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Generalized canonical states

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GENERALIZED CANONICAL STATES

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1. INTRODUCTION

In recent years, much attention has been devoted to the study of canonical Gibbs states or more general "conditional Gibbs states". One can distinguish several conceptions of canonical states : among them, the one developed by Lanford [9], Martin-Löf [10] and another one, by Georgii [6], Thompson [14], Aizenman, Goldstein, Lebowitz [1], Preston [12] and others. The difference of point of view is essentially the following : the first group of authors considers only one energy level on each finite box, while the second one does not initially single out any energy level but deals with all of them. Here we adopt the second point of view.

We introduce a generalization of the canonical specifications studied by Georgii [6] in a way that integrates also the more general concept considered by Thompson [14]. The set of all energy levels on a finite box is viewed as a partition of the set of all possible configurations on the box without reference to any "conditioning" potential.

In this general setting we prove a conditional variational principle to be satisfied by every translation invariant canonical state (§ 4) : those canonical states maximize a thermodynamic quantity which may be interpreted as the free energy of an "energy level". A formulation in term of information gain is also given (§ 5).

In § 7, we study the notion of irreductible family of potentials introduced by Thompson and show it is in some sense the biggest family that determines a given system of partitions.

In the sequel, we consider only systems of partitions defined by a family of potentials as used by Thompson, but the "main" potential is allowed to have infinite range unlike in Thompson's thesis where all potentials have finite range. By means of the variational principle, we generalize and prove in a much simpler way a theorem due to Thompson that says all locally positive canonical states are Gibbs states for some specified potential. This implies a kind of converse of the variational principle (§ 6).

2. NOTATIONS AND DEFINITIONS

a) Space of configurations

We consider the lattice gas with configurations space $\Omega = \{0, 1\}^{Z^\nu}$. The σ-algebra on Ω is the product of the discrete σ-algebras on $\{0, 1\}$. For any subset Λ of Z^ν, $\Omega(\Lambda) = \{0, 1\}^\Lambda$; $\mathcal{F}(\Lambda)$ is the product σ-algebra on $\Omega(\Lambda)$. $\mathcal{F}_\Lambda$ is the σ-algebra of measurable sets of configurations that depend only on coordinates in Λ.

A greek letter like ω or η will generally denote a configuration. The restriction of a configuration to a subset Λ of Z^ν is denoted by ω_Λ, η_Λ, ... In the sequel Λ denotes a finite subset of Z^ν and $\Lambda^c = Z^\nu \backslash \Lambda$

b) Potentials

We are given a potential φ, i.e. a real valued function defined on the configurations of the finite subsets of Z^ν. φ belongs to the Banach space $\mathcal{B}$ of translation invariant potentials that are null on the non identically 1 configurations and that are normed in the sense :

$$\|\varphi\| = \sum_{\substack{X \text{ finite} \\ 0 \in X \subset Z^\nu}} |\varphi(X)| < \infty$$

The energy potential V^φ associated to φ is

$$V^\varphi(\omega_\Lambda) = \sum_{X \subset \Lambda} \varphi(\omega_X)$$

(the superscript φ will be dropped when no confusion is possible.)

The interaction energy W^φ associated to φ is defined by :

$$W^\varphi(\omega_{\Lambda_1} | \omega_{\Lambda_2}) = \sum_{\substack{X \subset \Lambda_1 \cup \Lambda_2 \\ X \cap \Lambda_1 \neq \phi \neq X \cap \Lambda_2}} \varphi((\omega_{\Lambda_1} \omega_{\Lambda_2})_X) \text{ with } \Lambda_1 \cap \Lambda_2 = \phi$$

and

$$U^{\varphi}(\omega_{\Lambda_1}|\omega_{\Lambda_2}) = V^{\varphi}(\omega_{\Lambda_1}) + W^{\varphi}(\omega_{\Lambda_1}|\omega_{\Lambda_2})$$

Remark : due to the fact that φ is normed, one is allowed to consider

$$W^{\varphi}(\omega_{\Lambda}|\omega_{\Lambda^c}) = \sum_{\substack{X \text{ finite} \\ X\cap\Lambda\neq\phi\neq X\cap\Lambda^c}} \varphi((\omega_{\Lambda}\,\omega_{\Lambda^c})_X)$$

$$U^{\varphi}(\omega_{\Lambda}|\omega_{\Lambda^c}) = \sum_{\substack{X \text{ finite} \\ X\cap\Lambda\neq\phi}} \varphi((\omega_{\Lambda}\,\omega_{\Lambda^c})_X)$$

c) Gibbs states

Let us first define the local Gibbs state with external condition η_{Λ^c}

$$\nu_{\Lambda}(\omega_{\Lambda}, \eta_{\Lambda^c}) = \frac{\exp[-U^{\varphi}(\omega_{\Lambda}|\eta_{\Lambda^c})]}{Z^{\varphi}_{\Lambda}(\eta_{\Lambda^c})}$$

with $Z^{\varphi}_{\Lambda}(\eta_{\Lambda^c}) = \sum_{\zeta_{\Lambda}\in\Omega(\Lambda^c)} \exp[-U^{\varphi}(\zeta_{\Lambda}|\eta_{\Lambda^c})]$

A probability μ on $(\Omega,\mathcal{F})$ is said a Gibbs state associated to potential φ if for any finite Λ and any ω_{Λ} of $\Omega(\Lambda)$:

$$E_{\mu}[\mathbb{1}_{\{\omega_{\Lambda}\}} \mid \mathcal{F}_{\Lambda}](\eta) = \nu_{\Lambda}(\omega_{\Lambda}, \eta_{\Lambda^c}) \qquad \mu \text{ a.s. in } \eta.$$

3. GENERALIZED CANONICAL FORMALISM

a) Canonical σ-algebras

We consider a system $\pi = \{\pi(\Lambda) ; \Lambda \subset Z^\nu, \Lambda \text{ finite}\}$ of partitions of the spaces $\Omega(\Lambda)$ and we suppose that the partitions are translation invariant, i.e. for every x in Z^ν, $\pi(\Lambda) = \pi(\Lambda + x)$.

Let us define a coherent system of partitions : let $\Gamma = \Lambda \cup \Lambda'$ with $\Lambda \cap \Lambda' = \phi$ and $\Omega(\Gamma, i)$ be a class of partition $\pi(\Gamma)$; π is said coherent if for any finite Γ and i, $\Omega(\Gamma, i)$ is a reunion of cartesian products of the type $\Omega(\Lambda,j) \times \Omega(\Lambda',j')$ with $\Omega(\Lambda, j)$ (resp. $\Omega(\Lambda', j')$) a class of partition $\pi(\Lambda)$ (resp. $\pi(\Lambda')$).

Consider a non necessarily coherent system of partitions π. It is easy to associate to π a coherent system $\tilde{\pi}$. Denote $\tilde{\Omega}(\Lambda, \zeta_\Lambda)$, the class of $\tilde{\pi}(\Lambda)$ containing configuration ζ_Λ ; it is defined as the class of all configurations ω_Λ of $\Omega(\Lambda)$ such that for any finite Λ' in Z^ν, containing Λ, and for any configuration $\eta_{\Lambda'\setminus\Lambda}$ of $\Omega(\Lambda'\setminus\Lambda)$, $\Omega(\Lambda', \omega_\Lambda \eta_{\Lambda'\setminus\Lambda}) = \Omega(\Lambda', \zeta_\Lambda \eta_{\Lambda'\setminus\Lambda})$. Remark $\tilde{\pi} = \pi$ if π is coherent.

The canonical σ-algebra $\mathcal{G}_\Lambda(\pi)$ is the σ-algebra generated by the sets of the form :

$\Omega(\Lambda, i) \times A$ with $\Omega(\Lambda, i) \in \pi(\Lambda)$ and $A \in \mathcal{F}(\Lambda^c)$

(when no confusion is possible, we write $\mathcal{G}_\Lambda$.)

Properties of the canonical σ-algebras $\mathcal{G}_\Lambda(\pi)$

1. For any system π of partitions, $\mathcal{G}_\Lambda(\pi) \supset \mathcal{F}_{\Lambda^c}$

2. π is coherent if and only if for any finite Λ, Λ' with $\Lambda \subset \Lambda'$, $\mathcal{G}_\Lambda(\pi) \supset \ _{\Lambda'}(\pi)$.

A way of giving a system of partitions is to define it by means of a "vector" of potentials : $\bar{\Psi} = (\psi_1, \psi_2, \ldots, \psi_n)$. We suppose all ψ_i belong to $\mathcal{B}$. Let $\bar{m} \in R^n$ be a "value" taken by $V^{\bar{\Psi}}(.) = (V^{\bar{\Psi}i}(.))$; we denote $\Omega(\Lambda, \bar{m})$ the set of configurations ω_Λ of $\Omega(\Lambda)$ such that $V^{\bar{\Psi}}(\omega_\Lambda) = \bar{m}$. Further :

$$\Omega(\Lambda, \omega_\Lambda) = \Omega(\Lambda, V^{\bar{\Psi}}(\omega_\Lambda))$$

is the set of all configurations of $\Omega(\Lambda)$ that give the same value to $V^{\bar{\Psi}}$ as ω_Λ : $\Omega(\Lambda, \omega_\Lambda)$ is the "energy level" of ω_Λ. The system of partitions $\pi(\bar{\Psi})$ is given by :

$$\pi(\Lambda) = \{\Omega(\Lambda, \omega_\Lambda), \omega_\Lambda \in \Omega(\Lambda)\}$$

In general such a system is not coherent.

To get a coherent system, take $\bar{\Psi}$ like above and define :

$$\tilde{\Omega}(\Lambda, \bar{m}, \eta_{\Lambda^c}) = \{\omega_\Lambda \in \Omega(\Lambda) : U^{\bar{\Psi}}(\omega_\Lambda | \eta_{\Lambda^c}) = \bar{m}\}$$

$$\tilde{\Omega}(\Lambda, \omega_\Lambda) = \bigcap_{\eta_{\Lambda^c} \in \Omega(\Lambda^c)} \Omega(\Lambda, U^{\bar{\Psi}}(\omega_\Lambda | \eta_{\Lambda^c}), \eta_{\Lambda^c})$$

$$\tilde{\pi}(\Lambda) = \{\tilde{\Omega}(\Lambda, \omega_\Lambda), \omega_\Lambda \in \Omega(\Lambda)\}$$

$\tilde{\Omega}(\Lambda, \omega_\Lambda)$ is the set of configurations of $\Omega(\Lambda)$ that give the same value to $U^{\bar{\Psi}}$ as ω_Λ whatever the external condition η_{Λ^c} is. For the same set $\bar{\Psi}$ of potentials, we have the relation :

$$\tilde{\Omega}(\Lambda, \omega_\Lambda) \subset \Omega(\Lambda, \omega_\Lambda).$$

Examples

1. $n = 0$. There are no "conditioning potentials" so that we are in the "grand canonical" formalism.

2. $n = 1 : \psi_1 (\omega_\Lambda) = \begin{cases} 1 & \text{if } \omega_\Lambda = 1 \text{ and } |\Lambda| = 1 \\ 0 & \text{otherwise} \end{cases}$

It is what is usually called "canonical formalism" ; it has been studied e.g. by Georgii [6]. In this case, $\pi = \tilde{\pi}$.

3. $\Psi = (\psi_i)$ and all ψ_i are finite range potentials. This situation was studied by Thompson [14].

b) Local canonical states

$$\gamma_\Lambda (\omega_\Lambda | \eta) = 1_{\Omega (\Lambda, \eta_\Lambda)} (\omega_\Lambda) \frac{\exp (- U^\varphi (\omega_\Lambda | \eta_{\Lambda^c}))}{Z^\varphi_\Lambda (\eta)}$$

with the canonical partition function :

$$Z^\varphi_\Lambda (\eta) = \sum_{\zeta_\Lambda \in \Omega(\Lambda, \eta_\Lambda)} \exp (- U^\varphi (\zeta_\Lambda | \eta_{\Lambda^c}))$$

$\gamma_\Lambda (\omega_\Lambda | \eta)$ is to be interpreted as the conditional probability of finding ω_Λ in $\Omega (\Lambda)$, knowing that we are on the energy level of η_Λ and that the external condition is η_{Λ^c}.

Properties of the local canonical states

1. $\gamma_\Lambda (\omega_\Lambda | .)$ is $\mathcal{G}_\Lambda$ measurable

2. $\gamma_\Lambda (. | \eta)$ is a probability on $\Omega (\Lambda)$ carried by $\Omega (\Lambda, \eta_\Lambda)$

3. Relation to the local Gibbs states :

$$\gamma_\Lambda (\omega_\Lambda, (\omega_\Lambda \eta_{\Lambda^c})) = \frac{\nu_\Lambda (\omega_\Lambda, \eta_{\Lambda^c})}{\nu_\Lambda (\Omega (\Lambda, \omega_\Lambda), \eta_{\Lambda^c})}$$

c) Canonical states

A probability measure μ on $(\Omega, \mathcal{F})$ is a canonical state associated to potential φ and the system of partititions π, if for any Λ and any ω_Λ of $\Omega(\Lambda)$:

$$E_\mu [1_{\{\omega_\Lambda\}} | \mathcal{G}_\Lambda] (\eta) = \gamma_\Lambda (\omega_\Lambda | \eta) \qquad \mu\text{-a.s.}$$

The set of canonical states associated to φ and π is noted $\mathcal{C}(\varphi, \pi)$ and when π is determined by a family $\overline{\Psi}$ of potentials, we write $\mathcal{C}(\varphi, \overline{\Psi}, \pi)$ or $\mathcal{C}(\varphi, \overline{\Psi}, \tilde{\pi})$ ($\tilde{\pi}$ is used when the system of partitions is defined in the coherent way). As the system of partitions is not necessarily coherent, $\{\gamma_\Lambda\}$ is not necessarily a specification in the usual sense (see Preston [11]).

The following theorem solves the problem of existence of canonical states.

Theorem 3.1 - The set $\mathcal{G}(\varphi)$ of Gibbs states associated to φ is included in $\mathcal{C}(\varphi, \pi)$. More generally, if π is finer than π', then $\mathcal{C}(\varphi, \pi') \subset \mathcal{C}(\varphi, \pi)$. In particular, if $\overline{\Psi} = (\psi_1, \ldots, \psi_n)$ is a family of potentials, we have :

$$\mathcal{G}(\varphi) \subset \mathcal{C}(\varphi, \overline{\Psi}, \pi) \subset \mathcal{C}(\varphi, \overline{\Psi}, \tilde{\pi})$$

Proof : it is easily derived from the proof of Georgii [6], (1.10).

Let $\mathcal{C}_0(\varphi, \pi)$ denote the set of translation invariant states in $\mathcal{C}(\varphi, \pi)$.

Theorem 3.2 - The convex sets $\mathcal{C}(\varphi, \tilde{\pi})$ and $\mathcal{C}_0(\varphi, \tilde{\pi})$ have an H-sufficient σ-algebra in Dynkin's sense ; in particular any probability in these sets admits a unique integral representation in term of extreme points. Moreover, a state in $\mathcal{C}_0(\varphi, \tilde{\pi})$ is extremal if and only if it is ergodic, i.e. if it is 0-1 on the σ-algebra $\mathcal{J}$ of translation invariant sets of $\mathcal{F}$.

A fine proof of this result for $\mathcal{C}(\varphi, \tilde{\pi})$ is to be found in Dynkin [2] for instance. The corresponding result for $\mathcal{C}_0(\varphi, \tilde{\pi})$ is easily derived from Dynkin [2], § 3.5. and from the fact that the σ-algebra $\mathcal{J}$ is almost surely contained in $\mathcal{G}_\infty(\tilde{\pi})$, the intersection of all canonical σ-algebras $\mathcal{G}_\Lambda(\tilde{\pi})$, for any translation invariant probability (see Georgii [4]). This fact implies also the last assertion of the theorem.

4. A CONDITIONAL VARIATIONAL PRINCIPLE FOR CANONICAL STATES

In this section, we drop the superscripts in V^φ, W^φ, U^φ, as we consider general systems of partitions ; $\Omega(\Lambda, \eta)$ is set for $\Omega(\Lambda, \eta_\Lambda)$. In the sequel, "$\lim_{\Lambda \uparrow Z^\nu}$" means that we take the limit along the "rectangular" boxes. The results would also hold with limits in the sense of Van Hove.

Theorem 4.1 - Let φ be a potential belonging to $\mathcal{B}$ and π a system of canonical partitions. For any translation invariant probability measure μ on $(\Omega, \mathcal{F})$ and for any fixed configuration η,

$$\limsup_{\Lambda \uparrow Z^\nu} \frac{1}{|\Lambda|} \Big[- \sum_{\omega_\Lambda \in \Omega(\Lambda,\eta)} \frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda, \eta))} \ln \frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda, \eta))}$$

$$- \sum_{\omega_\Lambda \in \Omega(\Lambda,\eta)} \frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda, \eta))} V(\omega_\Lambda) - \ln \sum_{\zeta_\Lambda \in \Omega(\Lambda,\eta)} \exp(-V(\zeta_\Lambda))\Big] \leq 0$$

Moreover, if μ belongs to $\mathcal{C}(\varphi, \pi)$, then the limit of the above expression actually exists and is zero.

Proof : let $Z_\Lambda((\eta)_\Lambda, 0)$ denote $\sum_{\zeta_\Lambda \in \Omega(\Lambda,\eta)} \exp(-V(\zeta_\Lambda))$.

1. Using Jensen's inequality, we get

$$\sum_{\omega_\Lambda \in \Omega(\Lambda,\eta)} \frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda, \eta))} \Big[- \ln \frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda, \eta))} - V(\omega_\Lambda) - \ln Z_\Lambda((\eta)_\Lambda, 0)\Big]$$

$$= \sum_{\omega_\Lambda \in \Omega(\Lambda,\eta)} \frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda, \eta))} \ln \Big[\frac{\exp(-V(\omega_\Lambda))}{Z_\Lambda((\eta)_\Lambda, 0)} \cdot \Big(\frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda, \eta))}\Big)^{-1}\Big] \leq 0$$

This proves the first assertion of the theorem.

2. If μ belongs to $\mathscr{C}(\varphi, \pi)$,

$$\mu(\omega_\Lambda) = \int \gamma_\Lambda(\omega_\Lambda|\eta)\,\mu(d\eta)$$

$$= \int 1_{\Omega(\Lambda,\omega_\Lambda)}(\eta_\Lambda)\,\gamma_\Lambda(\omega_\Lambda, \omega_\Lambda\eta_{\Lambda^c})\,\mu(d\eta)$$

We fix an element $\Omega(\Lambda, i)$ of the partition $\pi(\Lambda)$ and denote by $Z_\Lambda(i)$ the corresponding canonical partition function with 0 external condition. As $-t\ln t$ is a concave function of t,

$$\sum_{\omega_\Lambda\in\Omega(\Lambda,i)} \left[-\frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda,i))}\ln\frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda,i))} - \frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda,i))}V(\omega_\Lambda)\right] - \ln Z_\Lambda(i) \qquad (1)$$

$$\geqslant \int \frac{\mu(d\eta)}{\mu(\Omega(\Lambda,i))}\,1_{\Omega(\Lambda,i)}(\eta_\Lambda) \sum_{\omega_\Lambda\in\Omega(\Lambda,i)} \gamma_\Lambda(\omega_\Lambda|\omega_\Lambda\eta_{\Lambda^c})\,[-\ln\gamma_\Lambda(\omega_\Lambda|\omega_\Lambda\eta_{\Lambda^c}) - V(\omega_\Lambda) - \ln Z_\Lambda(i)]$$

Let us work on the terms between brackets in the last expression :

$$-\ln\gamma_\Lambda(\omega_\Lambda|\omega_\Lambda\eta_{\Lambda^c}) - V(\omega_\Lambda) - \ln Z_\Lambda(i)$$

$$= W(\omega_\Lambda|\eta_{\Lambda^c}) + \ln \sum_{\omega_\Lambda\in\Omega(\Lambda,i)} \gamma_\Lambda(\omega_\Lambda, (\omega_\Lambda, 0))\exp(-W(\omega_\Lambda|\eta_{\Lambda^c}))$$

$$\geqslant W(\omega_\Lambda|\eta_{\Lambda^c}) - \sum_{\omega_\Lambda\in\Omega(\Lambda,i)} \gamma_\Lambda(\omega_\Lambda, (\omega_\Lambda, 0))\,W(\omega_\Lambda|\eta_{\Lambda^c})$$

According to the following lemma, expression (1), divided by $|\Lambda|$, tends to zero as Λ tends to Z^ν.

Lemma 4.1 $\dfrac{|W(\omega_\Lambda|\eta_{\Lambda^c})|}{|\Lambda|}$ tends to zero, uniformly in ω_Λ and η_{Λ^c} when Λ tends to Z^ν.

Proof : see Georgii [5], p. 76.

Corollary 4.1 - If μ is an invariant probability measure on $(\Omega, \mathscr{F})$,

$$s(\mu) - e(\mu,\varphi) \leqslant \liminf_{\Lambda\nearrow Z^\nu} \frac{1}{|\Lambda|} \sum_{\Omega(\Lambda,i)\in\pi(\Lambda)} \mu(\Omega(\Lambda,i))\ln\left(\frac{Z(\Lambda,i)}{\mu(\Omega(\Lambda,i))}\right)$$

where the specific entropy $s(\mu) = \lim_{\Lambda \nearrow Z^\nu} - \frac{1}{|\Lambda|} \sum_{\omega_\Lambda \in \Omega(\Lambda)} \mu(\omega_\Lambda) \ln \mu(\omega_\Lambda)$

and the specific energy $e(\mu, \varphi) = \lim_{\Lambda \nearrow Z^\nu} \frac{1}{|\Lambda|} \sum_{\omega_\Lambda \in \Omega(\Lambda)} \mu(\omega_\Lambda) V(\omega_\Lambda)$

(those two limits are known to exist : see, for instance, Ruelle [13]). Moreover, if μ belongs to $\mathcal{C}(\varphi, \pi)$,

$$\lim_{\Lambda \nearrow Z^\nu} \frac{1}{|\Lambda|} \sum_{\Omega(\Lambda, i) \in \pi(\Lambda)} \mu(\Omega(\Lambda, i)) \ln \frac{Z(\Lambda, i)}{\mu(\Omega(\Lambda, i))} = s(\mu) - e(\mu, \varphi)$$

Remark : $- \sum_i \mu(\Omega(\Lambda, i)) \ln \mu(\Omega(\Lambda, i))$ is the entropy of the partition $\pi(\Lambda)$.

Proof : just take the mean w.r.t. μ of the expression in theorem 4.1.

5. INFORMATION GAIN

Let λ and μ be two probability measures on $(\Omega, \mathcal{F})$ and η a configuration. The local information gain of λ on μ, $s(\Lambda, \eta; \lambda, \mu)$ is :

$$s(\Lambda, \eta; \lambda, \mu) = - \frac{1}{|\Lambda|} \sum_{\omega_\Lambda \in \Omega(\Lambda, \eta)} \frac{\lambda(\omega_\Lambda)}{\lambda(\Omega(\Lambda, \eta))} \ln \left(\frac{\lambda(\omega_\Lambda)}{\lambda(\Omega(\Lambda, \eta))} \cdot \left(\frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda, \eta))} \right)^{-1} \right)$$

Proposition 5.1 - Suppose λ and μ are two invariant probability measures on (Ω, Λ) and μ belongs to $\mathcal{C}(\varphi, \pi)$. If Λ is large enough, the absolute value of the difference between $s(\Lambda, \eta; \lambda, \mu)$ and

$$- \frac{1}{|\Lambda|} \sum_{\omega_\Lambda \in \Omega(\Lambda, \eta)} \frac{\lambda(\omega_\Lambda)}{\lambda(\Omega(\Lambda, \eta))} \left[\ln \frac{\lambda(\omega_\Lambda)}{\lambda(\Omega(\Lambda, \eta))} + V(\omega_\Lambda) + \ln Z_\Lambda ((\eta)_\Lambda, 0) \right]$$

is less than ε.

Corollary 5.1 λ verifies the conditional variational principle if and only if $\lim_{\Lambda \nearrow Z^\nu} s(\Lambda, \eta; \lambda, \mu) = 0$

Proof of proposition 5.1 : as μ belongs to $\mathcal{C}(\varphi, \pi)$,

$$\sum_{\omega_\Lambda \in \Omega(\Lambda,\eta)} \frac{\lambda(\omega_\Lambda)}{\lambda(\Omega(\Lambda,\eta))} \ln \frac{\mu(\omega_\Lambda)}{\mu(\Omega(\Lambda,\eta))}$$

$$= \sum_{\omega_\Lambda \in \Omega(\Lambda,\eta)} \frac{\lambda(\omega_\Lambda)}{\lambda(\Omega(\Lambda,\eta))} \ln \int \frac{\mu(d\,\zeta)}{\mu(\Omega(\Lambda,\eta))} 1_{\Omega(\Lambda,\eta)} (\zeta_\Lambda)\, \gamma_\Lambda (\omega_\Lambda | \eta_\Lambda\, \zeta_{\Lambda^c})$$

$$= \sum_{\omega_\Lambda \in \Omega(\Lambda,\eta)} \frac{\lambda(\omega_\Lambda)}{\lambda(\Omega(\Lambda,\eta))} [- V(\omega_\Lambda) - \ln Z_\Lambda ((\eta)_\Lambda, o)$$

$$+ \ln \int \frac{\mu(d\,\zeta)}{\mu(\Omega(\Lambda,\eta))} 1_{\Omega(\Lambda,\eta)} (\zeta_\Lambda) \left[\frac{\exp(- W(\omega_\Lambda | \zeta_{\Lambda^c}))}{\sum_{\xi_\Lambda \in \Omega(\Lambda,\eta)} \frac{\exp(- V(\xi_\Lambda))}{Z_\Lambda((\eta)_\Lambda, o)} \exp - W(\xi_\Lambda | \zeta_{\Lambda^c})} \right]]$$

When Λ is sufficiently large, the argument of the second logarithm in the last expression lies between $\exp(- \varepsilon\, |\Lambda|)$ and $\exp(+ \varepsilon\, |\Lambda|)$ (by lemma 4.1) ; so, the logarithm divided by $|\Lambda|$ tends to zero when Λ tends to Z^ν and the proposition is proved.

6. INVARIANT LOCALLY POSITIVE CANONICAL STATES

In view of the usual variational principle for translation invariant Gibbs states (see Lanford, Ruelle [8] and Föllmer [3]), it is natural to try to characterize the translation invariant canonical states by means of a variational principle, i.e. to prove some kind of converse of theorem 4.1.

We are not able to do it in general. So, we restrict ourselves to the case where the partitions $\tilde{\pi}$ are determined by a vector of finite range potentials $\overline{\Psi}$. We prove that ergodic locally positive states satisfying the variational equality are in fact Gibbs states for some potential $\varphi + \overline{v} \,.\, \overline{\Psi}$ ($\overline{v} \in R^n$) and consequently, are canonical states of $\mathcal{C}(\varphi, \overline{\Psi}, \tilde{\pi})$.

Let us first recall two wellknown results.

Proposition 6.1 - If μ is an invariant probability measure on $(\Omega, \mathcal{F})$, for any φ belonging to $\mathcal{B}$,

$$\lim_{\Lambda \nearrow Z^\nu} \frac{V^\varphi\ ((\omega)_\Lambda)}{|\Lambda|}$$

exists μ-almost surely in ω, is $\mathcal{I}$-measurable and its expectation value is e (μ, φ). Moreover, if μ is ergodic, the limit is μ-almost surely equal to e (μ, φ).

Proof : see Georgii [5], Satz (7.7).

Proposition 6.2 - If φ belongs to $\mathcal{B}$,

$$\lim_{\Lambda \nearrow Z^\nu} \frac{1}{|\Lambda|} \ln \sum_{\omega_\Lambda \in \Omega(\Lambda)} \exp\ (- V^\varphi\ (\omega_\Lambda))$$

exists and defines the "pressure" P (φ). P (φ) is a strictly convex function on $\mathcal{B}$. If φ and θ belong to $\mathcal{B}$,

$$P\ (\varphi - \theta) \leqslant \|\varphi - \theta\|$$

Proof : see Ruelle [13] and Griffiths, Ruelle [7].

Before stating two useful other results due to Thompson, let us define the notion of irreducible potentials. In § 7, we shall try to clarify this concept introduced by Thompson a bit mysteriously. After analysis, an irreducible vector potential appears, in some sense, as the "biggest" determining a given system of partitions.

Consider a vector $\overline{\psi} = (\psi_1, \ldots, \psi_n)$ of independent finite range potentials ; let $\overline{\theta} = (\theta_1, \ldots, \theta_k)$ be a vector of cluster potentials such that there is a linear surjective mapping $T : R^k \to R^n$ with $T\ \overline{\theta} = \overline{\psi}$. $\overline{\psi}$ is irreducible if there is no integer $l : n \leqslant l \leqslant k$ such that $T = T_1 T_2$ with $T_1 : R^l \to R^n$, $T_2 : R^k \to R^l$, two linear mappings sucht that T_2 is surjective and T_1 is not injective on R^1 but is injective on the set $T_2\ (Z^k)$.

Proposition 6.3 - Suppose the system of partitions $\pi\ (\overline{\psi})$ is determined by an irreducible vector $\overline{\psi} = (\psi_1, \ldots, \psi_n)$ of finite range, linearly independent potentials. Let Λ_k be a sequence of boxes tending to Z^ν in the sense of Van Hove ; suppose η is a configuration such that

$$\lim_{k\to\infty} \frac{V^{\overline{\psi}}\ ((\eta)_{\Lambda_k})}{|\Lambda_k|}$$

exists ; denote this limit $\overline{\rho}$ and assume there exists $\overline{v}$ in R^n such that

$$P\ (\varphi + \overline{v}\ .\ \overline{\psi}) + \overline{v}\ .\ \overline{\rho} = \min_{\overline{w} \in R^n}\ [P\ (\varphi + \overline{w}\ .\ \overline{\psi}) + \overline{w}\ .\ \overline{\rho}]$$

(with $\overline{v}\ .\ \overline{\psi}$ the euclidean scalar product in R^n). If φ has finite range :

$$\lim_{k\to\infty} \frac{1}{|\Lambda_k|} \ln Z^{\varphi}_{\Lambda_k}\ ((\eta)_{\Lambda_k}) =$$

$$\lim_{k\to\infty} \frac{1}{|\Lambda_k|} \ln Z^{\varphi}_{\Lambda_k}\ ((\eta)_{\Lambda k}, \xi_k) = P\ (\varphi + \overline{v}\ .\ \overline{\psi}) + \overline{v}\ .\ \overline{\rho} \qquad (2)$$

uniformly in the finite configuration ξ_k of Λ_k^c

with $Z^{\varphi}_{\Lambda_k}\ ((\eta)_{\Lambda_k}) = \sum_{\omega_{\Lambda_k} \in \Omega(\Lambda_k, \eta)} \exp\ [-\ V^{\varphi}\ (\omega_{\Lambda_k})]$

and $Z^{\varphi}_{\Lambda_k}\ ((\eta)_{\Lambda_k}, \xi_k) = \sum_{\omega_{\Lambda_k} \in \Omega(\Lambda_k, \eta_{\Lambda_k}, \xi_k)} \exp\ [-\ V^{\varphi}\ (\omega_{\Lambda_k}, \xi_k)]$

$$\Omega\ (\Lambda_k, \eta_{\Lambda_k}; \xi_k) = \{\omega_{\Lambda_k} \in \Omega\ (\Lambda_k) : U^{\overline{\psi}}\ (\omega_{\Lambda_k}, \xi_k) = U^{\overline{\psi}}\ (\eta_{\Lambda_k}, \xi_k)\}$$

Remark : the first term of (2) is a special case of the second one for $\xi_k \equiv 0$; moreover,

$$\Omega(\Lambda_k, \eta_{\Lambda_k}, \xi_k) = \{\omega_{\Lambda_k} \in \Omega(\Lambda_k) : V^{\overline{\psi}}(\omega_{\Lambda_k}, \xi_k) = V^{\overline{\psi}}(\eta_{\Lambda_k}, \xi_k)\}$$

Proposition 6.4 - Let $\overline{\psi}$ be as in proposition 6.3 and suppose φ has finite range. If a translation invariant probability μ is locally positive, i.e. gives a positive measure to any cylinder set, then there exists $\overline{v} \in R^n$ such that :

$$P(\varphi + \overline{v}.\overline{\psi}) + \overline{v}.e(\mu, \overline{\psi}) = \min_{\overline{w} \in R^n} [P(\varphi + \overline{w}.\overline{\psi}) + \overline{w}.e(\mu, \overline{\psi})]$$

Proof : see Thompson [14], theorem 2.2 and lemmas 3.7 (p. 69) and 2.4.

Proposition 6.3 can be extended to deal with the coherent system of partitions $\tilde{\pi}(\overline{\psi})$ associated to a family of potentials $\overline{\psi}$.

Proposition 6.5 - If the hypothesis of proposition 6.3 holds with $\pi(\overline{\psi})$ replaced by $\tilde{\pi}(\overline{\psi})$ as only change, then

$$\lim_{k\to\infty} \frac{1}{|\Lambda_k|} \ln \tilde{Z}^{\varphi}_{\Lambda_k}((\eta)_{\Lambda_k}) = P(\varphi + \overline{v}.\overline{\psi}) + \overline{v}.\overline{\rho}$$

with $\tilde{Z}^{\varphi}_{\Lambda_k}((\eta)_{\Lambda_k}) = \sum_{\omega_{\Lambda_k} \in \tilde{\Omega}(\Lambda_k, \eta)} \exp[-V^{\varphi}(\omega_{\Lambda_k})]$

Proof : let L be the maximum range of the potentials ψ_i ; denote by Λ_k^- the subset of points of Λ_k such that their distance to Λ_k^c is strictly greater than L ; according to Thompson's definition of Van Hove's convergence ([14], p. 9), Λ_k^- tends to Z^ν. As $\tilde{Z}^{\varphi}_{\Lambda_k}((\eta)_{\Lambda_k})$ is less than $Z^{\varphi}_{\Lambda_k}((\eta)_{\Lambda_k})$, proposition 6.3 yields :

$$\limsup_k \frac{1}{|\Lambda_k|} \ln \tilde{Z}^{\varphi}_{\Lambda_k}((\eta)_{\Lambda_k}) \leqslant P(\varphi + \overline{v}.\overline{\psi}) + \overline{v}.\overline{\rho}$$

On the other hand, if we denote $\Omega(\Lambda_k^-, \eta_{\Lambda_k})$ the set $\{\omega_{\Lambda_k^-} \in \Omega(\Lambda_k^-) : V^{\overline{\Psi}}(\omega_{\Lambda_k^-}, \eta_{\partial^-(\Lambda_k)}) = V^{\psi}(\eta_{\Lambda_k})\}$ and $\partial^-(\Lambda_k) = \Lambda_k \setminus \Lambda_k^-$:

$$\tilde{Z}^{\varphi}_{\Lambda_k}((\eta)_{\Lambda}) \geqslant \sum_{\omega_{\Lambda_k^-} \in \Omega(\Lambda_k^-, \eta_{\Lambda_k})} \exp[- V^{\varphi}(\omega_{\Lambda_k^-}, \eta_{\partial^-(\Lambda_k)})]$$

$$= \exp[- V^{\varphi}(\eta_{\partial^-(\Lambda_k)})] \, Z^{\varphi}_{\Lambda_k^-}((\eta)_{\Lambda_k}, (\eta)_{\partial^-(\Lambda_k)})$$

As $|V^{\varphi}(\eta_{\partial^-(\Lambda_k)})| \leqslant |\partial^-(\Lambda_k)| \, \|\varphi\|$, by proposition 6.3 :

$$\liminf_k \ln \tilde{Z}^{\varphi}_{\Lambda_k}((\eta)_{\Lambda_k}) \geqslant P(\varphi + \overline{v}.\overline{\Psi}) + \overline{v}.\overline{\rho}$$

and proposition 6.5 follows.

Theorem 6.1 - Suppose the following :

a) φ belongs to $\mathcal{B}$;

b) the system of partitions $\tilde{\pi}(\overline{\Psi})$ is given by an irreducible vector $\overline{\Psi} = (\psi_1, \ldots, \psi_n)$ of finite range, independent potentials ;

c) μ is ergodic, locally positive and for almost all configuration η :

$$s(\mu) - e(\mu, \varphi) = \lim_{\Lambda \nearrow Z^{\nu}} \frac{1}{|\Lambda|} \int \ln \frac{\tilde{Z}^{\varphi}_{\Lambda}((\eta)_{\Lambda})}{\tilde{\mu}(\Omega(\Lambda, \eta))} \mu(d\eta)$$

Then μ is a Gibbs state associated to potential $\varphi + \overline{v}.\overline{\Psi}$ and, consequently, μ belongs to $\mathcal{G}(\varphi, \overline{\Psi}, \tilde{\pi})$.

Remark : if partition $\tilde{\pi}(\overline{\Psi})$ is determined by a reducible vector of potentials, we'll see in § 7 there exists an irreducible vector determining the same partition.

Proof :

1. Let us first suppose φ has finite range. As μ is ergodic, $\frac{V^{\overline{\Psi}}((\eta)_{\Lambda})}{|\Lambda|}$ converges μ-a.s. to $e(\mu, \overline{\Psi})$ and $\frac{V^{\varphi}((\eta)_{\Lambda})}{|\Lambda|}$ to $e(\mu, \varphi)$ (proposition 6.1).

2. For μ-almost all η, $\lim_{\Lambda \nearrow Z^\nu} \frac{1}{|\Lambda|} \ln \tilde{Z}^\varphi_\Lambda ((\eta)_\Lambda) = P(\varphi + \bar{v}.\bar{\Psi}) + \bar{v}.e(\mu, \bar{\Psi})$

with $\bar{v}$ as in proposition 6.4. The functions $\frac{1}{|\Lambda|} \ln \tilde{Z}^\varphi_\Lambda ((\eta)_\Lambda)$ are uniformly bounded, equi-integrable and therefore, converge also in L^1-norm to $P(\varphi + \bar{v}.\bar{\Psi}) + \bar{v}.e(\mu, \bar{\Psi})$.

3. So we have :

$$0 = s(\mu) - e(\mu, \varphi) - \lim_{\Lambda \nearrow Z^\nu} \frac{1}{|\Lambda|} \int \ln \tilde{Z}^\varphi_\Lambda ((\eta)_\Lambda) \, \mu(d\eta) + \lim_{\Lambda \nearrow Z^\nu} \frac{1}{|\Lambda|} \sum_i \mu(\tilde{\Omega}(\Lambda, i)) \ln \mu(\tilde{\Omega}(\Lambda, i))$$

$$= s(\mu) - e(\mu, \varphi) - P(\varphi + \bar{v}.\bar{\Psi}) - \bar{v}.e(\mu, \bar{\Psi}) + \lim_{\Lambda \nearrow Z^\nu} \frac{1}{|\Lambda|} \sum_i \mu(\tilde{\Omega}(\Lambda, i)) \ln \mu(\tilde{\Omega}(\Lambda, i))$$

$$\leqslant s(\mu) - e(\mu, \varphi + \bar{v}.\bar{\Psi}) - P(\varphi + \bar{v}.\bar{\Psi}) \leqslant 0$$

The last inequality follows from the classical variational principle for Gibbs states. From $s(\mu) - e(\mu, \varphi + \bar{v}.\bar{\Psi}) - P(\varphi + \bar{v}.\bar{\Psi}) = 0$, we conclude that μ belongs to $\mathcal{G}(\varphi + \bar{v}.\bar{\Psi})$ and as readily seen to $\mathcal{C}(\varphi, \bar{\Psi}, \tilde{\pi})$. By the way, observe that :

$$\lim_{\Lambda \nearrow Z^\nu} \frac{1}{|\Lambda|} \sum \mu(\tilde{\Omega}(\Lambda, i)) \ln \mu(\tilde{\Omega}(\Lambda, i)) = 0$$

4. If φ has infinite range, consider an increasing sequence of boxes Λ_k tending to Z^ν and define a sequence of potentials φ_k with $\lim_{k \to \infty} \|\varphi_k - \varphi\| = 0$:

$$\varphi_k(\omega_\Lambda) = \begin{cases} \varphi(\omega_\Lambda) & \text{if there is a translate of } \Lambda \text{ included in } \Lambda_k \\ 0 & \text{otherwise} \end{cases}$$

Let $f_k(\overline{w}) = P(\varphi_k + \overline{w}.\overline{\Psi}) + \overline{w}.e(\mu, \overline{\Psi})$; proposition 6.2 and the fact that $|e(\mu, \varphi_k) - e(\mu, \varphi)| \leqslant \|\varphi_k - \varphi\|$ yield :

$$\lim_k f_k(\overline{w}) = f(\overline{w}) := P(\varphi + \overline{w}.\overline{\Psi}) + \overline{w}.e(\mu, \overline{\Psi})$$

As μ is locally positive, proposition 6.4 applies ; let $\overline{v}_k$ be such that

$$f_k(\overline{v}_k) = \min_{\overline{w} \in R^n} f_k(\overline{w})$$

5. To prove that $\overline{v}_k$ tends to $\overline{v}$ and $f(\overline{v}_k)$ to $f(\overline{v})$ as k tends to infinity, observe that the equicontinuous sequence f_k converges uniformly to f on all compact sets of R^n. Then use some topological or geometrical arguments.

6. Reasoning as in Ruelle [13], for the pressure, we have for any Λ, η, k :

$$\left| \frac{1}{|\Lambda|} \ln \tilde{Z}_{\Lambda}^{\varphi_k}((\eta)_{\Lambda}) - \frac{1}{|\Lambda|} \ln \tilde{Z}_{\Lambda}^{\varphi}((\eta)_{\Lambda}) \right| \leqslant \|\varphi_k - \varphi\|$$

7. Using successively 5, proposition 6.5, 2, 6, 3, we get :

$$P(\varphi + \overline{v}.\overline{\Psi}) + \overline{v}.e(\mu, \overline{\Psi}) = \lim_{\Lambda \nearrow Z^{\nu}} \frac{1}{|\Lambda|} \int \mu(d\eta) \ln \left(\frac{\tilde{Z}_{\Lambda}^{\varphi}((\eta)_{\Lambda})}{\mu(\tilde{\Omega}(\Lambda, \eta))} \right)$$

So, we may conclude as in 3.

As a by-product, we have :

Corollary 6.1 - Under the hypothesis of theorem 6.1, the specific entropy of the canonical partition $\tilde{\pi}(\overline{\Psi})$ is equal to zero :

$$\lim_{\Lambda \nearrow Z^{\nu}} \frac{1}{|\Lambda|} \Sigma \mu(\tilde{\Omega}(\Lambda, i)) \ln \mu(\tilde{\Omega}(\Lambda, i)) = 0$$

Theorem 6.2 - Let $\mathscr{C}_0(\varphi, \overline{\psi}, \tilde{\pi})$ be the set of translation invariant canonical states associated to a potential φ, with partition $\tilde{\pi}(\overline{\psi})$ determined by an irreducible vector $\overline{\psi}$ of finite range independent potentials. The subset of translation invariant locally positive probabilities in $\mathscr{C}_0(\varphi, \overline{\psi}, \tilde{\pi})$ is exactly the convex set $\bigcup_{\overline{v}\in R^n} \mathscr{G}(\varphi + \overline{v}.\overline{\psi})$.

7. REDUCIBLE POTENTIALS

a) In this chapter we study completely the concepts of reducible and irreducible potentials and relate them to the system of partitions they induce. Let us first give some notation. Denote vect $(\psi_1, \ldots \psi_n)$ the real vector space generated by the potentials ψ_i. Consider a finite range potential ψ as a linear combination of cluster potentials, i.e. θ is a cluster potential if there is a finite subset X of Z^ν such that

$$\theta(\omega_\Lambda) = \begin{cases} 1 \text{ if } \Lambda \text{ is a translate of } X \text{ and } \omega_\Lambda \equiv 1 \\ 0 \text{ otherwise.} \end{cases}$$

$C(\psi_1, \ldots, \psi_n)$ denotes the set of cluster potentials $\psi_1, \ldots \psi_n$ are made of. In the sequel, we abbreviate "independent finite range" potentials by i.f.r. potentials ; $\mathcal{V}(\overline{\psi})$ denotes the set of values taken by $V^{\overline{\psi}}$.

Just before proposition 6.3, we defined the concept of "irreducible vector $\overline{\psi} = (\psi_1, \ldots, \psi_n)$ of i.f.r. potentials". Thompson [14], p. 37, proves this definition independent of the choice of vector $\overline{\theta}$; in fact we can take for $\overline{\theta}$ a vector made of the cluster potentials arising in $\psi_1, \ldots \psi_n$. A vector $\overline{\psi}$ is reducible if not irreducible.

b) Let $\overline{\psi} = (\psi_1, \ldots, \psi_n)$ be a vector of i.f.r. potentials ; the vector $\overline{\psi}' = (\psi'_1, \ldots, \psi'_n)$ of i.f.r. potentials is a strict reduction of $\overline{\psi}$ if there are linear mappings

$$T_1 : R^l \to R^n$$
$$T_2 : R^k \to R^l$$

with $C(\psi'_1, \ldots, \psi'_l) = \{\theta_1, \ldots, \theta_k\}$

$$\overline{\theta} = (\theta_1, \ldots, \theta_k)$$
$$T_2 \overline{\theta} = \overline{\Psi}' \quad ; \quad T_1 \overline{\Psi}' = \overline{\Psi}$$

and T_1 is one-to-one on $T_2 Z^k$, but not on R^l. We call $\overline{\Psi}'$ a reduction of $\overline{\Psi}$ if $\overline{\Psi}'$ is a strict reduction of $\overline{\Psi}$ or if there is a linear permutation P of R^n such that $\overline{\Psi}' = P \overline{\Psi}$.

Remark 1 - If $\overline{\Psi}'$ is a strict reduction of $\overline{\Psi}$, $\overline{\Psi}$ is reducible and $l > n$.

Remark 2 - The relation "is a reduction of" is an order relation.

Remark 3 - If $\overline{\Psi}$ is reducible, it has an irreducible strict reduction $\overline{\Psi}' = (\psi'_1, \ldots, \psi'_l)$ with $\{\psi'_1, \ldots, \psi'_l\}$ included in vect $(C(\psi_1, \ldots \psi_n))$.

c) Thompson ([14], p. 35) proved that if $\overline{\Psi}'$ is a reduction of $\overline{\Psi}$, then $V^{\overline{\Psi}}(\omega_X) = V^{\overline{\Psi}}(\zeta_X)$ iff $V^{\overline{\Psi}'}(\omega_X) = V^{\overline{\Psi}'}(\zeta_X)$ for every finite part X of Z^ν and every configurations ω, ζ. This is because T_1 defines a bijection between $\mathcal{V}(\overline{\Psi}')$ and $\mathcal{V}(\overline{\Psi})$. It means that $\overline{\Psi}$ and $\overline{\Psi}'$ determine the same systems of partitions : in this case we'll call $\overline{\Psi}$ and $\overline{\Psi}'$ π-equivalent. We can prove a kind of converse of Thompson's result.

Theorem 7.1 - Let $\overline{\Psi} = (\psi_1, \ldots, \psi_n)$, $\overline{\Psi}' = (\psi'_1, \ldots, \psi'_l)$ be two π-equivalent vectors of i.f.r. potentials. If $l > n$, $\overline{\Psi}$ is reducible ; moreover either $\overline{\Psi}'$ is irreducible and is a reduction of $\overline{\Psi}$, either $\overline{\Psi}'$ is reducible and $\overline{\Psi}$, $\overline{\Psi}'$ admit a common irreducible strict reduction. If $l = n$, either $\overline{\Psi}$, $\overline{\Psi}'$ are both irreducible and there exists a linear permutation P of R^n with $\overline{\Psi}' = P \overline{\Psi}$, either $\overline{\Psi}$, $\overline{\Psi}'$ are both reducible and have a common irreducible strict reduction.

<u>Proof</u> :

1. With a view to prove the reducibility of $\overline{\Psi}$, if $l > n$, we may suppose $\{\psi_1, \ldots, \psi_n\} \subset \text{vect}\,(\psi'_1, \ldots, \psi'_l)$; otherwise, construct a family $\{\psi''_1, \ldots, \psi''_m\}$ with $\psi''_i = \psi'_i$ for $i = 1, \ldots, l$ and the other potentials choosed among $\{\psi_1, \ldots, \psi_n\}$ so that $\{\psi_1, \ldots, \psi_n\}$ is included in vect $(\psi''_1, \ldots, \psi''_m)$ and $\{\psi''_1, \ldots, \psi''_m\}$ is an independent family ; in that case, $\overline{\Psi}'' = (\psi''_1, \ldots, \psi''_m)$ and $\overline{\Psi}$ are π-equivalent.

2. Let $C\,(\psi'_1, \ldots, \psi'_l) = \{\theta_1, \ldots, \theta_k\}$ and $\overline{\theta}$ a corresponding vector. A linear mapping $T_2 : R^k \to R^l$ is defined by $T_2\,\overline{\theta} = \overline{\Psi}'$. Then define $T_1 : R^l \to R^n$, linear, such that $T_1\,\overline{\Psi}' = \overline{\Psi}$; T_1 is not injective as $l > n$. But T_1 is injective on the set $\mathcal{V}(\overline{\Psi}')$ because $T_1\,V^{\overline{\Psi}'}(.) = V^{\overline{\Psi}}(.)$ and $\overline{\Psi}$, $\overline{\Psi}'$ are π-equivalent. The set $\mathcal{V}(\overline{\Psi}')$ is included in $T_2\,Z^k$ as $\mathcal{V}(\overline{\theta})$ is included in Z^k.

3. From the study of $\mathcal{V}(\overline{\Psi}')$ we can infer that T_1 is injective on $T_2\,Z^k$. The set of values taken by $\overline{\Psi}'$ contains a basis of R^l (as the set of values of $\overline{\theta}$ does) ; Möbius' inversion formula implies that $\mathcal{V}(\overline{\Psi}')$ also contains a basis $\{v_1, \ldots, v_l\}$ of R^l. As all ψ'_i are finite range potentials, $\mathcal{V}(\psi')$ contains the set K of all linear combinations with non negative integer coefficients of $v_1, \ldots, v_l$. Suppose there exists v in $T_2\,Z^k$ with $v \neq 0 = T_1\,v$; it is always possible to find w in K such that $w + v$ is again in K ; then $T_1\,w = T_1\,(w + v)$ which is impossible. So, 0 is the only point of $T_2\,Z^k$ in the kernel of T_1 ; this implies T_1 is one-to-one on $T_2\,Z^k$.

4. If $\{\psi_1, \ldots, \psi_n\}$ is included in vect $(\psi'_1, \ldots, \psi'_l)$, $\overline{\Psi}'$ is clearly a reduction of $\overline{\Psi}$. Suppose $\overline{\Psi}'$ is irreducible. If $l = n$, vect $(\psi_1, \ldots, \psi_n) =$ vect $(\psi'_1, \ldots, \psi'_l)$ and $\overline{\Psi}$ is irreducible ; when $l > n$, the theorem is also proved. If $\overline{\Psi}'$ is reducible, consider an irreducible strict reduction of $\overline{\Psi}'$ (remark 7.b.3) ; it is also an irreducible strict reduction of $\overline{\Psi}$ (remark 7.b.2).

If $\{\psi_1, \ldots, \psi_n\}$ is not included in vect $(\psi'_1, \ldots, \psi'_l)$, consider a vector $\overline{\Psi}''$ as considered in part 1 of this proof ;

$\overline{\Psi}''$ is a common strict reduction of $\overline{\Psi}$, $\overline{\Psi}'$ and an irreducible strict reduction of both can be found.

Remark : point 3 of the proof shows that the condition for a vector of i.f.r. potentials to be reducible may be weakened without changing the content of the notion : it is sufficient to impose T_1 to be one-to-one on $\mathcal{V}(T_2\, \overline{\sigma})$.

Corollary 7.1 - If $\overline{\Psi} = (\psi_1, \ldots, \psi_n)$ is a vector of i.f.r. potentials, the set of all irreducible reductions of $\overline{\Psi}$ is the set of all basis of some linear subspace of vect $(C\,(\psi_1, \ldots, \psi_n))$; the set of all vectors of i.f.r. potentials π-equivalent to $\overline{\Psi}$ is included in that subspace.

Proof : there is always an irreducible reduction $\overline{\Psi}'$ of $\overline{\Psi}$ in vect $(C\,(\psi_1, \ldots, \psi_n))$; consider another irreducible reduction $\overline{\Psi}''$ of $\overline{\Psi}$; $\overline{\Psi}'$, $\overline{\Psi}''$ are π- equivalent ; by theorem 7.1, they are reductions of each other and have the same number of components.

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