ANNALES SCIENTIFIQUES
DE L'UNIVERSITÉ DE CLERMONT-FERRAND 2
Série Mathématiques

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Annales scientifiques de l'Université de Clermont-Ferrand 2, tome 51, série *Mathématiques*, nº 9 (1974), p. 31-34

<http://www.numdam.org/item?id=ASCFM_1974__51_9_31_0>

ON AN OSCILLATING RADOM WALK

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Let $\{Y_n, n \geqslant 0 ; Y_0 = x\}$ be a Markov chain with values in R and such that

$$\Pr(Y_{n+1} - y \in A \mid Y_n = y) = \mu(A) \quad \text{if } y < 0 ;$$
$$= \nu(A) \quad \text{if } y > 0 ;$$
$$= \alpha\,\mu(A) + \beta\,\nu(A) \quad \text{if } y = 0.$$

Here, μ and ν denote given probability measures on R, while α and β are non negative constants, $\alpha + \beta = 1$. We are interested in recurrence properties and exact formulae for the process $\{Y_n\}$.

The special case $\mu = \nu$ is precisely the ordinary random walk governed by the measure μ . The special case $\nu(A) = \mu(-A)$ will be called the anti-symmetric case. In that case one may identify the points y and -y and thus obtain the process $Y_{n+1} = |Y_n - X_{n+1}|$; here, $\{X_n\}$ denotes an i.i.d. sequence of random variables with a common distribution μ . If moreover the measure μ is carried by $[0, +\infty)$ one may speak of a one-sided antisymmetric case.For an application to the construction of electrical cables, see [6] and [4] p. 208.

For the case where both μ and ν are carried by $\{-1, 0, +1\}$, this process was already treated by Bhat [1], [2] . The general process has applications in statistics, see [3] , and in information theory, see [5].

Consider the measure

$$Q\,(A) = \sum_{n=0}^{\infty} t^n \Pr\,(Y_n \in A),$$

where t is a fixed number, $|t| < 1$. It satisfies the identity

$$(Q^- + \alpha\, Q^\circ)\,(\delta_0 - t\,\mu) + (Q^+ + \beta\, Q^\circ)\,(\delta_0 - t\nu) = \delta_x$$

Here, Q^- denotes the restriction of Q to $(-\infty, 0)$. Similarly, Q° and Q^+. Further, δ_x denotes the probability measure supported by $\{x\}$. Finally, the above equation is to be interpreted in terms of the Banach algebra of all finite measures, where the multiplication is taken as the ordinary convolution. Let

$$L_\mu^+ = \sum_{n=1}^{\infty} \frac{t^n}{n}\,(\mu^n)^+,$$

similarly, L_μ° , L_ν^- , etc. Using that

$$\delta_0 - t\mu = \exp\,(-\,L_\mu^- - L_\mu^\circ - L_\mu^+)\ ,$$

one easily finds that

$$Q^\circ = \lambda_{-x}\,(\alpha\, e^{-L_\mu^\circ} + \beta\, e^{-L_\nu^\circ})^{-1}\ ,$$

where λ_j is defined by

$$\lambda = e^{L_\mu^+}\, e^{L_\nu^-} = \sum_{j=-\infty}^{+\infty} \lambda_j\, \delta_j$$

If μ and ν are absolutely continuous (relative to Lebesgue measure) and $x \neq 0$ then one further has that

$$Q^- = (\delta_x\, \lambda)^-\ e^{L_\mu^- - L_\nu^-}\ ;$$

$$Q^+ = (\delta_x\, \lambda)^+\ e^{L_\nu^+ - L_\mu^+}\ .$$

These formulae have many applications.

We discuss in detail the case where both μ and ν are supported by the set Z of all integers. For instance, in the one-sided antisymmetric case the state 0 is recurrent if and only if

$$\int_{-\varepsilon}^{+\varepsilon} |1 - \hat{\mu}(\theta)|^{-2} d\theta = +\infty .$$

Here, $\hat{\mu}(\theta)$ denotes the Fourier transform of μ . In the general case, the state 0 is recurrent if and only if

$$\sum_{h=1}^{\infty} C_{\mu}^{+}(h)\, C_{\nu}^{-}(h) = +\infty .$$

Here,

$$C_{\mu}^{+}(h) = \sum_{n=1}^{\infty} P_{\mu}(S_n = h \; ; \; S_m > 0 \quad \text{if} \quad 1 \leq m \leq n) \; ;$$

$$C_{\nu}^{-}(h) = \sum_{n=1}^{\infty} P_{\nu}(S_n = -h \; ; \; S_m < 0 \text{ if } 1 \leq m \leq n).$$

Here, the index μ in P_{μ} indicates that $(S_n = X_1 + \ldots + X_n)$ is an ordinary random walk governed by the measure μ . The quantity $C_{\mu}^{+}(h)$ may also be interpreted as the renewal function associated with the random variable $Z_{\mu}^{+} = S_N$ with $N = \inf \{n > 0 \; ; \; S_n > 0\}$, see [7]. Similarly, for $C_{\nu}^{-}(h)$.

Let m_{μ} and σ_{μ}^{2} denote the mean (assumed finite) and variance associated with the measure μ ; similarly, m_{ν} and σ_{ν}^{2} . Then the following conditions are each sufficient for the state 0 to be recurrent.

(i) $0 < m_{\mu} < \infty \quad ; \; -\infty < m_{\nu} \leq 0$;

(ii) $0 \leq m_{\mu} < \infty \quad ; \; -\infty < m_{\nu} < 0$;

(iii) $m_{\mu} = 0$; $m_{\nu} = 0$; either $\sigma_{\mu}^{2} < \infty$ or $\sigma_{\nu}^{2} < \infty$.

Non-recurrent would be for instance the case where μ is supported by $[0, \infty)$ such that $\mu(\{n\}) \sim n^{-a-1}$, while ν is supported by $(-\infty, 0]$ such that $\nu(\{-n\}) \sim n^{-b-1}$, with a and b as positive constants such that $a + b < 1$. With a positive probability 0 will never be reached, though the process always moves in the direction of 0, occasionally making huge jumps across 0.

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