ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Probabilités et applications

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Operator theorems on L^p-convergence to zero $(1 \leqslant p < +\infty)$

Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 78, série *Probabilités et applications*, n° 2 (1984), p. 15-23
<http://www.numdam.org/item?id=ASCFPA_1984__78_2_15_0>

OPERATOR THEOREMS ON L^p-CONVERGENCE TO ZERO $(1 \leq p < +\infty)$

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1. Introduction.

Let (X,Σ,m) be a measure space (where m is a positive, σ-additive measure) and let $L^p(X,\Sigma,m)$, $1 \leq p \leq +\infty$ be the usual Banach spaces. A linear bounded operator $T: L^p(X,\Sigma,m) \to L^p(X,\Sigma,m)$ is called a positive contraction of $L^p(X,\Sigma,m)$ if it transforms non-negative functions into non-negative functions and if $\|T\|_p \leq 1$.

Our goal here is to prove that if T is simultaneously a positive contraction of $L^p(X,\Sigma,m)$ for every $1 \leq p \leq +\infty$ and if we consider the set $\Omega \subseteq R$,

$$\Omega = \{ \frac{1}{m(A)} \cdot \int_A T\, 1_A \, dm \;/\; A \in \Sigma, \quad 0 < m(A) < +\infty \}$$

then if $\inf \Omega > 0$ it follows that for every $1 \leq q < +\infty$ $\lim_{n\to+\infty} \|T^{n+1} - T^n\|_q = 0$.

For notational conveniences we will recall some definitions from [1].

By a partition $E = \{E_1,\dots,E_n\}$ of X we mean a finite partition of X such that $E_i \in \Sigma$, $i = 1,2,\dots,n$, $0 < m(E_1) < +\infty$ and such that only the first k sets $(1 \leq k \leq n)$ have finite non-zero measures. Let $l_p(k,\mu)$ $(1 \leq p \leq +\infty)$ be the finite dimensional L^p-space defined by $\mu_i = \mu(\{i\}) = m(E_i)$, $i = 1,2,\dots,k$ (that is $l_p(k,\mu) = L^p(\Gamma_k, P(\Gamma_k), \mu)$ where $\Gamma_k = \{1,2,\dots,k\}$ and the measure μ is generated by the real, non-zero, positive numbers $\mu_1, \mu_2,\dots,\mu_k$). The space $l_p(k,\mu)$ will be called the finite dimensional L^p-space generated by the partition $E = \{E_1, E_2,\dots,E_n\}$. The norm on $l_p(k,\mu)$ will be denoted by

$\| \ \|_{\overline{p}}$.

If E is a partition of X we will denote by T_E the conditional expectation operator as defined by Akcoglu [1].

2. Positive Contractions of L^1 or L^∞-spaces.

Proposition 1. Let T be a positive contraction of $L^1(X,\Sigma,m)$.

I) The following are equivalent:

a) There exists $\rho > 0$ such that if $A,B \in \Sigma$, $0 < m(A)$, $m(B) < +\infty$, $A \cap B = \phi$ then

$$\int (1_B - 1_A) T 1_A \, d\,m < (1 - \rho)\, m(A)$$

b) There exists $\eta > 0$ such that for every partition E of X $\|T_E T - I\|_{\overline{1}} < 2(1 - \eta)$.

II) If T satisfies a) (or b)) then $\lim\limits_{n\to+\infty} \|T^{n+1} - T^n\|_1 = 0$.

Proof. I) a) $\Rightarrow$ b) . Let $E = \{E_1,\dots,E_n\}$ be a partition of X and let $1_1(k,\mu)$ be the finite dimensional L^1-space generated by the partition E .

The operator $T_E T$ as a positive contraction of $1_1(k,\mu)$ has the matrix $\left(\frac{1}{m(E_j)} \int_{E_j} T\, 1_{E_i} \, d\,m\right)_{i,j=1,\dots,k}$.

It follows that

$$\|T_E T - I\|_{\overline{1}} = \max_{1\le i\le k} ((m(E_i))^{-1} \cdot (\sum_{\substack{j=1\\ j\ne i}}^{k} \frac{1}{m(E_j)} \cdot \int_{E_j} T\, 1_{E_i}\, dm \cdot m(E_j) +$$

$$+ (1 - \frac{1}{m(E_i)} \int_{E_i} T\, 1_{E_i} dm)\cdot m(E_i))) =$$

$$= \max_{1\le i\le k} ((\int_{\bigcup_{\substack{j=1\\ j\ne i}}^{k} E_j} T\, 1_{E_i}\, dm - \int_{E_i} T\, 1_{E_i}\, dm + m(E_i)) \cdot \frac{1}{m(E_i)})$$

Using a) we obtain that

$$\|T_E T - I\|_{\overline{1}} < \max_{1\le i\le k} (((1-\rho)m(E_i) + m(E_i)) \cdot \frac{1}{m(E_i)}) = 2(1 - \frac{\rho}{2})$$

and if we note $\eta = \frac{\rho}{2}$ we obtain b) .

b) $\Rightarrow$ a). Let $A, B \in \Sigma$, $0 < m(A)$, $m(B) < +\infty$ be two disjoint sets. We define the partition $E = \{A, B, X \setminus (A \cup B)\}$. It follows that $\|T_E T - I\|_{\overline{1}} < 2(1-\eta)$ and we obtain that

$$(1 - \frac{1}{m(A)} \int_A T\, 1_A\, dm)\, m(A) + \int_B T\, 1_A\, dm < 2(1-\eta)\, m(A) .$$

The last inequality implies that

$$\int (1_B - 1_A) \cdot T\, 1_A\, dm < (1 - 2\eta) m(A)$$

and a) follows as $\rho = 2\eta$.

II) Suppose T satisfies b) and let $f \in L^1(X,\Sigma,m)$ be such that $\|f\|_1 \le 1$. By lemma 3.1 from [1] it follows that there exists a partition E of X such that

$$\|f - T_E f\|_1 < \frac{\eta}{2} \quad \text{and} \quad \|Tf - T_E\, T\, T_E f\|_1 < \frac{\eta}{2} .$$

It follows that

$$\|(I - T)f\|_1 \leq \|f - T_E f\|_1 + \|(I - T_E T)T_E f\|_1 + \|T_E T\, T_E f - Tf\|_1 <$$

$$< \frac{\eta}{2} + 2(1 - \eta) + \frac{\eta}{2} = 2(1 - \frac{\eta}{2}) .$$

We obtain that $\| I - T\|_1 \leq 2(1 - \frac{\eta}{2})$ and the proof is completed by using the "zero-two" law for positive contractions in L^1-spaces. (Theorem 1.1 from [4]).

Proposition 2. Let T be a positive contraction of $L^\infty(X,\Sigma,m)$. The following are equivalent :

i) There exists $\alpha > 0$ such that if $A,B \in \Sigma$, $A \cap B = \phi$, $0 < m(A)$, $m(B) < +\infty$ then

$$\int_B T(1_A - 1_B)\, dm < (1 - \alpha)m(B) .$$

ii) There exists $\beta > 0$ such that for every partition E of X, $\|T_E T - I\|_\infty < 2(1 - \beta)$.

Proof. i) $\Rightarrow$ ii) Let E be a partition of X and let $1_\infty(k,\mu)$ be the finite dimensional L^∞-space generated by the partition E . The operator $T_E T$ thought as operator on $1_\infty(k,\mu)$ has the matrix $\left(\frac{1}{m(E_j)} \int_{E_j} T\, 1_{E_i}\, dm\right)_{i,j = 1,\ldots,k}$.

It follows that

$$\| T_E T - I \|_\infty = \max_{1\leq j\leq k} \Big(\sum_{\substack{i=1\\ i\neq j}}^{k} \frac{1}{m(E_j)} \int_{E_j} T\, 1_{E_i}\, dm + 1 - \frac{1}{m(E_j)} \int_{E_j} T\, 1_{E_j}\, dm\Big) =$$

$$= \max_{1\leq j\leq k} \left(\frac{1}{m(E_j)} \int_{E_j} T\left(\bigcup_{\substack{i=1\\ i\neq j}}^{k} E_i\right) dm - \frac{1}{m(E_j)} \int_{E_j} T\, 1_{E_j}\, dm + 1\right) .$$

Using i) we obtain that

$$\|T_E T - I\|_\infty < \max_{1\leq j\leq k} (1 - \alpha) + 1 = 2(1 - \tfrac{\alpha}{2}) .$$

Taking $\beta = \frac{\alpha}{2}$ the assertion follows.

ii) $\Rightarrow$ i) Let $A,B \in \Sigma$ be such that $A \cap B = \phi$, $0 < m(A)$, $m(B) < +\infty$.

We will define the partition $E = \{A,B,X \setminus (A \cup B)\}$.

Given that $\|T_E T - I\|_\infty < 2(1 - \beta)$ it follows that

$$1 - \frac{1}{m(B)} \int_B T\, 1_B dm + \frac{1}{m(B)} \int_B T\, 1_A dm < 2(1 - \beta)$$

and if we note $\alpha = 2\beta$ we obtain i).

3. The Main Results

Theorem 3. Let T be simultaneously a positive contraction of $L^q(X,\Sigma,m)$ for every $1 \leq q \leq +\infty$. If T satisfies condition a) of Proposition 1 and condition i) of Proposition 2, then for every $1 \leq p < +\infty$

$$\lim_{n\to+\infty} \|T^{n+1} - T^n\|_p = 0 .$$

Proof. We will assume $1 < p < +\infty$ for if $p = 1$ we obtain a special case of Proposition 1.

For every partition E of X we will note $S_E = T_E T$. By Proposition 2 it follows that there exists $\beta > 0$ such that for every partition E of X, $\|S_E - I\|_{\overline{\infty}} < 2(1 - \beta)$. If $1_\infty(k,\mu)$ is the finite dimensional L^∞-space generated by the partition E then its dual space will be $1_1(k, \mu_0)$ where μ_0 is the counting measure $(\mu_0(\{i\}) = 1, \ i = 1,2,\ldots,k)$. Using the proof of the "zero-two" law for positive contractions in L^1-spaces (Theorem 1.1 from [4]) it follows that given $\varepsilon > 0$ there exists $n_1 \in N$ (which depends only on β and ε) such that for every partition E of X and for every $n \geq n_1$ $\|S_E^{n+1} - S_E^n\|_{\overline{\infty}} = \|S_E^{*n+1} - S_E^{*n}\|_{\overline{1}} < \frac{\varepsilon}{3}$ where S_E^* is the dual operator of S_E (S_E^* is a positive contraction of $1_1(k, \mu_0)$).

By Proposition 1 and the proof of Theorem 1.1 from [4] for the same $\varepsilon > 0$ there exists $n_2 \in N$ (which depends on $\frac{\rho}{2}$ and ε) such that for every partition E of X and for every $n \geq n_2$ $\|S_E^{n+1} - S_E^n\|_{\overline{1}} < \frac{\varepsilon}{3}$.

By the Riesz convexity theorem (see for instance [2]) it follows that if $1 \leq p \leq +\infty$ and $n \geq \max\{n_1, n_2\}$ then for every partition E of X $\|S_E^{n+1} - S_E^n\|_{\overline{p}} < \frac{\varepsilon}{3}$.

If we put $n_0 = \max\{n_1, n_2\}$ then from the last inequality it follows that for every partition E of X $\|S_E^{n_0+1} T_E - S_E^{n_0} T_E\|_p < \frac{\varepsilon}{3}$.

Now let $f \in L^p(X,\Sigma,m)$ be such that $\|f\|_p \leq 1$. By lemma 3.1 from [1] it follows that there exists a partition E of X such that

$$\|T^{n_0+1}f - S_E^{n_0+1}T_E f\|_p < \frac{\varepsilon}{3}$$

and

$$\|T^{n_0}f - S_E^{n_0}T_E f\|_p < \frac{\varepsilon}{3},$$

It follows that

$$\|T^{n_o+1}f - T^{n_o}f\|_p \leq \|T^{n_o+1}f - S_E^{n_o+1}T_E f\|_p +$$

$$+ \|S_E^{n_o+1}T_E f - S_E^{n_o}T_E f\|_p + \|S_E^{n_o}T_E f - T^{n_o}f\|_p < \varepsilon .$$

We obtain that $\|T^{n_o+1} - T^{n_o}\|_p \leq \varepsilon$ and the theorem follows as $(\|T^{n+1} - T^n\|_p)_n$ is a non-increasing sequence.

Theorem 3 has the following consequence:

Corollary 4. Let T be simultaneously a positive contraction of $L^q(X,\Sigma,m)$ for every $1 \leq q \leq +\infty$.

If there exists $0 < \gamma < 1$ such that for every $A \in \Sigma$, $0 < m(A) < +\infty$, $\int_A T 1_A \, dm \geq \gamma m(A)$ then for every $1 \leq p < +\infty$

$$\lim_{n\to+\infty} \|T^{n+1} - T^n\|_p = 0 .$$

Proof. It is enough to prove that T satisfies condition a) of Proposition 1 and condition i) of Proposition 2.

Let $A,B \in \Sigma$ such that $0 < m(A), m(B) < +\infty$ and $A \cap B = \phi$. It follows that

$$\int (1_B - 1_A)\cdot T 1_A \, dm \leq \int T 1_A \, dm - \int_A T 1_A \, dm \leq (1 - \gamma)m(A) .$$

Taking $\rho = \quad \gamma$ it follows that condition a) of Proposition 1 is satisfied.

On the other hand

$$\int_B T(1_A - 1_B)\, dm \leq \int 1_B dm - \int_B T\, 1_B\, dm \leq (1 - \gamma)m(B)$$

(we used here that T is a positive contraction of $L^{\infty}(X,\Sigma,m)$ and therefore $1_B\, T\, 1_A \leq 1_B$).

Taking $\alpha = \quad \gamma$ it follows that condition i) of Proposition 2 is also satisfied.

Remarks. 1) The fact that the existence of $\gamma > 0$ implies the existence of $\rho > 0$ was noticed by Professor Foguel.

2) If T as a positive contraction of a finite dimensional L^1-space $l_1(k,\mu)$ is generated by the matrix $(a_{ij})_{i,j=1,\ldots,k}$ and if $a_{ii} \neq 0$ for every $i = 1,2,\ldots,k$ (that is there exists $\rho > 0$ such that all the elements of the diagonal of $(a_{ij})_{i,j}$ are greater than ρ) then by the "zero-two" law for positive contractions in L^1-spaces (Theorem 1.1 from [4]) we obtain that $\lim_n \|T^{n+1} - T^n\|_1 = 0$.

If we note that $a_{ii} = \frac{1}{\mu_i} \int_{\{i\}} T\, 1_{\{i\}}\, d\mu$ it follows that Corollary 4 can be thought of as an extension of the above observation.

References

1. Akcoglu, M.A.: "A pointwise ergodic theorem in L_p-spaces." Canadian J. of Math., Vol. XXVII, no. 5, 1975, 1075-1082.

2. Dunford, N., Schwartz, J.T.: "Linear operators" Part I., New York: Interscience 1958.

3. Neveu, J.: "Mathematical foundations of the calculus of probability", San Francisco, London, Amsterdam: Holden-Day 1965.

4. Ornstein D., Sucheston, L.: "An operator theorem on L_1 convergence to zero with applications to Markov kernels". Ann. Math. Statist. 1970, vol. 41, no. 5, 1631-1639.

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Reçu en Septembre 1983