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THE RADON-NIKODYM PROPERTY AND CONVERGENCE OF AMARTS IN FRECHET SPACES

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Résumé. On démontre que l'espace de Fréchet E possède la propriété de Radon-Nikodym si et seulement si tout amart uniformément intégrable à valeurs dans E est convergente pour la topologie de Pettis. On donne aussi des conditions nécessaires et suffisantes pour la convergence des amarts uniformément intégrables à valeurs dans les espaces de Fréchet généraux.

§ 0. INTRODUCTION.

The martingale limit theorems in the Banach spaces with the Radon-Nikodym property were considered in [10, 2, 8] and etc. A necessary and sufficient condition for the L^1-convergence of martingales in general Banach spaces was also obtained in [12], using the Radon-Nikodym theorem of Rieffel [9]. The main purpose of the paper is to extend some of these above results to amarts in Fréchet spaces (see Section 2). One of our main results says that a uniformly integrable amart in Fréchet spaces is convergent in the Pettis topology if and only if it satisfies "the Uhl's condition". This extension is not trivial because the semi-balls in Fréchet spaces are, in general unbounded. For terminology and notations we refer to Section 1.

§ 1. TERMINOLOGIES AND NOTATIONS.

Throughout the paper, let E be a Fréchet space, $\langle U_n \downarrow \rangle$ a fundamental decreasing sequence of closed absolutely convex sets which forms a o-neighborhood base for E, E' the topological dual of E, N the set of all positive integers and (Ω, A, P) a probability space. By U_n^o and $p_n(.)$, resp. we mean the polar and the continuous semi-norm, resp. associated with U_n $(n \in N)$. For any $x \in E$ and $A, B \in 2^E$, we define

$$d_n(x,B) = \inf \{p_n(x-y) \ ; \ y \in B\} \ ;$$

$$e_n(A,B) = \sup \{d_n(x,B) \ ; \ x \in A\} \ .$$

It is easily seen that the sequence $\langle e_n(.,.)\rangle$ has the following property :

<u>Property 1.1.</u> *Let* $x \in E$ *and* $A,B,C \in 2^E$.

(1) $e_n(A,C) \leqslant e_n(A,B) + e_n(B,C)$,

(2) $e_n(A,B) \leqslant 0 \quad (n \in N) \quad A \subset \overline{B}$,

(3) *If* A *is convex compact and* $\delta \geqslant 0$ *then*

$x \in A + U_n \Leftrightarrow e_n(x,A) < \delta$.

A function $f : \Omega \to E$ is called to be strongly measurable, if there is a sequence $\langle f_n\rangle$ of simple function such that for each $\varepsilon > 0$ there is some $T_\varepsilon \in \mathcal{A}$ with $P(T_\varepsilon) \geqslant 1-\varepsilon$ and

$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \quad , \quad \text{uniformly on } T_\varepsilon \ .$$

A strongly measurable function f is said to be Bochner integrable, write $f \in \mathcal{L}_1(E)$ if each $V_n(f) = \int_\Omega p_n(f)dP < \infty$. It is easy to check that for any $f,g \in \mathcal{L}_1(E)$, $f = g$ a.e. if and only if $V_n(f-g) = 0 \ (n \in N)$. Now let $L_1(E) = L_1(\Omega,\mathcal{A},E)$ be the space of all (equivalence classes of) Bochner integrable functions. Therefore, according to [3] , the class $L_1(E)$, endowed with the Bochner topology, generated by the sequence $\langle V_n(.)\rangle$ of continuous semi-norms, becomes also a Fréchet space. Further for $f \in L_1(E)$, we define

$$q_n(f) = \sup \{p_n(\int_A f \, d\, P) \quad ; \; A \in \mathcal{A}\},$$

$$S_n(f) = \sup \{\int_\Omega |\langle e, f\rangle| dP \; , \quad e \in U_n^o\} \quad .$$

Then $L_1(E)$, endowed with the Pettis topology, generated by the sequence $\langle S_n(.)\rangle$ of continuous semi-norms is a metrizable (but not necessarily complete) space. The proof of the following property is analogous to that for the Banach space-valued case.

Property 1.2. *Let* $f \in L_1(E)$ *and* $g \in L_1(\Omega, \mathcal{B}, E)$, *where* $\mathcal{B}$ *is a subs-field of* $\mathcal{A}$. *Then*

(1) $S_n(f) \leqslant V_n(f)$,

(2) $q_n(f) \leqslant S_n(f) \leqslant 4q_n(f) \qquad (n \in N)$,

(3) $q_n(g) \leqslant S_n(g) \leqslant 4q_n^{\mathcal{B}}(g) \qquad (n \in N)$,

where $q_n^{\mathcal{B}}(g) = \sup \{p_n(\int_B g \, d\, P) \quad ; \quad B \in \mathcal{B}\}$.

Finally, for definition and basis properties of vector-valued measures we refer to [6] .

§ 2. THE RADON-NIKODYM PROPERTY AND CONVERGENCE OF AMARTS.

Hereafter, we shall consider an increasing sequence $\langle \mathcal{A}_n \rangle$ of subσ-fields of $\mathcal{A}$ with $\Sigma = \bigcup_N \mathcal{A}_n$ and $\mathcal{A} = \sigma - (\Sigma)$. A sequence $\langle f_n \rangle$ in $L_1(E)$ is said to be adapted to $\langle \mathcal{A}_n \rangle$ if each f_n is $\mathcal{A}_n$-measurable. We shall consider only such sequences. Let T be the set of all bounded stopping times, relative to $\langle \mathcal{A}_n \rangle$. Given a sequence $\langle f_n \rangle$ and $\tau \in T$, let $\mathcal{A}_\tau$, f_τ and μ_τ be defined as in [8]. Thus $\langle \mathcal{A}_\tau \rangle_{\tau \in T}$ is an increasing family of subσ-fields of $\mathcal{A}$, $f_\tau \in L_1^\tau(E) = L_1(\Omega, \mathcal{A}_\tau, E)$ and $\mu_\tau(A) = \int_A f_\tau \, dP \quad (A \in \mathcal{A}_\tau$, $\tau \in T)$.

Call $\langle f_n \rangle$ a martingale if for all $m \geqslant n \in N$, $\int_A f_n dP = \int_A f_m dP \quad (A \in \mathcal{A}_n)$.

If this occurs then $\mu_{\sigma,\tau} = \mu_\sigma|_{\mathcal{A}_\tau} = \mu_\tau \quad (\sigma \geqslant \tau \in T)$. In what follows we shall need the following Fréchet version of Proposition IV-2-3 [8].

<u>Proposition 2.1.</u> *Let $\langle f_n \rangle$ be a martingale. Then the following conditions are equivalent :*

(1) $\langle f_n \rangle$ *is regular, i.e., there is some* $f \in L_1(E)$ *such that*

$$\int_A f_n \, dP = \int_A f \, dP \qquad (A \in \mathcal{A}_n, \quad n \in N).$$

(2) $\langle f_n \rangle$ *converges strongly almost surely to* $f \in L_1(E)$ *and* $\langle f_n \rangle$ *is uniformly integrable, i.e.*

$$\sup_N \int_{\{p_k(f_n) > a\}} p_k(f_n)\,dP \downarrow 0 \quad \text{as} \quad a \uparrow \infty \qquad (k \in N) .$$

(3) $\langle f_n \rangle$ *converges in the Bochner topology.*

(4) $\langle f_n \rangle$ *converges in the Pettis topology to some* $f \in L_1(E)$.

Proof. $(1 \to 2)$ Let $\langle f_n \rangle$ be a martingale satisfying (1) . Define $F = \overline{\text{span}}\,\{f_1(\Omega), f_2(\Omega), \ldots, f(\Omega)\}$. Then by lemma 1.1.[4] F is a separable closed subspace of E . Thus to show (2) , it suffices to suppose that E is itself separable. Let $e \in E'$, by (1) it follows that each $\langle e, f_n \rangle$ is a regular martingale. Hence

$$\lim_{n \to \infty} |\langle e, f_n \rangle| = |\langle e , f \rangle| \ , \quad \text{a.e.} \tag{2.1}$$

Further, for any but fixed $k \in N$, by ([11], III.4.7) the separability of E implies the separability of U_k^o in the $\sigma(E'E)$-topology. Let $\{e_i, i \in I_k\}$ be a countable $\sigma(E',E)$-dense family in U_k^o . By theorem II.18 in [1] , it follows that

$$\begin{aligned} \rho_k(f_n) &= \sup \{ |\langle e, f_n \rangle| \ ; \ e \in U_k^o \} \\ &= \sup \{ |\langle e_i, f_n \rangle| \ ; \ i \in I_k \} \qquad (n \in N) , \end{aligned}$$

and

$$\rho_k(f) = \sup \{ |\langle e_i , f \rangle| \ ; \ i \in I_k \} .$$

Consequently, by lemma V.2.9 in [8] , (2.1) yields

$$\lim_{n \to \infty} \rho_k(f_n) = \rho_k(f) \quad \text{a.e.} \tag{2.2}$$

This with the submartingale property of $<\rho_k(f_n)>$ implies that each $<\rho_k(f_n)>_{n=1}^{\infty}$ is uniformly integrable. Equivalently, $<f_n>$ is uniformly integrable. Further, given $a \in E$, by using the above argument to $<f_n - a>$, we infer that as (2.2) we get

$$\lim_{n \to \infty} p_k(f_n - a) = p_k(f - a) \quad , \quad \text{a.e.}$$

But since E is separable then the same argument used by Neveu in the proof of Proposition V.2.8 [8] shows that

$$\lim_{n \to \infty} p_k(f_n - f) = 0 \quad , \quad \text{a.e.}$$

Equivalently, $<f_n>$ converges, strongly almost surely, to f. It proves (2).

Finally, we note that $(2 \to 3)$ follows directly from Lemma IV.2.5 [8] and the implications $(3 \to 4 \to 1)$ are easy consequences of Property 1.2. Thus the proof of the proposition is completed.

In the opinion of the referee, the proof of the following proposition is classical hence omitted.

<u>Proposition 2.2.</u> *Let* $f \in L_1(E)$ *Then for each* $k \in N$ *there is a sequence* $<f_n^k>_{n=1}^{\infty}$ *of* Σ-*simple functions such that*

$$\lim_{n \to \infty} q_k(f_n^k - f) = 0 \tag{2.3}$$

Call $<f_n>$ an amart if the net $<\int_\Omega f_\tau dP>_{\tau \in T}$ converges strongly in E. It is easily seen that every martingale is an amart. Moreover, by Lemma 2.2[7] it follows that $<f_n>$ is an amart if and only if there is a finitely additive measure $\mu_\infty : \Sigma \to E$ such that

$$\lim_{\tau \in T} \sup_{A_\tau} p_k\left(\int_A f_\tau dP - \mu_\infty(A)\right) = 0 \qquad (k \in N) \tag{2.4}$$

In the sequel, μ_∞ will be called the limit measure associated with the amart $<f_n>$. The following result gives a relationship between an amart $<f_n>$ and its limit measure μ_∞.

<u>Proposition 2.3</u>. *An amart* $<f_n>$ *converges to an element in* $L_1(E)$ *in the Pettis topology if and only if the limit measure* μ_∞, *associated with it has the Radon-Nikodym derivative contained in* $L_1(E)$.

<u>Proof</u>. Let $<f_n>$ and μ_∞ be as in the hypothesis of the proposition. Suppose first that $<f_n>$ converges, in the Pettis topology, to some $f \in L_1(E)$. Then by the above remark, f must be a Radon-Nikodym derivative of μ_∞ i.e.

$$\mu_\infty(A) = \int_A f\, dP \qquad (A \in \Sigma) \tag{2.5}$$

Conversely, suppose that (2.5) is satisfied for some $f \in L_1(E)$. Given $k \in N$, let $<f_n^k>_{n=1}^\infty$ be the sequence of Σ-simple function, satisfying (2.3) in Proposition 2.2. Then given $\varepsilon > 0$ one can choose some $n(\varepsilon) \in N$ such that

$$q_k(f^k_{n(\varepsilon)} - f) \leqslant \frac{\varepsilon}{12} \tag{2.6}$$

Since $f^k_{n(\varepsilon)}$ is Σ-simple, one can choose some $n_1 \in N$ such that $f^k_{n(\varepsilon)}$ is A_{n_1}-measurable. Further, by (2.4) there is some $n_2 \geqslant n_1$ such that

$$\sup_{n \geqslant n_2} \sup_{A_n} p_k(\int_A f_n \, dP - \mu_\infty(A)) \leqslant \frac{\varepsilon}{12}$$

This with (2.6) and Property 1.2 yields

$$\begin{aligned} S_k(f_n - f) &\leqslant S_k(f_n - f^k_{n(\varepsilon)}) + S_k(f^k_{n(\varepsilon)} - f) \\ &\leqslant 4\, q_k^{A_n}(f_n - f^k_{n(\varepsilon)}) + 4\, q_k(f^k_{n(\varepsilon)} - f) \\ &\leqslant 4 \sup_{A_n} p_k(\int_A f_n \, dP - \int_A f dP) + 4 \sup_{A_n} p_k(\int_A f_{n(\varepsilon)} \, dP - \int_A f dP) + \frac{\varepsilon}{3} \\ &\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad (n \geqslant n_2) \quad . \end{aligned}$$

This implies that for each $k \in N$

$$\lim_{n \to \infty} S_k(f_n - f) = 0 \ .$$

It means that $\langle f_n \rangle$ converges to f in the Pettis topology. This completes the proof.

Theorem 2.1. *Let* E *be a Fréchet space. Then the following conditions are equivalent :*

(1) E *has the Radon-Nikodym property* .

(2) *Every uniformly integrable* E*-valued amart converges to some element of* $L_1(E)$ *in the Pettis topology.*

(3) *Every uniformly integrable* E*-valued martingale converges in the Bochner topology.*

Proof (1-2) follows from Proposition 2.3.

(2-3) follows from Proposition 2.1.

$(3 \to 1)$ Suppose that E fails to have the Radon-Nikodym Property. Hence by definition there is a measure $\mu : \mathcal{A} \to E$ which is absolutely P-continuous, of bounded variation but $\mu \not\equiv \mu_f$ for all $f \in L_1(E)$, where $\mu_f(A) = \int_A f\,dP \quad (A \in \mathcal{A})$.

Now define

$$f_\Pi = \sum_{A \in \Pi} P(A)^{-1}\ \mu(A)\ 1_A \quad , \quad \mathcal{A}_\Pi = \sigma\text{-}(\Pi) \qquad (\Pi \in \Pi(\Omega)) \ ,$$

where $\Pi(\Omega)$ denotes the indexed set of all finite partitions of Ω . Let $e \in E'$, it is known that the family $\{ < e , f_\Pi > , \Pi \in \Pi(\Omega) \}$ is a martingale (w.r.t. $< \mathcal{A}_\Pi >_{\Pi \in \Pi(\Omega)}$) . Consequently, if $\Pi \ll \Pi'$, one has

$$\int_A < e , f_\Pi > dP = \int_A < e , f_{\Pi'} > dP \qquad (A \in \mathcal{A}_\Pi) \ .$$

Equivalently,

$$< e, \int_A f_\Pi dP > \; = \; < e, \int_A f_{\Pi'} dP > \qquad (A \in A_\Pi) \quad .$$

But since $e \in E'$ was arbitrarily taken then

$$\int_A f_\Pi \, dP = \int_A f_{\Pi'} \, dP \qquad (A \in A_\Pi) \quad .$$

Since μ does not have any Radon-Nikodym derivative in $L_1(E)$ then the generalised martingale $\{f_\Pi, \Pi \in \Pi(\Omega)\}$ cannot be convergent in the Bochner topology. But endowed with the last topology, $L_1(E)$ is a Fréchet space, then by Lemma V.1.1 [8] , there is an increasing sequence $<\Pi_n>$ such that the martingale $< f_n = f_{\Pi_n}>$ cannot be convergent in the Bochner topology.

On the other hand, the measure μ is assumed to be absolutely P-continuous and of bounded variation then it is clear that the martingale $<f_n>$, constructed above is uniformly integrable. This contradicts (3) which completes the proof of the theorem.

We say that a finitely additive measure $\mu_\infty : \Sigma \to E$ satisfies the Uhl's condition, if for each $\varepsilon > 0$ there is a compact absolutely convex set $K_\varepsilon \subset E$ such that for each $k \in N$ and $\delta > 0$ there is some $A(\varepsilon,k,\delta) \in \Sigma$ with $P(A(\varepsilon,k,\delta)) \geqslant 1-\varepsilon$ and such that if $A \in \Sigma$ with $A \subset A(\varepsilon,k,\delta)$ then

$$\mu(A) \in P(A)K_\varepsilon + \delta U_k \quad . \qquad (2.7)$$

The following result is a Fréchet version of Proposition 1[12].

Proposition 2.4. *Let* $\mu_\infty : \Sigma \to E$ *be a finitely additive measure. Then* $\mu = \mu_f$ *for some* $f \in L_1(E)$ *if and only if the following conditions are satisfied :*

(1) μ_∞ *is of bounded variation.*

(2) μ_∞ *is absolutely* P*-continuous.*

(3) μ_∞ *satisfies the Uhl's condition on* Σ .

Proof. Let μ_∞ be a finitely additive measure on Σ . Suppose that $\mu_\infty = \mu_f$ for some $f \in L_1(E)$. Then it is clear that μ_∞ satisfies the conditions (1,2) . We shall show that μ_∞ satisfies also the Uhl's condition. For this purpose, let $\varepsilon > 0$ be any but fixed. It follows from Lemma 1.1.[7] that there is some $T_\varepsilon \in \Sigma$ such that $P(T_\varepsilon) \geqslant 1-\varepsilon$ and $f(T_\varepsilon)$ is precompact. Let K_ε be the closed absolutely convex hull of $f(T_\varepsilon)$. By ([11], II.4.3) it follows that K_ε is compact. Therefore, if we put $A(\varepsilon,k,\delta) = T_\varepsilon$ for all $k \in N$ and $\delta > 0$ then it is not hard to show that if $A \in \Sigma$ with $A \subset A(\varepsilon,k,\delta)$ then

$$\mu_\infty(A) = \mu_f(A) \in P(A)K_\varepsilon \subset P(A)K_\varepsilon + \delta U_k \quad .$$

This proves (2.7) and the necessity condition.

Conversely, suppose that μ_∞ satisfies the conditions (1-3). It is clear that by (1-2) , μ_∞ can be extended to a σ-additive measure on $\mathcal{A}$ which satisfies conditions (1.2) and will be still denoted by μ_∞. We shall show that μ_∞ satisfies the following condition

(3') For every $\varepsilon > 0$ there are a compact absolutely convex set K'_ε in E and $A_\varepsilon \in \mathcal{A}$ with $P(A_\varepsilon) \geqslant 1-\varepsilon$ and such that for each $A \in \mathcal{A}$ with $A \subset A_\varepsilon$ we have

$$\mu_\infty(A) \in P(A)K'_\varepsilon \quad .$$

Indeed, let ε be given. Take $K'_\varepsilon = K_\varepsilon$, where K_ε is borrowed from (2.7) . Define

$$A_\varepsilon = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A(\varepsilon, j, 2^{-j})$$

Obviously, $A_\varepsilon \in \mathcal{A}$ with $P(A_\varepsilon) \geqslant 1-\varepsilon$. We show now that the triple $(\varepsilon, K'_\varepsilon, A_\varepsilon)$ satisfies the condition (3') . For this purpose let $A \quad \Sigma$, $A \in A_\varepsilon$ and $k \in N$. For any but fixed $n \in N$ with $n \geq k$ we define

$$S_n^n = A(\varepsilon , n , 2^{-n})$$

$$S_{n+1}^n = A(\varepsilon , (n+1) , 2^{-(n+1)}) \setminus S_n^n \quad ,$$

$$\vdots$$

$$S_{n+\ell+1}^n = A(\varepsilon , n+\ell+1 , 2^{-(n+\ell+1)}) \setminus \bigcup_{j=n}^{n+\ell} S_j^n \quad ;$$

$$\vdots$$

Clearly, $< S_j^n >_{j=n}^{\infty}$ is a sequence of pairwise disjoint elements of Σ with

$$A \subset A_\varepsilon = \bigcup_{j=n}^{\infty} A(\varepsilon , j , 2^{-j}) = \bigcup_{j=n}^{\infty} S_j^n .$$

Therefore, $\langle A \cap S_j^n \rangle_{j=n}^{\infty}$ is a Σ-measurable partition of A and by Proposition 1.1. it follows that

$$e_k(\mu_\infty(A) , P(A)K'_\varepsilon) = e_k(\mu_\infty(A) , P(A)K_\varepsilon)$$

$$\leqslant e_k(\mu_\infty(A) , \mu_\infty(\bigcup_{j=n}^{n+\ell} A_j^n)) + e_k(\mu_\infty(\bigcup_{j=n}^{n+\ell} A \cap S_j^n) , P(\bigcup_{j=n}^{n+\ell} A \cap S_j^n)K_\varepsilon)$$

$$\leqslant p_k(\mu_\infty(\bigcup_{j=n+\ell+1}^{\infty} A \cap S_j^n)) + \sum_{j=n}^{n+\ell} e_k(\mu_\infty(A \cap S_j^n) , P(\bigcup_{j=n}^{n+\ell} A \cap S_j^n)K_\varepsilon)$$

$$\leqslant p_k(\mu_\infty(\bigcup_{j=n+\ell+1}^{\infty} \cap A \cap S_j^n)) + \sum_{j=n}^{n+\ell} e_j(\mu_\infty(A \cap S_j^n) , P(\bigcup_{j=n}^{n+\ell} A \cap S_j^n)K_\varepsilon)$$

since $P(\bigcup_{j=n}^{n+\ell} A \cap S_j^n)K_\varepsilon \subset P(A)K_\varepsilon$ and $e_j(x) \geqslant e_K(x)$ $(j \geqslant n \geqslant K , x \in E)$.

On the one hand, since μ_∞ satisfies the condition (2) then there is some $\ell \in N$ such that

$$p_k(\mu_\infty(\bigcup_{j=n+\ell+1}^{\infty} A \cap S_j^n) \leqslant \sum_{j=n}^{\infty} 2^{-j} .$$

On the other hand, since each $A \cap S_j^n \in \Sigma$ with $A \cap S_j^n \subset A(\varepsilon,j,2^{-j}) \in \Sigma$ then by condition (3) , (2.7) yields

$$\mu_\infty(A \cap S_j^n) \in P(A \cap S_j^n)K_\varepsilon + 2^{-j} U_j \qquad (j \geqslant n) .$$

This with Property 1.1. implies

$$\sum_{j=n}^{n+\ell} e_j(\mu_\infty(A \cap S_j^n), P(A \cap S_j^n)K_\varepsilon)$$

$$\leqslant \sum_{j=n}^{n+\ell} 2^{-j}$$

$$\leqslant \sum_{j=n}^{\infty} 2^{-j} \quad .$$

Consequently,

$$e_k(\mu_\infty(A), P(A)K'_\varepsilon) \leqslant 2 \sum_{j=n}^{\infty} 2^{-j} = 2^{-n} \downarrow 0 \qquad \text{as} \qquad n \uparrow \infty .$$

This shows that

$$e_k(\mu_\infty(A), P(A)K'_\varepsilon) \leqslant 0 \qquad (k \in N) .$$

It means that

$$\mu_\infty(A) \in P(A)K'_\varepsilon \qquad (A \in \Sigma, A \subset A_\varepsilon) .$$

Finally, since μ_∞ satisfies condition (2) , then the last conclusion remains valid for all $A \in \mathcal{A}$ with $A \subset A_\varepsilon$. This proves (3') . Consequently, the extended measure μ_∞ satisfies the Rieffel's conditions given in Theorem 2.1.[3] . Therefore, the last theorem guarantees the existence of some $f \in L_1(E)$ such that $\mu_\infty = \mu_f$. It completes the proof of the proposition.

We say that an amart $\langle f_n \rangle$ satisfies the Uhl's condition if for every $\varepsilon > 0$ there is a compact absolutely convex subset Q_ε in E such that for each $k \in N$ and $\delta > 0$ one can choose some

$N(\varepsilon,k,\delta) \in N$, $B(\varepsilon,k,\delta) \in A_{N(\varepsilon,k,\delta)}$ with $P(B(\varepsilon,k,\delta)) \geqslant 1-\varepsilon$ and such that if $n \geqslant N(\varepsilon,k,\delta)$ and $A \in A_n$ with $A \subset B(\varepsilon,k,\delta)$ then

$$\int_A f_n \, dP \in P(A)Q_\varepsilon + \delta U_k \ . \tag{2.8}$$

Proposition 2.5. *Let* $<f_n>$ *be an amart and* μ_∞ *its limit measure. Then* $<f_n>$ *satisfies the Uhl's condition if and only if so does* μ_∞ .

Proof. Since the proofs of the necessity and sufficiency conditions are symetric. Thus we shall give only a proof of, for example , the necessity condition.

Suppose that $<f_n>$ satisfies the Uhl's condition. Given $\varepsilon > 0$, take $K_\varepsilon = Q_\varepsilon$, where Q_ε exists in (2.8) . For each $k \in N$ and $\delta > 0$, let $A(\varepsilon,k,\delta) = B(\varepsilon,k,\frac{\delta}{2})$, where $B(\varepsilon,k,\frac{\delta}{2})$ exists in (2.8) . First, by (2.4) there is some $n_o \in N$ such that

$$\operatorname*{Sup}_{A_n} p_k\Big(\int_A f_n \, dP - \mu_\infty(A)\Big) \leqslant \frac{\varepsilon}{2} \qquad (n \geqslant n_o) \tag{2.9}$$

Given $A \in \Sigma$ with $A \subset A(\varepsilon,k,\delta)$ then $A \in A_n$ for some $n \geqslant n_o$. Therefore, by (2.8) and (2.9) we get

$$e_k(\mu_\infty(A), P(A)K_\varepsilon) = e_k(\mu_\infty(A), P(A)Q_\varepsilon)$$

$$\leqslant P_k\Big(\mu_\infty(A) - \int_A f_n dP\Big) + e_k\Big(\int_A f_n \, dP, P(A)Q_\varepsilon\Big)$$

$$\leqslant \frac{\delta}{2} + \frac{\delta}{2} = \delta \ .$$

Hence, by Property 1.1. the above inequality implies

$$\mu_{\infty}(A) \in \varphi(A)K_{\varepsilon} + \delta U_k .$$

This proves (2.7) and the proposition.

Note that in [5] , Daurès has defined also the Uhl's condition, where instead of the semiballs he used "a bounded set". Thus it is clear that if an amart satisfies the Uhl's condition then so does its limit measure. But the converse implication is not true. Therefore the author thinks that the definitions (2.7) and (2.8) are better suited for the Uhl's condition in Fréchet spaces.

Theorem 2.2. *Let* $\langle f_n \rangle$ *be a uniformly integrable amart. Then* $\langle f_n \rangle$ *converges, in the Pettis topology, to some element of if and only if it satisfies the Uhl's condition.*

Proof. It follows from Propositions 2.4, 2.5 and 2.3.

Combining the theorem with Proposition 2.1. it is easy to establish the following result :

Corollary 2.3. *Let* $\langle f_n \rangle$ *be a martingale. Then* $\langle f_n \rangle$ *converges in the Bochner topology, if and only if it is uniformly integrable and satisfies the Uhl's condition.*

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