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Existence theorems for ordinary problems of the calculus of variations (part II)

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EXISTENCE THEOREMS FOR ORDINARY PROBLEMS OF THE CALCULUS OF VARIATIONS

(PART II)

by EDWARD JAMES MCSHANE (Göttingen).

In the first part of this paper (1) we have studied the properties of ordinary integrals by means of the associated parametric integrals, and we have there proved that under certain conditions the associated parametric integral $J[C]$ is lower semi-continuous on the class $\overline{K}_a$. In the next two sections we shall establish theorems (7.1 and 8.1) on the existence of a minimizing curve for the integral $J[C]$ on the extended class $\overline{K}_a$. But these theorems do not represent the solution of our original problem. The chief purpose of this whole study is to find conditions under which the integral $I[y]$ attains its minimum on a complete class K_a of absolutely continuous functions. Theorems 7.1 and 8.1 do not solve this problem; they are to be regarded rather as basic lemmas in its solution, which we now proceed to investigate. By Theorems 7.1 and 8.1 we are assured of the existence of a minimizing curve C for the integral $J[C]$ in the extended class $\overline{K}_a$. We seek now to find conditions on $I[y]$ which will assure us that C lies not merely in $\overline{K}_a$, but in K_a itself. This proved, it follows at once that C is a minimizing curve for $J[C] \equiv I[y]$ on the class K_a.

Sections 9 to 13 of this study will be devoted to the search for such conditions. We readily obtain theorems (10.1 and 10.2) which include most of the known existence theorems for the ordinary problem, in particular that of NAGUMO (part I, footnote 2) and those of TONELLI cited in footnote 1. We then proceed to discuss problems in which the associated integrand G is bounded, and finally obtain existence theorems whose hypotheses are satisfied, for example, by integrals

$$\int \varphi(y) \sqrt{1+y'^2}\, dx,$$

where φ is positive and continuously differentiable.

(1) This volume, pp. 183-211. We retain all the definitions and notations of Part I, and number the sections of this part 7 to 13 to avoid confusion in references to the theorems in Part I.

§ 7. - Existence Theorem for the Associated Parametric Problem.

In order to construct an existence theorem for the parametric problem associated with a problem in ordinary form, we first need a lemma which will permit us to prove the existence of a convergent minimizing sequence. This lemma follows closely the lines of the well-known existence proof of Hahn, and is given here more for the sake of completeness than because of the need of new details.

Lemma 7.1. - *If*

a) A is a bounded closed set;

b) $I[y]$ is positive quasi-regular semi-normal on A;

c) $F(x, y, y')$ is bounded below for all (x, y) on A and all y';

d) there is a constant μ such that all the plane curves

$$C: \quad z^0 = \text{const.}, \qquad z^i = z^i(t), \qquad i = 1, 2, \qquad a \leqq t \leqq b$$

of A for which $J[C] = 0$ have lengths less than μ;

then for every positive M all (²) *the curves C of A such that $J[C] \leqq M$ have lengths less than a constant N depending only on M.*

By hypothesis there exists a constant K such that

$$F(x, y, y') \geqq -K;$$

or, in terms of the associated parametric integrand,

$$G(z, z') \geqq -Kz^{0\prime}.$$

The function $F + K$ satisfies all the hypotheses of the lemma, and $\int K dx$ is bounded, say $\leqq M'$, for all curves in A. Hence the class of curves for which

$$\int (G(z, z') + Kz^{0\prime}) dt \leqq M + M'$$

contains the class of curves for which $J[C] \leqq M$, and so we need only prove the conclusion for the integral $\int (G + Kz^{0\prime}) dt$. In other words, we may consider without loss of generality that

(7.1) $\qquad I[y]$ is positive semi-definite

to begin with.

Suppose now that the lemma is false. There then exists a sequence of curves C_n such that

(7.2) $$\lim L[C_n] = \infty, \qquad J[C_n] \leqq M.$$

From equation (7.2) we see that for every n we can choose one of the curves C_m whose length is greater than n^3; and we consider not the whole curve, but an

(²) Of course we suppose that C satisfies the condition $x'(s) \geqq 0$.

arc of it of length exactly n^3. We re-name this arc C_n, and thus have (using the positive semi-definiteness of $I[y]$):

$$L[C_n]=n^3, \qquad J[C_n]\leqq M. \tag{7.3}$$

Let us denote by a the greatest difference of x-coordinates of any two points of A. We first subdivise C_n into n equal arcs of length n^2; on at least one of these arcs the functional J has value at most $\frac{M}{n}$. This arc we again subdivide into n equal parts of length n; at least one of these arcs has a projection $[\alpha_n, \beta_n]$ on the x-axis whose length $\beta_n-\alpha_n$ is at most $\frac{a}{n}$. This last arc we call C_n^1; we then have

$$L[C_n^1]=n, \qquad J[C_n^1]\leqq\frac{M}{n}, \qquad \beta_n-\alpha_n\leqq\frac{a}{n}, \tag{7.4}$$

where α_n and β_n are the projections on the x-axis of the ends of C_n^1.

The points α_n have a point of accumulation α; from the C_n^1 we select a subsequence $C_{n_1}^1$, $C_{n_2}^1$,...., such that

$$\alpha_{n_k}\rightarrow\alpha; \tag{7.5a}$$

from (7.4) it follows that

$$\beta_{n_k}\rightarrow\alpha. \tag{7.5b}$$

Each of these curves has length $L[C_{n_k}^1]=n_k\geqq k$; from the curve $C_{n_k}^1$ we select an arc of length k and name this arc $\bar{C}_k$. For these arcs $\bar{C}_k$ the relationship

$$L[\bar{C}_k]=k, \qquad J[\bar{C}_k]\leqq\frac{M}{k}, \qquad \bar{\beta}_k-\bar{\alpha}_k\leqq\frac{a}{k} \tag{7.6}$$

hold; and since the projections $\bar{\alpha}_k$, $\bar{\beta}_k$ of the end points of $\bar{C}_k$ lie between α_{n_k} and β_{n_k}, from (7.5) we obtain

$$\lim\bar{\alpha}_k=\lim\bar{\beta}_k=\alpha. \tag{7.7}$$

We now subdivide $\bar{C}_k$ into k arcs $C_{k,1}$, $C_{k,2}$,...., of equal length, so that

$$L[C_{k,r}]=1, \qquad (k=1, 2, ...;\ r=1, 2, ..., k). \tag{7.8}$$

The arcs $C_{k,1}$ have uniformly bounded lengths, hence have a limit curve C_1^*. We choose a subsequence $\{C_k^{(1)}\}$ such that $C_{k,1}^{(1)}$ tends to C_1^*. For this subsequence the arcs $C_{k,2}^{(1)}$ have a limit curve C_2^*. We choose a subsequence $\{C_k^{(2)}\}$ of the sequence $C_k^{(1)}$ such that $C_{j,2}^{(2)}$ tends to C_2^*, and continue the process. Thus for each positive integer r we obtain a subsequence $C_k^{(r)}$ such that the r-th arc of $C_k^{(r)}$ satisfies the relation

$$\lim C_{k,r}^{(r)}=C_r^*. \tag{7.9}$$

The arcs C_1^*, C_2^*,...., thus defined have the property that the end point of each is the beginning point of the next. Hence they join together to form a curve C^*. By the closure of the set A, C^* lies in A.

For the whole curve $\bar{C}_k$, and a fortiori for its subarcs, the x-coordinate lies between $\bar{\alpha}_k$ and $\bar{\beta}_k$; hence by (7.7) the curve C^*, which consists of limit curves $C_n{}^*$ of arcs of the $\bar{C}_k$, must have its x-coordinate constantly equal to α. By its definition, each curve $C_k^{(r)}$ is one of the curves $\bar{C}_h$ with $h \geqq k$, so that by (7.6)

$$J[C_{k,r}^{(r)}] \leqq J[C_k^{(r)}] = J[\bar{C}_h] \leqq \frac{M}{h} \leqq \frac{M}{k}.$$

From this, together with equation (7.9) and Theorem 6.1, we see that

$$0 \leqq J[C_r{}^*] \leqq \liminf \frac{M}{k} = 0.$$

Therefore the curve C^*, pieced together out of the $C_r{}^*$, is such that

$$J[C^*] = 0. \tag{7.10}$$

Thus by our hypothesis d) we find that C^* has finite length. Since

$$L[C^*] = L[C_1{}^*] + L[C_2{}^*] + \ldots,$$

it is possible to choose a subsequence $\{C_{n_k}^*\}$ such that

$$\lim_{k \to \infty} L[C_{n_k}^*] = 0. \tag{7.11}$$

From this subsequence we choose a further subsequence, for which we retain the same symbol, such that the initial point of C_{n_k} tends to a unique limit point P: (α, c^1, c^2) of A. By lemma 6.1 we can find an $\varepsilon > 0$ and constants v^0, v^1, v^2 such that

$$G(z, z') + v^\lambda z^{\lambda'} \geqq \varepsilon [z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}} \tag{7.12}$$

for all z' with $z^{0'} \geqq 0$ and all points z of A in a neighborhood U of P. This neighborhood U we take to be a sphere with P at center and radius 3δ, $\delta > 0$.

Since P is a limit point of the initial points of the $C_{n_k}^*$, we can find infinitely many of the $C_{n_k}^*$ whose initial points are less than δ distant from P. By (7.11), all except at most a finite number of these curves lie entirely in the sphere about P with radius 2δ. Each curve $C_{n_k}^*$ is a limit curve of the arcs C_{m,n_k} (by 7.9, recalling that the $C_{m,r}^{(r)}$ are selected from the $C_{m,r}$). Hence for each $C_{n_k}^*$ we can find an arc C_{m_k,n_k} at a distance less than $\frac{\delta}{n_k}$ and hence entirely within the sphere U. Each of these arcs C_{m_k,n_k} has length 1 (by 7.8), so we may suppose that it is given by the equations

$$C_{m_k,n_k}: \quad z = \zeta_k(s), \qquad 0 \leqq s \leqq 1.$$

By (7.12) we obtain

$$\int_0^1 G(\zeta_k(s), \zeta_k'(s))ds \geqq \int_0^1 \varepsilon \cdot 1 \cdot ds + v^\lambda[\zeta_k^\lambda(1) - \zeta_k^\lambda(0)]. \tag{7.13}$$

The last expression tends to zero; for the points $\zeta_k(0)$ and $\zeta_k(1)$ have distance less than $\frac{\delta}{n_k}$ from the initial and terminal points of $C^*_{n_k}$ respectively and the length of $C^*_{n_k}$ tends to 0 by (7.11). Hence (7.13) implies that

$$\liminf J[C_{m_k, n_k}] \geqq \varepsilon.$$

But $C^*_{m_k, n_k}$ is an arc of $\overline{C}_{m_k}$, so that by (7.6) we have

$$\lim J[C_{m_k, n_k}] = 0.$$

This contradiction establishes the lemma.

As an aid to the applicability of lemma 7.1 we have

Lemma 7.2. - *In lemma* 7.1, *hypothesis d) is satisfied whenever it is true that*

d′) for each constant value of z^0, *the inequality*

$$G(z, z') > 0 \tag{7.14}$$

holds for all non-vanishing z' *with* $z^{0\prime}=0$ *and all* z^1, z^2 *except at most those belonging to a denumerable set* $E[z^0]$.

For let C be a plane curve $z=z(s)$ with $z^0(s) \equiv z^0 = \text{const.}$ and with $L[C] > 0$, the parameter s being the length of arc on C. Omitting the set of measure 0 on which $z'(s)$ is undefined or is equal to (0, 0, 0), the remaining values of s for which $z(s)$ coincides with a specific point of $E[z^0]$ form an isolated, hence denumerable, set, and so the values of s for which $z(s)$ is in $E[z^0]$ form a set of measure 0. For all remaining s we have $G > 0$, so that $J[C] > 0$. It follows that the curves C with $z^0 = \text{const.}$ for which $J[C] = 0$ all have length 0, so that hypothesis $d)$ of lemma 7.1 is satisfied.

Remark. - If in the statement $d')$ we replace (7.14) by

$$G(z, z') = +\infty, \tag{7.15}$$

then the same argument shows that for every plane curve C: $z=z(s)$, with $z^0(s) = \text{const.}$, we have $J[C] = +\infty$ unless C has length 0.

Let us suppose that K_a is any class of absolutely continuous (3) functions $y(x)$ such that the curves $y=y(x)$ lie in A. From K_a we form the class $\overline{K}_a$ by adjoining to K_a all the rectifiable curves C: $z=z(t)$, $z^{0\prime}(t) \geqq 0$, $a \leqq t \leqq b$, which are limit curves of sequences of curves of K_a. We say that the class K_a is *complete* if every absolutely continuous function $y(x)$ which is a limit of functions of K_a is itself a function of the class K_a. But whether or not K_a is closed, the class $\overline{K}_a$ is is necessarily closed, in the sense that every rectifiable curve C which is a limit

(3) The subscript a connotes the absolute continuity of the functions, which is later assumed without specific mention. For brevity we sometimes say that « K_a is a class of curves in A », meaning exactly what is here stated.

curve of curves of $\overline{K}_a$ is itself a member of the class $\overline{K}_a$. The proof is quite simple. With this terminology we state

THEOREM 7.1. - *If* $I[y]$ *satisfies the hypotheses of lemma* 7.1, *and* K_a *is any class of curves in* A, *then in the class* $\overline{K}_a$ *there exists a minimizing curve* ([4]) *for the associated parametric functional* $J[C]$.

For, first, since F is bounded below there exists a constant c such that

$$G(z, z') + cz^{0'} \geqq 0;$$

hence $J[C]$ is bounded below. Let i be the greatest lower bound of $J[C]$ for all curves C of $\overline{K}_a$. We choose a sequence $\{C_n\}$ of curves such that

$$J[C_n] \leqq i + \frac{1}{n}. \tag{7.16}$$

By lemma 7.1 there exists a constant N such that every curve C_n has length

$$L[C_n] < N.$$

Hence we can select a subsequence of the $\{C_n\}$ (for which we retain the same notation) which converges to a limit curve C_0. Since for every C_n we have

$$z_n^{0'}(t) \geqq 0,$$

the same is true for C_0; and by the completeness of $\overline{K}_a$, C_0 is a curve of $\overline{K}_a$. Hence

$$J[C_0] \geqq i. \tag{7.17}$$

On the other hand, by Theorem 6.1 and inequality (7.14) we have

$$J[C_0] \leqq \liminf J[C_n] \leqq i. \tag{7.18}$$

Comparing (7.17) and (7.18), we have

$$J[C_0] = i,$$

and the theorem is established.

The hypothesis *d*) of lemma 7.1 has a somewhat artificial appearance. Nevertheless, if it is not fulfilled there may be no minimizing curve for $J[C]$, as is shown by the example ([5])

$$I = \int [2(zy' - yz') + (1 + y^2 + z^2)\sqrt{1 + y'^2 + z'^2}]dx,$$

where the set A is any closed set containing the cylinder $0 \leqq x \leqq 1$, $y^2 + z^2 \leqq 1$,

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([4]) We assume here and in all later existence theorems that in the class K_a there exists a set of functions y such that $I[y]$ is finite; otherwise the problem is meaningless.

([5]) At the suggestion of Dr. RELLICH, who has constructed a somewhat similar example, I have modified the integrand in such a way as simultaneously to show that the theorem of LEWY (cf. footnote ([7])) can not be extended directly to problems in space.

and we seek a minimizing curve for $J[C]$ in the class of all rectifiable curves C: $x=x(t)$, $y=y(t)$, $z=z(t)$, $x'(t)\geqq 0$, joining the points $(0, 1, 0)$ and $(1, 1, 0)$. The associated parametric integrand is

$$G=2(zy'-yz')+(1+y^2+z^2)\sqrt{x'^2+y'^2+z'^2}.$$

By use of the elementary inequality $1+(y^2+z^2)\geqq 2[y^2+z^2]^{\frac{1}{2}}$ and also of the inequality of SCHWARZ we find

$$G\geqq 2[zy'-yz'+(y^2+z^2)^{\frac{1}{2}}(x'^2+y'^2+z'^2)^{\frac{1}{2}}]\geqq 0, \tag{7.19}$$

the equality signs holding only when all the conditions

$$y^2+z^2=1, \qquad x'=0, \qquad yy'+zz'=0$$

are satisfied. This shows that the integral $I[y]$ is positive definite, and it is also easily seen to be positive quasi-regular semi-normal (6). Yet no minimizing curve exists. For inequality (7.19) shows that $i\geqq 0$; and in fact we find that $i=0$ when we consider the curves

$$y=\cos 2n\pi x, \qquad z=\sin 2n\pi x, \qquad 0\leqq x\leqq 1 \tag{7.20}$$

joining $(0, 1, 0)$ and $(1, 1, 0)$. For these curves we have

$$J[C]=I[y]=2\int_0^1(-2n\pi+[1+4n^2\pi^2]^{\frac{1}{2}})dx,$$

which tends to 0 with $\frac{1}{n}$. But for no curve C joining $(0, 1, 0)$ and $(1, 1, 0)$ is the relation $J[C]=0$ satisfied; for there must exist a set of values of t of positive measure on which $x'(t)=0$ and y' and z' are finite, and for such values of t we know by (7.19) that G is positive; hence $J[C]>0$.

Likewise, if hypothesis *c*) is not fulfilled the lower bound of $J[C]$ may be $-\infty$. To show this we need only replace the term $2(zy'-yz')$ in the above example by $a(zy'-yz')$, $a>2$, and consider the same family of curves (7.20).

However, for problems in the plane matters are essentially simpler. We can for instance show that in the plane hypothesis *d*) is a consequence of hypotheses *a*) and *b*). But as a matter of fact hypothesis *c*) also proves to be unnecessary. The proof that in the plane hypotheses *a*) and *b*) are adequate to imply the conclusion of lemma 7.1 requires a somewhat different type of proof, to which we devote the next section.

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(6) And in fact positive regular, in the usual sense of the term.

§ 8. - Existence Theorem for the Associated Parametric Problem in the Plane.

For problems in the plane lemma 7.1 can be replaced by

Lemma 8.1. - *If $I[y]$ is positive quasi-regular semi-normal on the bounded closed set A, then for every M all the rectifiable curves C of A such that*

$$J[C] \leqq M$$

have length less than a constant N, depending only on M.

To prove this we need only to show that there are constants $m_1 > 0$ and m_2 such that

$$J[C] \geqq m_1 L[C] + m_2. \tag{8.1}$$

By lemma 6.1, to every point $(\bar{x}, \bar{y})$ of A there correspond constants a, b and $k > 0$ such that the associated parametric integrand G satisfies the inequality

$$G(x, y, x', y') + ax' + by' \geqq k[x'^2 + y'^2]^{\frac{1}{2}} \tag{8.2}$$

for all (x, y) in a neighborhood of $(\bar{y}, \bar{x})$ and all x', y' such that $x' \geqq 0$. Let Q: $-c \leqq x < c$, $-c \leqq y < c$ be a square containing A. By standard devices we can subdivide Q into equal smaller squares

$$q_j: \quad x_j \leqq x < x_j + \delta, \qquad y_j \leqq y < y_j + \delta, \qquad j = 1, 2, \ldots, r, \tag{8.3}$$

such that the inequality

$$G(x, y, x', y') + a_r x' + b_r y' \geqq k_r [x'^2 + y'^2]^{\frac{1}{2}} \tag{8.4}$$

is satisfied for all (x, y) of A on q_r and all x', y' with $x' \geqq 0$; the a_r, b_r, k_r are here constants, and $k_r > 0$. Moreover, since $x' \geqq 0$, if we denote the greatest of the a_r by a and the smallest of the k_r by $k > 0$ we have from (8.4)

$$G(x, y, x', y') \geqq k[x'^2 + y'^2]^{\frac{1}{2}} - ax' - b_r y', \tag{8.5}$$

holding for the same arguments as before.

Let us suppose that C is represented in the form $x = x(s)$, $y = y(s)$, $0 \leqq s \leqq L$, with length of arc s as parameter. We subdivide the interval $[0, L]$ into the subsets $C_1, \ldots, C_r$, where C_j is the set of values of s for which $(x(s), y(s))$ belongs to the square q_j. On integrating both sides of inequality (8.5) we obtain

$$J[C] \geqq kL[C] - a[x(L) - x(0)] - \sum_{j=1}^{r} \int_{C_j} b_j y' ds. \tag{8.6}$$

The term $a[x(L) - x(0)]$ is bounded for all curves C in A; hence if we can show that each of the r terms of the sum on the right is also bounded, (8.6) has the form of inequality (8.1), and the lemma is proved.

Consider then the integral over C_j. Denoting by α and β the lower and upper bounds of C_j, we see that

$$x_j \leqq x(s) \leqq x_j + \delta \quad \text{for } \alpha \leqq s \leqq \beta, \tag{8.7}$$

because $x(s)$ is a monotonic increasing function of s. On the interval $[\alpha, \beta]$ we define $\eta(s)$ by the relations

$$\begin{cases} \eta(s) = y(s) & \text{where } y_j \leqq y(s) < y_j + \delta, \\ \eta(s) = y_j & \text{where } y(s) < y_j, \\ \eta(s) = y_j + \delta & \text{where } y_j + \delta \leqq y(s). \end{cases} \tag{8.8}$$

Since $y(s)$ is absolutely continuous, so is $\eta(s)$. On the set C_j we have $\eta(s) = y(s)$, so that for almost all points of C_j the equation

$$\eta'(s) = y'(s)$$

holds. (The points of C_j at which $\eta'(s)$ and $y'(s)$ both exist but are unequal are isolated points of C_j). The set $[\alpha, \beta] - C_j$ subdivides into the set on which $\eta = y_j$ and the set on which $\eta = y_j + \delta$; on each of these sets we have $\eta'(s) = 0$ almost everywhere. Hence

$$\int_\alpha^\beta \eta'(s)ds = \int_{C_j} y'(s)ds. \tag{8.9}$$

But the integral on the left has the value $\eta(\beta) - \eta(\alpha)$ and by the definition of η this difference is at most δ in absolute value. Hence the inequality

$$\left| b_j \int_{C_j} y' ds \right| \leqq |b_j| \delta$$

holds for all rectifiable curves C, and the proof of the lemma is complete.

This leads us to

THEOREM [7] 8.1. - *If $I[y]$ is an integral in the plane which satisfies the hypotheses of lemma* 8.1, *and K_a is any class of curves in A, then in the class $\overline{K}_a$ there exists a minimizing curve for the associated parametric problem $J[C]$.*

The proof of this is the same as that of Theorem 7.1, the reference to lemma 7.1 being replaced by a reference to lemma 8.1.

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(7) H. LEWY has proved a theorem (Math. Annalen, 98, pp. 107-124), which in our terminology may be thus stated: If $I[y]$ is a regular integral in the plane, and A is a convex region, and K_a is the class of all curves with continuous derivatives joining two points of A, then there exists a curve C of $\overline{K}_a$ such that $J[C] \leqq I[y]$ for all y in K_a, and the derivatives with respect to arc length of the functions defining C are continuous. The methods used by LEWY seem very appropriate for regular problems, but apparently do not extend to the class of problems here considered.

§ 9. - First Existence Theorem for the Ordinary Problem.

From the preceding theorems we can easily obtain an existence theorem for the ordinary problem by adding to the hypotheses of lemma 7.1 the following assumptions: $I[y]$ is positive quasi-regular, and its associated parametric integrand $G(z, z')$ has the value $+\infty$ for all z on A and all unit vectors z' with $z^{0'}=0$. The proof is quite simple. But these added hypotheses in fact imply the hypotheses *b*), *c*) and *d*) of lemma 7.1, as we shall establish in two lemmas.

Lemma 9.1. - *If at the point z the function*

$$\mathcal{E}(z, z'_u, \bar{z}'_u) \geqq 0 \tag{9.1}$$

for all unit vectors z'_u *and* $\bar{z}'_u$, *and if for all unit vectors* z'_u *with* $z_u^{0'}=0$ *one of the equations*

$$G(z, z'_u) = +\infty \tag{9.2}$$

or

$$G_0(z, z'_u) = -\infty \tag{9.3}$$

holds,

then there exists a unit vector Z'_u *such that*

$$\mathcal{E}(z, Z'_u, z'_u) > 0 \tag{9.4}$$

for all unit vectors $z'_u \neq Z'_u$.

We recall that in § 2 we have seen that $G_{00}(z, z'_u) \geqq 0$ for all z'_u, so that for unit vectors z'_u with $z_u^{0'}=0$ we must have $G_0(z, z'_u)$ either finite or $-\infty$, and likewise $G(z, z'_u)$ either finite or $+\infty$.

By (9.1)

$$\begin{aligned}\Gamma(z, z') \equiv G(z, z') - G(z, 1, 0, 0) - (z^{0'} - 1) G_0(z, 1, 0, 0) + \\ + z^{0'} - z^{\alpha'} G_\alpha(z, 1, 0, 0) = \mathcal{E}(z, 1, 0, 0, z') + z^{0'} \geqq z^{0'} \geqq 0\,;\end{aligned} \tag{9.5}$$

the index α here takes the values 1, 2. Since Γ and G differ only by a function linear in z', they have the same $\mathcal{E}$-function. Also equations (9.2) and (9.3) remain valid for Γ as well as for G. Hence we need only to prove (9.4) for Γ.

Let us consider the surface defined in polar coordinates by the equation

$$r = r(z'_u) = \Gamma(z, z'_u), \qquad z_u^{0'} \geqq 0. \tag{9.6}$$

Since $r \geqq 0$ by (9.5), there exists a unit vector Z'_u such that $r(Z'_u)$ is a minimum. For this vector we must have $Z_u^{0'} > 0$; for if $Z_u^{0'}=0$, then either $r=\infty$ by (9.2) or else by (9.3) there exist neighboring unit vectors z'_u for which $r(z'_u) < r(Z'_u)$. Hence the point with coordinates Z'_u, $r(Z'_u)$ lies on the differentiable part of the surface (9.6). Since $r(Z'_u)$ is a minimum the normal to the surface (9.6) at Z'_u has the same direction as the radius vector, that is, the same direction as Z'_u itself; hence

$$\Gamma_j(z, Z_u) = a Z_u^{j'}, \qquad j = 0, 1, 2, \ldots., \tag{9.7}$$

where a is a constant. Moreover, $a > 0$, since by (9.5) and (9.7)

$$0 < \Gamma(z, Z'_u) = Z_u^{\lambda'} \Gamma_\lambda(z, Z'_u) = a Z_u^{\lambda'} Z_u^{\lambda'} = a. \tag{9.8}$$

If we now write the $\mathcal{E}$-function for Γ, we obtain by use of (9.7)

$$\mathcal{E}(z, Z'_u, z'_u) = \Gamma(z, z'_u) - z_u^{\lambda'} \Gamma_\lambda(z, Z'_u) = \Gamma(z, z'_u) + a z_u^{\lambda'} Z_u^{\lambda'}. \tag{9.9}$$

Let ϑ be the angle between the vectors z'_u and Γ_j, or what is the same thing the angle between z'_u and Z'_u; by (9.8), equation (9.9) can then be written

$$\mathcal{E}(z, Z'_u, z'_u) = \Gamma(z, z'_u) - a \cos \vartheta = \Gamma(z, z'_u) - \Gamma(z, Z'_u) \cos \vartheta.$$

Since $\Gamma(z, Z'_u)$ is the least value of Γ for all unit vectors z'_u with $z_u^{0'} \geqq 0$, this last inequality shows that if Z'_u and z'_u are distinct vectors inequality (9.4) holds. The lemma is thus established.

Lemma 9.2. - *If $I[y]$ is positive quasi-regular on the bounded closed set A, and at every point z of A the inequality*

$$G(z, z'_u) > 0 \tag{9.10}$$

holds for all unit vectors z'_u with $z_u^0 = 0$, then $F(x, y, y')$ is bounded below for all (x, y) on A and all y'.

Since for every constant C the parametric integrand associated with $F(x, y, y') + C$ has the form

$$G(z, z') + C z^{0'}, \tag{9.11}$$

we need only to show that there exists a C such that the expression (9.11) is non-negative.

By lemma 2.1, $G(z, z')$ is lower semi-continuous; hence from (9.10) we see that for each argument $(\bar{z}, \bar{z}')$ in the bounded closed set

$$z \text{ in } A; \qquad |z^{1'}| \leqq 1, \qquad |z^{2'}| \leqq 1, \qquad z^{0'} = 0 \tag{9.12}$$

we can find a neighborhood

$$|z^j - \bar{z}^j| < \delta, \qquad |z^{j'} - \bar{z}^{j'}| < \delta \tag{9.13}$$

on which

$$G(z, z') > 0. \tag{9.14}$$

A finite number of these neighborhoods cover the set (9.12); letting δ be the smallest of the values of δ in (9.13) corresponding to these neighborhoods, we find that (9.14) holds for all arguments z, z' such that

$$z \text{ is in } A; \qquad 0 \leqq z^{0'} < \delta, \qquad |z^{i'}| \leqq 1 \qquad (i = 1, 2). \tag{9.15}$$

On the bounded closed set

$$[z \text{ in } A; \qquad \delta \leqq z^{0'} \leqq 1, \qquad |z^{1'}| \leqq 1, \qquad |z^{2'}| \leqq 1] \tag{9.16}$$

the function G has a lower bound; that is, there is a positive number M such that $G > -M$ on the set (9.16). Hence on this set

$$G(z, z') + Mz^{0'}/\delta \geqq 0. \tag{9.17}$$

A fortiori, from (9.14) we see that (9.17) continues to hold on the set (9.15). Hence (9.17) holds for all z in A and all unit vectors z', and by homogeneity for all vectors z', and the lemma is established.

We are now in a position to prove

THEOREM 9.1. - *If $I[y]$ is positive quasi-regular on a bounded closed set A, and for every point (x, y) of A and every $y' \neq (0, 0)$ the relationship*

$$\lim_{x' \to \infty} x' F\left(x, y, \frac{y'}{x'}\right) = \infty \tag{9.18}$$

holds,

then in every complete class K_a of absolutely continuous functions in A there exists (8) *a function $y_0(x) = (y_0^1(x), y_0^2(x))$ for which $I[y]$ assumes its least value on K_a.*

Equation (9.18), expressed in terms of the associated parametric integrand G, states that

$$G(z, z'_u) = \infty \tag{9.19}$$

for all z in A and all unit vectors z'_u with $z_u^{0'} = 0$. In lemma 7.1, hypothesis *a*) is satisfied; hypothesis *b*) follows from lemma 9.1; hypothesis *c*) from lemma 9.2, and hypothesis *d*) is satisfied by lemma 7.2. Hence by Theorem 7.1 there exists a rectifiable minimizing curve C_0 for $J[C]$ in the class $\overline{K_a}$. By (9.19), at almost every point $z_0(s)$ on C_0 at which $z_0^{0'} = 0$ the equation

$$G(z_0(s), z_0'(s)) = \infty$$

holds. Hence from the finiteness of $J[C_0]$ we see that the set of such values of s has measure 0. By lemma 2.4, this implies that C_0 can be represented in the form $y = y_0(x)$, $a \leqq x \leqq b$, with absolutely continuous functions $y_0(x)$. Since the class K_a is complete, the functions $y_0(x)$ belong to K_a. But by lemma 2.7, for every set of functions $y(x)$ in K_a defining a curve C the integrals $J[C]$ and $I[y]$ are the same; hence

$$I[y_0] = J[C_0] \leqq J[C] = I[y]$$

for all functions y of K_a, and the functions $y_0(x)$ therefore minimize $I[y]$ in the class K_a.

From Theorem 9.1 there readily follows

Corollary (9) *to Theorem* 9.1. - *The conclusion of Theorem* 9.1 *is valid*

(8) Here and in all succeeding theorems we assume without mention that the class K_a actually contains a curve for which the integral is defined and finite.

(9) NAGUMO, loc. cit. (3), part I.

if the class K of functions has the properties there described, $I[y]$ is positive quasi-regular, and there exists a continuous monotonic increasing function $\Phi(p)$ such that

$$F(x, y, y') \geqq \Phi([y^{\alpha'} y^{\alpha'}]^{\frac{1}{2}})$$

for all (x, y) in A and all y', and

$$\lim_{p \to \infty} \Phi(p)/p = +\infty.$$

For then, denoting $[y^{\alpha'} y^{\alpha'}]^{\frac{1}{2}}$ by $|y'|$, we have

$$\lim_{x' \to 0} x' F\left(x, y, \frac{y'}{x'}\right) = \lim_{x' \to 0} x' \cdot \frac{|y'|}{x'} \left[\Phi\left(\frac{|y'|}{x'}\right) \cdot \frac{x'}{|y'|}\right] = +\infty.$$

In particular, if we take $\Phi(p) = p^{1+a}$, $a > 0$, we obtain for plane problems a theorem of TONELLI (loc. cit. ([1])) and for space problems a theorem of GRAVES (loc. cit. ([2]), part I).

§ 10. - Second Existence Theorem for the Ordinary Problem.

In this section we shall relax the hypotheses of Theorem 9.1 by allowing an exceptional set E on which (9.21) is not fulfilled.

THEOREM 10.1. - *If*

a) A is a bounded closed set of points;

b) K_a is a complete class of absolutely continuous functions lying in A;

c) $I[y]$ is positive quasi-regular semi-normal on A;

d) the equation

(10.1) $$\lim_{x' \to 0} x' F\left(x, y, \frac{y'}{x'}\right) = \infty$$

holds for all $y' \neq (0, 0)$ and for all (x, y) in $A - E$, the exceptional set E consisting of the points lying on a finite or denumerably infinite set of absolutely continuous curves

$$C_n^*: \quad y = y_n^*(x), \qquad a_n \leqq x \leqq b_n;$$

e) for all (x, y) on a neighborhood U of the set E and for all y' the integrand $F(x, y, y')$ is bounded below;

then there exists a function $y = y_0(x) = (y_0^1(x), y_0^2(x))$ in the class K_a for which $I[y]$ assumes its least value.

On the closed set $A - U$ the hypotheses of lemma 9.2 are satisfied, hence for all (x, y) on this set and all y' the integrand $F(x, y, y')$ is bounded below. Therefore by hypothesis $e)$ F is bounded below for all (x, y) in A and all y'. Hence hypotheses $a)$, $b)$, $c)$ of lemma 7.1 are satisfied. Hypothesis $d)$ is fulfilled as is shown by lemma 7.2.

Theorem 7.1 now assures us that there exists a curve $C_0: z=z_0(s)$, $0 \leqq s \leqq L$ of the class $\bar{K}_a$ such that $J[C_0]$ is a minimum. It remains only to prove that C_0 belongs to K_a.

Since this can be quite readily proved in case the functions $y_n{}^*(x)$ defining the exceptional set E each satisfy a Lipschitz condition, we first prove this special case, and then take up independently the general result. Denote then by M_n the set of values of s such that $z_0(s)$ lies on the curve $C_n{}^*$. From this set M_n we reject the subset on measure 0 on which $z_0(s)$ fails to exist or is equal to (0, 0, 0), and from the remaining set we reject the isolated points. This gives us a subset $M_{0,n}$ of M_n. Select any point s_0 of $M_{0,n}$, and choose a sequence s_n of points of $M_{0,n}$ approaching s_0. Since the points $z_0(s_0)$ and $z_0(s_n)$ lie on $C_n{}^*$, we have

$$|z_0^i(s_n)-z_0^i(s_0)| \leqq K_n |z_0^0(s_n)-z_0^0(s_0)|,$$

where K_n is the Lipschitz constant of $C_n{}^*$. Hence

$$|z_0^{i'}(s_0)| \leqq K_n |z_0^{0'}(s_0)|,$$

and since the three derivatives are not all 0 we must have $z_0^{0'}(s_0)>0$. Hence only on a subset of M_n of measure 0 can we have $z_0^{0'}(s)=0$. Denoting $M_1+M_2+\ldots$ by M, for only a subset of M of measure 0 can $z_0^{0'}(s)=0$.

For the values of s which do not belong to M, the point $z_0(s)$ lies in $A-E$, and by equation (10.1) for at most a subset of measure 0 can we have $z_0^{0'}(s)=0$. Hence $z_0^{0'}(s)>0$ almost everywhere, and by lemma 2.4 the curve C_0 can be represented in the form

$$y=y_0(x), \qquad a \leqq x \leqq b$$

with absolutely continuous functions $y_0(x)$. By the same argument as that which concluded the proof of Theorem 9.1, C_0 is the minimizing curve sought.

For the general case, in which the curves

$$C_n{}^*: \quad y=y_n{}^*(x), \qquad a_n \leqq x \leqq b_n$$

are absolutely continuous, we argue as follows: The curve C_0 can be represented in the form

$$C_0: \quad y=y_0(x), \qquad a \leqq x \leqq b.$$

For if to one x_0 there correspond two distinct points (x_0, y_1^1, y_1^2) and (x_0, y_2^1, y_2^2) of C_0, on the whole arc of length >0 joining these points we have $x_0(s)=x_0=\text{const.}$, and by the remark after lemma 7.2 this would imply $J[C_0]=\infty$.

Moreover, the functions $y_0(x)$ are continuous. For if as $x \to x_0$ there exist two distinct limit points (x_0, y_1^1, y_1^2) and (x_0, y_2^1, y_2^2) of the points $z_0(s)$, these points both belong to the curve C_0, and as before $J[C_0]=\infty$. It remains to show that the functions $y_0(x)$ are absolutely continuous.

First, the $y_0^i(x)$ are of limited total variation, since they define a curve C_0 of finite length. Second, they satisfy Lusin's condition N; that is to every set of values of x of measure 0 corresponds a set of values of y^i of measure 0. For let E^* be any set of values of x of measure 0. We subdivide E^* into the subset E_0, in which the point $(x, y_0(x))$ lies on $A-E$, and the sets E_n (possibly overlapping) for which $(x, y_0(x))$ lies on the curves C_n^*. If the set of values of s for which $x_0(s) \equiv z_0^0(s)$ lies in E_0 has positive measure, it is easy to show that $x_0'(s)=0$ for almost all such s, and hence by (10.1) $J[C_0]=\infty$. Since $J[C_0]<\infty$, the measure of this set of values of s is 0, and a fortiori the measure of [10] $y_0^1(E_0)=0$. For x on E_n, the functions $y_0^1(x)$ and $y_n^{*1}(x)$ coincide, hence

$$m[y_0^1(E_n)]=m[y_n^{*1}(E_n)]=0,$$

for by hypothesis y_n^* is absolutely continuous and hence satisfies condition N. Hence summing for all n,

$$0 \leqq m[y_0^1(E^*)] \leqq \sum_{n=0}^{\infty} m[y_0^1(E_n)]=0.$$

This proves that $y_0^1(x)$ satisfies condition N. Together with the facts that $y_0^1(x)$ is continuous and of limited total variation, this implies [11] that $y_0^1(x)$ is absolutely continuous. Likewise $y_0^2(x)$ is absolutely continuous, and the theorem is established.

The corresponding theorem for curves in the plane requires fewer hypotheses:

THEOREM 10.2. - *If*

a) A *is a bounded closed set of points in the plane;*

b) K_a *is a complete class of absolutely continuous functions lying in* A;

c) $I[y]=\int F(x, y, y')dx$ *is positive quasi-regular semi-normal on* A;

d) the equation

$$\lim_{x' \to 0} x'F\left(x, y, \frac{y'}{x'}\right)=\infty \tag{10.1}$$

holds for all $y' \neq 0$ *and all* (x, y) *in* $A-E$, *the exceptional set* E *consisting of the points lying on a finite or denumerably infinite set of absolutely continuous curves*

$$C_n^*: \quad y=y_n^*(x), \qquad a_n \leqq x \leqq b_n;$$

then there exists a function $y_0(x)$ *in the class* K_a *for which* $I[y]$ *assumes its least value.*

The hypotheses of lemma 8.1 are satisfied, hence by Theorem 8.1 there exists a curve C_0: $z=z_0(s)$, $0 \leqq s \leqq L$ of $\overline{K}_a$ such that $J[C_0]$ is a minimum. From here we can follow the proof of Theorem 10.1, making no changes except to suppress all references to $y_0^2(x)$.

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(10) By $y_0^1(M)$ we mean the set of values of $y^1(x)$ corresponding to the values of x on the set M.

(11) S. BANACH: *Sur les lignes rectifiable*, etc Fund. Math.. 7 (1925), p. 229.

Corollary [12]. - *In Theorems* 10.1 *and* 10.2, *hypothesis d) can be replaced by*

d') there exist positive constants a, M, b such that

$$F(x, y, y') \geqq b(y^{\alpha'} y^{\alpha'})^{(1+a)/2}$$

for all (x, y) *in* $A-E$ *and all* y' *such that* $y^{\alpha'} y^{\alpha'} \geqq M$; *the set E having the same properties as in hypothesis d).*

The proof is the same as that of the corollary to Theorem 9.1.

§ 11. - Extension to Unbounded Fields.

In all our existence theorems we have made the assumption that the field A is bounded. We shall now establish a lemma which will enable us to replace that hypothesis by certain hypotheses on the integrand F. For the statement of these conditions it is convenient to define $\|y\| = \sqrt{(y^1)^2 + (y^2)^2}$. Then

Lemma [13] **11.1.** - *If*

a) the set A lies between the planes $x = -c$ *and* $x = c$;

b) the associated parametric integrand $G(z, z')$ *is non-negative* [14] *for all z on A and all vectors z' with* $z^{0'} \geqq 0$;

c) there exist positive constants h, a, b such that for all y for which $\|y\| \geqq h$ *and for all unit vectors* (x'_u, y'_u) *with* $x'_u \geqq 0$ *the relation*

$$G(x, y, x'_u, y'_u) > a/\|y\| \tag{11.1}$$

holds whenever

$$x'_u \leqq b/\|y\|; \tag{11.2}$$

d) A^* *is a bounded subset of A;*

then for every number M the class of all curves C in A having at least one point in A^* *and satisfying the inequality*

$$J[C] < M \tag{11.3}$$

lies in a bounded portion of (x, y^1, y^2) *space.*

In case it is desired to have the hypotheses stated in terms of the integrand F instead of in terms of the associated integrand G, we have only to notice that for every unit vector $(x'_u, y^{1'}_u, y^{2'}_u)$ with $x'_u > 0$ the relationship

$$1/x'_u = [1 + (y^{\alpha'}_u y^{\alpha'}_u)/x'^2_u)]^{\frac{1}{2}}$$

holds; hypotheses $b)$ and $c)$ then transform respectively into

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[12] Tonelli: *Fondamenti di Calcolo delle Variazioni.* Vol. II, pp. 287-307.

[13] This lemma includes the theorem of Tonelli: *Op. cit.* [12], Vol. II, pp. 307-310.

[14] It is in fact sufficient to assume that there exists a constant d such that $G + dx'$ is non-negative.

b′) the function $F(x, y, y')$ *is non-negative* (15) *for all* (x, y) *on* A *and all* y';

c′) there exist positive constants h, a, b *such that for all* y *for which* $\|y\| \geqq h$ *the relation*

$$F(x, y, y') \geqq a[1+y^{\alpha'}y^{\alpha'}]^{\frac{1}{2}}/\|y\| \tag{11.3}$$

holds wherever

$$[1+y^{\alpha'}y^{\alpha'}]^{\frac{1}{2}} \geqq \|y\|/b. \tag{11.4}$$

We now take up the proof of the lemma. Let H be a number greater than h, whose value we shall later specify. We may assume without loss of generality that the constant h is large enough so that the cylinder $\|y\| \leqq h$ includes the bounded set A^* of hypothesis *d*). Suppose then that $\bar{C}$: $x=x(t)$, $y=y(t)$, is a curve having a point in common with A^* and a point outside of the cylinder $\|y\|=H$. We can choose an arc C of $\bar{C}$ with initial point on the cylinder $\|y\|=h$, with all its other points outside of that cylinder, and with length exactly $H-h$. This arc we represent in the form

$$C: \quad x=x(s), \qquad y=y(s), \qquad h \leqq s \leqq H, \tag{11.6}$$

where s is the length of arc plus h. For $\|y\|$ we have the obvious inequality

$$\|y(s)\| \leqq s, \tag{11.7}$$

since $\|y(h)\|=h$ and $\frac{d}{ds}\|y(s)\| \leqq 1$; and also

$$J[\bar{C}] \geqq J[C] \tag{11.8}$$

by hypothesis *b*).

Let us suppose, as we may without loss of generality, that in hypothesis *d*) the inequality $a \leqq b$ holds. Then for almost every point s of the interval $[h, H]$ either $x'(s) > a/\|y\|$ or $G(x(s), y(s), x'(s), y'(s)) > a/\|y\|$; and thus, by inequality (11.7),

$$\int_h^H [x'(s)G+(x, y, x', y')]ds > \int_h^H \frac{ads}{\|y\|} \geqq \int_h^H \frac{ads}{s} = a \log (H/h).$$

Hence

$$J[C] > a \log (H/h) - [x(H)-x(h)]. \tag{11.9}$$

The last term in (11.9) has value at most equal to $2c$, since both ends of C lie in A. Hence we can choose H large enough so that the right member of inequality (11.9) has a value greater than M. A fortiori, by (11.8),

$$J[\bar{C}] \geqq J[C] > M.$$

(15) It is sufficient to assume that F is bounded below, since a constant a added to F changes $I[y]$ by at most $2ac$ for all $y=y(x)$ in A.

Thus every curve C with a point in A^* and with $J[C] \leqq M$ must lie completely interior to the cylinder $\|y\| \leqq H$, and the lemma is proved.

It is obvious that hypothesis c) could be replaced by

c_1) there exists a constant h and a function $\varphi(x)$, continuous, positive, monotonic decreasing, and not summable from h to ∞, such that for all y for which $\|y\| \geqq h$ and for all unit vectors x'_u, y'_u with $x'_u \geqq 0$ the inequality

$$G(x, y, x'_u, y'_u) > \varphi(\|y\|)$$

holds whenever

$$x'_u \leqq \varphi(\|y\|).$$

For then

$$\int_h^H [G(x, y, x', y') + x'(s)]ds > \int_h^H \varphi(\|y\|)ds \geqq \int_h^H \varphi(s)ds,$$

and we need only choose H large enough so that the integral on the right is greater than $M + 2c$ to insure

$$J[\bar{C}] > M.$$

For example, we can take $\varphi(x) = \frac{a}{x \log x}$ or

$$\varphi(x) = \frac{a}{x \log x \log \log x}, \qquad a > 0.$$

Examples of integrands $F(x, y, y')$ satisfying the hypotheses of lemma 11.1 are

$$\sqrt{\frac{1 + y'^2 + z'^2}{1 + z^2}}, \qquad \left|\frac{\sqrt{1 + y'^2}}{y}\right|^k, \qquad (k \geqq 1), \qquad \frac{|y'|^k}{|y|^l}, \qquad (k \geqq l \geqq 1).$$

From lemma 11.1 we have at once

THEOREM 11.1. - *The conclusions of Theorems* 7.1, 8.1, 9.1, 10.1 *and* 10.2 *remain valid, if in the hypotheses of those theorems we remove the assumption that A is bounded and assume instead that the hypotheses of lemma* 11.1 *are satisfied.*

For let C_1 be any curve of A for which $J[C_1]$ is finite. Since we seek a minimum for J, we need consider only curves C for which $J[C] < J[C_1] + 1$. By lemma 11.1, all of these lie in a bounded closed portion of A. If we restrict our attention to this portion of A, the hypotheses of the theorems are satisfied.

§ 12. - Third Existence Theorem for Integrands in Ordinary Form.

The preceding existence theorems have all contained the assumption that $G(z, z'_u) = \infty$ if $z_u^{0'} = 0$, at least at almost points of A. We now turn our attention to integrands in which this condition may fail to hold. Naturally we are obliged to strengthen our other hypotheses; in fact, we find it necessary to add stronger conditions on the integral, on the field and on the curves considered. In order to state these new conditions, we define, with TONELLI, an *arc of indifference* of a curve. We say that an arc $\overline{C}$ of a curve C, lying in a set A and belonging to a class K_a of absolutely continuous curves in A is an *arc of indifference* with respect to K_a and A provided that every absolutely continuous curve C' of A lying in a sufficiently small neighborhood ([16]) of C and coinciding with C except along the arc $\overline{C}$ is also a curve of K_a. When (as is the case in all the theorems following) only one set A and only one family K_a enter the discussion, we abbreviate the expression and say simply that C is an arc of indifference. An analogous definition holds for classes $\overline{K}_a$ of rectifiable curves with $z^{0\prime} \geqq 0$; we need only to replace the symbol K_a by $\overline{K}_a$ and the words « absolutely continuous curve » by « rectifiable curve with $z^{0\prime} \geqq 0$ » in the definition above.

With this terminology, we state

Lemma 12.1. - *If*

a) the integral $\int F(x, y, y')dx$ *is positive quasi-regular on a set* A;

b) there exist positive constants M_1, M_2, δ *such that* ([17])

$$|G_{z^0}(\bar{z}^0, z^1, z^2, z'_u)| \leqq M_1 G(z^0, z^1, z^2, z'_u) + M_2 \tag{12.1}$$

for all z *in* A, *all* z'_u *with* $z_u^{0\prime} \geqq 0$, *and all* $\bar{z}^0$ *such that* $|\bar{z}^0 - z^0| < \delta$;

c) for all z *in* A *and for all* z'_u *with* $z_u^{0\prime} = 0$, *the equation*

$$G_0(z, z'_u) = -\infty \tag{12.2}$$

holds;

d) $\overline{K}_a$ *is a class of rectifiable curves* $z = z(t)$ *with* $z^{0\prime}(t) \geqq 0$, *lying in* A;

e) for the curve C: $z = z(s)$, $0 \leqq s \leqq L$, *the associated parametric integral* $J[C]$ *assumes its least value on* $\overline{K}_a$;

f) the arc C_0: $z = z(s)$, $a \leqq s \leqq b$, *of* C *is interior to* A *and is an arc of indifference with respect to* $\overline{K}_a$ *and to* A, *and* $z^0(a) < z^0(b)$;

then $z^{0\prime}(s) > 0$ *for almost all values of* s *in the interval* $[a, b]$.

In proving this lemma we find it convenient to return to the (x, y) notation. Suppose then that the theorem is false, and that the set E_1 on which $x'(s) = 0$ has positive measure. Since $x(b) > x(a)$, there is a set of positive measure on which $x'(s) > 0$; hence we can find a constant $k > 0$ and a set E_2 of positive measure such that

$$x'(s) \geqq k \quad \text{for } s \text{ on } E_2. \tag{12.3}$$

([16]) That is, having a sufficiently small distance from C, in the sense of § 1.

([17]) G is as usual the parametric integrand associated with F. We assume that the derivative G_{z^0} exists.

Denoting by $\chi_i(s)$ the function which has the value 1 for s on E_i and the value 0 for s not on E_i $(i=1,2)$, we define

$$(12.4)\qquad \varphi(s)=\varphi(s;\,E_i,\,E_2)=\int_a^s [m(E_2)\cdot\chi_1(s)-m(E_1)\cdot\chi_2(s)]ds.$$

We have at once

$$(12.5)\qquad \varphi(a)=\varphi(b)=0,$$

and remembering that

$$\frac{d}{ds}\int_a^s \chi_i(s)ds$$

has the value 1 for almost all points of E_i and the value 0 for almost all points of the complement of E_i, we find

$$(12.6)\qquad \begin{cases} \varphi'(s)=-m(E_1) & \text{for almost all points of } E_2,\\ \varphi'(s)=m(E_2) & \text{for almost all points of } E_1,\\ \varphi'(s)=0 & \text{for almost all points of } C[E_1+E_2], \end{cases}$$

where $C[E_1+E_2]$ is the complement in $[a,b]$ of the set E_1+E_2.

We now define C_α by the equations

$$(12.7)\qquad C_\alpha:\quad x=x_\alpha(s)=x(s)+\alpha\varphi(s),\qquad y=y(s),\qquad a\leqq s\leqq b,\qquad \alpha\geqq 0.$$

Since C is interior to A, so is C_α for all sufficiently small values of α. Moreover, by (12.6), $x'_\alpha(s)=x'(s)+\alpha\varphi'(s)\geqq x'(s)\geqq 0$ for almost all s not belonging to E_2, while for almost all points s of E_2 we have

$$x'_\alpha(s)=x'(s)+\alpha\varphi'(s)\geqq k-\alpha m(E_1)>0$$

provided that $\alpha<k/m(E_1)$. Hence for almost all s we have $x'_\alpha(s)\geqq 0$; and since $x_\alpha(s)$ is absolutely continuous, this implies that it is monotonic and that

$$(12.8)\qquad x'_\alpha(s)\geqq 0 \quad\text{whenever it exists.}$$

Thus for all sufficiently small values of α the curve formed by substituting the arc C_α for the arc C_0 is a curve of $\overline{K}_a$, and so the inequality

$$(12.9)\qquad J[C_\alpha]\geqq J[C_0]$$

holds for all sufficiently small α.

On the other hand, let us write the identity

$$(12.10)\qquad G(x_\alpha,y,x'_\alpha,y')-G(x,y,x',y')=[G(x_\alpha,y,x'_\alpha,y')-G(x_\alpha,y,x',y')]+ \\ +[G(x_\alpha,y,x',y')-G(x,y,x',y')].$$

For every value of s for which x' and y' are defined and not equal to $(0,0,0)$

the last bracket can be transformed by the theorem of the mean (since by hypothesis *b*) the derivative as to x exists) into

$$G_x(\bar{x}, y, x', y') \cdot \alpha\varphi(s);$$

$\bar{x}$ between x and x_α. By the same hypothesis this shows that

$$(12.11) \quad |G(x_\alpha, y, x', y') - G(x, y, x', y')| \leqq \alpha \max \varphi(s)[M_1 G(x, y, x', y') + M_2]$$

for all sufficiently small α.

Again for almost every value of s not in the set E_1 we have

$$(12.12) \qquad |G(x_\alpha, y, x'_\alpha, y') - G(x_\alpha, y, x', y')| < P\alpha,$$

where P is a suitable chosen constant. For at all points of E_2 we have $x'(s) \geqq k$, so that $x'_\alpha(s) \geqq \frac{k}{2}$ if α is small enough; and for unit vectors (x', y') with $x' \geqq \frac{k}{2}$ the derivative $G_0 \equiv G_{x'}$ is bounded, say $\leqq P_1$ in absolute value, and the expression on the left of (12.12) can be written in the form

$$|G_0(x_\alpha, y, \bar{x}', y') \cdot \alpha\varphi'(s)| \leqq P_1 \cdot \alpha m(E_1).$$

And for almost all points of $C[E_1 + E_2]$ we have $x'_\alpha = x'$, so that the expression on the left in (12.12) is zero.

We still have to consider the points of E_1.

For this purpose we first notice that if $\overline{U}$ is a bounded closed neighborhood of the points of C lying in A, then for every N there exists a $\gamma > 0$ such that

$$(12.13) \qquad G_0(x, y, x', y') < -N$$

for every (x, y) in $\overline{U}$, every x' less than γ, and every y' such that $y^{\alpha'} y^{\alpha'} = 1$. For by lemma 2.2 and equation (12.2), to each point $(\bar{x}, \bar{y}, 0, y')$ with $(\bar{x}, \bar{y})$ in A and $y^{\alpha'} y^{\alpha'} = 1$ there corresponds a neighborhood

$$|x - \bar{x}| < \delta, \qquad |y^i - \bar{y}^i| < \delta, \qquad 0 \leqq x' < \delta, \qquad |y^{i'} - \bar{y}^{i'}| < \delta$$

on which (12.13) holds. A finite number of these neighborhoods cover the set $[(x, y)$ in $\overline{U}$, $y^{\alpha'} y^{\alpha'} = 1$, $x' = 0]$, and we need only to choose for γ the smallest of the values of δ for these neighborhoods.

Suppose now that we have chosen a positive N, and that α has been restricted to be small enough so that C_α lies in $\overline{U}$. By (12.4) and (12.5), for almost all points of E_1 we have $x'(s) = 0$, whence $y^{\alpha'} y^{\alpha'} = 1$, and we also have $x_\alpha'(s) = \alpha m(E_2)$, so that for these points

$$(12.14) \quad G(x_\alpha, y, x'_\alpha, y') - G(x_\alpha, y, x', y') = \int_0^{\alpha m(E_2)} G_0(x_\alpha, y, \xi, y') d\xi < -N\alpha m(E_2).$$

Combining (12.10), (12.14), (12.12) and (12.11), we obtain

$$J[C_a]-J[C_0]<\int_b^{} + \int_{E_1}^{}\int_{CE_1}^{}[G(x_a,y,x'_a,y')-G(x_a,y,x',y')]ds+$$
$$+\int_a^{}[G(x_a,y,x',y')-G(x,y,x',y')]ds\leqq -Nam(E_2)m(E_1)+Pa[b-a]+$$
$$+a\max\varphi\{M_1J[C_0]+M_2[b-a]\},$$

valid for all sufficiently small a. But since N can be chosen arbitrarily large, this is inconsistent with (12.9), and the lemma is proved.

Let us define a *cylindrical set* in the following way:

(12.15) *The set A is a cylindrical set if it consists of all points (x, y^1, y^2) such that x belongs to an interval $k_1\leqq x\leqq k_2$ and (y^1, y^2) belongs to a set B in the (y^1, y^2)-plane.*

With this terminology we state that *if the field A is a cylindrical set, the hypothesis that the arc C_0 is interior to A can be omitted.* For in the family of curves (12.5) the functions $y(s)$ are independent of a, and for all sufficiently small values of a we know by (12.8) that $x(a)=x_a(a)\leqq x_a(s)\leqq x_a(b)=x(b)$, so that C_a lies in A.

This leads to

THEOREM 12.1. - *If*

a) the set A is a bounded closed cylindrical set $[c\leqq x\leqq d, (y^1, y^2)$ in $B]$;

b) the integral $I[y]$ is positive quasi-regular on A;

c) the integrand $F(x, y, y')$ is bounded below for all (x, y) on A and all y';

d) hypothesis b) of Lemma 12.1 is satisfied;

e) hypothesis c) of lemma 12.1 is satisfied;

f) K_a is the class of all absolutely continuous curves in A joining two points (18) *P_1 and P_2 of A;*

then there exists a curve of the class K_a for which $I[y]$ assumes its least value.

The extended class $\bar{K}_a$ here consists of all rectifiable curves $x=x(s)$, $y=y(s)$ in A joining P_1 and P_2 and having $x'(s)\geqq 0$. We first show that there exists a curve C of $\bar{K}_a$ for which $J[C]$ assumes its least value.

By lemma 9.1, $I[y]$ is semi-normal on A. So referring to lemma 7.1, we find that hypotheses a), b) and c) are satisfied. To prove that hypothesis d) is also satisfied, we notice that since $F(x, y, y')$ is bounded below, there exists an m such that

(12.16) $$G(z, z')+mz^{0'}\geqq 0$$

(18) As an obvious generalization, we could define K_a to be the class of all absolutely continuous curves joining two closed subsets P_1, P_2 of A, each point of P_1 having a smaller x-coordinate than every point of P_2.

for all z such that $z^{0'} \geqq 0$. If now there existed a point z of A and a unit vector z'_n with $z^{0'}=0$ such that $G(z, z'_u)=0$, by (12.16) we would have

$$G_0(z, z'_u) \geqq -m,$$

contradicting hypothesis *e*). Hence for all z in A and all z'_u with $z_u^{0'}=0$ we have $G(z, z'_u)>0$. Lemma 7.2 then shows that hypothesis *d*) of lemma 7.1 is satisfied. Hence by Theorem 7.1 there exists a curve C: $x=x(s)$, $y=y(s)$, $0 \leqq s \leqq L$, in $\bar{K}_a$ for which $J[C]$ has minimum value. It remains to show that the curve C belongs to the class K_a.

Since the hypotheses of lemma 12.1 are all satisfied for the curve C, except perhaps that C is not entirely interior to A, from lemma 12.1 and the remark following it we find that

$$x'(s)>0$$

for almost all values of s. Hence by lemma 2.4 the curve C can be represented in the form

$$C: \quad y=\bar{y}(x), \qquad x_1 \leqq x \leqq x_2, \tag{12.17}$$

with absolutely continuous functions $\bar{y}(x)$; and by lemma 2.7 we have $I[y]=J[C]$, and the curve (12.17) is the solution sought.

For problems in the plane hypothesis *c*) can be omitted, since this hypothesis was used only in proving, by way of Theorem 7.1, that a minimizing curve for $J[C]$ exists. For plane problems we can refer instead to Theorem 8.1, in which hypothesis *c*) is not needed.

It is possible also to allow an exceptional set E^* consisting of a finite or denumerable set of absolutely continuous curves $y=y_n^*(x)$, on which hypothesis *e*) (i. e., hypothesis *c*) of lemma 2.1) is not fulfilled, similarly to what was done in Theorems 10.1 and 10.2. But we shall not enter into these matters.

Finally, the extension to unbounded cylindrical fields can be made by use of lemma 11.1.

An example of a function satisfying the hypotheses of Theorem 12.16 is

$$F(x, y, y')=[1+y'^2]^{\frac{1}{2}}-[1+y'^2]^{\frac{1}{4}},$$

for which the associated parametric integrand

$$G(x, y, x', y')=[x'^2+y'^2]^{\frac{1}{2}}-[x']^{\frac{1}{2}}[x'^2+y'^2]^{\frac{1}{4}}$$

is bounded on the class of all unit vectors.

As a corollary to Theorem 12.1 we have:

Corollary ([19]). - If for a plane problem hypotheses *a*), *b*), *d*), *f*) of 12.1 are fulfilled, and in addition there exist positive numbers α, M_1, M_2 such that

$$|F-y'F_{y'}| \geqq M_1 |y'|^{\alpha}-M_2, \tag{12.18}$$

([19]) TONELLI, II, p. 370. The theorem of TONELLI has alternative hypotheses [n.° 116 *d*) o *f*)], so that our corollary covers only half the theorem [n.° 116 *f*)].

then there exists a curve of the class K_a for which $I[y]$ assumes its least value. For hypothesis *c*) is not needed for plane problems, and inequality (12.18) can be written

$$|G_0(x, y, x', y')| \geqq M_1 |y'/x'|^\alpha - M_2,$$

whence $|G_0(x, y, 0, y')| = +\infty$, which can only be the case if $G_0 = -\infty$. Hence hypothesis *e*) is fulfilled.

§ 13. - Existence Theorems for Integrals for which G_0 is Bounded.

In applications of the Calculus of Variations a particularly frequently occurring integrand is that of the type

$$F(x, y, y') = \varphi(y)[1 + y^{\alpha'} y^{\alpha'}]^{\frac{1}{2}}.$$

For example, the least length problem and the brachistochrone problem are of this type. But for such integrals we find that the associated parametric integrand G has the partial derivative

$$G_0(y, x', y') = \varphi(y) x' [x'^2 + y'^2]^{-\frac{1}{2}},$$

which fails to tend to $-\infty$ as x' tends to zero. Thus none of our previous theorems are applicable. But for problems in the plane such integrands form a special case of a class treated by Tonelli, in the other half of the theorem cited in ([19]). This theorem overlaps our Theorem 13.2. We now proceed to prove several theorems referring to such integrands.

Theorem 13.1. - *If*

a) *the field A is a bounded closed cylindrical set $[c \leqq x \leqq d, (y^1, y^2)$ in $B]$;*

b) *the integral $I[y]$ is positive quasi-regular semi-normal on A;*

c) *the integrand is a function $F(y, y')$ of y and y' alone;*

d) *$F(y, y')$ is bounded below for all (x, y) on A and all y';*

e) *hypothesis d) of lemma 7.1 is satisfied;*

f) *for all (x, y) in A and all $y' \neq (0, 0)$ the equation*

$$G_0(y, x', y') = 0 \tag{13.1}$$

holds if and only if $x' = 0$;

g) *K_a is the class of all absolutely continuous curves joining two points* ([20]) *P_1 and P_2 of A;*

then there exists a curve C: $y = y(x)$ of the class K_a for which $I[y]$ assumes its least value on the class K_a, and the functions $y(x)$ are Lipschitzian.

([20]) Cf. footnote ([18]).

From equation (13.1) we first draw two conclusions concerning the integrand G. First, for all points (x, y) of A and all unit vectors x'_u, y'_u the integrand $G(y, x', y')$ is bounded. For if $|y^{i'}| \leqq 1$ $(i=1, 2)$, then

$$G(y, 1, y') = F(y, y') \leqq M, \tag{13.2}$$

where the existence of the constant M follows from the continuity of F and the boundedness of the arguments. Since by § 2 G_0 is a monotonic increasing function of x', we have by (13.1)

$$G_0(y, x', y') \geqq 0 \tag{13.3}$$

for all arguments; hence if $0 \leqq x' \leqq 1$, from (13.2) and (13.3) follows

$$G(y, x', y') \leqq G(y, 1, y') \leqq M,$$

and G is bounded above. On the other hand, we already know from lemma 2.3 that G is bounded below.

Second, for fixed (y, y') the derivative $G_0(y, x', y')$ is a continuous function of x'. The derivative G_0 exists for $x'=0$ by hypothesis f), and for $x'>0$ by § 2; hence if for fixed (y, y') it assumes two distinct values μ, ν, it assumes all [21] values between μ and ν, and so can not have jump discontinuities (discontinuities of the first kind). But for fixed (y, y') the function G_0 is monotonic in x' and can have no discontinuities except jump discontinuities. Hence it can have no discontinuities at all.

These facts established, we begin with the proof of the theorem. By Theorem 7.1 there exists a curve C: $x=x(s)$, $y=y(s)$, $a \leqq s \leqq b$ of the extended class $\overline{K}_a$ such that $J[C]$ is a minimum. We must prove that C belongs to K_a; that is, according to lemma 2.1, we must prove that $x'(s)>0$ for almost all values of s.

Suppose that this is false; there then exists a set E_1 of positive measure such that

$$x'(s) = 0 \qquad (s \text{ on } E_1). \tag{13.4}$$

Clearly $x(b)-x(a)>0$, for otherwise P_1 and P_2 would have the same abscissa and the class K_a would be empty. Hence $x'(s)>0$ on a set of positive measure, and we can therefore find a $k>0$ and a set E_2 of positive measure such that

$$x'(s) \geqq k > 0 \qquad (s \text{ on } E_2). \tag{13.5}$$

We now define $\varphi(s)$ by equation (12.4); equation (12.5) and (12.6) then follow. Likewise we define C_α by the equations

$$C_\alpha: \quad x = x_\alpha(s) = x(s) + \alpha\varphi(s), \qquad y = y(s), \qquad a \leqq s \leqq b; \qquad \alpha \geqq 0. \tag{13.6}$$

(21) De la Vallée Poussin: *Cours d'Analyse*, p. 97.

The integral

$$J[C_a]=\int_a^b G(y, x'_a, y')ds \tag{13.7}$$

exists, for G is measurable and bounded. By (12.8), C_a belongs to $\bar{K}_a$ for all small values of a, and so for all small a we have

$$J[C_a]\geqq J[C]. \tag{13.8}$$

Omitting a set of values of s of measure 0, the integrand in (13.7) is a differentiable function of a. Moreover, its derivative

$$G_0(y, x'_a, y')\cdot\varphi'(s)$$

is bounded if $a\leqq 1/m(E_2)$; for then x'_a never exceeds 1, and since G_0 is monotonic

$$0\leqq G_0(y, x'_a, y')\leqq G_0(y, 1, y'),$$

and $\varphi'(s)$ is bounded. Hence ([22]) we can differentiate under the integral sign. On setting $a=0$, this yields

$$J'[C_0]=\int_a^b G_0(y, x', y')\cdot\varphi'(s)\cdot ds. \tag{13.9}$$

For almost all s in the complement of E_1+E_2 the factor φ' has the value 0 by (12.6); for all s in E_1 the factor G_0 vanishes by (13.4) and (13.1); hence, using (12.6),

$$J'[C_0]=-\int_{E_2} m(E_1)\cdot G_0(y, x', y')ds. \tag{13.10}$$

But on E_2 we have $x'\geqq k>0$; hence on this set $G_0\neq 0$, and by (13.4) we have $G_0>0$, so that

$$J'[C_0]<0. \tag{13.11}$$

Hence for all sufficiently small positive values of a we have

$$J[C_a]<J[C_0].$$

This contradicts inequality (13.8). Therefore the assumption that $x'=0$ on a set of positive measure leads to a contradiction; hence $x'(s)>0$ for almost all s, and the curve C belongs to K_a.

We have yet to prove that the functions $y=y(x)$ defining C are Lipschitzian. For this purpose we use the Du Bois-Reymond relation which has elsewhere ([23])

([22]) Carathéodory, p. 664.

([23]) E. J. McShane: *The Du Bois-Reymond Relation in the Calculus of Variations*, to be published in Math. Annalen. In particular, we make use of corollary 1, § 5.

been proved to be valid for absolutely continuous minimizing curves under the present hypotheses. This states that there exists a constant c_3 such that the equation

$$G_0(y, x', y') = c_3 \tag{13.12}$$

holds for almost all s. Also, for almost all s we have $x'(s) > 0$. Letting s_0 be a value at which $x'(s) > 0$ and (13.12) holds, we have by (13.1)

$$c_3 = G_0(y(s_0), x'(s_0), y'(s_0)) > 0.$$

There exists a $\delta > 0$ such that for all (x, y) in A and all unit vectors (x'_u, y'_u) with $x'_u < \delta$ the inequality

$$G_0(y, x'_u, y'_u) < c_3$$

holds. For otherwise we could find a sequence of points (x_n, y_n) of A and of unit vectors x'_n, y'_n such that $x'_n \to 0$ and

$$G_0(y_n, x'_n, y'_n) \geqq c_3.$$

Let (x_0, y_0, x_0', y_0') be a limit point of the (x_n, y_n, x'_n, y'_n); then $x_0' = 0$. By the upper semi-continuity of G_0 (lemma 2.2) we would have

$$G_0(x_0, y_0, 0, y_0') \geqq c_3 > 0,$$

contradicting (13.1).

Hence for almost all values of s at which (13.12) holds we have $x'(s) \geqq \delta$. Thus if x_1 and $x_2 > x_1$ be any two abscissas of the curve and s_1 and s_2 the corresponding parameters, we have

$$x_2 - x_1 \geqq (s_2 - s_1)\delta ;$$

a fortiori,

$$\left| \frac{y_1^i - y_2^i}{x_2 - x_1} \right| \leqq \frac{1}{\delta}$$

and so the functions $y^i(x)$ are Lipschitzian.

In the above proof hypotheses *d*) and *e*) were used only in proving, by way of Theorem 7.1, that a minimizing curve for $J[C]$ exists. For problems in the plane this can be established by use of Theorem 8.1, in which hypotheses *d*) and *e*) do not occur. Hence for plane problems we have

THEOREM 13.2. - *If*

a) *the field A is a bounded closed cylindrical set* $[c \leqq x \leqq d,\ y \text{ in } B]$;

b) *the integral $I[y]$ is positive quasi-regular semi-normal on A*;

c) *the integrand is a function $F(y, y')$ of y and y' alone;*

d) *for all (x, y) in A the equation*

$$G_0(y, x', y') = 0$$

holds if and only if $x' = 0$;

e) K_a is the class of all absolutely continuous curves joining two points [24] *P_1 and P_2 of A;*

then there exists a curve C: $y=y(x)$ of the class K_a for which $I[y]$ assumes its minimum value, and the functions $y(x)$ defining C are Lipschitzian.

As a final generalization of Theorem 13.1, we state

THEOREM 13.3. - *Let the hypotheses of Theorem* 13.1 (*or, if the problem is a plane problem, those of Theorem* 13.2) *be satisfied, and let $\psi(y)$ be a non-negative differentiable function of the variable y. Denote by N the set of y for which $\psi(y)=0$. Let the inequality*

$$(13.13) \qquad G(y, x', y') > 0$$

be satisfied for all unit vectors (x', y') with $x' \geqq 0$ and all y on a neighborhood U of the set N.

Then there exists a curve C: $y=y(x)$ of the class [25] *K_a for which*

$$(13.14) \qquad I^*[y] = \int_a^b (\psi(y))^{-1} F(y, y')\,dx$$

assumes its minimum value on K_a.

The integral associated with I^* is

$$J^*[C] = \int_C (\psi(y))^{-1} G(y, x', y')\,ds.$$

We may assume (diminishing U if necessary) that (13.13) holds on the closure $\overline{U}$ of U. Let $\psi_1(y)$ be a continuous function coinciding with $\psi(y)$ on the complement of U, everywhere positive, and such that $\psi_1 \geqq \psi$ on U; and let

$$J_1[C] = \int_C (\psi_1(y))^{-1} G(y, x', y')\,ds.$$

Applying lemma 7.1 to J_1, the curves for which $J_1[C] \leqq M$ have uniformly bounded lengths. But this is *a fortiori* true of the curves for which $J^*[C] \leqq M$, since $J^* \geqq J_1$. Hence every minimizing sequence for J^* has a limit curve

$$C: \quad x=x(s), \qquad y=y(s), \qquad 0 \leqq s \leqq L$$

in the class $\overline{K}_a$.

Let N_1 be the set of values of s for which the point $y(s)$ lies in N. For all such s we have $\psi^{-1}G = +\infty$, so that N_1 has measure 0. Moreover, N is closed because of the continuity of $\psi(y)$, so that N_1 is also closed. Therefore if ε be any positive number, we can enclose N_1 is a finite set of closed intervals $\delta_1, \ldots, \delta_r$ with the properties

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(24) Cf. footnote (18).

(25) Cf. footnote (8). We consider the integrand in (13.14) to be $+\infty$ when $\psi(y)=0$.

a) The arcs C^k: $x=x(s)$, $y=y(s)$, s on δ_k are interior to U.

b) The measure of the set $\delta_1+ \dots +\delta_r$ is so small that

$$\left| \int_{\sum \delta_k} \psi^{-1} G(y(s), x'(s), y'(s)) ds \right| < \varepsilon. \tag{13.16}$$

The remaining parts of $[0, L]$ form a finite number of intervals; to these we add their end points, and call these closed intervals $\Delta_1, \dots, \Delta_p$. Every arc $\bar{C}^k$: $x=x(s)$, $y=y(s)$, s on Δ_k has a positive distance from N, and hence can be enclosed in a closed neighborhood on which $\psi>0$. If $\{C_n\}$ is a sequence of curves tending to C, we subdivide each C_n into subarcs $C_n^1, \dots, C_n^r$ and $\bar{C}_n^1, \dots, \bar{C}_n^p$ in such a way that $\lim_{n\to\infty} C_n^k = C^k$ and $\lim_{n\to\infty} \bar{C}_n^k = \bar{C}^k$. For sufficiently large n the arc C_n^k lies in U, and by (13.13) $G>0$, so that

$$\sum \int_{C_n^k} (\psi(y))^{-1} G(y, x', y') ds \geqq 0. \tag{13.17}$$

On each $\bar{C}^k$ the hypotheses of Theorem 6.3 are satisfied by $\psi^{-1}G$, and so

$$\begin{aligned} \liminf \sum \int_{\bar{C}_n^k} \psi^{-1} G(y, x', y') ds \geqq \\ \geqq \sum \liminf \int_{\bar{C}_n^k} \psi^{-1} G(y, x', y') ds \geqq \sum \int_{\bar{C}^k} \psi^{-1} G(y, x', y') ds. \end{aligned} \tag{13.18}$$

By (13.16), (13.17) and (13.18) we have

$$\liminf \int_{C_n} \psi^{-1} G(y, x', y') ds \geqq J^*[C] - \varepsilon.$$

This being true for every positive ε, the ε can be omitted from the right of the inequality. If in particular $\{C_n\}$ is a minimizing sequence for J^*, it follows that C minimizes J^*.

Since $J^*[C]$ is finite, $(\psi(y))^{-1}$ is summable along C; for on the arcs C^k the factor $G(y, x', y')$ is bounded from zero, and on the arcs $\bar{C}^k$ the function $(\psi(y))^{-1}$ is bounded. Hence the proof of Theorem 13.1 (from equation (13.7) to bottom of page) can be applied with only trivial modifications to $J^*[C]$ to show that the curve C can be represented in the form $y=y(x)$ with absolutely continuous functions $y(x)$.

An example coming under Theorem 13.3 is the problem of the brachistochrone: to minimize

$$\int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{y_1-y}} dx$$

in the class of all curves joining the points (a, y_1) and (b, y_2), $y_2<y_1$.