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ABSTRACT EUCLIDEAN SPACES WITH INDEPENDENTLY POSTULATED ANALYTICAL AND GEOMETRICAL METRICS (*)

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Introduction.

In this memoir we initiate the study of abstract « Euclidean » spaces, whose analysis and topology are those of a normed vector space ([1]), and whose geometry is mainly developed from a Hermitean bilinear function, which, unlike the inner product of HILBERT space and some previously considered Euclidean ([2]) spaces, is not necessarily used to define the norm. The essential novelty of these considerations is realized only in spaces of infinite dimensionality, for in a space of finite dimensions the analytical and geometrical metrics lead to equivalent theories of limits, and hence there is no need for considering an independently postulated norm.

The Hermitean bilinear function (x, y) is positive definite. However, there are interesting examples of spaces in which this requirement is not satisfied, for instance, the space of special relativity, and the usual space of continuous functions of two variables, with

$$(x, y) = \int_a^b ds \int_a^b x(s, t) y(t, s) dt.$$

We have called such spaces *indefinite* Euclidean spaces.

The main emphasis of the paper, after the preliminary work on the postulates, is on a theory of rotations and the illustration of this theory by examples chosen from various functional spaces. In the discussion of groups of rotations we have made use of the work of MICHAL and ELCONIN on abstract transformation groups with abstract parameters ([3]), and a number of rotation groups have been

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([1]) See Corollary to Theorem 2.5.

([2]) H. LÖWIG, Acta Scientiarum Mathematicarum, vol. 7 (1934), pp. 1-33; F. RELLICH, Mathematische Annalen, vol. 110 (1934), pp. 342-356.

([3]) A. D. MICHAL, Annali di Matematica (in press); A. D. MICHAL and V. ELCONIN, American Journal of Math. (in press); see also A. D. MICHAL and E. W. PAXSON, Bulletin of American Math. Soc. (August, 1936).

characterized by their completely integrable differential equations (4) in FRÉCHET differentials.

The first two sections of the paper are devoted to the postulates for a definite Euclidean space and the derivation of their immediate consequences. We then prove, in § 3, the consistency and independence of the postulates, and observe that the well known space of continuous functions $x(s)$, forms a Euclidean space with

$$(x, y) = \int_a^b x(s) y(s) ds.$$

The definitions of analytical and geometrical metrics are contained in § 4. Motions and rotations, which are transformations leaving geometrical distance invariant, are defined in § 5. In a real complete space it is proved that a rotation is linear; this is to be contrasted with § 12, where, in a complete indefinite Euclidean space, we give an example of a non-linear rotation. The rotations which are defined by a skew-symmetric linear function are defined in § 6, while the next section brings into evidence a useful relation between rotations and adjoints, which, in the case of a real space, affords an alternative definition of rotations. The first part of the paper concludes with some examples of rotations: in the space of functions $x(s)$ with a continuous derivative, and metrics given by (8.3); and in the space of functions $f(z)$, analytic in the unit circle S, and with

$$(f, g) = \iint_S f(z) \bar{g}(z) d\omega.$$

Part II deals with indefinite Euclidean spaces, the postulates for which are given in § 9. Rotations of the form (10.2) in the space of continuous functions $x(s, t)$ with metrics defined by (10.1) are considered in § 10, and the differential equations characterizing the group of these rotations and its first parameter group are obtained. In § 11 we discuss rotations of FREDHOLM and VOLTERRA type in a subspace E_4 of E_3, consisting of functions of one variable. In the concluding section we employ a theorem of TONELLI on non-linear functional equations to obtain non-linear rotations in E_4.

(4) A. D. MICHAL and V. ELCONIN, Acta Mathematica (in press).

PART I.

Spaces with Definite Geometrical Metric.

§ 1. - The Postulates.

Let E be a class of elements $x, y, z, \ldots$; and let A denote either the real number system R or the complex number system C. Let there be given in E a binary relation called equality, and denoted by $=$, such that given an ordered pair of elements x, y from E, then either x bears the relation to $y(x=y)$, or it doesn't $(x \neq y)$. Let $x+y$ be a function defined on E^2; $a \cdot x$ a function defined on AE; (x, y) a function defined on E^2; $\|x\|$ a function defined on E. The universe of discourse composed of the class E, the relation $=$, and the above four functions will be called an abstract Euclidean space if it satisfies the following two groups of postulates.

I.

1. There exists at least one element $x \varepsilon E$.
2. If $x, y \varepsilon E$ then $x+y \varepsilon E$.
3. If $a \varepsilon A$ and $x \varepsilon E$ then $a \cdot x \varepsilon E$.
4. If $x, y, -1 \cdot y, x+-1 \cdot y \varepsilon E$, and if for each element $u \varepsilon E$, $(x+-1 \cdot y)+u=u$, then $x=y$.
5. If $x, y \varepsilon E$ then $(x, y) \varepsilon A$.
6. If $x, y, z, x+y \varepsilon E$, and $(x+y, z)$, (x, z), $(y, z) \varepsilon A$, then $(x+y, z)=(x, z)+(y, z)$.
7. If $x, y \varepsilon E$ and (x, y), $(y, x) \varepsilon A$, then $(x, y)=\overline{(y, x)}$.
8. If $x, y, a \cdot x \varepsilon E$ and $(a \cdot x, y)$, $(x, y) \varepsilon A$, then $(a \cdot x, y)=a(x, y)$.
9. If $x \varepsilon E$ and $(x, x) \varepsilon R$, then $(x, x) \geqslant 0$.
10. If $x \varepsilon E$ and $(x, x)=0$, then $x+y=y$ for each element $y \varepsilon E$.
11. If $x, y \varepsilon E$ and $x=y$, and if $u \varepsilon E$ is such that (x, u), $(y, u) \varepsilon A$, then $(x, u)=(y, u)$.

II.

12. If $x \varepsilon E$ then $\|x\| \varepsilon R$.
13. If $x, y, x+y \varepsilon E$, and $\|x+y\|$, $\|x\|$, $\|y\| \varepsilon R$, then $\|x+y\| \leqslant \|x\|+\|y\|$.
14. If $x, a \cdot x \varepsilon E$, $a \varepsilon A$, and $\|a \cdot x\|$, $\|x\| \varepsilon R$, then $\|a \cdot x\|=|a|\,\|x\|$.
15. If $x, y \varepsilon E$ and $\|x\|$, $\|y\| \varepsilon R$ and $x=y$, then $\|x\|=\|y\|$.
16. There exists a positive constant M such that if $x, y \varepsilon E$ and $(x, y) \varepsilon A$ and $\|x\|$, $\|y\| \varepsilon R$, then $|(x, y)| \leqslant M\|x\|\,\|y\|$.

The first group of postulates (1-11) form an independent set, and define a linear space in which a metric can be constructed from the function (x, y) ([5]).

([5]) A. E. TAYLOR, Amer. Math. Soc. Bulletin, vol. 41 (1935), pp. 439-448.

It will be understood that the number M of postulate 16 is the lower bound of all such numbers as will satisfy the inequality.

§ 2. - Some Immediate Consequences of the Postulates.

In this paragraph we shall enumerate the theorems which link our postulates with the usual properties of a normed linear space.

THEOREM 2.1. - *The relation $=$ is reflexive, symmetric, and transitive.*

THEOREM 2.2. - *If $x_1=y_1$ and $x_2=y_2$, then $x_1+x_2=y_1+y_2$.*

If $x=y$ and $a \varepsilon A$, then $ax=ay$.

THEOREM 2.3 (6). - *The space E is a linear space: that is,*

(1) *The class E forms an Abelian group under addition; we denote the unique zero element in E, as well as the zero number, by 0.*

(2) $1 \cdot x=x$

$a(x+y)=ax+ay$

$(a+b)x=ax+bx$

$a(bx)=(ab)x$

$0 \cdot x=0$

$a \cdot 0=0.$

Definition. - For the element $-1 \cdot x$ we shall write $-x$; for the element $x+(-1) \cdot y$ we shall write $x-y$.

THEOREM 2.4 (7) - *The function (x, y) has the following properties:*

(1) $(x, y+z)=(x, y)+(x, z)$.

(2) $(x, az)=\bar{a}(x, z)$.

(3) $|(x, y)| \leqq (x, x)^{\frac{1}{2}}(y, y)^{\frac{1}{2}}$, *the equality holding if and only if there exist numbers a, b not zero, such that $ax+by=0$.*

(4) $(x+y, x+y)^{\frac{1}{2}} \leqq (x, x)^{\frac{1}{2}}+(y, y)^{\frac{1}{2}}$

$(ax, ax)^{\frac{1}{2}}=|a|(x, x)^{\frac{1}{2}}$

$(x, x)^{\frac{1}{2}}=0$ *if and only if* $x=0$.

THEOREM 2.5. - *The function $\|x\|$ has the properties*

(1) $\|x\|=0$ *if and only if* $x=0$.

(2) $\|x\| \geqq 0$.

(3) $|\|x\|-\|y\|| \leqq \|x \pm y\|$.

Proof: If $\|x\|=0$, then $(x, x)=0$ by postulate 16, and it follows from Theorem 2.4 (4) that $x=0$. Since $0 \cdot x=0$ it is obvious that $\|0\|=0$. For the

(6) For linear spaces and other abstract spaces see BANACH: *Théorie des Opérations Linéaires*, 1932. For the above theorems 2.1-2.3, which depend on postulates 1-11, see A. E. TAYLOR, loc. cit.

(7) M. H. STONE: *Linear Transformations in Hilbert Space.* Colloquium Pub. of A. M. S., V. 15, 1932, pp. 4-5.

proof of (2) we have

$$||x|| = ||x + (x - x)|| \leqslant 2\,||x|| + ||x||$$

whence the result follows. The proof of (3) is well-known, and will not be repeated here.

Corollary. - E is a normed linear space, with $||x||$ as norm. In case A is the real number system. E has all the properties of a Banach space (8) *save that of completeness.*

§ 3. - Special Instances and Independence Proofs.

The space of continuous functions $f(t)$ (real or complex-valued, according as A is R or C) defined on the closed interval (a, b) forms an interesting instance of a Euclidean space. The inner product (f, g) of two functions $f(t)$, $g(t)$, is defined as the integral

$$\int_a^b f(t)\bar{g}(t)\,dt,$$

and the norm is defined:

$$||f|| = \max_{(a,\,b)} |f(t)|.$$

Addition, multiplication, and equality are defined as usual.

Another example of a Euclidean space is furnished by the class of functions of a complex variable, analytic in a regular region S of the complex plane, and continuous on its boundary. The inner product of $f(z)$, $g(z)$ is given by

$$\iint_S f(z)\bar{g}(z)\,d\omega,$$

the double integral taken over the region S; the norm is defined: $||f|| = \max_S |f(z)|$.

An example of a Euclidean space which is definitely not a HILBERT Space that is, a space in which the norm $||x||$ *cannot* be generated by a bilinear function, is afforded by the Euclidean plane, with points $x = (x_1, x_2)$, $y = (y_1, y_2)$,.... Addition, multiplication, and equality are defined as usual, and

$$||x|| = |x_1| + |y_1|; \qquad (x, y) = x_1y_1 + x_2y_2.$$

The sixteen examples given below serve to establish the independence of the system of postulates 1-16. The verification of the postulates is left to the reader, as it presents no difficulties. Most of the examples are modelled on the HILBERT space H_0 of infinite one-rowed matrices of complex numbers (9).

(8) BANACH, loc. cit. Chap. IV.

(9) M. H. STONE, loc. cit. p. 14.

Example 1. - Let E be any null class.

Example 2. - Let G be a positive real number, and consider the class of all infinite one-rowed matrices of complex numbers $x=(x_1, x_2, ...)$ in which at most G elements are distinct from zero. This is a sub-class of H_0, and $=$, $+$, $\cdot$, $(,)$ are all defined as in H_0. The norm is defined: $\|x\|=\left(\sum |x_i|^2\right)^{\frac{1}{2}}$. Then $M=1$, and all the postulates are satisfied except (2).

Example 3. - Consider the class of infinite one-rowed matrices of complex numbers $x=(x_1, x_2, ...)$, where $\sum |x_i|^2$ is convergent, and $x_k=r_{1k}+ir_{2k}$, the r's being rational numbers. Let the definitions be as in the HILBERT space H_0, and define $\|x\|=\left(\sum |x_i|^2\right)^{\frac{1}{2}}$.

Example 4. - Consider the class of complex-valued functions of the form

$$F+\sin nx, \qquad -\infty<x<\infty$$

where F is a complex number, and n is a non-negative integer. Then, if

$$f\equiv F+\sin nx$$

and

$$g\equiv G+\sin mx,$$

we define:

$$f=g \text{ if and only if } F=G \text{ and } n=m$$
$$f+g\equiv F+G+\sin mx$$
$$a\cdot f\equiv aF+\sin nx$$
$$(f, g)=F\overline{G}$$
$$\|f\|=|F|.$$

Each element in the class is determined by a complex number and a non-negative integer; the representation is unique. Postulate 4 is not satisfied, for if the arbitrary element h is $H+\sin px$, the hypothesis of postulate 4 allows us to infer that $F=G$, but we can say nothing about n and m.

Example 5. - Consider the class H_0, with $+$, $\cdot$, $=$ defined as usual. Let the inner product (x, y) be the *matrix* $(x_1\overline{y}_1, x_2\overline{y}_2, ...)$ and let the norm be defined: $\|x\|=\left(\sum |x_i|^2\right)^{\frac{1}{2}}$.

Example 6. - Consider the class H_0, with equality, addition, and multiplication defined as usual. The inner product is defined $(x, y)=x_i\overline{y}_k$, where

$$|x_i|=\max\{|x_1|, |x_2|, ...\}$$

and

$$|y_k|=\max\{|y_1|, |y_2|, ...\}.$$

The norm is defined: $\|x\|=|x_i|$ (the maximum modulus).

Example 7. - Consider the class H_0, with $+$, $\cdot$, $=$ defined as usual. The inner product is defined:

$$(x, y) = \Big(\sum_i x_i \bar{y}_i\Big) f(y)$$

where

$$f(y) = \frac{1}{1+\sum\limits_i |y_i|^2} + 1.$$

The norm is defined $\|x\| = \big(\sum |x_i|^2\big)^{\frac{1}{2}}$. Since $0 < f(y) \leq 2$ for all y, it is sufficient to take $M=2$.

Example 8. - Consider the class H_0, with $+$, $=$, $(\,,\,)$ as usual. Define

$$a \cdot x = (I(a)x_1,\ I(a)x_2, \ldots)$$

where $I(a) = |a|$ if a is *not* a negative integer, and $I(a) = a$ if a *is* a negative integer. The norm is defined $\|x\| = \big(\sum |x_i|^2\big)^{\frac{1}{2}}$ and we take $M=1$.

Example 9. - Consider the class H_0, with $+$, $\cdot$, $=$ as usual. Define $(x, y) = -\sum x_i \bar{y}_i$ and $\|x\| = \big(\sum |x_i|^2\big)^{\frac{1}{2}}$.

Example 10. - Consider the class H_0, with $+$, $\cdot$, $=$ as usual. Define $(x, y) \equiv 0$ and $\|x\| = \big(\sum |x_i|^2\big)^{\frac{1}{2}}$.

Example 11. - Consider the class H_0, with $+$, $\cdot$, $(,)$ as usual. Define $x = y$ if and only if $\sum |x_i|^2 = \sum |y_i|^2$; and let $\|x\| = \big(\sum |x_i|^2\big)^{\frac{1}{2}}$.

Example 12. - Consider the class H_0, and define $\|x\| = 0$ if $x = (0, 0, \ldots)$. In all other cases define $\|x\| = x$. The other definitions are as usual.

Example 13. - Consider the class H_0, and define $\|x\| = -\big(\sum |x_i|^2\big)^{\frac{1}{2}}$. Other definitions as usual.

Example 14. - Consider the class H_0, and define $\|x\| = 1 + \big(\sum |x_i|^2\big)^{\frac{1}{2}}$. Other definitions as usual. Then $M=1$ will serve the conditions of postulate 16.

Example 15. - Consider the class of functions $f(t)$ defined, complex-valued, bounded and of LEBESGUE integrable square over $a \leq t \leq b$. Define $\|f\| = \underset{(a,b)}{\text{l. u. b.}} |f(t)|$. Other definitions as in the well-known HILBERT space of the L^2 functions.

Example 16. - Consider the class H_0. Then define $\|x\| = |x_1|$, where $x \equiv (x_1, x_2, \ldots)$. Other definitions as usual.

§ 4. - Analytical and Geometrical Metrics.

In HILBERT space the norm $\|x\|$ is *defined* in terms of the postulated form (x, y) by means of the simple relation

$$\|x\| = (x, x)^{\frac{1}{2}}$$

while in our abstract Euclidean space E, the norm $\|x\|$ and the form (x, y) are independently postulated functions, so that a relation of the above type does not necessarily hold. We are thus led to the following definition.

Definition. - The value of $\|x_1-x_2\|$ will be called the *analytical* distance between the elements x_1 and x_2 of E, while the value of $(x_1-x_2, x_1-x_2)^{\frac{1}{2}}$ will be called the *geometrical* distance between these elements.

The choice of our terminology is justified by the following remarks. Convergergence of a sequence to a limit, continuity of functions, differentials of functions, and other analytical and topological concepts will be defined in the usual way with respect to the analytical metric $\|x_1-x_2\|$. Orthogonality, angle, length of curve, motions, rotations, and other geometrical notions, will be defined by means of the geometrical metric.

Definition. - If x, y are elements of E, the real quantity $\dfrac{Re(x,y)}{(x,x)^{\frac{1}{2}}(y,y)^{\frac{1}{2}}}$ is defined to be the cosine of the angle which the elements x, y form with the zero element 0.

This definition is such as to preserve the law of cosines. That is, given three « points » P, Q, R represented by elements p, q, r then

$$\overline{PQ}^2=\overline{PR}^2+\overline{RQ}^2-2\overline{PR}\cdot\overline{RQ}\cos(PRQ)$$

where

$$\overline{PQ}=(p-q, p-q)^{\frac{1}{2}}$$

and

$$\cos(PRQ)=\frac{Re(p-r, q-r)}{(p-r, p-r)^{\frac{1}{2}}(q-r, q-r)^{\frac{1}{2}}}.$$

Definition. - A sequence $\{x_n\}$ such that $(x_n, x_m)=0$ when $n\neq m$ is said to form an orthogonal set. If $(x_n, x_n)=1$, the set is said to be normalized, and is then called an *orthonormal* set.

By a well known process we may deduce from a finite or denumerably infinite set an orthonormal set forming a basis for the original set.

THEOREM 4.1. - *If E is a space of finite dimensions, a necessary and sufficient condition that* $\lim\limits_{n\to\infty}\|x_n-x\|=0$ *is that* $\lim\limits_{n\to\infty}(x_n-x, x_n-x)=0$.

Proof: We may assume a finite orthonormal basis for the space E, consisting of elements $\Phi_1, \Phi_2, \ldots, \Phi_n$. Then

$$x=\sum_{i=1}^{n} a_i\Phi_i$$

$$x_k=\sum_{i=1}^{n} a_{ik}\Phi_i$$

$$(x_k-x, x_k-x)=\sum_{i=1}^{n}|a_{ik}-a_i|^2$$

and

$$\lim_{k\to\infty} (x_k - x,\ x_k - x) = 0$$

is equivalent to $\lim_{k\to\infty} | a_{ik} - a_i | = 0$, $(i=1, 2, \ldots, n)$. But this latter statement implies that $\lim_{k\to\infty} \| x_k - x \| = 0$, which in turn implies that $\lim_{k\to\infty} (x_k - x, x_k - x) = 0$. This proves the theorem.

Definition. - The set of points in E defined by $x = x(t)$, where $x(t)$ is a function of the real variable t, to the space E, continuous in the interval $t_0 \leqslant t \leqslant t_1$, is called an arc, or curve, in the space E. The curve will be said to be of class C' if the derivative $\frac{dx(t)}{dt}$ exists and is continuous in the interval.

The length of a curve is defined as usual, as the upper bound of the lengths of « polygonal lines » constructed on subdivisions of the curve, the distance between consecutive subdivision points being

$$(x(t_i) - x(t_{i-1}),\ x(t_i) - x(t_{i-1}))^{\frac{1}{2}}.$$

A curve for which this upper bound is finite is called rectifiable. It is clear that if the curve is of class C' it is rectifiable and its length is given by the RIEMANN integral

$$\int_{t_0}^{t_1} \left(\frac{dx(t)}{dt},\ \frac{dx(t)}{dt} \right)^{\frac{1}{2}} dt.$$

We shall say that the function $x(t)$ is of bounded variation in the interval (t_0, t_1) if it satisfies the usual condition, the norm being used instead of the familiar absolute value of real analysis. It is then easy to prove the following proposition.

THEOREM 4.2. - *The curve defined by the function $x(t)$ is rectifiable if $x(t)$ is of bounded variation in (t_0, t_1). This condition is also necessary if the space E is finite-dimensional.*

The problem of minimal arcs of class C' in a real Euclidean space may be treated by a slight extension of the methods of the classical calculus of variations. To minimize the integral

$$\int_{t_0}^{t_1} \left(\frac{dx(t)}{dt},\ \frac{dx(t)}{dt} \right)^{\frac{1}{2}} dt,$$

we consider curves

$$x(t) = \bar{x}(t) + \varepsilon\eta(t) \cdot \xi,$$

where ξ is arbitrary in E, $\eta(t)$ is a real function of class C', vanishing at t_0, t_1 and ε is a real number, and $\bar{x}(t)$ is assumed to define a minimal curve. In the usual way, using DU BOIS-REYMONDS lemma, we obtain the equation

$$\left(\frac{d\bar{x}(s)}{ds},\ \xi \right) = C(\xi),$$

where s is the arc length along the curve from x_0. $C(\xi)$ depends only on ξ. From this we infer that

$$\bar{x}(s)=\frac{s}{l}(x_1-x_0)+x_0,$$

where x_0, x_1 are the end points, and l is the total length of the curve.

THEOREM 4.3. - *If* $|(f,g)|=M\|f\|\,\|g\|$ *then f and g are linearly dependent.*

Proof: From the SCHWARTZ inequality we have

1) $|(f,g)|\leq(f,f)^{\frac{1}{2}}(g,g)^{\frac{1}{2}}$ and also;

2) $(f,f)^{\frac{1}{2}}\leq M^{\frac{1}{2}}\|f\|$, $(g,g)^{\frac{1}{2}}\leq M^{\frac{1}{2}}\|g\|$.

Using 1) and the hypothesis

3) $M\|f\|\,\|g\|\leq(f,f)^{\frac{1}{2}}(g,g)^{\frac{1}{2}}$.

Using 2) and 3),

$$M\|f\|\,\|g\|\leq(f,f)^{\frac{1}{2}}(g,g)^{\frac{1}{2}}\leq M\|f\|\,\|g\|.$$

4) Therefore $(f,f)^{\frac{1}{2}}(g,g)^{\frac{1}{2}}=M\|f\|\,\|g\|=|(f,g)|$ and the result follows by application of theorem 2.4 (3).

§ 5. - Motions and Rotations in E.

Let $U(x)$ be a function on E to E. Then $U(x)$, regarded as a transformation of the elements of E, will be called a *motion of the space E* if

(1) $U(x)$ is a biunivocal function taking E into itself;

(2) $U(x)$ preserves geometrical distance in E:

$$(U(x_1)-U(x_2),\ U(x_1)-U(x_2))=(x_1-x_2,\ x_1-x_2)$$

for all elements x_1, x_2 in E.

A motion which leaves the zero point unaltered will be called a *rotation* about the origin. Thus, if $U(x)$ defines a motion, and $U(0)=0$, then $U(x)$ is a rotation.

A transformation of the form $T(x)=x+x_0$, where x_0 is a fixed element in E, is said to be a translation of the space E. Evidently a translation is a motion.

THEOREM 5.1. - *A motion consists of a rotation and a translation.*

For let $U(x)$ define a motion, and consider the transformation defined by

$$f(x)=U(x)-U(0).$$

Then $f(0)=0$, and $f(x_1)-f(x_2)=U(x_1)-U(x_2)$. Therefore $f(x)$ defines a rotation. This establishes the theorem.

THEOREM 5.2 [10]. - *A rotation* $f(x)$ *is additive*: $f(x+y)=f(x)+f(y)$.

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[10] BANACH, loc. cit., p. 166. Since $(x,x)^{\frac{1}{2}}$ has the properties of a norm, the theorem carries over, but we *do not* infer continuity. The complex multiplier domain does not play any part in this proof.

THEOREM 5.3. - *If $f(x)$ is a rotation, a necessary and sufficient condition that $(f(x), f(y))=(x, y)$ for all x, y is that $f(ix)=if(x)$. In all cases,*

$$Re(f(x), f(y))=Re(x, y).$$

Proof: Since $f(x)$ is a rotation,

$$(f(x)-f(y), f(x)-f(y))=(x-y, x-y)$$

or, since $f(0)=0$, and therefore $(f(x), f(x))=(x, x)$,

$$Re(f(x), f(y))=Re(x, y).$$

Let $f(ix)=if(x)$. Then replacing x by ix in the above relation we readily obtain

$$(f(x), f(y))=(x, y).$$

Conversely, if $(f(x), f(y))=(x, y)$ for all $x, y \varepsilon E$, then

$$(f(ix), f(y))=(ix, y)=i(x, y)=(if(x), f(y)).$$

But $f(x)$ is biunivocal. We can therefore determine y so that $f(y)=if(x)-f(ix)$. This gives us the equation

$$(f(ix)-if(x), f(ix)-if(x))=0.$$

Therefore

$$f(ix)=if(x).$$

Corollary. - A rotation leaves the cosine invariant.

THEOREM 5.4. - *If $f(x)$ is a rotation, and if a is real, then*

$$f(ax)=af(x).$$

Proof:

$$(f(ax)-af(x), f(ax)-af(x))=(f(ax), f(ax))-2Re(f(ax), af(x))+a^2(f(x), f(x))$$

$$\begin{aligned}(f(ax)-af(x), f(ax)-af(x)) &= 2a^2(x, x)-2aRe(f(ax), f(x))\\ &= 2a^2(x, x)-2aRe(ax, x)\\ &= 0.\end{aligned}$$

Therefore

$$f(ax)=af(x).$$

THEOREM 5.5. - *A motion in closed* [11].

Proof: Let $f(x)$ be a motion, and let

$$x_n \to x_0, \qquad f(x_n) \to y_0.$$

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[11] A function $f(x)$ defined on $S \subset E$ to E is said to be closed if $x_n \to x_0$, $f(x_n) \to u_0$ implies $x_0 \varepsilon S$ and $f(x_0)=u_0$.

Then

$$(x_n - x_0, x_n - x_0) \leq M \| x_n - x_0 \|^2$$
$$(x_n - \bar{x}, x_n - \bar{x}) = (f(x_n) - y_0, f(x_n) - y_0) \leq M \| f(x_n) - y_0 \|^2,$$

where $\bar{x}$ is an element such that $f(\bar{x}) = y_0$. From the hypothesis of the theorem, and the continuity of the inner product, it is evident that $\bar{x} = x_0$. Therefore $f(x_0) = u_0$, and $f(x)$ is closed.

THEOREM 5.6. - *If the space E is complete, and $f(x)$ is a rotation in E, then $f(x)$ is continuous.*

Proof: Since $f(x)$ is closed and additive, it satisfies the hypothesis of Theorem 7, p. 41, in BANACH: *Opérations Linéaires*, and is therefore continuous.

Corollary. - If E is a complete space, a motion $U(x)$ is continuous.

§ 6. - Generating Functions for Rotations.

It is well known that in an ordinary Euclidean space of n dimensions the group of rotations can be generated by an infinitesimal linear transformation with a skew-symmetric matrix. The question then arises as to whether this is the situation in an abstract Euclidean space. We are able to show that if E is complete, a large class of rotations can be generated by means of linear, skew-symmetric functions.

Definition 6.1. - A function $S(x)$ on E to $T \subset E$ will be called skew-symmetric if

$$(S(x), y) + (x, S(y)) = 0$$

for all x, y in E.

If E is a real Euclidean space this condition is equivalent to

$$S(x+y) = S(x) + S(y), \qquad (S(x), x) = 0$$

for all x in E. If E is a complex space, it is easily seen that $S(x) = iH(x)$, where

$$(H(x), y) = (x, H(y)).$$

$H(x)$ is said to be Hermitean symmetric, or merely Hermitean. We can readily show that the skew-symmetric function $S(x)$ is additive and homogeneous of degree one. Moreover, if $x = y$, then $S(x) = S(y)$. See theorems 7.1 and 7.2.

Let $S(x)$ be a continuous, skew-symmetric function, and let a function $L(x)$ be defined in terms of $S(x)$ as follows

$$L(x) = e^S(x) \equiv x + S(x) + \frac{S^2(x)}{2!} + \ldots.$$

where

$$S^0(x) = x, \qquad S^{n+1}(x) = S(S^n(x)).$$

THEOREM 6.1. - *If E is a complete space, $L(x)$ is a rotation. The mo-*

dulus m_L of $L(x)$ and the modulus m_S of $S(x)$ satisfy the relations

$$e^{-m_S} \leqslant m_L \leqslant e^{m_S}.$$

Proof: Since $\| S^n(x) \| \leqslant m_S^n \| x \|$, we conclude by a theorem of BANACH ([12]) that $L(x)$ is linear-that is, additive, continuous, and homogeneous of degree one. It is then at once evident that m_L exists and satisfies the inequality $m_L \leqslant e^{m_S}$.

By a repeated application of the skew-symmetry relation we easily verify that

$$(S^m(x), S^n(x)) = (-1)^{\frac{m-n}{2}} (S^{\frac{m+n}{2}}(x), S^{\frac{m+n}{2}}(x)), \qquad m+n \text{ even}$$

$$(S^m(x), S^n(x)) = (-1)^{\frac{m-n-1}{2}} (S^{\frac{m+n+1}{2}}(x), S^{\frac{m+n-1}{2}}(x)), \qquad m+n \text{ odd.}$$

Since the inner product is a continuous function of its arguments we can expand $(L(x), L(x))$ term by term, and rearrange the resulting double series, which is absolutely convergent.

$$\begin{aligned}(L(x), L(x)) &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!} \frac{1}{q!} (S^p(x), S^q(x)) \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!} \frac{1}{q!} (S^p(x), S^q(x)), \qquad p+q \text{ even} \\ &+ \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{p!} \frac{1}{q!} (S^p(x), S^q(x)), \qquad p+q \text{ odd.}\end{aligned}$$

In the first summation let $p+q=2m$, $p-q=2(p-m)$; in the second let $p+q=2n+1$. We then have

$$\begin{aligned}(L(x), L(x)) = (x, x) &+ \sum_{m=1}^{\infty} (-1)^m (S^m(x), S^m(x)) \sum_{p=0}^{2m} (-1)^{p-2m} \frac{1}{p!\,(2m-p)!} + \\ &+ \sum_{n=0}^{\infty} (-1)^n (S^{n+1}(x), S^n(x)) \sum_{p=0}^{2n+1} (-1)^{p-2n-1} \frac{1}{p!\,(2n+1-p)!}.\end{aligned}$$

The inner summations represent respectively the expansions

$$\frac{(-1+1)^{2m}}{2m!}, \qquad \frac{(-1+1)^{2n+1}}{(2n+1)!}$$

and are hence zero.

Therefore

$$(L(x), L(x)) = (x, x).$$

The function $L(x)$ admits a unique inverse $L^{-1}(x)$:

$$L^{-1}(x) = e^{-S}(x) \equiv x - S(x) + \frac{S^2(x)}{2!} - \frac{S^3(x)}{3!} + \ldots.$$

([12]) S. BANACH, Fundamenta Mathematicae, t. III (1922), p. 157.

defined throughout the space E. From this we easily conclude that $L(x)$ is a biunique transformation of E into itself. $L(x)$ is therefore a rotation. It is clear that

$$\| L^{-1}(x) \| \leqslant e^{m_S} \| x \|.$$

But $L^{-1}(L(x))=x$, and therefore

$$\| x \| = \| L^{-1}(L(x)) \| \leqslant e^{m_S} \| L(x) \| \leqslant e^{m_S} e^{m_S} \| x \|$$

or

$$e^{-m_S} \leqslant m_L \leqslant e^{m_S}.$$

Corollary. - If E is a complete space and λ is a real number, the function $e^{\lambda S}(x)$ defines a rotation. We obtain in this manner a one-parameter family of rotations.

THEOREM 6.2. - *Let E be a complete space. If $S_1(x)$, $S_2(x)$ are two continuous, skew-symmetric functions which are permutable, that is, $S_1(S_2(x))=S_2(S_1(x))$, then $e^{S_1}(e^{S_2}(x))=e^{S_1+S_2}(x)$, and $e^{S_1+S_2}(x)$ is a rotation.*

Proof: S_1+S_2 is continuous and skew-symmetric, so that $e^{S_1+S_2}(x)$ is a rotation, by Theorem 6.1. The remainder of the proof presents no difficulty. As a consequence of this we have the following theorem.

THEOREM 6.3. - *Let E be a complete space. Let Σ be a class of continuous, skew-symmetric functions $\sigma(x)$, permutable amongst themselves, and forming a linear set. (That is, with σ_1, σ_2, Σ contains $\sigma_1+\sigma_2$, $a \cdot \sigma$). If P is the class of rotations generated by $e^{\sigma}(x)$, where σ is in Σ, then P is an Abelian subgroup of the group of rotations in E. Σ is an additive Abelian group and it is isomorphic with P.*

In commenting on this theorem we note merely that if $\sigma_1(x)$ corresponds to $e^{\sigma_1}(x)$, and $\sigma_2(x)$ corresponds to $e^{\sigma_2}(x)$, then $e^{\sigma_1+\sigma_2}(x)$ corresponds to $\sigma_1(x)+\sigma_2(x)$. The zero transformation in Σ corresponds to the identity transformation in P.

The transformation group P may be characterized by a completely integrable equation in FRÉCHET differentials. Here, and later in the paper, we shall employ some results of a recent general theory of abstract transformation groups ([13]). For details of theorems, definitions, and notations the reader is referred to paper D. The abstract parameter of the group P is a variable over the class Σ. To obtain the group space we proceed as follows.

Let O be the BANACH space of linear functions on E to E, with $\| a \|$=modulus of the linear function $a(x)$. The linear space Σ defined in Theorem 6.3 is a sub

([13]) A. D. MICHAL and V. ELCONIN: *Differential Properties of Abstract Transformation Group with Abstract Parameters*, Amer. Jour. Math., 1936 (in press). This paper will henceforth be referred to as D.

BANACH space of O when normed in the same manner. Elements of Σ will be denoted by small Greek letters.

To conform with the notation of paper D, the rotations $e^{a}(x)$ of the group P will be written in the form

$$\bar{x} = T_a x. \tag{6.1}$$

The differential equation in question is then

$$d_\xi^a T_a x = U(T_a x, V(T_a x, \Omega(a, \xi))), \tag{6.2}$$

where,

$$\left\{ \begin{array}{l} U(x, z) = [\text{inverse in } z \text{ of } -d_z^x T_a x]_{a=\delta}, \\ V(x, \xi) = [d_\xi^a T_a x]_{a=\delta}, \\ \Omega(a, \xi) = \underset{\xi}{\overset{a}{d}} \sigma(\sigma' a \delta), \end{array} \right. \tag{6.3}$$

and δ is an arbitrary fixed element of Σ. In the present case $a\beta$ stands for $a+\beta$, σ' for $-\sigma$ (the group operations in Σ). When these functions are calculated for the transformation group P, we obtain the following theorem.

THEOREM 6.4. - *The group P of transformations* (6.1) *is characterized by the completely integrable Fréchet differential equation,*

$$d_\xi^a \bar{x} = \xi(\bar{x}), \tag{6.4}$$

with the initial condition $\bar{x} = x$ when $a = 0$ (the identically zero function in Σ).

§ 7. - Adjoints and Rotations.

The unitary transformations of HILBERT space have the property that their inverses are their adjoints. It is therefore to be expected that the concept of adjointness will play an important role in the theory of rotations. This is indeed the case, and we shall therefore devote some space to a development of the theory of adjoint transformations in Euclidean spaces ([14]).

Definition 7.1. - Let $T_1(x)$ and $T_2(x)$ be transformations with domains $\mathfrak{D}_1$, $\mathfrak{D}_2$, and ranges R_1, R_2, respectively. Then T_1 and T_2 are said to be adjoint if

$$(T_1(x), y) = (x, T_2(y))$$

for every x in $\mathfrak{D}_1$ and every y in $\mathfrak{D}_2$.

Definition 7.2. - A set $\mathfrak{D}$ of elements in E is said to span the space E if the closure of the linear manifold determined by $\mathfrak{D}$ *is* the space E.

THEOREM 7.1. - *If $T(x)$ is a transformation whose domain $\mathfrak{D}$ spans the space E, there exists a uniquely defined transformation $T^*(x)$, adjoint*

([14]) The basic theorems differ but slightly from those in HILBERT space, cf. for instance M. H. STONE, loc. cit. pp. 41-49.

to $T(x)$, *with the property that if* $T_1(x)$ *is also adjoint to* $T(x)$, *then* $T^*(x)$ *is an extension of* $T_1(x)$.

Proof: The domain $\mathfrak{D}^*$ of $T^*(x)$ is the set of elements y such that

$$(T(x), y) = (x, y^*)$$

is a true relation for every x in $\mathfrak{D}$, and at least one y^* in E. $T^*(y)$ is then defined by $T^*(y) = y^*$. It is evident that $\mathfrak{D}^*$ contains the zero element, and $T^*(0) = 0$. The transformation $T^*(y)$ is uniquely defined, for corresponding to a given y there can be at most one y^*. Finally, if $T_1(x)$ is adjoint to $T(x)$, and if y is in the domain of T_1, we have the relation

$$(T(x), y) = (x, T_1(y))$$

from which we conclude that y is in $\mathfrak{D}^*$, and $T^*(y) = T_1(y)$. When $T^*(x)$ exists it will be called *the* adjoint of $T(x)$.

THEOREM 7.2. - *If* $T(x)$ *is a transformation whose domain* $\mathfrak{D}$ *spans the space* E, *then* $T^*(x)$ *is closed, additive, and homogeneous of degree one.*

Proof: Let y_1, y_2 be in $\mathfrak{D}^*$. Then

$$(T(x), a_1 y_1 + a_2 y_2) = (x, a_1 T^*(y_1) + a_2 T^*(y_2))$$

for all x in $\mathfrak{D}$. Therefore T^* is additive and homogeneous of degree one. Let $\{y_n\}$ be a sequence in $\mathfrak{D}^*$ such that $y_n \to y$, and $T^*(y_n) \to y^*$. Then, for all x in $\mathfrak{D}$,

$$(T(x), y_n) = (x, T^*(y_n)),$$

and

$$\begin{aligned} |(T(x), y) - (x, y^*)| &= |(T(x), y - y_n) + (x, T^*(y_n) - y^*)| \\ &\leq \| T(x) \| . \| y - y_n \| + M \| x \| \, \| T^*(y_n) - y^* \|. \end{aligned}$$

From this we conclude that y is in $\mathfrak{D}^*$, and that $T^*(y) = y^*$. Therefore $T^*(x)$ is a closed transformation.

THEOREM 7.3. - *Let* $T(x)$ *be a transformation such that its domain and range each span the space* E, *and let the inverse* $T^{-1}(x)$ *exist. Then* $T^*(x)$, $T^{-1*}(x)$ *exist and are mutually inverse. If the domain* $\mathfrak{D}^*$ *of* $T^*(x)$ *spans the space* E, *the adjoint* $T^{**}(x)$ *of* $T^*(x)$ *exists, and* $T^{**}(x)$ *is a closed extension of* $T(x)$, *with adjoint* $T^*(x)$.

The proof of this theorem is exactly like that of Theorems 2.7 and 2.9 in STONE's book, and will not be given here.

The following theorem brings into evidence a useful relation between rotations and adjoints which, in the case that E is a real space, affords an alternative definition of rotations.

THEOREM 7.4. - *Let* $L(x)$ *define a homogeneous rotation in* E. *Then the inverse of* $L(x)$ *is its adjoint:* $L^{-1}(x) \equiv L^*(x)$. *Conversely, if* $T(x)$ *is a* *transformation defined throughout E, with adjoint $T^*(x)$ likewise defined throughout E, and if the equation $y=x+T(x)$ has the unique solution $x=y+T^*(y)$, then the transformation $x+T(x)$ is a rotation.*

Proof: Since $L(x)$ is a homogeneous rotation we have

$$(L(x),\ L(y))=(x,\ y)$$

for all x, y in E, by Theorem 5.3. The inverse $L^{-1}(x)$ exists, and

$$(L(x),\ y)=(L(x),\ L(L^{-1}(y)))=(x,\ L^{-1}(y)).$$

This shows that $L^*(x)$ exists and coincides with $L^{-1}(x)$. We observe that the adjoint of $L(x)-x$ is $L^{-1}(x)-x$, and that the solution of $y=x+(L(x)-x)$ is $x=y+(L^{-1}(y)-y)$.

It is easily seen that $x+T(x)$ is a biunique transformation of E into E. We must therefore prove that

$$(x+T(x),\ x+T(x))=(x,\ x).$$

Now let $U(x)=x+T(x)$. Then $U^{-1}(x)=x+T^*(x)$, and $U^*(x)=x+T^*(x)$. Consequently

$$(U(x),\ U(x))=(x,\ U^*(U(x)))=(x,\ x)$$

as was to be proved.

A rotation $L(x)=x+T(x)$ will be called *proper* if

1°) $L(x)$ is homogeneous of degree one

2°) $y=x+\frac{1}{2}T(x)$ has a unique solution x for every y in E, thus defining an inverse $x=y-\frac{1}{2}\Gamma(y)$.

Proper rotations give rise to some interesting theorems, as we shall presently show. From the geometrical point of view it is worthy of notice that improper rotations (i. e. those which are not proper) include the reflections in the special, finite dimensional instances. There is a close connection between proper rotations and skew-symmetric transformations, as is evidenced by the following theorem.

THEOREM 7.5. - *Let $L(x)=x+T(x)$ be a proper rotation and let $\Gamma(x)$ be the transformation associated with it (as in the definition). Then $\Gamma(x)$ is skew-symmetric. Conversely if $\Gamma(x)$ is a skew-symmetric transformation, such that $x=y-\frac{1}{2}\Gamma(y)$ has the unique solution $y=x+\frac{1}{2}T(x)$ for each x in E, then $x+T(x)$ is a proper rotation.*

The proof of this theorem rests on two propositions concerning resolvents. We shall present them as lemmas.

Lemma *A*. - *Let $T(x)$ and $\Gamma(x,\lambda)$ be additive, homogeneous functions defined on E to E, where λ is a numerical parameter. Let the equation*

$y=x-\lambda T(x)$ *have the unique solution* $x=y+\lambda\Gamma(y, \lambda)$, *for* $\lambda=\lambda_0$ *and* $\lambda=\lambda_0+\mu$. *Then the equation* $y=x-\mu\Gamma(x,\lambda_0)$ *has the unique solution* $x=y+\mu\Gamma(y,\lambda_0+\mu)$.

Proof: By hypothesis we have

$$x=x-\lambda T(x)+\lambda\Gamma(x, \lambda)-\lambda^2\Gamma(T(x), \lambda)$$
$$x=x+\lambda\Gamma(x, \lambda)-\lambda T(x)-\lambda^2 T(\Gamma(x, \lambda))$$

for all x, and $\lambda=\lambda_0$, $\lambda_0+\mu$. Therefore $(\lambda\neq 0)$,

$$T(x)=\Gamma(x, \lambda)-\lambda T(\Gamma(x, \lambda))=\Gamma(x, \lambda)-\lambda\Gamma(T(x), \lambda).$$

Then let u be defined

$$u=y+\mu\Gamma(y, \lambda_0+\mu)-\mu\Gamma(y, \lambda_0)-\mu^2\Gamma(\Gamma(y, \lambda_0+\mu), \lambda_0)$$
$$u-\lambda_0 T(u)=y-\lambda_0 T(y)+\mu\Gamma(y, \lambda_0+\mu)-\lambda_0\mu T(\Gamma(y, \lambda_0+\mu))-\mu\Gamma(y, \lambda_0)+$$
$$+\mu\lambda_0 T(\Gamma(y, \lambda_0))-\mu^2\Gamma(\Gamma(y, \lambda_0+\mu), \lambda_0)+\lambda_0\mu^2 T(\Gamma(\Gamma(y, \lambda_0+\mu), \lambda_0))$$
$$u-\lambda_0 T(u)=y-\lambda_0 T(y)$$

as we find with the aid of the above relations. But since the solutions are unique, $u=y$. That is, the solution of $y=x-\mu\Gamma(x, \lambda_0)$ is $x=y+\mu\Gamma(y, \lambda_0+\mu)$.

Lemma *B*. - *Let* $L(x)$ *define a homogeneous rotation in E, and let the equation* $y=\lambda x+(1-\lambda)L(x)$ *have the unique solution* $x=y+(\lambda-1)\Gamma(y)$ *for each y in E. Then the adjoint* $\Gamma^*(y)$ *is defined throughout E, and the equation*

$$y=(1-\bar\lambda)x+\bar\lambda L(x)$$

has the unique solution

$$x=y+\bar\lambda\Gamma^*(y)$$

for each y in E.

Proof: By Theorem 7.4 and Lemma *A* the equation $y=x+\lambda(L^{-1}(x)-x)$ has the solution $x=y+\lambda\Gamma(y)$, so that

$$x=x+\lambda(L^{-1}(x)-x)+\lambda\Gamma(x)+\lambda^2\Gamma(L^{-1}(x)-x)$$
$$0=L^{-1}(x)-x+(1-\lambda)\Gamma(x)+\lambda\Gamma(L^{-1}(x)).$$

Γ^* exists, defined everywhere, by Theorem 7.3, so that, taking adjoints on both sides ([15]), we obtain

$$0=L(x)-x+(1-\bar\lambda)\Gamma^*(x)+\bar\lambda L(\Gamma^*(x))$$

or

$$x=(1-\bar\lambda)(x+\bar\lambda\Gamma^*(x))+\bar\lambda L(x+\bar\lambda\Gamma^*(x)).$$

This proves the lemma.

([15]) It is easily verified that when all the transformations are defined throughout E,

$$(T_2T_1)^*=T_1^*T_2^*, \qquad (T_1+T_2)^*=T_1^*+T_2^*, \qquad (aT)^*=\bar a T^*.$$

We now turn to the proof of Theorem 7.5. It is clear, from Theorem 7.3, that Γ^* is defined throughout E, and by lemma B, the solution of

$$y=\frac{1}{2}x+\frac{1}{2}L(x)=x+\frac{1}{2}T(x)$$

is

$$x=y+\frac{1}{2}\Gamma^*(y).$$

Therefore $\Gamma^*(y)=-\Gamma(y)$, or $(\Gamma(x), y)=-(x, \Gamma(y))$. That is, $\Gamma(x)$ is skew-symmetric.

Conversely, if $\Gamma(x)$ is skew-symmetric, $\Gamma^*=-\Gamma$, and, by hypothesis and Theorem 7.3 we conclude that T^* is defined throughout E. But

$$y=y+\frac{1}{2}T(y)-\frac{1}{2}\Gamma(y)-\frac{1}{4}\Gamma(T(y))$$
$$x=x-\frac{1}{2}\Gamma(x)+\frac{1}{2}T(x)-\frac{1}{4}T(\Gamma(x)).$$

Taking adjoints on both sides we obtain

$$y=y+\frac{1}{2}T^*(y)+\frac{1}{2}\Gamma(y)+\frac{1}{4}T^*(\Gamma(y))$$
$$x=x+\frac{1}{2}\Gamma(x)+\frac{1}{2}T^*(x)+\frac{1}{4}\Gamma(T^*(x)).$$

In other words, $x+\frac{1}{2}\Gamma(x)$ and $x+\frac{1}{2}T^*(x)$ are mutually inverse. By Lemma A we conclude that $x+T(x)$ and $x+T^*(x)$ are mutually inverse. Therefore, by Theorem 7.4, $x+T(x)$ is a rotation. It is clear that it is *proper*.

§ 8. - Rotations in Special Spaces.

If E is the space of real valued continuous functions $x(t)$, defined in § 3, then the necessary and sufficient condition that the FREDHOLM transformation with continuous kernel,

(8.1) $$y(s)=x(s)+\int_a^b K(s, t)x(t)dt,$$

be a rotation is that the following equivalent conditions hold ([16])

(8.2) $$\begin{cases} K(s, t)+K(t, s)+\int_a^b K(s, u)K(t, u)du=0 \\ K(s, t)+K(t, s)+\int_a^b K(u, s)K(u, t)du=0 \end{cases}$$

([16]) KOWALEWSKI, Vienna Sitzungsberichte, 1911. See also DELSARTE: *Memorial des Sciences Math.* (1932).

This classical FREDHOLM rotation group has been studied extensively; we shall see that conditions 8.2 play a fundamental role in defining a rotation group in the following functional space:

Let E consist of the class of real-valued functions $x(t)$ which have a continuous derivative $x'(t)$ in the interval (a, b), with analytic and geometric metrics defined by

$$(8.3)\qquad \begin{cases} \|x\| = |x(a)| + \max\limits_{a \leqslant t \leqslant b} |x'(t)| \\ (x, y) = x(a)y(a) + \int\limits_a^b x'(t)y'(t)dt. \end{cases}$$

With the other operations defined as usual this is clearly a complete real Euclidean space, which we shall denote by E_1.

THEOREM 8.1. - *In* (8.1) *let* $\frac{\partial K(s,t)}{\partial s}$ *be continuous. Then a sufficient condition that* (8.1) *be a rotation in* E_1 *is that* $K(s, t)$ *be of the form,*

$$(8.4)\qquad K(s, t) = -\int\limits_a^s \frac{\partial}{\partial t} H(u, t)du,$$

where $H(s, t)$ *satisfies the conditions,*

1°) $H(s, t)$ *is continuous,*

2°) $\frac{\partial H(s, t)}{\partial t}$ *is continuous,*

3°) $H(s, a) = H(s, b) = 0$,

4°) $H(s, t)$ *satisfies* (8.2).

The kernel $k(s, t)$ *of the inverse of* (8.1) *is then*

$$(8.5)\qquad k(s, t) = -\int\limits_a^s \frac{\partial H(t, u)}{\partial t} du.$$

Proof: We observe that $x(a)$ and $x'(t)$ are independent variables.

Expressing the rotation conditions of Theorem 5.3 solely in terms of these variables, we obtain the following sufficient conditions on $K(s, t)$:

i) $\int\limits_a^b \frac{\partial}{\partial s} K(s, t)dt = 0,$

ii) $K(a, t) = 0,$

iii) $\int\limits_s^b \frac{\partial}{\partial t} K(t, u)du + \int\limits_t^b \frac{\partial}{\partial s} K(s, u)du + \int\limits_a^b dw \int\limits_s^b \frac{\partial}{\partial w} K(w, u)du \int\limits_t^b \frac{\partial}{\partial w} K(w, v)dv = 0.$

Let us define

$$(8.6)\qquad H(s, t) = \int\limits_t^b \frac{\partial}{\partial s} K(s, u)du.$$

Then conditions 1°)-4°) are satisfied because of i)-iii), and $K(s, t)$ is given by (8.4). Conversely, it is easy to verify that i)-iii) are consequences of (8.4) and 1°)-4°). Since $H(s, t)$ satisfies (8.2) we see that $\frac{\partial H(s,t)}{\partial s}$ exists and is continuous. By the use of DIRICHLET's lemma and simple integrations we get

$$K(s, t)+k(s, t)+\int_a^b K(s, u)k(u, t)du=$$
$$=-\int_a^s \frac{\partial}{\partial t}\left[H(u, t)+H(t, u)+\int_a^b H(u, v)H(t, v)dv\right]du=0,$$

and hence that (8.5) is indeed the kernel reciprocal to $K(s, t)$.

THEOREM 8.2. - *The totality of rotations considered in theorem* 8.1 *form a group. If two rotations of kermels $K_1(s, t)$, $K_2(s, t)$, defined by $H_1(s, t)$, $H_2(s, t)$ as in the preceding theorem, are carried out in succession, the resulting rotation has the kernel*

$$K_3(s, t)=-\int_a^s \frac{\partial}{\partial t} H_3(u, t)du, \tag{8.7}$$

where

$$H_3(s, t)=H_1(s, t)+H_2(s, t)+\int_a^b H_1(s, u)H_2(u, t)du.$$

Thus this group is a representation of a sub-group of the parameter group of the classical Fredholm rotation group.

The proof of this theorem is readily achieved.

Let E_2 be the Euclidean space of analytic functions considered in § 3 where we take the regular region S to be the unit circle with center at the origin. Consider the equation

$$\Phi(z)=f(z)+\lambda f(0)=T_\lambda(f), \tag{8.8}$$

regarded as a functional transformation in E_2.

We first investigate the solvability of this equation. If two solutions exist, they differ by at most a constant, for if

$$\Phi(z)=T_\lambda(f_1)=T_\lambda(f_2)$$

then

$$f_1(z)-f_2(z)=\lambda\{f_2(0)-f_1(0)\}.$$

Since T_λ is a distributive operation, f_1-f_2 is a solution of $T_\lambda(g)=0$, and so

$$\lambda(1+\lambda)\{f_2(0)-f_1(0)\}=0.$$

Three cases arise:

1°) $\lambda=0$. Then T_λ is the identity, and there is but one solution.

2°) $\lambda+1=0$. This is an exceptional value, for which the equation admits many solutions, each pair differing by a constant (that is, if it admits at least one solution).

3°) $\lambda(1+\lambda)\neq 0$. In this case $f_2(0)=f_1(0)$ and the solution, if it exists, is unique.

Let us attempt to find a solution in the form

$$f(z)=\Phi(z)+l\Phi(0).$$

The condition is that

$$\Phi(0)\,[l(1+\lambda)+\lambda]=0.$$

Again we distinguish two cases.

Case 1°). $\Phi(0)=0$. Then $f(z)=\Phi(z)$ is a solution. If $1+\lambda\neq 0$, it is the only solution. Otherwise $f(z)=\Phi(z)+C$ is a solution for every value of C.

Case 2°). $\Phi(0)\neq 0$. Then

$$l=-\frac{\lambda}{1+\lambda}$$

provided that $1+\lambda\neq 0$. In this case the unique solution is

$$f(z)=\Phi(z)-\frac{\lambda}{1+\lambda}\,\Phi(0).$$

If $1+\lambda=0$ and $\Phi(0)\neq 0$ no solution of the above form exists. Indeed *no* solution exists, for the equation requires that

$$\Phi(0)=(1+\lambda)f(0)$$

which is impossible when $(1+\lambda)=0$, $\Phi(0)\neq 0$.

In working out the condition that (8.8) be a rotation we observe that

$$\iint_S f(z)\,d\omega=\pi f(0)$$

and using this we find that the condition on λ is

$$\lambda+\bar{\lambda}+\lambda\bar{\lambda}=0. \tag{8.9}$$

This means that λ is a point on the circle C_1 of unit radius with center at $z=-1$ in the complex plane.

The inverse is then

$$f(z)=\Phi(z)+\bar{\lambda}\Phi(0),$$

since

$$\frac{\lambda}{1+\lambda}=-\bar{\lambda}.$$

The point λ may be conveniently represented in the form

$$\lambda+1=e^{i\theta}.$$

If the rotations with parameters θ_1, θ_2 corresponding to λ_1, λ_2 are carried out in succession, the resultant rotation has the parameter $\theta_1+\theta_2$ as is easily verified.

The transformation $T_\lambda(f)$ is generated by the skew-symmetric transformation (see § 6)

$$S_\lambda(f)=[\log(1+\lambda)]f(0)=i\theta f(0),$$

for

$$e^S(f)=f(z)+(e^{i\theta}-1)f(0)=f(z)+\lambda f(0).$$

The family of rotations defined by (8.8), as λ ranges over the circle C_1, contains one *improper* rotation, namely, that one for which $\lambda=-2$, for

$$\Phi=T_{\frac{1}{2}\lambda}(f)$$

is not uniquely solvable in that case (see § 7 for proper and improper rotations). It may be remarked that the skew-symmetric transformation $\Gamma(f)$ associated with $T_\lambda(f)$, as in Theorem 7.5, is

$$\Gamma(f)=\left[2i\tan\frac{1}{2}\theta\right]f(0).$$

Summing up the foregoing observations, and using the theory of abstract continuous groups referred to in § 7, we have the theorem.

THEOREM 8.3. - *The rotations*

$$\Phi(z,\theta)=f(z)+(e^{i\theta}-1)f(0)$$

form a one parameter group of period 2π *with real, continuous parameter* θ. *This group is characterized by the functional differential equation*

$$\frac{\partial\Phi(z,\theta)}{\partial\theta}=i\Phi(0,\theta)$$

subject to the initial condition

$$\Phi(z,0)=f(z).$$

Although there exist FREDHOLM rotations in the space of continuous functions C of § 3, there are none of VOLTERRA type, as is shown in the following theorem.

THEOREM 8.4. - *The necessary and sufficient conditions that the Volterra transformation,*

$$y(s)=x(s)+\int_a^s K(s,t)x(t)dt$$

with continuous kernel, be a rotation in C, *is that* $K(s,t)$ *vanish identically,* $a\leqslant t\leqslant s\leqslant b$, *that is, that it be the identity transformation.*

Proof: The necessary sufficient conditions may be easily calculated from the rotation condition, and with the aid of the fundamental lemma of the calculus of variations, they reduce to

$$K(s,t)+\int_s^b K(u,s)K(u,t)du=0. \tag{8.10}$$

The only continuous solution of this homogeneous VOLTERRA equation is $K(s,t)=0$, $a \leqslant t \leqslant s \leqslant b$.

A similar result holds for the Euclidean space of continuous functions $x(s,t)$, with analytical and geometrical metrices defined by

$$\|x\|=\max|x(s,t)|$$

$$(x_1, x_2)=\int_a^b ds \int_a^b x_1(s,t)x_2(s,t)dt.$$

THEOREM 8.5. - *The only rotations of the form*

$$y(s,t)=x(s,t)+\int_s^t K(s,u)x(u,t)du,$$

where $K(s,t)$ *is continuous, are those for which* $K(s,t)=0$, $a \leqslant s \leqslant t \leqslant b$.

PART II.

Spaces with Indefinite Geometrical Metric.

§ 9. - The Postulates for an Indefinite Space.

In the foregoing we have considered spaces with a positive definite geometrical metric. If we drop the requirement that $(x,x)=0$ if and only if $x=0$, the resulting space will be termed an « indefinite Euclidean space ». Many important theorems, true for the definite space, cannot be proved in the indefinite case. For example, a rotation is not necessarily linear, and in fact an example of a non-linear rotation is given in § 12.

The indefinite Euclidean space is defined more precisely as follows: it is a normed linear vector space (real or complex), with a geometrical metric determined by an inner product (x,y), satisfying the conditions:

1°) $(x,y)\,\varepsilon\, C$

2°) $(x+y,z)=(x,z)+(y,z)$

3°) $(a\cdot x,y)=a(x,y)$

4°) $(x,y)=\overline{(y,x)}$

5°) There exists a positive M such that

$$|(x, y)| \leq M \|x\| \, \|y\|$$

6°) If $x_1 = x_2$ and $y_1 = y_2$ then $(x_1, y_1) = (x_2, y_2)$.

We note that (x, x) is real. By the distance $(x_1 - x_2, x_1 - x_2)^{\frac{1}{2}}$. We shall understand always a positive pure imaginary, or a non-negative real number.

The definitions of rotations and motions are made the same as in Part I, § 5. For real spaces, an equivalent definition of a rotation $L(x)$ is: $L(x)$ takes the whole space into itself biuniquely, $L(0) = 0$, and $(L(x), L(y)) = (x, y)$.

As an example of an indefinite Euclidean space we mention the space of special relativity, with inner product

$$(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 - c^2 t\tau$$

and normed in any one of a variety of ways, for example

$$\|x\| = (x_1^2 + x_2^2 + x_3^2 + c^2 t^2)^{\frac{1}{2}}$$
$$\|x\| = |x_1| + |x_2| + |x_3| + c|t|.$$

§ 10. - Groups and Sub-groups of Rotations in an Indefinite Space E_3.

A second example of an indefinite space is furnished by E_3, the space of functions of two variables $x(s, t)$ defined and continuous in the range $a \leq s, t \leq b$, with analytical and geometrical metrics defined in terms of the following operations:

$$(10.1) \qquad \begin{cases} \|x\| = \max |x(s, t)| \\ x * y = \int_a^b x(s, u) y(u, t) du \quad \text{(composition)} \\ [x] = \int_a^b x(s, s) ds \quad \text{(trace)} \\ (x, y) = [x * y]. \end{cases}$$

Let $a(s, t)$ be an element of E_3 whose FREDHOLM determinant $D(a) \neq 0$ and whose resolvent kernel $a'(s, t)$ is therefore in E_3. The totality of such elements a forms an open set Δ in the space E_3.

THEOREM 10.1. - *The transformations of the form,*

$$(10.2) \qquad T_a x = x + a * x + x * a' + a * x * a',$$

in E_3, where a ranges over Δ, form a group of rotations. The inverse of $T_a x$ is $T_{a'} x$, and $T_\beta T_a x = T_{a\beta} x$; where $a\beta = a + \beta + \beta * a$ and $(a\beta)' = a' + \beta' + a' * \beta'$.

Proof: Since $D(a\beta) = D(a) D(\beta)$ and $D(a') = 1/D(a)$, the transformations (10.2) form a group. That the transformations are rotations is shown by a direct calculation using the well known resolvent relations,

$$\text{(10.3)} \qquad \begin{cases} a+a'+a*a'=0 \\ a+a'+a'*a=0. \end{cases}$$

In fact,

$$T_a(x*y)=T_ax*T_ay, \qquad [T_ax]=[x],$$

and therefore

$$(T_ax, T_ay)=[T_ax*T_ay]=[T_a(x*y)]=[x*y]=(x, y).$$

In case we consider complex-valued functions $x(s, t)$, a complex indefinite Euclidean space is obtained by a modification of (10.1)

$$(x, y)=[x*\bar{y}],$$

the other definitions being as before. For the rotations we take

$$T_ax=x+a*x+x*\bar{a}'+a*x*\bar{a}'.$$

Theorem 10.1 holds for this family of transformations, but the details of the proof are slightly different since

$$[T_ax]\neq[x].$$

We shall now turn to a consideration of the differential equations of the transformation group (10.2). Our results are embodied in the following theorem.

THEOREM 10.2. - *The completely integrable differential system characterizing the transformation group* (10.2) *is*

$$\text{(10.4)} \qquad \begin{cases} d^a_\xi T_ax=(\xi+\xi*a')*T_ax-(T_ax)*(\xi+\xi*a') \\ T_0x=x, \qquad (\xi \text{ in } E_3). \end{cases}$$

The first parameter group

$$\text{(10.5)} \qquad \Lambda_\beta a=a+\beta+\beta*a$$

is characterized by the differential system,

$$\text{(10.6)} \qquad \begin{cases} d^\beta_\eta \Lambda_\beta a=\eta+\eta*\beta'+\eta*\Lambda_\beta a+\eta*\beta'*\Lambda_\beta a \\ \Lambda_0 a=a, \qquad (\eta \text{ in } E_3), \end{cases}$$

and the structural function is

$$\Gamma(a, \beta)=a*\beta-\beta*a.$$

Proof: The space Σ of the parameters is in this case E_3, so all increments in the differentials will be understood to lie in E_3. To compute the functions (6.3) for the case at hand, we need to show the existence of $d^a_\beta a'$.

Let ξ range over Σ and a over Δ and define

$$L(a, \xi)=\xi+\xi*a, \qquad M(a, \xi)=\xi+\xi*a'.$$

Clearly $L(a, \xi)$ is a solvable linear function of ξ with $M(a, \xi)$ as inverse. Now $d_\beta^a L(a, \xi)$ exists, continuous in a and so the hypotheses of a theorem proved elsewhere are satisfied ([17]). An application of this theorem shows that $d_\beta^a M(a, \xi)$ exists and is given by

$$d_\beta^a M(a, \xi) = -M(a, \underset{\xi}{\overset{a}{d}}\sigma L(\sigma, M(a, \xi))).$$

A simple calculation shows that $d_\beta^a(\xi * a')$ exists,

$$(10.7) \qquad d_\beta^a(\xi * a') = -\xi * \beta - \xi * a' * \beta - \xi * \beta * a' - \xi * a' * \beta * a'.$$

From the definition of a FRÉCHET differential we have the inequality

$$(10.8) \quad \| \xi * (a+\beta)' - \xi * a' + \xi * \beta + \xi * a' * \beta + \xi * \beta * a' + \xi * a' * \beta * a' \| \leq \\ \leq \varepsilon \| \beta \| \quad \text{for} \quad \| \beta \| < \delta.$$

(Let us observe that the FREDHOLM determinant $D(a)$ is a continuous functional of a in Δ. Consequently if a, β are in Δ then $a+\beta$ is in Δ for δ sufficiently small). In (10.8) choose $\xi = a+\beta$ and use the resolvent relations (10.3). This yields the inequality

$$\| (a+\beta)' - a' + \beta + \beta * a' - (a+\beta+a * a' + \beta * a') * (\beta + \beta * a') \| \leq \varepsilon \| \beta \| \quad \text{for} \quad \| \beta \| < \delta.$$

Finally we obtain

$$\| (a+\beta)' - a' + \beta + a' * \beta + \beta * a' + a' * \beta * a' \| \leq \varepsilon \| \beta \| + \| \beta \|^2 k(a).$$

This shows that $d_\beta^a a'$ exists and is given by

$$(10.9) \qquad d_\beta^a a' = -\beta - a' * \beta - \beta * a' - a' * \beta * a'.$$

In (6.3) take $\delta = 0$, then $\delta' = 0$. On using (10.9) we find that

$$(10.10) \qquad \begin{cases} U(x, z) = -z \\ V(x, \xi) = \xi * x - x * \xi \\ \Omega(a, \xi) = -\xi - \xi * a'. \end{cases}$$

Referring to (6.2) and using the results of paper D we obtain the results (10.4) of the theorem.

The differential equations of the first parameter group are in general (see paper D)

$$(10.11) \qquad d_\eta^\beta \Lambda_\beta a = \Omega'(\Lambda_\beta a, \Omega(\beta, \eta)),$$

where $\Omega'(a, \xi)$ is the inverse in ξ of $\Omega(a, \xi)$. The remainder of the theorem follows from (10.10), (10.11) and the definition of the structural function Γ in paper D.

([17]) A. D. MICHAL and V. ELCONIN: *Completely Integrable Differential Equations in Abstract Spaces*, Acta Mathematica, loc. cit.

Equation (10.5) also defines the first parameter group of the classical total FREDHOLM group, and so results similar to those of Theorem 10.2 may be obtained for this group.

The totality of elements $x(s, t)$ of E_3 which are functions of the first variable s alone form a complete Euclidean sub-space E_4 of E_3. Those elements of Δ which lie in E_4 form a sub-group Δ_1 of Δ. Transformations of the form (10.2) operating in E_4, with a in Δ_1, constitute a rotation group in E_4. In this case the transformations reduce to

$$T_a x = \frac{x + a * x}{1 + [a]}, \tag{10.12}$$

since

$$a' = \frac{-a}{1 + [a]}. \tag{10.13}$$

It is interesting to observe that the only member of Δ_1 which gives a non-zero kernel for a classical FREDHOLM rotation is

$$a(s) = \frac{-2}{b - a}.$$

This kernel, together with the zero kernel (also in Δ_1), yields a subgroup of two elements of the classical FREDHOLM rotation group.

Remarks similar to the above may be made for the sub-space of E_3 consisting of functions $x(s, t)$ which depend on the second variable t alone.

§ 11. - Linear Rotations of Fredholm and Volterra Type in the Space E_4.

In the Euclidean space E_4, a sub-space of E_3 which was defined in § 10, a transformation with continuous kernel,

$$y(s) = x(s) + \int_a^b a(s, t) x(t) dt, \tag{11.1}$$

is a rotation if and only if $D(a) \neq 0$, and

$$\int_a^b a(s, t) ds = 0 \quad \text{or} \quad -2; \tag{11.2}$$

for by application of the rotation condition of § 9 we find

$$\Phi(s) + \Phi(t) + \Phi(s)\Phi(t) = 0, \qquad \Phi(s) = \int_a^b a(u, s) du.$$

According as the integral in (11.2) has the value 0 or -2, we shall call a, and the corresponding transformation (11.), even or odd respectively.

The product of two transformations of the same type is an even transformation, whereas the product of an even and an odd transformation is odd.

THEOREM 11.1. - *A necessary and sufficient condition that* (11.1) *be a rotation in E_4 is that $a(s,t)$ be expressible in one of the forms,*

$$(11.3)\qquad \begin{cases} a(s,t)=\dfrac{\partial L(s,t)}{\partial s}, & \alpha \text{ even} \\ \quad\;\; =\dfrac{\partial L(s,t)}{\partial s}-\dfrac{2}{b-a}, & \alpha \text{ odd,} \end{cases}$$

where $L(s,t)$, $\frac{\partial L(s,t)}{\partial s}$ are continuous, $L(a,t)=L(b,t)$, and in either case $D(\alpha)\neq 0$. These rotations form a group, with a sub-group composed of the even rotations.

Proof: The proof of the sufficiency is obvious. The necessity of the condition (of the theorem) is evident if we define $L(s,t)$ by

$$(11.4)\qquad \begin{cases} L(s,t)=\displaystyle\int_a^s a(u,t)du+f(t), & \alpha \text{ even} \\ L(s,t)=\displaystyle\int_a^s a(u,t)du+f(t)+\frac{2(s-a)}{b-a}, & \alpha \text{ odd,} \end{cases}$$

where $f(t)$ is an arbitrary continuous function. Q. E. D.

If for the case of even transformations we set the arbitrary function $f(t)$ equal to zero, the lew of composition of the functions $L(s,t)$ corresponding to the product of two transformations (11.1) is

$$(11.5)\qquad L_3(s,t)=L_1(s,t)+L_2(s,t)+\int_a^b L_2(s,u)\frac{\partial L_1(u,t)}{\partial u}du.$$

Although there were no non-trivial rotations of VOLTERRA type in the *definite* Euclidean space C considered in § 8, there do exist such rotations in E_4. In fact, if we consider the VOLTERRA transformation with continuous kernel,

$$(11.6)\qquad y(s)=x(s)+\int_a^s a(s,t)x(t)dt,$$

we get the rotation condition,

$$(11.7)\qquad \int_t^b a(s,t)ds=0.$$

These rotation kernels are given by $a(s,t)=\frac{\partial L(s,t)}{\partial s}$, where $L(s,t)$, $\frac{\partial L(s,t)}{\partial s}$ are continuous, and $L(b,t)=L(t,t)$. The details of the above deductions are very similar to those of the proof of Theorem 11.1.

To obtain the differential equations of the group of rotations (11.6), and of its first parameter group, let Σ be the BANACH space of continuous functions $a(s, t)$, $a \leqslant s$, $t \leqslant b$, satisfying (11.7), with $\|a\| = \max |a(s, t)|$. Let us rewrite (11.6) in the form

(11.8) $$T_a x = x + x \cdot a.$$

The transformations of the first parameter group are

(11.9) $$\Lambda_\beta a = a\beta$$

where Σ is a group with respect to the operation $a\beta$ defined by

(11.10) $$\begin{cases} a\beta = a + \beta + \beta * a \\ \beta * a = \int_t^s \beta(s, u) a(u, t) du. \end{cases}$$

The first parameter group (11.9) is abstractly identical with the first parameter group (10.5) of the rotation group (10.2). By arguments similar to those employed in the proof of theorem 10.2 we draw the following conclusions.

THEOREM 11.2. - *The group of rotations* (11.8) *in the space* E_4 *and its first parameter group* (11.9) *are characterized respectively by the differential systems*

$$\begin{cases} d_\xi^a T_a x = T_a x \cdot \xi + T_a x \cdot (\xi * a') \\ T_0 x = x \end{cases}$$

and

$$\begin{cases} d_\eta^\beta \Lambda_\beta a = \eta + \eta * \beta' + \eta * \Lambda_\beta a + \eta * \beta' * \Lambda_\beta a \\ \Lambda_0 a = a. \end{cases}$$

The structural function Γ *is*

$$\Gamma(a, \beta) = a * \beta - \beta * a.$$

There is a corresponding theorem for the group of *even* rotations mentioned in Theorem 11.1.

Still another example of an indefinite Euclidean space may be obtained from the *definite* Euclidean space E_1 of § 8, by omitting the term $x(a)y(a)$ from the equation (8.3) defining the inner product. From the sufficient conditions of Theorem 8.1 for rotations of FREDHOLM type in E_1, we see that these transformations also form a group of rotations in the *indefinite* space just defined.

§ 12. - An Example of a Non-Linear Rotation.

In contrast to the fact that in definite Euclidean spaces rotations are additive functions, as was proved in theorem 5.2, there are indefinite Euclidean spaces in which there exist non-linear rotations. For example, consider the transformation

$$(12.1) \qquad y(s)=x(s)+\int_0^s K(s,\ t,\ x(t))dt$$

in the space E_4, where we assume, for simplicity, $a=0$, $b=1$.

THEOREM 12.1. - *If the function $K(s, t, z)$ satisfies the following conditions, the transformation* (12.1) *is a rotation in the space E_4.*

1°)

$$K(s,\ t,\ z)=\sum_0^\infty K_n(s,\ t)z^n,$$

where $K_n(s, t)$ is defined and continuous for $0\leqq s, t\leqq 1$, and the series $\sum_0^\infty a_n z^n$ converges for all values of z, where $a_n=\max|K_n(s, t)|$.

2°)

$$\int_t^1 K_n(s,\ t)ds=0 \quad \textit{for all } n.$$

3°) *There is a positive constant M such that*

$$\left|\frac{\partial K(s,\ t,\ z)}{\partial z}\right|<M, \quad 0\leqslant s,\ t\leqslant 1, \quad \textit{all } z.$$

Proof: To show that 12.1 is a rotation we must prove that it is uniquely solvable throughout E_4, and that it is a transformation leaving the inner product invariant. For the latter condition, it is sufficient that

$$\int_0^1 y(s)ds=\int_0^1 x(s)ds.$$

This requirement leads to condition 2°) of the theorem.

To show that (12.1) is solvable throughout E_4 we make use of the following powerful theorem of TONELLI ([18]). Let $A[s,\ x_{t_1}^{t_2}(t)]$ be a realvalued functional, defined in the range $0\leqslant s\leqslant 1$, $0\leqslant t_1\leqslant t_2\leqslant 1$, for all functions $x(t)$ continuous in (0, 1), and satisfying the conditions:

i) To an arbitrary positive integer m, there corresponds an M_m such that

$$|A[s,\ x_{t_1}^{t_2}(t)]|\leqslant M_m(t_2-t_1)$$

when $|x(t)|\leqslant m$ in (0, 1).

([18]) L. TONELLI: *Sulle Equazioni Funzionali del Tipo di Volterra*, Bulletin Calcutta Mathematical Society, Vol. 20, p. 31 (1930).

ii) To an arbitrary positive integer m there corresponds an M'_m such that

$$|A[s, x_{t_1}^{t_3}(t)] - A[s, x_{t_1}^{t_2}(t)]| \leqslant M'_m(t_3 - t_2),$$

when $0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant 1$, and $|x(t)| \leqslant m$ in $(0, 1)$.

iii) To an arbitrary positive integer m and an arbitrary positive ε there corresponds a positive ϱ_m such that

$$|A[s_1, x_1{}_{t_1}^{t_2}(t)] - A[s_2, x_2{}_{t_1}^{t_2}(t)]| \leqslant \varepsilon(t_2 - t_1),$$

when $s_1 \leqslant s_2$, $s_2 - s_1 \leqslant \varrho_m$; $|x_1(t)| \leqslant m$, $|x_2(t)| \leqslant m$ in $(0, 1)$; $|x_1(t) - x_2(t)| \leqslant \varrho_m$ in (s_1, s_2).

iv) There exists a number M'' such that

$$|A[s, x_1{}_{t_1}^{t_3}(t)] - A[s, x_2{}_{t_1}^{t_3}(t)]| \leqslant$$
$$\leqslant M'' \{ (t_2 - t_1) \max_{(t_1, t_2)} |x_1(t) - x_2(t)| + (t_3 - t_2) \max_{(t_2, t_3)} |x_1(t) - x_2(t)| \},$$

when $0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant 1$.

Under these four hypotheses, the equation

$$y(s) = x(s) - A[s, x_0^s(t)],$$

where $y(s)$ is continuous in $(0, 1)$, admits one and only one solution $x(s)$, continuous in $(0, 1)$.

In applying this theorem to equation (12.1) we observe that the functional A is in this case

$$-\int_{t_1}^{t_2} K(s, t, x(t))dt.$$

Conditions i) to iii) are immediate consequences of the continuity of $K(s, t, z)$, and iv) is seen to be satisfied when we note that for the above functional,

$$A[s, x_{t_1}^{t_3}(t)] = A[s, x_{t_1}^{t_2}(t)] + A[s, x_{t_2}^{t_3}(t)],$$

and then use the law of the mean in conjunction with condition 3° of Theorem 12.1. Hence (12.1) is uniquely solvable in E_4, and is therefore a rotation in this space.

A specific example of a non-linear rotation of the type (12.1) is offered by

$$y(s) = x(s) + \int_0^s (2s - t - 1) \sin x(t) dt.$$

The linear rotations (11.6) in E_4 also fulfill the conditions of theorem 12.1.