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AMARTS OF FINITE ORDER AND PETTIS CAUCHY SEQUENCES OF BOCHNER INTEGRABLE FUNCTIONS IN LOCALLY CONVEX SPACES (*)

Dinh Quang LUU

Résumé. Dans cet article, on introduit et examine la classe des amarts d'ordre fini à valeurs dans les espaces de Hausdorff localement convexes quasicomplets. On donne une condition nécessaire et suffisante aux termes des amarts d'ordre fini pour qu'une suite adaptée des fonctions fortement intégrables soit de Cauchy pour la topologie de Pettis.

Summary. In the paper we introduce and examine the class of amarts of finite ordre, taking values in Hausdorff locally convex quasi-complete spaces. We give in terms of amarts of finite order a necessary and sufficient condition under which an adapted sequence of Bochner integrable functions is Cauchy in the Pettis topology.

(★) The paper was written during the author's visit at the University of Sciences and Technics of Languedoc 1984.

§ 1 INTRODUCTION.

The extension of Bochner integrals in Banach spaces to locally convex spaces (l.c.s's) are well-known (see, e.g. [1,5,7]) . Let E be a l.c.s. and $L_1(A,E)$ the l.c.s. of all E-valued Bochner integrable functions (see [5]) , defined on some probability space (Ω,A,P) . Suppose that $<A_n>$ is an increasing sequence of subσ-fields of A and $<f_n>$ a sequence in $L_1(A,E)$, adapted to $<A_n>$. Call $<f_n>$ an amart of finite order, if for each $d \in N = \{1,2,\ldots,\}$, the net $<\int_\Omega f_\tau \, dP>_{\tau \in T^d}$ converges in E , where T^d denotes the set of all bounded stopping times each of which takes essentially at most d values. In particular, if the net is convergent for $d = \infty$ with $T^\infty = \bigcup_N T^d$, then $<f_n>$ is called, as usual an amart. Obviously, every amart is that of finite order. A simple remark shows that there is a real-valued amart of finite order which fails to be an amart. The main purpose of the paper is to give some characterizations of amarts of finite order in Hausdorff quasi-complete l.c.s's and to find in terms of amarts of finite order, a necessary and sufficient condition under which an adapted sequence in $L_1(A,E)$ is Cauchy in the Pettis topology. The most important result used in the paper is Theorem 9 [8] which can be applied just to Hausdorff quasi-complete l.c.s's.

§ 2 PREMIMINARIES.

In the sequel, let (Ω,A,P) be a probability space, E a Hausdorff quasi-complete l.c.s. with the 0-neighborhood base U(E) .

Given $U \in \mathcal{U}(E)$, let U° and p_U denote the polar and the continuous seminorm, associated with U, resp. For every vector measure $\mu : \mathcal{A} \to E$ and $U \in \mathcal{U}(E)$, let define

$$V_U(\mu) = \sup \{ \sum_{j=1}^{k} p_U(\mu(A_j)) \mid \langle A_j \rangle_{j=1}^{k} \in \Pi(\mathcal{A},\Omega) \} ,$$

where $\Pi(\mathcal{A},\Omega)$ denotes the collection of all finite $\mathcal{A}$-measurable partitions of Ω,

$$S_U(\mu) = \sup \{ |\langle e,\mu \rangle|(\Omega) \mid e \in U^\circ \} .$$

Let $V(\mathcal{A},E)$ or $S(\mathcal{A},E)$, resp. denote the space of (all V-equivalence or S-equivalence classes, resp. of) E-valued V-bounded or S-bounded measures $\mu : \mathcal{A} \to E$, resp. Thus the V-topology or the S-topology of $V(\mathcal{A},E)$ or of $S(\mathcal{A},E)$, resp. is generated by the family of seminorms $(V_U \mid U \in \mathcal{U}(E))$ or $(S_U \mid U \in \mathcal{U}(E))$, resp.
In what follows we shall need the following result whose proof is similar to those, given in [12, 1.2.4 and 1.3.4] for the spaces $(\ell_N^1(E)$, ε-topology) and $(\ell_N^1\{E\}$, Π-topology).

<u>Lemma</u> 2.1 . <u>Let</u> E <u>be a Hausdorff quasi-complete l.c.s.</u> <u>Then so are spaces</u> $(V(\mathcal{A},E)$, V-topology) <u>and</u> $(S(\mathcal{A},E)$, S-topology).

Let $L_1(\mathcal{A},E)$ be the l.c.s. of all (equivalence classes of) E-valued Bochner integrable functions $f : \Omega \to E$ (see, [5]), topologized by the family of seminorms

$$B_U(f) = \int_\Omega p_U(f)dP \quad (f \in L_1(\mathcal{A},E) ,\ U \in \mathcal{U}(E)) .$$

Call the topology to be the Bochner topology of $L_1(\mathcal{A},E)$.
It is known that every $f \in L_1(\mathcal{A},E)$ is Pettis integrable. Therefore, one can regard $(L_1(\mathcal{A},E)$, B-topology) as a linear subspace of $V(\mathcal{A},E)$ using the following identification

$$f \mapsto \mu_f : \mathcal{A} \to E : \mu_f(A) = \int_A f\, dP \quad (A \in \mathcal{A}).$$

Furthermore, $L_1(\mathcal{A},E)$ with the Pettis topology, defined by the following family of seminorms

$$P_U(f) = \sup \{\int_\Omega |\langle e,f\rangle|\, dP \mid e \in U^\circ\} \quad (f \in L_1(\mathcal{A},E)\ ,\ U \in \mathcal{U}(E))$$

is a linear subspace of $S(\mathcal{A},E)$. Other properties of $L_1(\mathcal{A},E)$ are not known. But using the arguments, analogous to those, given in [2] for Banach spaces we can establish easily the following result :

Lemma 2.2 . Let $\mu \in S(\mathcal{A},E)$, $U \in \mathcal{U}(E)$ and $f \in L_1(\mathcal{B},E)$ for some subσ-field $\mathcal{B}$ of $\mathcal{A}$. Then

(1) $S_U(\mu) \leqslant V_U(\mu)$

(2) $q_U(\mu) = \sup \{p_U(\mu(A)) \mid A \in \mathcal{A}\} \leqslant S_U(\mu) \leqslant 4\, q_U(\mu)$

(3) $q_U(\mu_f) \leqslant S_U(\mu_f) \leqslant 4\, q_U^{\mathcal{B}}(\mu_f) = 4 \sup \{\int_A p_U(f) \mid A \in \mathcal{B}\}$.

§ 3 AMARTS OF FINITE ORDER AND PETTIS CAUCHY SEQUENCES IN $L_1(\mathcal{A},E)$.

Hereafter, let $\langle \mathcal{A}_n \rangle$ be an increasing sequence of subσ-fields of $\mathcal{A}$ with $\Sigma = \bigcup_N \mathcal{A}_n$ and $\mathcal{A} = \sigma(\Sigma)$ and T^∞ the set of all bounded stopping times. A sequence $\langle \mu_n \rangle$ in $S(\mathcal{A},E)$ or $\langle f_n \rangle$ in

$L_1(\mathcal{A},E)$, resp. is said to be adapted to $<\mathcal{A}_n>$ if each $\mu_n \in S(\mathcal{A}_n,E)$ or $f_n \in L_1(\mathcal{A}_n,E)$, resp. We shall consider only such sequences. Let $<\mu_n>$ and $<f_n>$ be given and $\tau \in T^\infty$. Define

$$\mathcal{A}_\tau = \{A \in \mathcal{A} \mid \forall n \in N \quad A \cap \{\tau = n\} \in \mathcal{A}_n\} ,$$

$$\mu_\tau : \mathcal{A}_\tau \to E \quad : \quad \mu_\tau(A) = \sum_N \mu_n(A \cap \{\tau = n\}) , \quad (A \in \mathcal{A}_\tau) ,$$

$$f_\tau : \Omega \to E \quad : \quad f_\tau(\omega) = f_{\tau(\omega)}(\omega) , \quad (\omega \in \Omega) .$$

It is known that $<\mathcal{A}_n>$ is an increasing family of subσ-fields (see, [11]) , $\mu_\tau \in S(\mathcal{A}_\tau,E)$ and $f_\tau \in L_1(\mathcal{A}_\tau,E)$.

In the sequel, <u>a sequence</u> $<f_n>$ <u>in</u> $L_1(\mathcal{A},E)$ <u>is said to have a property</u> (★) , <u>if so has the sequence</u> $<\mu_n>$ <u>of measures, associated with</u> $<f_n>$, <u>given by</u>

$$\mu_n(A) = \int_A f_n \, dP \quad (A \in \mathcal{A}_n , n \in N) .$$

Clearly,

$$\mu_\tau(A) = \int_A f_\tau \, dP \quad (A \in \mathcal{A}_\tau , \tau \in T^\infty) .$$

<u>Definition</u> 3.1 . <u>A sequence</u> $<\mu_n>$ <u>is said to be a martingale if</u> $\mu_{m,n} = \mu_m|_{\mathcal{A}_n} = \mu_n \quad (m \geq n \in N)$.

<u>Definition</u> 3.2 . <u>A sequence</u> $<\mu_n>$ <u>is said to be an amart of finite order, if for each</u> $d \in N$, <u>the net</u> $<\mu_\tau(\Omega)>_{\tau \in T^d}$ <u>converges in</u> E , <u>where</u> T^d <u>is the set of all bounded stopping times each of which</u>

takes essentially at most d values. Further, if the net converges for $d = \infty$, then as usual, $<\mu_n>$ is called an amart (see, [4], [2]). Obviously, every amart is that of finite order. The simple remark given at the end shows that the reverse is false even for the case, where $E = \mathbb{R} = (-\infty, \infty)$.

Lemma 3.3. Let $<\mu_n>$ be a sequence in $S(A, E)$. Then the following conditions are equivalent :

(1) $<\mu_n>$ is an amart of finite order.

(2) $\lim_{n \in N} \sup_{m \geq n} S_U^n(\mu_{m,n} - \mu_n) = 0$, $(U \in U(E))$.

(3) $<\mu_n>$ can be written in a form

$$\mu_n = \alpha_n + \beta_n \qquad (n \in N),$$

where $<\alpha_n>$ is a martingale and $<\beta_n>$ a Pettis potential, i.e.

$$\lim_N S_U^n(\beta_n) = 0 \quad , \quad (U \in U(E)).$$

(4) There is a finitely additive measure $\mu_\infty : \Sigma \to E$, call μ_∞ the limit measure associated with $<\mu_n>$, such that each $\mu_{\infty,n} = \mu_\infty \big|_{A_n} \in S(A_n, E)$ and

$$\lim_N S_U^n(\mu_n - \mu_{\infty,n}) = 0 \quad , \quad (U \in U(E)).$$

Proof. $(1 \to 2)$ Suppose that $<\mu_n>$ is an amart of finite order. Then in particular, the net $<\mu_\tau(\Omega)>_{\tau \in T^2}$ converges in E. Let $U \in U(E)$ and $\varepsilon > 0$ be given. Then there is some $\tau(\varepsilon) \in T^2$ such that if $\sigma,\tau \in T^2$ with $\sigma,\tau \geqslant \tau(\varepsilon)$ then

$$p_U(\mu_\sigma(\Omega) - \mu_\tau(\Omega)) \leqslant 4^{-1}\,\varepsilon \tag{3.1}$$

Let $m,n \in N$ with $m \geqslant n \geqslant \tau(\varepsilon)$ and $A \in \mathcal{A}_n$. Let define $\sigma = m\, 1_\Omega$ and $\tau = n\, 1_A + m\, 1_{\Omega \setminus A}$, where 1_B denotes the characteristic function of $B \in \mathcal{A}$. Obviously, $\sigma,\tau \in T^2$ with $\sigma \geqslant \tau \geqslant \tau(\varepsilon)$. Hence by (3.1) , it follows that

$$p_U(\mu_m(A) - \mu_n(A)) = p_U(\mu_\sigma(\Omega) - \mu_\tau(\Omega)) \leqslant 4^{-1}\,\varepsilon \ .$$

Consequently, by Lemma 2.2 , we get

$$S_U^n(\mu_{m,n} - \mu_n) \leqslant 4\, q_U^{\mathcal{A}_n}(\mu_{m,n} - \mu_n)$$

$$= 4 \sup\, \{p_U(\mu_m(A) - \mu_n(A)) \mid A \in \mathcal{A}_n\} \leqslant \varepsilon \ .$$

This proves (2) .

(2 → 3) Suppose that $<\mu_n>$ satisfies (2) . It is easily checked that for any but fixed $n \in N$, the sequence $<\mu_{m,n}>_{m=n}^\infty$ is Cauchy in the S^n-topology of $S(\mathcal{A}_n,E)$. Then by virtue of Lemma 2.1 $<\mu_{m,n}>_{m=n}^\infty$ must be convergent to some $\alpha_n \in S(\mathcal{A}_n,E)$. It is not hard to prove that $<\alpha_n>$ is just a martingale. Moreover, if we put $\beta_n = \mu_n - \alpha_n$ $(n \in N)$, then the convergence of $<\mu_{m,n}>_{m=n}^\infty$ to α_n and (2) imply

$$\lim_N S_U^n(\beta_n) = 0 \qquad (U \in U(E)) \ .$$

This proves (3) .

(3 → 4) Suppose that $<\mu_n>$ satisfies (3) . Let define

$$\mu_\infty : \Sigma \to N : \mu_\infty(A) = \alpha_n(A) \quad (n \in N \ , \ A \in A_n) \ .$$

Obviously, μ_∞ satisfies (4) .

(4 → 1) Suppose that $<\mu_n>$ satisfies (4) . Let $d \in N$ be given. For each $U \in U(E)$ and $\varepsilon > 0$, by (4) there is some $n(\varepsilon) \in N$ such that

$$\sup_{n \geqslant n(\varepsilon)} S_U^n(\mu_n - \mu_{\infty,n}) \leqslant d^{-1} \varepsilon \ .$$

Let $\tau \in T^d$ with $\tau \geqslant n(\varepsilon)$. The last inequality with Lemma 2.2 yields

$$p_U(\mu_\tau(\Omega) - \mu_\infty(\Omega)) \leqslant \sum_{n \geqslant n(\varepsilon)} p_U(\mu_n(\{\tau = n\}) - \mu_\infty(\{\tau = n\}))$$

$$\leqslant d \ . \ \sup q_U^{A_n}(\mu_n - \mu_{\infty,n}) \leqslant \varepsilon \ .$$

This implies that the net $<\mu_\tau(\Omega)>_{\tau \in T^d}$ converges in E (just to $\mu_\infty(\Omega)$) . Since $d \in N$ was arbitrarily taken, by definition, $<\mu_n>$ must be an amart of finite order. This completes the proof.

Remark. The inspection of the proof shows that $<\mu_n>$ is an amart of finite order if and only if for some $d \geqslant 2$, the net $<\mu_\tau(\Omega)>_{\tau \in T^d}$ converges in E .

Suppose now that E has the Radon-Nikodym property and $<f_n>$ a Bochner uniformly integrable amart of finite order in $L_1(A,E)$. Clearly, there is some $f \in L_1(A,E)$ such that

$$\mu_\infty(A) = \int_A f\,dP \qquad (A \in \mathcal{A}) ,$$

where μ_∞ is the limite measure, associated with $<f_n>$. Moreover, the martingale $<\alpha_n>$, associated with $<f_n>$ which exists in Lemma 3.3 has a form

$$\alpha_n(A) = \int_A E^{\mathcal{A}_n}(f)\,dP \qquad (n \in N , A \in \mathcal{A}) ,$$

where $E^{\mathcal{A}_n}(.)$ is the $\mathcal{A}_n$-conditional expectation operator on $L_1(\mathcal{A},E)$ (see, [5]) . This implies that the margingale $<E^{\mathcal{A}_n}(f)>$ is regular, i.e.

$$\int_A E^{\mathcal{A}_n}(f)\,dP = \int_A f\,dP \qquad (n \in N , A \in \mathcal{A}) .$$

But we note that as Proposition 2.1 in [10] is easily extended to regular martingales in Hausdorff quasi-complete l.c.s's , the regular martingale $< E^{\mathcal{A}_n}(f)>$ must be convergent to f in the (Bochner) Pettis topology. This with Lemmas 3.3 and 2.2 proves the following theorem :

Theorem 3.4 . Let E be a Hausdorff quasi-complete l.c.s. with the Radon-Nikodym property (see, [5]) and $< f_n>$ a Bochner uniformly integrable amart of finite order in $L_1(\mathcal{A},E)$. Then $< f_n>$ converges to some $f \in L_1(\mathcal{A},E)$ in the Pettis topology.

Now let $\mathcal{P}_f(E)$ denote the space of all closed bounded nonempty subsets of E . For each $x \in E$, $U \in \mathcal{U}(E)$ and $A,B \in \mathcal{P}_f(E)$, we define

$$d_U(x,B) = \inf \{p_U(x-y) \mid y \in B\} ,$$

$$e_U(A,B) = \sup \{d_U(x,B) \mid x \in A\} ,$$

$$h_U(A,B) = \max \{e_U(A,B) , e_U(B,A)\} .$$

Then as in [3] , the Hausdorff topology of $\mathcal{P}_f(E)$ is defined by the family $\langle h_U \mid U \in U(E) \rangle$ of semi-distances. Next, a subset F of E is said to be totally bounded if for each $U \in U(E)$, there is a finite sequence $\langle x_j \rangle_{j=1}^{k}$ in F such that

$$F \subset \bigcup_{j=1}^{k} \{x_j + U\} .$$

Thus using the same proof of Theorem II.4 in [3] , we can establish easily the following result.

<u>Lemma</u> 3.5 . <u>Let</u> $\mathcal{P}_{cb}(E)$ <u>be the space of all closed totally bounded non empty subsets of</u> E . <u>Then with the Hausdorff topology,</u> $\mathcal{P}_{cb}(E)$ <u>is a closed subspace of</u> $\mathcal{P}_f(E)$.

In connection with Theorem 2.4 , we note that even for Banach spaces E , $(L_1(\mathcal{A},E)$, Pettis topology) is complete if and only if $\dim E < \infty$. Therefore it is useful to look for necessary and sufficient conditions under which a sequence in $L_1(\mathcal{A},E)$ is Cauchy in the Pettis topology. This is the aim of the following main result which extends Theorem 2 [13] to any directions.

Theorem 3.6 . A sequence $\langle f_n \rangle$ in $L_1(\mathcal{A},E)$ is Cauchy in the Pettis topology if and only if $\langle f_n \rangle$ is an amart of finite order and the limit measure associated with it has a relatively compact range.

Proof ($\rightarrow$) Let $\langle f_n \rangle$ be a sequence in $L_1(\mathcal{A},E)$. Suppose first that $\langle f_n \rangle$ is Cauchy in the Pettis topology and $\langle \mu_n \rangle$ the sequence of measures associated with $\langle f_n \rangle$, i.e.

$$\mu_n(A) = \int_A f_n \, dP \qquad (n \in N \ , \ A \in \mathcal{A}_n) \ .$$

Then for every $U \in U(E)$ and $\varepsilon > 0$, one can choose some $n(\varepsilon) \in N$ such that for all $m,n \in N$ with $m \geq n \geq n(\varepsilon)$, one has

$$P_U(f_m - f_n) \leq \varepsilon \ .$$

This with Lemma 2.2 yields

$$\sup_{m \geq n \geq n(\varepsilon)} S_U^n(\mu_{m,n} - \mu_n) \leq \sup_{m \geq n \geq n(\varepsilon)} P_U(f_m - f_n) \leq \varepsilon \ .$$

Therefore, by Lemma 3.3 , $\langle f_n \rangle$ must be an amart of finite order. Now, let μ_∞ be the limit measure associated with $\langle \mu_n \rangle$. Define

$$E_n = \mathrm{cl} \, \{\mu_n(A) \mid A \in \Sigma\} = \mathrm{cl} \, \{\int_A f_n \, dP \mid A \in \Sigma\}$$

and

$$E_\infty = \mathrm{cl} \, \{\mu_\infty(A) \mid A \in \Sigma\} \ ,$$

where "cl" is the closure operator in E .

Then by Lemma 1.1 [5], every E_n $(n \in N)$ is a compact subset in E. We shall show that $\langle E_n \rangle$ is convergent to E_∞ in the Hausdorff topology of $\mathcal{P}_f(E)$. Indeed, let $U \in \mathcal{U}(E)$ and $\varepsilon > 0$ be given. Since $\langle f_n \rangle$ is Cauchy in the Pettis topology, one can choose some $n(\varepsilon) \in N$ such that for all $m \geqslant n \geqslant n(\varepsilon)$ we have

$$p_U(f_n - f_m) \leqslant \varepsilon \ .$$

Consequently,

$$\sup_{A \in \Sigma} \{p_U(\int_A f_n dP - \int_A f_m dP) \leqslant \varepsilon \ .$$

Thus, one get

$$\sup_{A \in \Sigma} \{p_U(\int_A f_n dP) - \mu_\infty(A)\} \leqslant \varepsilon \ ,$$

by letting $m \uparrow \infty$.

Therefore,

$$h_U(E_n, E_\infty) = \max \{e_U(E_n, E_\infty) \ , \ e_U(E_\infty, E_n)\}$$

$$= \max \{ \sup_{A \in \Sigma} d_U(\int_A f_n dP , E_\infty) \ , \ \sup_{B \in \Sigma} d_U(\mu_\infty(B) , E_n)\}$$

$$\leqslant \max \{ \sup_{A \in \Sigma} p_U(\int_A f_n dP - \mu_\infty(A)) \ , \ \sup_{B \in \Sigma} p_U(\mu_\infty(B) - \int_B f_n dP)\}$$

$$= \sup_{A \in \Sigma} p_U(\int_A f_n dP - \mu_\infty(A)) \leqslant \varepsilon \ .$$

It means that the sequence $<E_n>$ in $\mathcal{P}_{cb}(E)$ converges to $E_\infty \in \mathcal{P}_f(E)$ in the Hausdorff topology. Thus by Lemma 3.5, it follows that $E_\infty \in \mathcal{P}_{cb}(E)$. On the other hand, as E is a Hausdorff quasi-complete l.c.s., each $F \in \mathcal{P}_{cb}(E)$ is compact. Then so is E_∞. Equivalently, μ_∞ has a relatively compact range. This completes the proof of the necessity condition.

(⇐). Suppose now that $<f_n>$ is an amart of finite order and the limit measure associated with it has a relatively compact range. Then using Theorem 9 [8], the same arguments, given by Uhl in the proof of Theorem 2 in [13] and Lemmas 2.2 and 3.3 one can prove easily the sufficiency condition, noting that $<f_n>$ <u>is Cauchy in the Pettis topology if and only if for each</u> $U \in U(E)$

$$\lim_{n\to\infty} \sup_{m \geqslant n} S_U(f_m - f_n) = 0 .$$

Thus the proof is completed.

Combining the theorem with Lemma 1.1 [5], we get the following corollary :

<u>Corollary</u> 3.7. <u>A sequence</u> $<f_n>$ <u>in</u> $L_1(A,E)$ <u>is convergent to some</u> $f \in L_1(A,E)$ <u>in the Pettis topology if and only if</u> $<f_n>$ <u>is an amart of finite order and the limit measure associated with it has a Radon-Nikodym derivative in</u> $L_1(A,E)$.

<u>Remarks</u>. A sequence $<f_n>$ in $L_1(A,E)$ is said to be a L^1-potential, if $<f_n>$ converges to 0 in the Bochner topology. Thus by Corollary 3.7

and Lemma 2.2 , every L^1-potential is an amart of finite order. Further, a sequence $<f_n>$ in $L_1(A,E)$ is called an approximate martingale if net $<\int_\Omega f_\tau \, dP>_{\tau \in T^\infty}$ is bounded. It is known (see [6]) that every real-valued amart is an approximate martingale. Thus by [9], there is a nonnegative real-valued amart of finite order which fails to be an amart. Further, a sequence $<f_n>$ in $L_1(A,E)$ is said to have a Riesz decomposition if $<f_n>$ can be written in a form

$$f_n = g_n + h_n \quad (n \in N) ,$$

where $<g_n>$ is a martingale in $L_1(A,E)$ and $<h_n>$ a L^1-potential. Thus Lemma 2.3 shows that a sequence $<f_n>$ in $L_1(A,\mathbb{R})$ is an amart of finite order if and only if it has a Riesz decomposition, noting that on $L_1(A,\mathbb{R})$ the Pettis topology is equivalent to the Bochner topology. This seems to be a new characterization of sequences in $L_1(A,\mathbb{R})$ having a Riesz decomposition.

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