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ABSOLUTELY SUMMING OPERATORS AND MEASURE AMARTS IN FRECHET SPACES (*)

D. QUANG LUU

Résumé. Dans cet article on démontre quelques théorèmes de caractérisation pour les opérateurs absolument sommants et on donne diverses applications à la convergence des martingales asymptotiques vectorielles dans les espaces de Fréchet.

Summary. In the paper, we prove some characterization theorems for the absolutely summing operators and we give various applications to convergence of vector-valued asymptotic martingales in Fréchet spaces.

(*) This paper was partly written during the author's stay at the University of Sciences and Technics in Montpellier 1984.

§ 0. INTRODUCTION.

The Radon-Nikodym property and convergence of vector-valued asymptotic martingales (amarts) in Fréchet spaces have been extensively studied in recent years by many authors, see, [6,7,17,3,12,13,8,9] and etc. The purpose of the paper is to continue these above investigations. Namely, after stating some needed notations and definitions in Section 1, we shall prove in Section 2 some representation theorems for the absolutely summing operators in Fréchet spaces which are different from those, given in [16] . And finally, in Section 3 we shall give some applications of the results in Section 2 to convergence and boundedness problems of vector-valued amarts in Fréchet spaces.

§ 1. NOTATIONS AND DEFINITIONS.

In the paper we shall use the notations and definitions, given in [9] and introduce some other one's concerning measures in Fréchet spaces. Namely, let E be a Fréchet space, $U(E)$ a fundamental countable family of closed absolutely convex sets which form a 0-neighborhood base for E , E' the topological dual of E and (Ω,A,P) a probability space. Given $U \in U(E)$ the polar U° and the continuous seminorm p_U , associated with U are given by

$$U^{\circ} = \{e \in E' \mid |\langle x,e\rangle| \leqslant 1\} ,$$

$$p_U(x) = \inf \{\alpha > 0 \mid \alpha^{-1}x \in U\} , \quad (x \in E) .$$

For a σ-additive measure $\mu : A \to E$ and $U \in U(E)$ we define the semivariation (or the total variation, resp.) seminorm $S_U(\mu)$ (or $V_U(\mu)$, resp.) as follows

$$S_U(\mu) = \sup \{|<\mu,e>|(\Omega) \quad e \in U^\circ\} ,$$

$$V_U(\mu,A) = \sup \{\sum_{j=1}^{k} p_U(\mu(A_j)) | <A_j>_{j=1}^{k} \in \Pi(A)\} , \quad (A \in \mathcal{A}) ,$$

$$V_U(\mu) = V_U(\mu,\Omega) \quad ,$$

where $\Pi(A)$ denotes the set of all finite measurable partitions of $A \in \mathcal{A}$.

By $S(E) = S(\Omega,\mathcal{A},P,E)$ (or $V(E) = V(\Omega,\mathcal{A},P,E)$,resp.) we mean the space of all S-equivalence (or V-equivalence, resp.) classes of S-bounded (or V-bounded, resp.) σ-additive measures $\mu : \mathcal{A} \to E$. Then by using the same argument given in [16] for the spaces $\ell_I^1(E)$ and $\ell_I^1\{E\}$, one can establish easily the following property.

<u>Property</u> 1.1 . <u>Both</u> (S(E),S-topology) <u>and</u> (V(E) , V-topology) <u>are Fréchet spaces</u>.

Now, for definition of strong measurability and Bochner integrability of vector-valued functions $f : \Omega \to E$, we refer to [6,7] and let $L_1(E) = L_1(\Omega,\mathcal{A},P,E)$ denote the space of all V-equivalence classes of Bochner integrable functions $f : \Omega \to E$, where $V_U(f) = \int_\Omega p_U(f(\omega))dP(\omega)$ $(U \in \mathcal{U}(E))$. Then according to [6] , one can regard $(L_1(E)$,V-topology) as a closed subspace of $V(E)$ with the following identification : $L_1(E) \ni f \mapsto \mu_f \in V(E) : \mu_f(A) = \int_A f\,dP$ $(A \in \mathcal{A})$.

Note that with the above identification, $L_1(E)$ becomes a (not necessarily closed) subspace of $S(E)$. Finally, as in the Banach space case (see, e.g. [4]) , the following property remains true.

<u>Property</u> 1.2 . <u>Let</u> $\mu' \in S(E)$, $\mu \in V(E)$, $U \in \mathcal{U}(E)$ <u>and</u> $f \in L_1(\Omega, \mathcal{B}, P, E)$ <u>for some sub σ-field</u> $\mathcal{B} \subset \mathcal{A}$. <u>Then</u>

(1) $S_U(\mu) \leqslant V_U(\mu)$.

(2) $q_U(\mu') \leqslant S_U(\mu') \leqslant 4\, q_U(\mu')$,

<u>where</u> $q_U^{\mathcal{B}}(\mu) \overset{df}{=} \sup \{p_U(\mu(A)) : A \in \mathcal{A}\}$ <u>and</u> $q_U(\mu) = q_U^{\mathcal{A}}(\mu)$,

(3) $q_U(f) \overset{df}{=} q_U(\mu_f) \leqslant S_U(f) \overset{df}{=} S_U(\mu_f) \leqslant A\, q_U^{\mathcal{B}}(\mu_f)$.

§ 2. ABSOLUTELY SUMMING OPERATORS AND MEASURES AMARTS.

Let E and F be Fréchet spaces, $\ell_N^1(E)$ (or $\ell_N^1\{E\}$, resp.) the space of all summable (or absolutely summable, resp.) sequences (x_n) in E . Thus the ε-topology for $\ell_N^1(E)$ and the Π-topology for $\ell_N^1\{E\}$ are defined as in [16] . A linear continuous operator $T : E \to F$, write $T \in \mathcal{L}(E,F)$, is sait to be absolutely summing if it maps $\ell_N^1(E)$ into $\ell_N^1\{F\}$.

<u>Theorem</u> 2.1 . <u>Let</u> E,F <u>be Fréchet spaces and</u> $T \in \mathcal{L}(E,F)$. <u>Then the following conditions are equivalent</u> :

(1) T <u>is absolutely summing.</u>

(2) <u>For every probability space</u> (Ω, A, P) , <u>the operator</u> $T^\circ : S(E) \to V(E)$, <u>defined by</u>

$$(T^\circ \mu)(A) = T(\mu(A)) \quad (\mu \in S(E) \ , \ A \in A) \ ,$$

<u>is linear continuous</u>.

(3) <u>For every probability space</u> (Ω, A, P) , <u>the operator</u> $T^1 : (L_1(E)$, S-topology) $\to$ $(L_1(F)$, V-topology) , <u>defined by</u>

$$(T^1 f(\omega) = T(f(\omega)) \ \ (f \in L_1(E) \ , \ \omega \in \Omega) \ ,$$

<u>is linear continuous</u>.

(4) For only special probability space $(N, P(N), \gamma)$, where N is the set of all positive integers, $P(N)$ the class of all subsets of N and $\gamma(\{n\}) = 2^{-n}$ $(n \in N)$, T^1 is linear continuous.

<u>Proof</u> (1 $\to$ 2) . Let E , F be Fréchet spaces and $T \in \mathcal{L}(E,F)$ an absolutely summing operator. We shall show that for each $C \in U(F)$ there are some $U \in U(E)$ and $\beta(C,U) > 0$ such that for all finite sequences $\langle x_j \rangle_{j=1}^{k} \subset E$, we have

$$\sum_{j=1}^{k} p_C(Tx_j) \leq \beta(C,U) \sup\{ \sum_{j=1}^{k} |\langle x_j, e \rangle| \mid e \in U^\circ \} \ . \qquad (2.1)$$

Indeed, first of all applying ([16], 2.1.3) to T , we infer that the operator $T_N : \ell_N^1(E) \to \ell_N^1(F)$, given by

$$T_N(\langle x_n \rangle) = \langle Tx_n \rangle \qquad (\langle x_n \rangle \in \ell_N^1(E)) \ ,$$

is linear continuous. Therefore, by Theorem 1 in ([19], I.6) for each $C \in U(F)$ there is some $U \in U(E)$ and $\beta(C,U) > 0$ such that

$$\Pi_C(<Tx_n>) \leqslant \beta(C,U)\varepsilon_U(<x_n>) \quad (<x_n> \in \ell^1_N(E)) .$$

Equivalently,

$$\sum_N p_C(Tx_n) \leqslant \beta(C,U)\sup\{\sum_N |<x_n,e>| \mid e \in U^\circ\} \quad (<x_n> \in \ell^1_N(E)) .$$

Further, since for every finite sequence $<x_j>_{j=1}^k \subset E$, the sequence $\ll x_j>_{j=1}^k,0,0,\ldots> \in \ell^1_N(E)$, then the last inequality implies (2.1).

Now let $\mu \in S(E)$, $C \in U(F)$ and $<A_j>_{j=1}^k \in \Pi(\Omega)$. Applying (2.1) to the finite sequence $<\mu(A_j)>_{j=1}^k \subset E$, we get

$$\begin{aligned}\sum_{j=1}^k p_C((T^\circ\mu)(A_j)) &\leqslant \beta(C,U)\sup\{\sum_{j=1}^k |<\mu(A_j),e>| \mid e \in U^\circ\}\\ &\leqslant \beta(C,U)\sup\{|<\mu,e>|(\Omega) \mid e \in U^\circ\}\\ &= \beta(C,U)S_U(\mu) .\end{aligned}$$

Hence,

$$\begin{aligned}V_C(T^\circ\mu) &= \sup\{\sum_{j=1}^k p_C((T^\circ\mu)(A_j)) \mid <A_j>_{j=1}^k \in \Pi(\Omega)\}\\ &\leqslant \beta(C,U)S_U(\mu) . \qquad (2.2)\end{aligned}$$

Finally, again applying Theorem 1 in ([19], I.6) to the operator $T^\circ : S(E) \to V(F)$, it is clear that $T^\circ \in \mathcal{L}(S(E),V(F))$. This proves (2), taking into account that the linearity of T° is naturally satisfied.

(4 → 1) . Suppose that T fails to be absolutely summing. Then by definition, there is some $\langle x_n \rangle \in \ell^1_N(E)$ such that $\langle Tx_n \rangle \notin \ell^1_N(F)$, i.e. there is some $C \in U(F)$ such that

$$\sum_N p_C(Tx_n) = \infty \quad .$$

For convinience, we can always suppose that $p_C(Tx_n) \neq 0 \quad (n \in N)$. Now, choose a strictly increasing subsequence $\langle n_k \rangle$ of N such that

$$\sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geq k \qquad (k \in N)$$

and define $f_k : N \to E \quad (k \in N)$, by

$$f_k = \sum_{j=n_k+1}^{n_{k+1}} 2^j x_j 1_{\{j\}} \qquad (k \in N) ,$$

where 1_A is the characteristic function of $A \in \mathcal{A}$.
It is clear that by ([16], 1.3.6) the sequence $\langle f_k \rangle$ in $L_1(E)$ is S-convergent to 0 . On the other hand, as

$$\int_N p_C(T^1 f)d\gamma = \sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geq k \qquad (k \in N) ,$$

the sequence $\langle T^1 f_k \rangle$ in $L_1(F)$ fails to be V-convergent. It contradicts (4) . Finally since the implications (2 → 3 → 4) are trivial, the proof of the theorem is completed. Now, by Theorem 4.2.5 in [16], the Fréchet space E is nuclear if and only if the identification operator is absolutely summing so that the following corollary is an easy consequence of the above theorem.

Corollary 2.2. For a Fréchet space E, the following conditions are equivalent :

(1) E is nuclear.

(2) On V(E), the S-topology is equivalent to the V-topology.

(3) On $L_1(E)$, the S-topology (the Pettis topology) is equivalent to the V-topology (the Bochner topology).

Remark. (1) Theorem 2.1 was first partly proved by Ghoussoub in [14] for Banach spaces and later completed by Bru-Heinich in [4], using directly Proposition 2.2.1 in [16] which can be applied to only normed spaces.

(2) Egghe [12] has applied however Proposition 4.1.5 in [16] to obtain the equivalence $(1 \leftrightarrow 3)$ in the corollary.

In order to give some probability characterizations of absolutely summing operators in Fréchet spaces we give now some additional notations and definitions. Indeed, hereafter we shall fix an increasing sequence $<A_n>$ os subσ-fields of A such that $A = \sigma(\bigcup_N A_n)$. Let

$$S(<A_n>,E) = \{<\mu_n> \mid \forall n \quad \mu_n \in S^n(E) = S(\Omega,A_n,P,E)$$

$$V(<A_n>,E) = \{<\mu_n> \mid \forall n \quad \mu_n \in V^n(E) = V(\Omega,A_n,P,E)$$

$$L_1(<A_n>,E) = \{<f_n> \mid \forall n \quad f_n \in L_1^n(E) = L_1(\Omega,A_n,P,E)$$

and T the set of all bounded stopping times. Given $\tau \in T$, $<\mu_n> \in S(<A_n>,E)$ and $<f_n> \in L_1(<A_n>,E)$ we define

$$A_\tau = \{A \in A \mid A \cap \{\tau = n\} \in A_n \quad \forall n\} ,$$

$$\mu_\tau : A_\tau \to E \quad : \quad \mu_\tau(A) = \sum_N \mu_n(A \cap \{\tau = n\}) \quad (A \in A_\tau)$$

$$f_\tau : \Omega \to E \quad : \quad f_\tau(\omega) = f_n(\omega) \quad (\omega \in \{\tau = n\} , n \in N) .$$

It is known (cf. [15]) that $<A_\tau ; \tau \in T>$ is increasing family of subσ-fields of A ; $\mu_\tau \in S^\tau(E) = S(\Omega, A_\tau, P, E)$ and $f_\tau \in L_1^\tau(E) = L_1(\Omega, A_\tau, P, E)$. Moreover, if $<\mu_n> \in V(<A_n>, E)$ then $\mu_\tau \in V^\tau(E) = V(\Omega, A_\tau, P, E)$.

Definition 2.1 . Call $<\mu_n> \in S(<A_n>, E)$ to be a martingale if $\mu_{m,n} = \mu_m \mid A_n = \mu_n \quad (m \geq n \in N)$.

Note that if $<\mu_n> \in S(<A_n>, E)$ is a martingale then $\mu_{\sigma,\tau} = \mu_\sigma \mid A_\tau = \mu_\tau \quad (\sigma \geq \tau \in T)$. Hence $\mu_\tau(\Omega)$ does not depend upon the choice of $\tau \in T$. Thus $<\mu_n> \in S(<A_n>, E)$ is said to be an amart if the net $<\mu_\tau(\Omega), \tau \in T>$ is convergent in E .

We note that as for the amarts in Banach spaces (see, [5] , [10] , [18] , [4]) , the following basis lemma is obtained.

Lemma 2.3 . Let $<\mu_n> \in S(<A_n>, E)$. Then the following conditions are equivalent :

(1) $<\mu_n>$ is an amart.

(2) μ_n has a Riesz decomposition : $\mu_n = \alpha_n + \beta_n \quad (n \in N)$, where $<\alpha_n> \in S(<A_n>, E)$ is a martingale and $<\beta_n>$ is a potential, i.e.

$$\lim_{\tau \in T} S_U^\tau(\beta_\tau) = 0 \quad (U \in U(E)) ,$$

where $S_U^\tau(.)$ is defined as S_U with respect to the probability space (Ω, A_τ, P) .

(3) There is a finitely additive measure $\mu_\infty : \bigcup_N A_n \to E$ such that $\mu_\infty | A_n = \mu_{\infty,n} \in S^n(E)$ $(n \in N)$ and

$$\lim_{\tau \in T} S_U^\tau(\mu_\tau - \mu_{\infty,\tau}) = 0 \quad (U \in U(E)) .$$

We shall call μ_∞ the limit measure associated with $<\mu_n>$.

Definition 2.2 . Call $<\mu_n> \in V(<A_n>,E)$ to be a uniform amart if the following condition is satisfied

$$\lim_{\tau \in T} \sup_{\sigma \geq \tau} V_U^\tau(\mu_{\sigma,\tau} - \mu_\tau) = 0 \quad (U \in U(E)) ,$$

where V_U^τ is defined as V_U with respect to (Ω, A_τ, P) .

It is clear that by Lemma 2.3 and Property 1.2, every uniform amart is an amart. Moreover, as for the uniform amarts in Banach spaces (cf. [2], [4]) we get the following.

Lemma 2.4 . Let $<\mu_n> \in V(<A_n>,E)$. Then the following conditions are equivalent :

(1) $<\mu_n>$ is a uniform amart.

(2) $<\mu_n>$ has a Riesz decomposition : $\mu_n = \alpha_n + \beta_n$ $(n \in N)$, where $<\alpha_n>$ is a martingale in $V(<A_n>,E)$ and $<\beta_n>$ a uniform potential, i.e.

$$\lim_{\tau \in T} V_U^\tau(\beta_\tau) = 0 \qquad (U \in U(E)) .$$

(3) There is a finitely additive measure $\mu_\infty : \bigcup_N A_n \to E$ such that each $\mu_{\infty,n} \in V^n(E)$ and

$$\lim_{\tau \in T} V_U^\tau(\mu_\tau - \mu_{\infty,\tau}) = 0 \qquad (U \in U(E)) .$$

Note that Property 1.1 is needed in the proofs of Lemmas 2.3 and 2.4 .

Finally, we say that a sequence $<f_n>$ in $L_1(<A_n>,E)$ has a property (★) if so has the sequence $<\mu_n = \mu_{f_n}>$, associated with $<f_n>$.

Theorem 2.5 . Let E,F be Fréchet spaces and $T \in \mathcal{L}(E,F)$. Then the following conditions are equivalent :

(1) T is absolutely summing

(2) T° maps amarts in $S(<A_n>,E)$ into uniform amarts in $V(<A_n>,F)$.

(3) For each S-bounded amart $<f_n>$ in $L_1(<A_n>,E)$ and $C \in U(F)$, the sequence $<p_C(T^1 f_n)>$ is a uniform amart of nonnegative real-valued functions.

(4) T^1 maps every V-convergent amart $<f_n>$ in $L_1(<A_n>,E)$ into a sequence $<T^1 f_n>$ of class (B) , i.e.

$$\sup_{\tau \in T} \int_\Omega p_C(g_\tau)\,dP < \infty \qquad (C \in U(F)) ,$$

where $g_n \overset{df}{=} T^1 f_n \quad (n \in N)$.

Proof. Let E,F be Fréchet space and $T \in \mathcal{L}(E,F)$. Suppose first

(α_n) is a martingale in $S(\langle A_n \rangle, E)$. It is clear that $\langle T^\circ \alpha_n \rangle$ is also a martingale in $S(\langle A_n \rangle, F)$. Now if T is absolutely summing and (β_n) is a potential in $S(\langle A_n \rangle, E)$, by (2.2) in the proof of Theorem 2.1 it follows that the sequence $\langle T^\circ \beta_n \rangle$ is a uniform potential in $V(\langle A_n \rangle, F)$ and $\langle T^\circ \alpha_n \rangle \in V(\langle A_n \rangle, F)$, noting that if $\gamma_n = T^\circ \beta_n$ $(n \in N)$ then $\gamma_\tau = T^\circ \beta_\tau$ $(\tau \in T)$. Therefore, by Lemmas 2.3 and 2.4 we get $(1 \rightarrow 2)$.

$(1 \rightarrow 3)$. To prove $(1 \rightarrow 3)$, we suppose first that $\langle \gamma_n \rangle$ is a uniform amart in $V(\langle A_n \rangle, F)$, γ_∞ the limit measure associated with $\langle \gamma_n \rangle$ and $C \in U(F)$. Then by Lemma 2.4 , it follows that

$$\lim_{\tau \in T} V_C^\tau(\gamma_\tau - \gamma_{\infty,\tau}) = 0 .$$

This with properties of seminorms in $V^\tau(E)$ implies

$$\lim_{\tau \in T} |V_C^\tau(\gamma_\tau) - V_C^\tau(\gamma_{\infty,\tau})| = 0 \qquad (2.3)$$

Further, if $\langle \gamma_n \rangle$ is V-bounded, it is easily checked that

$$\uparrow \lim_{\tau \in T} V_C^\tau(\gamma_{\infty,\tau}) = V_C^\Sigma(\gamma_\infty) < \infty ,$$

where $\Sigma = \bigcup_N A_n$,

$$V_C^\Sigma(\gamma_\infty) = \sup \{ \sum_{j=1}^{k} p_C(\gamma_\infty(A_j)) \mid \langle A_j \rangle_{j=1}^{k} \in \Pi(\Sigma, \Omega) \}$$

and $\Pi(\Sigma, \Omega)$ is the set of all finite Σ-measurable partitions of Ω . Consequently, by (2.3) we get

$$\lim_{\tau \in T} |V_C^\tau(\gamma_\tau) - V_C^\Sigma(\gamma_\infty)|$$

$$\leqslant \lim_{\tau \in T} \{ |V_C^\tau(\gamma_\tau) - V_C^\tau(\gamma_{\infty,\tau}) + V_C^\tau(\gamma_{\infty,\tau}) - V_C^\Sigma(\gamma_\infty)| = 0 \qquad (2.4)$$

Now we suppose that T is absolutely summing, $<f_n>$ a S-bounded amart in $L_1(<A_n>,E)$, $<\mu_n>$ the measure amart associated with $<f_n>$ and μ_∞ the limit measure associated with $<\mu_n>$. IT is clear that if we define

$$\gamma_n = T^\circ \mu_n \quad (n \in N) ,$$

then by $(1 \to 2)$, $<\gamma_n>$ is a uniform amart in $V(<A_n>,F)$ and by (2.2) in the proof of Theorem 2.1, $<\gamma_n>$ is V-bounded. Therefore, for any but fixed $C \in U(F)$, the uniform amart $<\gamma_n>$ must satisfy (2.4). Moreover, if we define

$$g_n = p_C(T^1 f_n) \quad (n \in N) ,$$

then

$$\int_\Omega g_\tau \, dP = V_C^\tau(\gamma_\tau) \qquad (\tau \in T) .$$

This with (2.4) proves that the sequence $<p_C(T^1 f_n)>$ is a uniform amart (of nonnegative real-valued functions), taking into account that in (2.4), $V_C^\Sigma(\gamma_\infty)$ is a finite number. It completes the proof of $(1 \to 3)$.

$(2 \to 4)$ Suppose that $<f_n>$ is a V-convergent amart in $L_1(<A_n>,E)$. Then given $C \in U(F)$, the sequence $<p_C(T^1 f_n)>$ must be V-bounded. Thus as in the proof of $(1 \to 3)$, the V-boundedness of $<p_C(T^1 f_n)>$ with (2) shows that $<p_C(T^1 f_n)>$ must be a uniform amart. Therefore the V-boundedness of $<p_C(T^1 f_n)>$ is equivalent to

$$\sup_{\tau \in T} \int_\Omega p_C(T^1 f_\tau) dP < \infty ,$$

i.e. $<p_C(T^1 f_n)>$ is of class (B). This proves (4).

Because $(3 \to 4)$ is similarly established, it remains to prove $(4 \to 1)$. For this purpose, suppose that T is not absolutely summing. Returning to the sequence $<x_n>$ in the example, given in the proof of Theorem 2.1 we take $(\Omega, A, P) \equiv (N, P(N), \gamma)$ and define

$$f_j = 2^j x_j 1_{\{j\}} \qquad (j \in N) ;$$

$$A_j = \sigma(f_1, f_2, \ldots, f_n) \qquad (j \in N) ;$$

$$\underline{\tau} = \min \{n \in N \mid P(\{\tau = n\}) > 0\} \quad ;$$

$$\overline{\tau} = \max \{n \in N \mid P(\{\tau = n\}) > 0\} \qquad (\tau \in T) .$$

Then

$$\int_N f_\tau \, dP = \int_N \sum_{j=\underline{\tau}}^{\overline{\tau}} 2^j x_j 1_{\{j\}} \, dP = \sum_{j=\underline{\tau}}^{\overline{\tau}} x_j .$$

Consequently, by Theorem 1.3.6 in [16], the sequence $<f_j>$ defined above is a potential (hence an amart). Further, since

$$\int_N p_U(f_j) dP = p_U(x_j) \qquad (j \in N , U \in U(E))$$

the sequence $<f_j>$ is V-convergent to 0. On the other hand, if we put

$$\iota_k(j) = \begin{cases} j & \text{if} \quad n_k + 1 \leqslant j \leqslant n_{k+1} \\ n_{k+1} + 1 , & \text{if} \quad j \notin \{n_k + 1, \ldots, n_{k+1}\} \end{cases}$$

for all $k \in N$, then

$$\int_N p_C(T^1 f_{\tau_k})\,dP \geq \sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geq k .$$

Therefore, the V-convergent (to 0) amart $<f_j>$ cannot be of class (B) which contradicts (4) and completes the proof of the theorem.

Note that since the sequence $<p_C(T^1 f_j)>$ is V-convergent (hence V-bounded) and is not of class (B), $<p_C(T^1 f_n)>$ cannot neither be an amart. Further, since a Fréchet space E is nuclear if and only if the identical operator is absolutely summing, the following corollary is an easy consequence of the theorem.

Corollary 2.6 . *For a Fréchet space, E , the following conditions are equivalent*

(1) E *is nuclear.*

(2) *Every amart in* $S(<A_n>,E)$ *is uniform.*

(3) *For every* S-*bounded amart* $<f_n>$ *in* $L_1(<A_n>,E)$ *and* $U \in U(E)$, *the sequence* $<p_U(f_n)>$ *is a uniform amart in* $L_1(<A_n>,\mathbb{R})$.

(4) *For every* V-*convergent amart* $<f_n>$ *in* $L_1(<A_n>,E)$ *and* $U \in U(E)$, *the sequence* $<p_U(f_n)>$ *is an amart in* $L_1(<A_n>,\mathbb{R})$.

Remark. Since every L_1-bounded real-valued uniform amart must be of class (B) , Theorem 2 in | 13 | is hence easily established from Corollary 2.6 . Note that for the proof of Theorem 2 in | 13 | , Egghe has needed the Radon-Nikodym property of nuclear Fréchet spaces. So that his proof cannot be applied to Theorem 2.5 .

§ 3. CONVERGENCE AND BOUNDEDNESS OF AMARTS.

In this section, we shall apply the results in Section 2 to convergence and boundedness problems of amarts in Fréchet spaces. We begin with

Theorem 3.1 . Let E,F be Fréchet spaces and $T \in \mathcal{L}(E,F)$. Then the following properties are equivalent :

(1) T is absolutely summing.

(2) T maps potentials in $L_1(\langle A_n \rangle,E)$ into F-valued sequences, strongly convergent to 0 , almost everywhere (a.e.) .

(3) T maps potentials in $L_1(\langle A_n \rangle,E)$ into F-valued sequences, weakly convergent, to 0 , a.e.

(4) T maps V-convergent potentials in $L_1(\langle A_n \rangle,E)$ into F-valued sequences, strongly bounded, a.e.

Proof. $(1 \to 2)$ Suppose that $T \in \mathcal{L}(E,F)$ is absolutely summing. By Theorem 2.5 T maps potentials in $L_1(\langle A_n \rangle,E)$ into uniform potentials in $L_1(\langle A_n \rangle,F)$. Thus to prove (2) it is sufficient to show that every uniform potential $\langle g_n \rangle$ in $S(\langle A_n \rangle,F)$ is strongly convergent, a.e. Indeed, $\langle g_n \rangle$ is a uniform potential, by definition we get

$$\lim_{\tau \in T} \int_\Omega p_C(g_\tau)\,dP = 0 \qquad (C \in U(F)) .$$

Hence, also by definition, the sequence $\langle p_C(g_n) \rangle$ is a uniform potential of real-valued functions. Hence it must be convergent to 0 , a.e. (cf. [2], [4]) . We note that since $U(F)$ is countable, $\langle g_n \rangle$ must converge itself strongly a.e. to 0 . This proves (2) . Here, it is worth to note that in [13] , Egghe has hardly proved that every uniform potential in nuclear Fréchet spaces converges strongly, a.e. to 0 . But as we have just shown, this fact is clear even for uniform potentials in general Fréchet spaces F .

Returning to the proof of the theorem we see that the implications $(2 \to 3 \to 4)$ are easy. Thus we have to prove only

$(4 \to 1)$ Suppose that T fails to be absolutely summing. And let $\langle x_n \rangle$ and C be as in the example given in the proof of Theorem 2.1 . By (Ω, A, P) we mean the Lebesgue probability space on $[0,1)$. Since

$$\sum_N p_C(Tx_n) = \infty$$

~~and (Ω, A, P) has no atoms,~~ we can choose a subsequence $n < n_2 < \ldots < n_k < \ldots$ of N such that

$$\alpha_k = \sum_{j=n_k+1}^{n_{k+1}} p_C(Tx_j) \geqslant k \qquad (k \in N) .$$

Next, for each $k \in N$, find $\langle A_{k,j} \rangle_{j=n_k+1}^{n_{k+1}} \in \Pi(\mathcal{B}_{[0,1)}, [0,1))$ with

$$\alpha_{k,j} = P(A_{k,j}) = \alpha_k^{-1} \, p_C(Tx_j) \qquad (n_k+1 \leqslant j \leqslant n_{k+1})$$

and define

$$f_j = \alpha_{k,j}^{-1} \, x_j \, 1_{A_{k,j}} \qquad (k \in N \ , \ n_k+1 \leqslant j \leqslant n_{k+1}) \ ,$$

$$A_j = \sigma(f_1, f_2, \ldots, f_j) \qquad (j \in N) .$$

Finally, given $k,j \in N$ we put

$$\delta_{k,j} = \begin{cases} 1 & \text{if } j \in \{n_k+1, n_k+2, \ldots, n_{k+1}\} \\ 0 & \text{for otherwise} \end{cases}$$

and for each $\tau \in T$, we define

$$\beta_{k,j} = P(A_{k,j} \cap \{\tau = j\}) \quad ;$$

$$k(\underline{\tau}) = \max \{k \in N \mid \underline{\tau} \geq n_k+1\} \quad ;$$

$$k(\overline{\tau}) = \min \{k \in N \mid \overline{\tau} \leq n_{k+1}\} \quad .$$

Therefore, with the above notations, one get

$$\int_0^1 f_\tau \, dP = \int_0^1 \sum_{j=\underline{\tau}}^{\overline{\tau}} f_j \, 1_{\{\tau=j\}} dP = \sum_{k=k(\underline{\tau})}^{k(\overline{\tau})} \sum_{j=\underline{\tau}}^{\overline{\tau}} \delta_{k,j} \, \alpha_{k,j}^{-1} \, \beta_{k,j} \, x_j \ .$$

But note that $0 \leq \delta_{k,j} \, \alpha_{k,j}^{-1} \, \beta_{k,j} \leq 1$ $(k,j \in N)$ and $(\underline{\tau} \to \infty)$ implies $(k(\underline{\tau}) \to \infty)$. Consequently, by Theorem 1.3.6 in [16], the summability of $<x_n>$ implies that the net $<\int_0^1 f_\tau \, dP>_{\tau \in T}$ converges to 0 . It means that

(a) $<f_j>$ is a potential in $L_1(<A_n>, E)$.

Next, since

$$\int_0^1 p_U(f_j) dP = p_U(x_j) \qquad (j \in N \ , \ U \in U(E))$$

(b) $<f_j>$ is V-convergent to 0 .

Finally, for each $\omega \in [0,1)$, $k \in N$, one can choose some j_k such that $n_k+1 \leq j_k \leq n_{k+1}$. This yields

$$p_C(T^1 f_{j_k}(\omega)) = \alpha_{k,j_k}^{-1} \, p_C(Tx_{j_k}) = \alpha_k \quad .$$

Therefore,

$$\sup_N \, p_C(Tf_j(\omega)) = \infty \qquad (\omega \in [0,1)) \ .$$

Consequently, the sequence $<f_j>$ with the properties (a-b-c) contradicts (4) which completes the proof of $(4 \to 1)$ and the theorem.

Now suppose that E is nuclear and $<f_n>$ a S-bounded amart in $L_1(<A_n>,E)$. Then by [6] , E has the Radon-Nikodym property, $<L_1(E),\text{V-topology}>$ is a Fréchet space and by Corollary 2.6 $<f_n>$ is a V-bounded uniform amart. Hence, $<f_n>$ has a more precise Riesz decomposition : $f_n = g_n + h_n$ $(n \in N)$, where $<g_n>$ is a V-bounded martingale in $L_1(<A_n>,E)$ and $<h_n>$ a uniform potential. Thus the proof of Theorem 3.1 shows that $<h_n>$ converges strongly a.e. to 0 . Further, since every nuclear Fréchet space is a projective limit of a sequence of Hilbert spaces, the martingale limit theorem in Hilbert spaces shows that $<g_n>$ must converge strongly a.e. Therefore, the following corollary is an easy consequence of the theorem.

<u>Corollary</u> 3.2 . <u>For a Fréchet space, the following properties are equivalent</u> :

(1) E <u>is nuclear</u>.

(2) <u>Every</u> S-<u>bounded amart in</u> $L_1(<A_n>,E)$ <u>is convergent strongly, a.e.</u>

(3) Every S-bounded amart in $L_1(<A_n>,E)$ is convergent weakly, a.e.

(4) Every V-convergent potential in $L_1(<A_n>,E)$ is strongly bounded, a.e.

Remark. The equivalence $(1 \leftrightarrow 2)$ was first proved by Bellow [1] for Banach spaces. This result has been recently extended to Fréchet spaces by Egghe in [13]. Also $(1 \leftrightarrow 3)$ has been proved by Edgar-Sucheston in [11] for Banach spaces.

We say that a sequence $<f_n>$ in $L_1(<A_n>,E)$ is S-uniformly integrable, if $<f_n>$ is S-bounded and for every $U \in U(F)$,

$$\lim_{P(A) \to 0} \sup \{\int_A |<f_n,e>| dP \mid e \in U^\circ , n \in N\} = 0$$

It is clear that if E is a nuclear Fréchet space then every S-uniformly integrable sequence $<f_n>$ in $L_1(<A_n>,E)$ is V-uniformly integrable, i.e. for each $U \in U(E)$, the sequence $<p_U(f_n)>$ is uniformly integrable. Conversely, if E fails to be nuclear then as in the proof of Theorem 2.1, one can construct a potential $<f_k>$ in $L_1(<A_n>,E)$ such that $<f_k>$ fails to be V-bounded. Therefore, the following corollary can be deduced easily from Corollary 3.2.

Corollary 3.3. For a Fréchet space E, the following conditions are equivalent:

(1) E is nuclear.

(2) Every S-uniformly integrable amart in $L_1(<A_n>,E)$ is V-convergent.

(3) Every potential in $L_1(<A_n>,E)$ is V-bounded.

Remark. The equivalence $(1 \leftrightarrow 2)$ was first proved by Egghe in [12], where he gave a very complecated example in order to prove $(2 \to 1)$. The implication $(3 \to 1)$ in the corollary seems to be new even for Banach spaces.

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