lines through O conditions the one-one nature of the correspondence between the real branches of $\mathfrak{C}$ and $\mathfrak{C}_\varepsilon$ established by the projection from O. For, otherwise, either $\mathfrak{C}$ or $\mathfrak{C}_\varepsilon$ would protrude outside one of the real tangents to the other from O, and then the protruding arc, say of $\mathfrak{C}_\varepsilon$, could have no corresponding real arc on $\mathfrak{C}$.

([17]) We suppose the notation arranged so that R contains the point Q, while the conjugate semi-surface contains the point $\overline{Q}$.

II. - Complex analytic method.

§ 12.

GREEN's function $G(z, w)$, being a harmonic function, can be expressed as the real part of an analytic function of the complex variable z. This expression has the form

$$G(z, w) = -\Re[S(z, w) - S(z, \overline{w})], \tag{12.1}$$

where $\overline{w}$ denotes the conjugate point to w on the symmetric RIEMANN surface R.

We derive from $S(z, w)$ the two other important functions

$$Z(z, w) = \frac{\partial}{\partial z} S(z, w), \qquad P(z, w) = \frac{\partial^2}{\partial z \partial w} S(z, w). \tag{12.2}$$

S, Z, P are analogous to log σ, ζ, $\wp$ in the theory of elliptic functions.

The solution of the DIRICHLET problem for the semi RIEMANN surface R, with given boundary values $\mathbf{g}(z)$ on C, can be expressed in the form

$$\mathbf{x} = \Re \mathbf{F}(w), \tag{12.3}$$

$$\mathbf{F}(w) = \frac{1}{\pi i} \int_C \mathbf{g}(z) Z(z, w) dz - \frac{1}{2\pi i} \int_C \mathbf{g}(z)[Z(z, w_0) + Z(z, \overline{w}_0)] dz. \tag{12.4}$$

Here $\mathbf{F}(w)$ is to be considered as a vector with n components $F_i(w)$. The second term in (12.4) is a constant, arranged so that, for a suitable branch of the multiform function [18] $\mathbf{F}(w)$,

$$\Im \mathbf{F}(w_0) = 0 \tag{12.5}$$

at the arbitrarily chosen particular point w_0. The complex analytic formula (12.4) for solving the DIRICHLET problem is the analogue of the solution (10.1) in real terms.

From (12.4) we derive, for the first member of the condition (5.6) for a minimal surface,

$$\mathbf{F}'^2(w) = \frac{1}{2\pi^2} \int_C \int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2 P(z, w) P(\zeta, w) dz d\zeta. \tag{12.6}$$

We also obtain, by a series of calculations, the following explicit formula for $A(\mathbf{g}, R)$:

$$A(\mathbf{g}, R) = \frac{1}{4\pi} \int_C \int_C [\mathbf{g}(z) - \mathbf{g}(\zeta)]^2 P(z, \zeta) dz d\zeta. \tag{12.7}$$

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[18] The periods of $F(w)$ are all pure imaginary, so that $x = \Re F(w)$ is a uniform function on R.

Although complex elements intervene in this formula, they combine in such a way that the value of $A(\mathbf{g}, R)=D(\mathbf{H})$ is positive real. We may remark also that the integral is improper, since $P(z, \zeta)$ becomes infinite like $\frac{1}{(z-\zeta)^2}$ when $z=\zeta$. Accordingly, we must integrate first with $|z-\zeta|\geqq\varepsilon$, and then allow $\varepsilon\to 0$. A similar remark applies to the real form (10.2) of the formula for $A(\mathbf{g}, R)$.

§ 13.

The GREEN's function of the semi RIEMANN surface R can be simply expressed in terms of the θ-functions pertaining to the complete RIEMANN surface $\mathfrak{R}$ through the intermediary of the related function $S(z, w)$ of (12.1).

In the case where the genus h of R is zero, we have simply

$$S(z, w)=\log\theta(\nu(z, w)). \tag{13.1}$$

This gives, by (12.2),

$$Z(z, w)=\zeta_j(\nu(z, w))\nu_j'(z), \qquad P(z, w)=\wp_{jk}(\nu(z, w))\nu_j'(z)\nu_k'(w) \tag{13.2}$$

(*systematically, the summation convention as to repeated indices will be applied to* $j, k, l, m=1, 2,\dots, p$).

Thus, the formulas (12.7, 6) become in the case $h=0$:

$$A(\mathbf{g}, R)=\frac{1}{4\pi}\int_C\int_C[\mathbf{g}(z)-\mathbf{g}(\zeta)]^2\wp_{jk}(\nu(z, \zeta))d\nu_j(z)d\nu_k(\zeta), \tag{13.3}$$

$$\mathbf{F}'^2(w)=\frac{1}{2\pi^2}\int_C\int_C[\mathbf{g}(z)-\mathbf{g}(\zeta)]^2\wp_{jl}(\nu(z, w))\wp_{km}(\nu(\zeta, w))\nu_j'(z)\nu_k'(\zeta)\nu_l'(w)\nu'_m(w)dz d\zeta. \tag{13.4}$$

§ 14.

Given any symmetric RIEMANN surface $\mathfrak{R}$ upon which Q, $\bar{Q}$ are any two conjugate points, a symmetric RIEMANN surface $\mathfrak{R}'$ can be found, conformally equivalent to $\mathfrak{R}$ with preservation of conjugate points, and upon which the points Q', $\bar{Q}'$ corresponding to Q, $\bar{Q}$ are branch-points. For, as in the statement of the Variational Theorem of § 12, let $\mathfrak{C}$ be any real algebraic curve ($P(x, y)=0$, with real coefficients) having $\mathfrak{R}$ for its RIEMANN surface, and let O be the real point of intersection of the conjugate imaginary tangents t, $\bar{t}$ to $\mathfrak{C}$ at Q, $\bar{Q}$. Then any real projective transformation of the plane of $\mathfrak{C}$ which sends O to the infinite point in the direction of the y-axis is evidently a real birational transformation of $\mathfrak{C}$, or a conformal transformation of $\mathfrak{R}$ with preservation of conjugate points, such that the points Q', $\bar{Q}'$ corresponding to Q, $\bar{Q}$ are branch-points of the transformed surface $\mathfrak{R}'$.

Accordingly, since conformally equivalent RIEMANN surfaces are identical for all our purposes, no generality is lost if we suppose an arbitrarily chosen point Q or w of $\mathfrak{R}$ to be a branch-point of that surface. We may also assume the order of the branch-point to be the first, like that of $\sqrt{z}$, since this means that the tangent t at Q has the usual two-point contact with $\mathfrak{C}$, and any higher contact can easily be avoided by a preliminary birational transformation (inversion).

If a branch-point w of the RIEMANN surface $\mathfrak{R}$ is displaced by the complex vector ε, then, as was proved by the author, the abelian integrals of the first kind undergo the variation (19)

$$\delta v_j(z) = -\frac{1}{2}\varepsilon v_j'(w)t(z, w). \tag{14.1}$$

Since the period of $t(z, w)$ as to the circuit B_k is $-2\pi i v_k'(w)$ (see end § 4), it follows that the periods τ_{jk} undergo at the same time the variation

$$\delta\tau_{jk} = \varepsilon \cdot \pi i v_j'(w) v_k'(w). \tag{14.2}$$

However, to vary w alone disturbs the symmetry of $\mathfrak{R}$. In order to keep $\mathfrak{R}$ symmetric, we must simultaneously subject the conjugate branch-point $\overline{w}$ to the conjugate complex displacement $\overline{\varepsilon}$. This gives, by (14.1), the variation

$$\delta v_j(z) = -\frac{\varepsilon}{2} v_j'(w)t(z, w) - \frac{\overline{\varepsilon}}{2} v_j'(\overline{w})t(z, \overline{w}). \tag{14.3}$$

Now, first let $\varepsilon = \overline{\varepsilon} = \lambda$, a real quantity, and second let $\varepsilon = i\lambda$, $\overline{\varepsilon} = -i\lambda$. Then we have the two variations

$$\delta_1 v_j(z) = -\frac{\lambda}{2} v_j'(w)t(z, w) - \frac{\lambda}{2} v_j'(\overline{w})t(z, \overline{w}), \tag{14.4}$$

$$\delta_2 v_j(z) = -\frac{i\lambda}{2} v_j'(w)t(z, w) + \frac{i\lambda}{2} v_j'(\overline{w})t(z, \overline{w}). \tag{14.5}$$

Formally, as we have shown [3, § 11], there is no objection to considering the variation δ which results by complex linear combination of δ_1 and δ_2:

$$\delta = \delta_1 - i\delta_2. \tag{14.6}$$

Under this permissible formal variation δ, we have, by (14.4, 5),

$$\delta v_j(z) = -\lambda v_j'(w)t(z, w). \tag{14.7}$$

Correspondingly, for the periods τ_{jk},

$$\delta\tau_{jk} = \lambda \cdot 2\pi i v_j'(w) v_k'(w). \tag{14.8}$$

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(19) It will be observed that in all the following formulas, δ denotes a *differential*, whereas in the formulas of method I, it denoted a *derivative*. See [2], § 4 and [3], § 11 for a proof of (14.1).

In summary, the variations (14.1, 2) are permissible in our formulas, even with due regard to the necessity of keeping $\mathfrak{R}$ symmetric.

§ 15.

An important and extensive part of our theory is concerned with proving that, under the variation defined by the last two formulas, we have

$$\delta A(\mathbf{g}, R) = \lambda \cdot \frac{\pi}{2} \mathbf{F}'^2(w). \tag{15.1}$$

When this is shown, it follows — the attainment of the minimum of $A(\mathbf{g}, R)$ for a particular $(\mathbf{g}^*, R^*)$ having been disposed of — that the defining condition $\mathbf{F}'^2(w) = 0$, for a minimal surface M is obeyed by the harmonic surface $\mathbf{H}^*$ determined by $(\mathbf{g}^*, R^*)$ according to the formulas (12.3, 4).

The proof of the relation (15.1) rests principally on a certain identity in θ-functions, first arrived at by the author precisely as an essential element in the present theory of the PLATEAU problem. This identity — which plays in the complex analytic method the same role as the variational formula (10.5) for GREEN's function in the real harmonic method — is, in its simplest form, the following:

$$\begin{aligned}
&[\wp_{jk}(\nu(z, w)) + \wp_{jk}(\nu(z, \zeta)) + \wp_{jk}(\nu(\zeta, w))]\, \nu_j'(w)\nu_k'(w) \\
&= [\zeta_j(\nu(z, w)) - \zeta_j(\nu(z, \zeta)) - \zeta_j(\nu(\zeta, w))] \\
&\cdot [\zeta_k(\nu(z, w)) - \zeta_k(\nu(z, \zeta)) - \zeta_k(\nu(\zeta, w))]\, \nu_j'(w)\nu_k'(w) + R(w).
\end{aligned} \tag{15.2}$$

Here $R(w)$ denotes some (undetermined) function of w alone, rational on $\mathfrak{R}$, i. e., uniform and with only polar singularities.

In the simplest case, where the genus of $\mathfrak{R}$ is $p = 1$, this identity becomes the following classic addition theorem for the elliptic functions ([20]):

$$\wp(u+v) + \wp(u) + \wp(v) = [\zeta(u+v) - \zeta(u) - \zeta(v)]^2. \tag{15.3}$$

It should be observed, however, as a matter of notation, that in the last formula the functions ζ, $\wp$ are derived from the WEIERSTRASS function σ by the definitions (4.3), rather than from the JACOBI function θ, which would be more in line with the definitions (4.12) of ζ_j, $\wp_{jk}$ that apply in (15.2). However, as pointed out in (4.2), the function σ is only an inessentially modified form of θ. With the use of σ, the term $R(w)$ in (15.2) reduces to zero, and the factor $v_1^{2\prime}(w)$ cancels from both sides, which gives exactly (15.3) after we write $u = \nu_1(z, \zeta)$, $v = \nu_1(\zeta, w)$, therefore $u + v = \nu_1(z, w)$.

([20]) See GOURSAT-HEDRICK: *A Course in Mathematical Analysis*, v. 2, pt. 1 (1916), p. 167.

A more elementary preliminary form of the identity (15.2) than the one involving elliptic functions is the following trigonometric identity:

$$(15.4)\qquad \csc^2(u+\nu)+\csc^2 u+\csc^2\nu=[\cot(u+\nu)-\cot u-\cot\nu]^2.$$

A still more simple version is the algebraic identity

$$(15.5)\qquad \frac{1}{(u+\nu)^2}+\frac{1}{u^2}+\frac{1}{\nu^2}=\left[\frac{1}{u+\nu}-\frac{1}{u}-\frac{1}{\nu}\right]^2.$$

The actual form in which we use the identity (15.2) is not exactly the one originally given, but rather the following, which results by applying the operator $\frac{\partial^2}{\partial z\partial\zeta}$; among other desirable features, this procedure gets rid of the undetermined function $R(w)$.

$$(15.6)\qquad \begin{aligned} \Big\{ &\wp_{jkl}(\nu(z,\zeta))[-\zeta_m(\nu(z,w))+\zeta_m(\nu(\zeta,w))] \\ &+\wp_{kl}(\nu(z,\zeta))\wp_{jm}(\nu(z,w))+\wp_{jm}(\nu(z,\zeta))\wp_{kl}(\nu(\zeta,w)) \\ &+2\pi i\,\frac{\partial\wp_{jk}(\nu(z,\zeta))}{\partial\tau_{lm}}\Big\}\,\nu_j'(z)\nu_k'(\zeta)\nu_l'(w)\nu'_m(w) \\ &=\wp_{jl}(\nu(z,w))\wp_{km}(\nu(\zeta,w))\nu_j'(z)\nu_k'(\zeta)\nu_l'(w)\nu'_m(w). \end{aligned}$$

Here, as notation,

$$(15.7)\qquad \wp_{jkl}(u)=-\frac{\partial^3\log\theta(u)}{\partial u_j\partial u_k\partial u_l}.$$

The partial differential equation (4.11) of the θ-function is used to derive the last term in the first member of (15.6).

§ 16.

In the general case, when the genus of R is $h>0$, the formula for GREEN's function acquires a complementary term, due to the acquisition of such a term by the function $S(z, w)$ in the expression (12.1); namely, for $h>0$,

$$(16.1)\qquad S(z,w)=\log\theta(\nu(z,w))-T_{\alpha\beta}[\nu_\alpha(z)-\nu_{\alpha'}(z)][\nu_\beta(w)-\nu_{\beta'}(w)],\quad (^{21})$$

in contrast to (13.1), where, h being zero, the complementary term is not present.

Due to this addition to the expression for GREEN's function, the formula (13.3) for $A(\mathrm{g}, R)$ also acquires a complementary term:

$$(16.2)\qquad \mathfrak{C}[A(\mathrm{g},R)]=\frac{1}{4\pi}\int_C\int_C[\mathrm{g}(z)-\mathrm{g}(\zeta)]^2K(z,\zeta)dzd\zeta,$$

where

$$(16.3)\qquad K(z,\zeta)=-T_{\alpha\beta}[\nu_\alpha'(z)-\nu'_{\alpha'}(z)][\nu_\beta'(\zeta)-\nu'_{\beta'}(\zeta)].$$

(21) Refer back to § 3 for the notation.

The summation convention as to repeated indices applies to α, β (and later λ, μ) $=1, 2, \dots, h$, while the symmetric indices vary *simultaneously* with α, β; i. e., always $\alpha'=\alpha+h$, $\beta'=\beta+h$ (see (3.7, 8)).

At the same time the formula (13.4) for $\mathbf{F}'^2(w)$ acquires the complementary term

$$\mathcal{C}[\mathbf{F}'^2(w)]=\frac{1}{2\pi^2}\int\limits_{C}\int\limits_{C}[\mathbf{g}(z)-\mathbf{g}(\zeta)]^2 L(z, \zeta;\ w)\,dz\,d\zeta, \tag{16.4}$$

where

$$\begin{aligned} L(z, \zeta;\ w) = &-T_{\alpha\beta}[\nu_\alpha'(\zeta)-\nu_{\alpha'}'(\zeta)][\nu_\beta'(w)-\nu_{\beta'}'(w)]\wp_{jk}(\nu(z, w))\nu_j'(z)\nu_k'(w) \\ &-T_{\alpha\beta}[\nu_\alpha'(z)-\nu_{\alpha'}'(z)][\nu_\beta'(w)-\nu_{\beta'}'(w)]\wp_{jk}(\nu(\zeta, w))\nu_j'(\zeta)\nu_k'(w) \\ &+T_{\alpha\lambda}T_{\beta\mu}[\nu_\alpha'(z)-\nu_{\alpha'}'(z)][\nu_\beta'(\zeta)-\nu_{\beta'}'(\zeta)][\nu_\lambda'(w)-\nu_{\lambda'}'(w)][\nu_\mu'(w)-\nu_{\mu'}'(w)]. \end{aligned} \tag{16.5}$$

We find that, under the variation δ defined by (14.7, 8), we have exactly

$$\delta K(z, \zeta)=\lambda\cdot L(z, \zeta;\ w). \tag{16.6}$$

This comes out by straightforward calculations, ([22]) no such rather deep-lying identity as (15.2) or (15.6) being involved.

Accordingly, by the formulas (16.2, 4, 6), we have

$$\delta\mathcal{C}[A(\mathbf{g}, R)]=\lambda\cdot\frac{\pi}{2}\,\mathcal{C}[\mathbf{F}'^2(w)]. \tag{16.7}$$

If we add this to the precisely similar relation (15.1), which applies in the case $h=0$, it results that also in the general case $h>0$ we have

$$\delta A(\mathbf{g}, R)=\lambda\cdot\frac{\pi}{2}\,\mathbf{F}'^2(w). \tag{16.8}$$

It follows that, in the general case likewise, the variational condition, $\delta A(\mathbf{g}, R)=0$, in the problem $A(\mathbf{g}, R)=\text{min.}$ implies the defining condition $\mathbf{F}'^2(w)=0$ for a minimal surface.

§ 17.

As a concluding comparison between the methods I and II, it may be remarked that I — not involving any explicit formula for the GREEN's function of R, but only the existence of this function — proceeds in the same way in all cases, regardless of the particular topological form of R. Indeed, R may even have an infinite number of boundaries and infinite connectivity; i. e., it may be the semi RIEMANN surface of any real analytic — not necessarily algebraic — curve; all the fundamental formulas of method I apply unchanged to this highly general case (see [5, 6]).

([22]) See [6], pp. 350-351, and [3], § 12, art. 2.

In the complex analytic method II, on the other hand, an explicit formula (12.1), (13.1), (16.1) is employed for GREEN's function, involving principally the θ-functions on the corresponding RIEMANN surface. This formula acquires certain complementary terms when $h>0$, whose treatment is of quite a minor order of difficulty as compared with that of the principal terms, which alone are present when $h=0$.

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