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Theory of gravitation. The problem of stationary states, and the apparatus of the echo phenomenon in radiographic transmission

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THEORY OF GRAVITATION. THE PROBLEM OF STATIONARY STATES, AND THE APPARATUS OF THE ECHO PHENOMENON IN RADIOGRAPHIC TRANSMISSION

by OLIVER E. GLENN (Lansdowne, Pennsylvania).

I. - Introduction.

A central orbit is said to be stable if the potential constrains the rotating body to traverse it continuously. If such an orbit is perturbed by a small amount it has a gyroscopic power to right itself. From this hypothesis the author has derived the following as the general form of the central force when the orbit is stable:

$$P(r)=\gamma^2\lambda^2[2(p(r))^2-rp(r)p'(r)+r^2/\lambda^2]/r^5\equiv\gamma^2\lambda^2a^2\Gamma(r)/r^5,$$

where $p(r)=ar^{n-1}+br^{n-2}+ \dots +k$; γ, λ, $a,\dots, k$ are constants and r is the distance from the center of force to the rotating body. The expanded form of $\Gamma(r)$ is,

$$(1)\quad \Gamma(r)=-\{(n-3)r^{2n-2}+A_1'r^{2n-3}+\dots+A'_{2n-7}r^5\}+L'r^3+M'r^2+U'r+V',$$

where A_i', L', M', U', V' are rational, integral polynomials in $b/a,\dots, k/a$, and $n>2$. In the case $n=6$, sufficiently general for the problems of practical astronomy, we find

$$(2)\quad \begin{cases} A_1'=5ab/a^2, \quad A_2'=2(b^2+2ac)/a^2, \quad A_3'=3(ad+bc)/a^2, \\ A_4'=(c^2+2ae+2bd)/a^2, \quad A_5'=(af+be+cd)/a^2, \\ L'=(cf+de)/a^2, \quad M'=(e^2+2df+\lambda^{-2})/a^2, \quad U'=3ef/a^2, \quad V'=2f^2/a^2. \end{cases}$$

It is known that, in (2), $a<b<c$ and that these three numbers are small. In the problem of the planets they may be taken as zero, ($n=6$). It is often useful to regard the respective cases $n=5$, $n=4$, $n=3$, as emanating from $P(r)$, $n=6$, by means of the particularizations (approximative), $a=0$, $b\neq 0$; $a=b=0$, $c\neq 0$; $a=b=c=0$, $d\neq 0$.

If $n>3$ and r is sufficiently large, the force $P(r)$, being negative, is repellent; which suggests that it may be practically impossible for an « outside star » to enter the Solar System unless by projection.

II. - The force $P(r)$ as a function of both distance r and mass m.

Any system Q of contiguous curves of fairly uniform general direction and of assigned length, may be called a field. Let the numerical coordinates of n

chosen points on a representative segment $a'b'$ be (r_i, θ_i), $(i=1,...., n)$. Then, by known processes, we can determine the equation of $a'b'$ in the polynomial form,

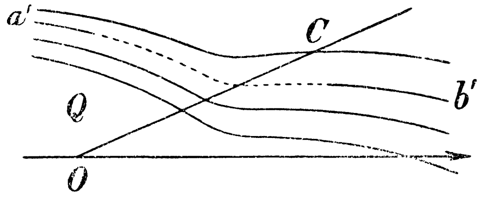

Fig. 1.

(3) $\theta = \nu r^{n-1} + \xi r^{n-2} + ... + \sigma.$

If q curves of Q are thus considered, we obtain q equations,

(4) $\theta = \nu_j r^{n-1} + \xi_j r^{n-2} + ... + \sigma_j$ $(j=1,...., q).$

These, however, may all be comprised in one equation which involves a parameter m. For, consider the n coefficients $\nu,....,$ σ in (3) to be functions of m,

$$\nu = \nu(m),...., \quad \sigma = \sigma(m),$$

such that ν_j, $(j=1,...., q)$, are the values of $\nu(m)$ for $m=m_1,...., m_q$, respectively; and likewise,

$$\xi_j = \xi(m_j),...., \quad \sigma_j = \sigma(m_j),$$
$$(j=1,...., q;\ m_j \neq m_i \text{ if } i \neq j).$$

This is possible since we then have q determinations of each function, necessary and sufficient to give each as a polynomial of order $q-1$ in m. Thus,

(5) $$\nu(m) = \sum_{k=1,...,q} \nu_{k1} m^{q-k},...., \quad \sigma(m) = \sum_{k=1,...,q} \nu_{kn} m^{q-k},$$

and at once this gives, to the polynomial (3), the following form:

(6) $$\theta = \sum_{i=1,...,n} (\nu_{1i} m^{q-1} + \nu_{2i} m^{q-2} + ... + \nu_{qi}) r^{n-i}.$$

Now (6) reduces to (4) when $m=m_1,...., m_q$, and if m is varied continuously over the interval from the smallest to the largest number m_j, $a'b'$ sweeps over the whole area preempted by the field, undergoing, at the same time, the deformations required by the equation (6).

A field Q of orbits is given by the integral curves of the differential equation of plane orbital motion for an arbitrary central force F [1],

(7) $$d^2u/d\theta^2 + u = F/\gamma^2 u^2, \qquad (u = 1/r).$$

The integral curves may be written,

(8) $$\theta + c = f(r, m, s, v, \beta),$$

where m is the mass of the planet N and c one of the constants of integration. The dependent set s, v, β, arises from convenient initial conditions: s is the

[1] O. E. GLENN: *The mechanics of the stability of a central orbit.* Annali della R. Scuola Normale Superiore di Pisa, ser. 2, vol. 2 (1933, XI). Cf. also vol. 4 (1935, XIII), p. 241.

distance from an initial position I of N to the center of force O (origin), v is the initial velocity and β the angle between OI and the vector v. In (8), if we fix all constants but m and allow m to vary over a set $m_1, \ldots, m_q$, we obtain again the forms (3), (4), m being now a mass. Here the curves Q are concurrent, and have a common tangent, at I. If both n and q are large enough, the system obtained by allowing m to vary continuously from m_1 to m_q in (6), is, within Q, the two-parameter system of integral curves (orbits). Then, by substituting (6) in (7) we obtain the force $F(r)$ in rational form.

A necessary and sufficient condition that Q, thus interpreted, be a field of *stable* orbits, is that the rational F should have the functional form of $P(r)$.

An assumption that there are local perturbations, that the masses rotating in a stable Q perturb each other's paths, is interchangable with the process of adding a perturbing function of small value to $P(r)$, sufficient to warp Q into the actual perturbed field Q'. Assuming Q' to be also stable, this can always be done so as to preserve $P(r)$ in the functional form corresponding to stable orbits. Moreover the explicit new force can be derived from Q'. We have only to regard the latter as a Q, reconstruct (6) and calculate F from (7). The total effect of altering Q to Q' consists of small changes in the ν_{ki}.

Another form additional to (6), for the equation of a typical orbit of the stable Q, is known, viz.,

$$\text{(9)} \qquad \lambda\theta+\mu=\int dr/p(r), \quad \text{or,} \quad \lambda d\theta=dr/p(r), \quad (\lambda,\ \mu \text{ const}).$$

When (9_2) is made identical with the differential of (6), we get,

$$\text{(10)} \qquad 1=\lambda p(r)\sum_{i=1,\ldots,n}(\nu_{1i}m^{q-1}+\ldots+\nu_{qi})(n-i)r^{n-i-1}.$$

With m constant and r variable (10) is equivalent to $2(n-1)$ linear equations in $a, \ldots, k$, determining the latter; also $n-2$ rational relations in the functions $\nu(m), \ldots, \varrho(m)$, to be satisfied within the approximations. The latter are the conditions that F be a function $P(r)$. The equations are,

$$\text{(11)} \qquad \left\{\begin{array}{l} a(n-1)\nu(m)=0, \\ a(n-2)\xi(m)+b(n-1)\nu(m)=0, \\ a(n-3)O(m)+b(n-2)\xi(m)+c(n-1)\nu(m)=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a(n-(n-1))\varrho(m)+2b\pi(m)+\ldots+j(n-1)\nu(m)=0, \\ b\varrho(m)+2c\pi(m)+\ldots+k(n-1)\nu(m)=0, \\ c\varrho(m)+2d\pi(m)+\ldots+k(n-2)\xi(m)=0, \\ d\varrho(m)+2e\pi(m)+\ldots+k(n-3)O(m)=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ j\varrho(m)+2k\pi(m)=0, \\ k\varrho(m)=1/\lambda. \end{array}\right.$$

This system may be solved readily if we begin with the last equation and proceed upward. Thus we obtain,

$$(12)\qquad \begin{cases} k=1/\lambda\varrho(m), \\ j=-2\pi(m)/\lambda\varrho(m)^2, \\ i=[4\pi(m)^2-3O(m)\varrho(m)]/\lambda\varrho(m)^3, \\ h=[-8\pi(m)^3+12O(m)\pi(m)\varrho(m)-4\xi(m)\varrho(m)^2]/\lambda\varrho(m)^4, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

For illustration we give, also, the $n-2=2$ rational relations which hold when $n=4$:

$$(13)\qquad \begin{cases} [16\xi(m)^4-36\xi(m)^2O(m)\nu(m)+9O(m)^2\nu(m)^2]/\lambda O(m)^4 \doteq 0, \\ [2\xi(m)^3\nu(m)-3\xi(m)O(m)\nu(m)^2]/\lambda O(m)^4 \doteq 0, \qquad (O\equiv\varrho). \end{cases}$$

From the result of substituting in equation (1) according to (12) we obtain, therefore, the following.

THEOREM. - *The coefficients* A_i, $L,\ldots,$ V, $(A_i=a^2A_i',\ldots,$ $V=a^2V')$, *of the force function* $P(r)$ *of stable central orbits, are numerical bi-rational fractions in* m. *The denominators of these fractions are all powers of one and the same polynomial, viz.,*

$$(14)\qquad \varrho(m)=\nu_{1n-1}m^{q-1}+\nu_{2n-1}m^{q-2}+\ldots.+\nu_{qn-1}.$$

This expression for $P(r)$ in terms of both mass and distance is the generalization of the newtonian formula $P=km'm/r^2$.

III. - Properties of the function $p(r)$.

In (6), considered as the equation of stable orbits in Q, which proceed from a point I according to identical initial conditions, if we fix θ as by OC in figure 1, the distances from O to the paths of the respective masses may be written $r_1,\ldots, r_q$. We can then calculate m as a function of r in Q, by use of the determinations (r_i, m_i), $(i=1,\ldots., q)$, viz.,

$$(15)\qquad m=ar^{q-1}+br^{q-2}+\ldots.+k.$$

If we use this as a case of $m=p(r)$, as in $P(r)$, $(n=q)$, it is necessary to note that the variation of r in (15), transverse to the curves of Q, is equivalent to the variation of the radius vector of $a'b'$ (fig. 1) as it's extremity traces the dotted segment of $a'b'$. Then m varies, as this segment is described, throughout the interval between the extreme numbers of the set $m_1,\ldots., m_q$.

We note parenthetically that the equation of the orbit in Q is then,

$$\int dr/m=\lambda\theta+\mu.$$

The maximum number of positives in the total set of roots $\beta_1,\ldots, \beta_{q-1}$ of

$$E(r)\equiv[-2p(r)+rp'(r)]/a=0,$$

is $q-2$. For, suppose this many, viz., $\beta_2,\ldots, \beta_{q-1}$ are positive. Since the coefficient of r^2 in $E(r)$ is zero, we have $\sum\beta_1\ldots\beta_{q-3}=0$. Hence β_1 is negative and is given in terms of symmetric functions by

$$\beta_1=-\sum\beta_2\beta_3\ldots\beta_{q-2}/\sum\beta_2\beta_3\ldots\beta_{q-3}.$$

Lemma 1. - *The roots* β_i $(i=2,\ldots, q-1)$, *assumed to be positive, separate the roots of* $p(r)=0$, *and, in fact, if the latter are* a_i, $(i=1,\ldots, q-1)$, *with* $a_1<a_2<\ldots<a_{q-1}$, *we have,*

$$a_1<\beta_2<a_2<\beta_3<a_3<\beta_4<\ldots<\beta_{q-1}<a_{q-1}. \tag{16}$$

In proof, since

$$p(r)=a(r-a_1)(r-a_2)\ldots(r-a_{q-1}),$$

then,

$$p'(r)=a\sum(r-a_1)\ldots(r-a_{q-2}).$$

Hence,

$$E(a_i)=a_i(a_i-a_1)(a_i-a_2)(\ldots)(a_i-a_{q-1}), \qquad [\text{Omit } a_i-a_i],$$

and consequently,

$$E(a_1)=+\sigma,\quad E(a_2)=-\sigma,\quad E(a_3)=+\sigma,\ldots,\quad E(a_{q-1})=(-1)^q\sigma,$$

where $\sigma=(-1)^q$. This proves the lemma.

IV. - The reality described by the force function $G=P(r)$.

1. *Niels Bohr's law.* - It follows from (14) that the real, positive roots of $\varrho(m)=0$, and only these numbers, are singular values of the planetary mass for which the force becomes infinite and the motion therefore unstable. This conclusion is a mathematical prediction, though lacking in some measure experimental and numerical verification, of BOHR's celebrated law of stationary states. This results from the following considerations.

According to Sir J. J. THOMPSON's concept of the atom of an element, an atom consists of a central, positively charged nucleus around which negatively charged electrons rotate, the general model being like an astronomical solar system. The resultant of the forces which give rise to the potential, causes the electron to behave as a material particle of mass m, which enjoys stable motion. Hence the central force within the atom, in it's effect upon electrons, has the functional form $P(r)$.

It has been previously noted that, when influences introduced from without affect the force and cause perturbations of the orbits of the electrons, many modifications of material can result. Let us assume that such influences are introduced systematically, as when a HITTORF tube is attached to an induction coil. As the exciting force is increased the electron's motion is disturbed beyond mere perturbation. Clearly, however, we must assume that the orbital motion of the electron keeps, or quickly regains the property of stability. Hence, in any emergency, the central force keeps the functional form $P(r)$. Thus the total mathematical effect of the excitation is to alter the values of the arithmetical parameters ν_{ki}, λ (cf. (2), (12)).

As $\nu_{1n-1},\ldots, \nu_{qn-1}$ are thus altered $\varrho(m)$ may become such that m is a root. At this juncture, therefore, the electron's motion becomes unstable. It catastrophically recedes to an orbit farther from the nucleus ([2]). The new orbit will also be of a field Q but it's position is such that the corresponding $\nu_{1n-1},\ldots, \nu_{qn-1}$ (cf. (6)) do not make m a root of $\varrho(m)$. The motion at the new level is therefore stable. If the exciting force is further increased this new orbit may become unstable as a result of the same mathematical process.

When, suddenly, the exciting force is released, the electron may fall all the way to the home orbit or it may be detained at a higher level which is also consistent with the original central force of the atom ([3]).

2. *The motions of the large planets.* - Other types of reality will be described by $P(r)$ according as we make different possible choices of the quantities $a,\ldots, k, \lambda$. When $P(r)$ is the relativistic approximation to NEWTON's formula of inverse squares, $P(r)$ describes the motions of the large planets, as is well known.

3. *Saturn's rings.* - If $1/a\lambda$ ($\doteq 0$) is small, while r is of a magnitude comparable to the distance from the center of Saturn to it's rings, $P(r)$ describes the rings.

4. *The comets.* - Inaccuracies in the computed periods of comets, such as the three day error in the perihelion passage of HALLEY's comet, in COWELL and CROMMELIN's Preisschrift, *Essay on the return of Halley's comet* (1910), as well as other imperfect data on the motions of these bodies, suggest that it may be necessary to study anew the motions of comets, from the standpoint of a force function $P(r)$ more general than that of NEWTON.

5. *Masses in motion in spiral nebulae.* - The author has previously shown that, with $n=4$, formula (15) may be employed to reduce $P(r)$ to the form,

$$G=\gamma^2\lambda^2[-cm_0/r^2+\lambda^{-2}/r^3+em_0/r^4+2fm_0/r^5],$$

where m_0 is the mean of the values of the mass in the motion over the dotted segment in figure 1. If the second term λ^{-2}/r^3 is to be the significant term so that the motion is in an elementary spiral, m_0 is small, and it will be small below

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([2]) If, during the change of levels, m also changes, the theory remains consistent.

([3]) For an arithmetical example of a singular mass cf. GLENN, *loc. cit.*, first paper, p. 308.

a definite limit, as is evidently proved with some obvious restrictions when we prove the following:

Lemma 2. - *If $P(r)$ vanishes for $\nu=2n-3$ real positive values of r, there is an upper limit but no lower limit to the values of the positive number λ^{-2}.*

Independently of the hypothesis of the lemma, $\Gamma(r)$, of (1), vanishes for a large positive, and a large negative value of r, by the principle of continuity. When $n=3$ the discriminant of $\Gamma(r)$ can be expressed in the form (cf. (2) *et seq.*),

$$\Delta=f^2[(e^2-4df-2\lambda^{-2})^3-27e^2\lambda^{-4}].$$

The condition for three real roots is $\Delta>0$. Therefore,

$$\lambda^{-2}<\frac{1}{2}(e^2-4df).$$

This proves the lemma for $n=3$ and shows that the discriminant of $p(r)$ is positive in this case.

With n literal, if we bring λ^{-2} arbitrarily near to zero, the roots of $\Gamma(r)=0$ are brought arbitrarily near to the respective roots of the pair $p(r)=0$, $E(r)=0$. By hypothesis all of these roots are real. Hence without proof it is evident that no lower limit can be assigned to λ^{-2}. Passing to the question of an upper limit, we write $\Gamma(r)=y^w\Gamma(x/y)$, $(w=2n-2)$, introducing homogeneous variables and coefficients. The discriminant Δ is the dialytic eliminant of $\partial\Gamma/\partial x$ and $\partial\Gamma/\partial y$ and from this determinant it is clear, first, all terms of Δ contain either U or V as a factor. Second, if we set $V=0$ in Δ, all remaining terms contain the factor U^2. LAPLACE's expansion method shows that w is the highest exponent of M in Δ. If $s_1,\ldots, s_w$ are the roots of $\Gamma(r)=0$,

$$\Delta=(-1)^{\frac{1}{2}w(w-1)}[(3-n)a^2]^{w-2}\Gamma'(s_1)\ldots\Gamma'(s_w), \qquad (n>3).$$

Hence a term of Δ of degree w in M is,

$$W=(-1)^{\frac{1}{2}w(w-1)}2^w[(3-n)a^2]^{w-3}VM^w,$$

V being $2k^2$. The sign of W is negative or positive according as n in odd or even. It's significance is in the fact that M is the only coefficient of $\Gamma(r)$ which contains λ^{-2}. Any other term t of Δ, involving M^w would contain another factor of degree $w-2$ and weight w, but such a term could not contain V without coinciding with W, on account of the weight. Hence it would contain U and therefore U^2. Since the weight of U^2 is $2w-2>w$, it follows that t does not exist in Δ, W being thus the only term which involves M^w.

A necessary condition for the postulated reality of all roots of $\Gamma(r)=0$ is $\Delta>0$. Writing $\Delta=\delta(z)$, δ being an integral polynomial in $z=\lambda^{-2}$, we note that δ is of order w in z, and when n is odd, the term in z^w has the negative sign. At $z\doteq0$,

Δ is positive. Hence, in the case n odd, the graph of $\Delta=\delta(z)$ shows readily that the greatest real positive root of $\delta(z)=0$ is an upper limit to λ^{-2}, $(\Delta>0)$.

Theoretically we can always choose n odd and remain within acceptable limits of error in astronomical practice. However when n is even the graph of $\Delta=\delta(z)$ shows that, if $\delta(z)=0$ possesses any real positive roots, there will be upper limits to λ^{-2} upon properly chosen intervals $(z_1,\ldots, z_2)$, $(\Delta>0)$.

6. *The Heaviside layer.* - We shall prove with the aid of the form (2) of $\Gamma(r)$ that, not only is there an accumulation (the saturnian rings), of small particles near to a large planet, but at least two concentric spherical walls constituted of such particles will form outside of the region of the rings. These walls, the HEAVISIDE Layers [4], are probably composed of a mixture of gases and particles because the mass m in certain ones of our formulas may approach zero; (Cf. G, § IV (5)).

V. - The Heaviside layers and the « Gegenschein » optical phenomenon.

The problem of the rest of this paper is to account for these walls, which are the sources of the echo phenomenon of long-wave radio signals, by means of the force function $P(r)$.

We choose $1/a\lambda$ as follows. Let

$$p(r)=a(r-a_{q-1})(r^{q-2}+p_1r^{q-3}+\ldots+p_{q-2})=a\zeta(r, a_{q-1}). \tag{17}$$

Assume $\alpha>\xi=a_{q-1}$, with $\alpha-\xi\doteq 0$. If we employ $\zeta(\alpha, \xi)$ as $p(\alpha)$, then $\Gamma(\alpha)$, assumed to vanish, becomes

$$2\zeta(\alpha, \xi)^2-\alpha\zeta(\alpha, \xi)\zeta'(\alpha, \xi)+\alpha^2/a^2\lambda^2=0, \qquad (\zeta'=\partial\zeta/\partial\alpha). \tag{18}$$

Solving (18) for $1/a^2\lambda^2$, the result is positive and of the same order of magnitude as $\alpha-\xi$. Abbreviating this result as $-\eta(\alpha, \xi)/\alpha^2$, the value of $-\eta/\alpha^2$, substituted for $1/a^2\lambda^2$ in $\Gamma(r)$ brings the $2n-2$ roots of the latter arbitrarily near to β_1, a_1, β_2, $a_2,\ldots$, β_{q-1}, a_{q-1}, respectively (cf. (16)). The actual roots differ from those of the above set by increments ε which conform in value to the variation of a single parameter α. We use primes in representing a root plus the corresponding ε;

$$a_i'=a_i+\varepsilon_i(\alpha), \qquad \beta_i'=\beta_i+\varepsilon_i'(\alpha). \tag{19}$$

We next revert to the case $n=6$ in which $\Gamma(r)=0$ is an equation of degree 10. The vicinity will be that of Saturn's gravitational field. Five of the nine positive

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[4] Their existence was first inferred by O. HEAVISIDE (1850-1925). Cf. G. VANNI: *Osservazioni sulle teorie della propagazione delle onde hertziane etc.*, Atti del Congresso Internaz. dei Matematici, tomo 6, Bologna, 1928 (VI). The latter author prefers the nomenclature KENNELLY-HEAVISIDE layer.

roots of $\Gamma(r)=0$ are respectively the radii of the outer bounding circles of the three parts of Saturn's ring. From astronomical measurements, approximations to the radii are $\alpha_1=7.1$, $\alpha_2=8.1$, $\alpha_3=8.6$, and two determined from the latter by a calculation based upon the case $n=4$ of $\Gamma(r)$. These two are $\beta_2=7.4664$, $\beta_3=8.3671$. The change from $n=4$ to $n=6$ (in Γ) alters α_1, $\beta_2, \ldots, \alpha_3$ by small amounts δ_i. (The unit of distance is 10000 miles).

The curve $G=P(r)$, $(n=6)$, intersects (r) at α_1', α_2', α_3', β_2', β_3', the width of CASSINI's division being $\beta_2'-\alpha_1'$ and that of ENCKE's division $\beta_3'-\alpha_2'$.

THEOREM. - *If a planet has a saturnian ring in which there are actually two divisions, it is surrounded by a Heaviside layer.*

We include in the hypothesis the requirement that the graph of (15) shall be a fairly smooth curve. In proof, the roots of $\Gamma(r)=0$ are all real if those of $p(r)=0$ are all real and distinct. The converse is true, assuming that $1/a\lambda$ is sufficiently small. With $n=5$ the conditions for reality of the roots of the quartic $p(r)=0$ are $T_1>0$, $T_2>0$, $T_3>0$, in which (cf. (2)),

$$\begin{aligned}
T_1 &= b(256b^2f^3-192bcef^2+144bde^2f-128bd^2f^2-27be^4-80cd^2ef \\
&\qquad +16d^4f+144c^2df^2+18cde^3-4d^3e^2-6c^2e^2f)+c^2R, \\
T_2 &= b(16bdf+14cde-4d^3-18be^2-6c^2f)+c^2S, \\
T_3 &= b(-8d)+3c^2, \\
R &= d^2e^2-27c^2f^2+18cdef-4d^3f-4ce^3, \\
S &= d^2-3ce.
\end{aligned}$$

Now R is the discriminant of

$$q(r)=cr^3+dr^2+er+f=0.$$

A necessary and sufficient condition that all three roots of $q(r)=0$ be real is $R>0$. If all are positive the planet will have a ring with two divisions. When $R>0$ then $S>0$. Hence, when the divided ring exists, $T_1>0$, $T_2>0$, $T_3>0$ provided

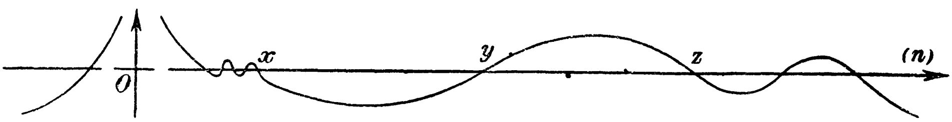

Fig. 2.

only that b is small enough: But, assuming the graph of (15) to be smooth, the smallness of b is at our choice. Hence the roots of the chosen $p(r)$ are all real. When this result is combined with the separation lemma 1, it is clear that $G=P(r)$, $(n=5)$, intersects (r) in two points additional to the five corresponding to the rings. Since b is small the additional pair is considerably distant from the original five, (shown as y, z, in fig. 2).

In the interval $y-\alpha_3'$ the force repels particles, while from y to z it is an attraction. Hence particles accumulate at y and this is true at all points of the distance of y from the center O of the planet. Hence the existence of a HEAVISIDE layer has been proved.

We can generalize by proving also that the succession of possible HEAVISIDE layers beyond z will not be interrupted by the occurrence of any imaginary roots in $P(r)=0$.

Let the conditions for reality of all roots of

$$q(r)=br^{n-2}+cr^{n-3}+\dots+k=0,$$

(which conditions always consist of inequalities, without equalities), be satisfied in the form,

(20) $$q_i(b, c, \dots, k)>0, \qquad (i=1, \dots, \iota),$$

where q_i is a rational, integral polynomial, and write the corresponding conditions, that all roots be real, for the equation,

$$p(r)=ar^{n-1}+q(r)=0,$$

as,

$$t_j(a, b, \dots, k)>0, \qquad (j=1, \dots, \varkappa).$$

Necessarily the set t_j $(0, b, \dots, k)$ is only a redundant form of the set,

$$\lambda_i b^{\nu_i} q_i(b, c, \dots, k), \qquad (\lambda_i,\ \nu_i \text{ positive integers}).$$

Hence, assuming (20),

$$t_j=aF_j(a, b, \dots, k)+t_j(0, b, \dots, k)>0, \qquad (j=1, \dots, \varkappa),$$

will be satisfied provided only that a is small enough. Since an induction begins with $n=5$, the curve $G=P(r)$, with n literal, therefore intersects (r) in $2n-8$ real points beyond (including) y, which was to be proved.

The force $P(r)$ $(n=6)$ is, for small masses, a repulsion within five intervals on (r) and an attraction within five. Partly for local reasons, such as the fact that a projectile fired eastward in the northern hemisphere is deflected toward the equator, the configuration of particles corresponding to the inner group of positive intersections is a ring. The spherical form of the HEAVISIDE layer evidently would not be affected much by influences local to the planet, due to the greater distance.

Note that $\alpha_1', \dots, \alpha_5'$ are the radii of the points at which, with r increasing, the force changes from attraction to repulsion. Hence the radii of the central spheres about which the two consecutive walls $(n=6)$ accumulate are β_4', β_5', the first and third of the outer abscissas.

From α_1', α_2', α_3', β_2', β_3', the outer dimensions α_4', α_5', β_4', β_5' are determined to close approximations. For, we can use $\Gamma(r)$ as if in the form $-a^{-1}p(r)E(r)$ instead of

(21) $$\Gamma(r)=-a^{-1}p(r)E(r)-\eta(a, \xi),$$

and leave all results within small limits of error, since $\eta \doteq 0$ as $a-\xi \doteq 0$. Hence we divide $x-a_i'$, $(i=1, 2, 3)$, out of

$$a^{-1}p(r)=r^5+br^4/a+cr^3/a+dr^2/a+er/a+f/a,$$

obtaining three vanishing remainders which are linear non-homogeneous equations in the five unknowns $b/a, \ldots, f/a$. Two additional equations are obtained by dividing $r-\beta_2'$, $r-\beta_3'$ from

$$E(r)=3r^5+2br^4/a+cr^3/a-er/a-2f/a. \tag{22}$$

The five linear equations determine $a^{-1}p(r)$, $E(r)$ and therefore five additional roots which approximate to a_4', a_5', β_4', β_5', together with β_1' which is negative and without physical interpretation.

In arithmetical practice this method is indeterminate, requiring us to assign five numerical increments $\varepsilon(a)$ to the set of numbers a_1, a_2, a_3, β_2, β_3, respectively, such that the increments conform to the single parameter variation of the set. If the numbers δ_i are not properly chosen, the above linear system in $b/a, \ldots, f/a$ will be inconsistent. The method is thus one of trial and error and, furthermore, small changes in the increments ε result in large alterations of the four outer abscissas.

A method of trial and error in which the values of only two, instead of five parameters, need to be manipulated in order to reach accuracy, is the following which solves our practical problem within limits of error which are good in view of the degree of exactness thus far attained in actual measurements of Saturn's rings. We regard $a_1(=a_1')=7.1$, $a_2=8.1$, $a_3=8.6$ as correct dimensions. We then assign (by trial) the numbers a_4, a_5. This determines the coefficients of $a^{-1}p(r)$ and therefore the roots (including β_2, β_3 approximately) of $E(r)=0$. We manipulate a_4, a_5 until we obtain by this process $\beta_2=7.4664$, $\beta_3=8.3671$, to a close approximation. With $a_4=34$, $a_5=46$ we find, in fact, $\beta_2' \doteq 7.4606$, $\beta_3' \doteq 8.3651$. We then regard

$$a_1=7.1, \qquad \beta_2'=7.4606, \qquad a_2=8.1, \qquad \beta_3'=8.3651, \qquad a_3=8.6,$$

as acceptable dimensions (radii) of the boundaries within the ring, and complete the solution of the equation $E(r)=0$ which they, with $a_4=34$, $a_5=46$, determine. The three additional solutions (one negative) include approximately the distances β_4', β_5', the dimensions sought.

A summary of the computation is as follows: Determining $a^{-1}p(r)$ from it's roots 7.1, 8.1, 8.6, 34, 46, we get,

$$p(r)=a(r^5-103.8r^4+3656.23r^3-52776.186r^2+333958.6r-773532.504).$$

Then,

$$E(r)=3r^5-207.6r^4+3656.23r^3-333958.6r+1547065.008.$$

Applying to $E(r)=0$ the method of RUFFINI and HORNER,

$$\beta_1' \doteq -9.264156, \qquad \beta_2' \doteq 7.460584, \qquad \beta_3' \doteq 8.365125,$$
$$\beta_4' \doteq 21.8878, \qquad \beta_5' \doteq 40.7507$$

The conclusion, therefore, is the following;

THEOREM. - *In the gravitational field of Saturn there exist at least two spherical Heaviside layers concentric with the planet. They are composed of particles, including gases. The distances from the center of the planet to the central spheres of the layers are, respectively, 218878 miles and 407507 miles.*

We see from this result that both of the corresponding HEAVISIDE layers of Saturn come between the moons Tethys and Titan, while the moons Dione and Rhea rotate between the layers. Also the values β_4', β_5' are in satisfactory correspondence with the distances from the earth to it's walls of radiographic echos, as experimentally determined.

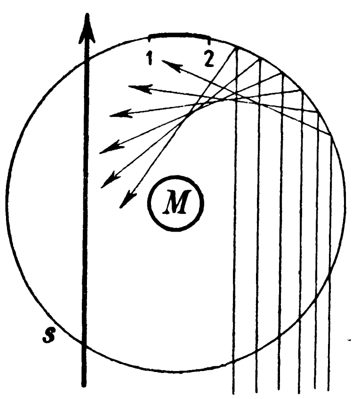

Fig. 3.

Addendum. - The « Gegenschein » is a small illuminated area visible at night on the ecliptic in the region opposite the Sun with respect to the earth. The following considerations make it rather evident that this is a view of the HEAVISIDE layer. Let s be a section of the layer by the plane of the ecliptic, M being an observer whose zenith is in (1, 2). (The relative size of the earth is magnified). Rays of sunlight, directed as the large arrow, would pass through the layer on the side near to the Sun and be reflected against the opposite interior along the small arrows. The latter envelop a caustic (in space, a horn angle). Evidently the observer will see reflected light from (1, 2) only, this region being somewhat extended by the principle of diffusion. The region is in the earth's shadow but it appears that the corona would be a sufficient source of light. Some indirect illumination would come from the earth's atmosphere.