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GEOMETRY AND COSMOLOGY

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The union between Geometry and Physics has always been very happy, each partner gaining much from its contact with the other, and in Relativity it has been particularly fruitful. In Relativistic Cosmology we find perhaps some of its more interesting products because the geometry of cosmology involves so many things - axiomatics, differential geometry, global geometry and Lie group theory. In my lecture I propose to say something of these various aspects of geometry in cosmology, and although the results I shall mention are taken mostly from earlier papers of mine I hope that the survey will be of some interest in relation to problems under consideration today.

The geometrical model now generally accepted as a good model of the universe as a whole is the topological space $T \times C_3$ where T is the real number (time parameter) continuum and C_3 is a space of constant curvature K which may be positive, negative or zero. This space is endowed with a Riemannian metric :

$$dt^2 - R^2 d\sigma^2$$

where R is a function of t only and $d\sigma^2$ is the metric of C_3, and most theories agree on this form although they differ as to the sign of K or the significance of the function R(t).

The first problem I wish to say something about is that of finding a set of axioms which lead to a model of this form, and in doing this I wish to avoid the usual assumptions, that time can be described by a numerical parameter and that space-time is a differentiable manifold. This can be done because these features are found to be consequences of certain assumptions involving order, denseness and symmetry. Also, in order to avoid the difficult question of communication between observers, I want all axioms, definitions, etc., to be expressible in terms of observations made by just one observer. Such an axiomatic system has been described fully elsewhere(1), and I will therefore now only describe it in outline in order to show what ideas are involved.

The primitives of the axiomatic system are (1) *events*, (2) certain sets of events called *particles*, one of which, the *observer* O, is preferred, (3) a total order relation on the set O, denoted by the symbol < and the word "before", and (4) *light-mappings*, giving a one-one mapping of any particle A onto any particle B, denoted by (A, B) [and thought of as given by light signals from A to B]. An *observable* is defined as a mapping $O \longrightarrow O$ composed of light-mappings and inverse light-mappings. For example one observable f determined by a particle A is given by f = (O, A) (A, O), composition being taken on the right. [If a light signal is sent by O at the event x, is reflected at A, and returns to O at the event y, then y = f(x)].

Although O is the only observer we can conveniently use *relative observables*, an observable relative to a particle A being a mapping $A \longrightarrow A$ composed of light-mappings and their inverses ; for every observable $g : A \longrightarrow A$ relative to A there is a proper observable $(O,A)\, g\, (O,A)^{-1}$. A total order relation can also be induced in A by the mapping (O, A).

For any particles A, B, C an important example of a relative observable is :

$$g = (A,B)\ (B,C)\ (A,C)^{-1}$$

and two of our axioms are that $x \leqslant g(x)$ for all x and that g is strictly increasing, i. e. that $x < y$ implies $g(x) < g(y)$. Taking C = A we deduce that the relative observable f = (A, B) (B, A) has the same

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(1) "Axioms for Cosmology". (Symposium on the Axiomatic method, Berkeley, 1958).

properties as g. If f(x) = x for some x we say that A and B *coincide* at the event x, and one of our axioms restricting the set of particles is that no two particles coincide at any event. [We now think of our particles as "fundamental" particles, corresponding to galaxies in the universe].

The case of equality in the first of the above axioms on g leads to the definition of collinear particles and of a "between-ness" relation, and a further axiom ensures that a linear system of particles is determined by any two of its members. Later, when the idea of distance between particles is established, the first axiom on g ensures that this distance satisfies the triangular inequality and so is a *metric*.

Denseness in a linear system of particles can now be defined, and an axiom is adopted to make every linear system everywhere dense. From this it follows that in the observer set O there is a countable subset of events which is everywhere dense in O, i.e. such that any two events of O are separated by an event of the subset. This property implies that the set O is ordinally similar to a set of real numbers, and it follows that we can parametrize O so that order is preserved. Such a parametrisation is called a "clock" ; it is clearly not unique since it can be "regraduated" by means of any continuous strictly increasing function.

The next axiom may be called the axiom of equivalence since it is derived from the idea of equivalence developed by E. A. Milne in his kinematical theory of relativity. I do not propose to go into details here but will mention only the important consequence that all the observables relative to a particle A, composed of light-mappings between particles of a collinear system containing A, are commutative. From this and certain properties of sets of commutative functions it may be deduced that the particles of any collinear system can be provided with clocks relative to which all light-mappings between these particles are linear functions. This leads to a definition of distance between particles of a collinear system, and from the next axiom, that of symmetry about each particle, it follows that distance measures in different collinear systems can be compared. We now have a *metric* on the set of particles and we find that all the axioms of a metric space are satisfied. We further have collinear systems satisfying Busemann's criteria for geodesics, and the set of particles has thus been endowed with the structure of a *geodesic metric space*. The assumption of symmetry about each particle together with another axiom which has the effect of limiting the dimensions now ensure that this space is three dimensional and is either spherical, projective, euclidean or hyperbolic.

The spherical and projective cases, i.e. those in which the space C_3 of particles has positive curvature, are in fact ruled out by our assumption that all light-mappings are one-one, but our axioms can be modified, or localized, so that these cases are admitted. I shall not go into details here, but this leads to the next problem I wish to discuss, the global effect of local restrictions, particularly the assumption of local symmetry.

Suppose now that space-time has a 4-dimensional Riemannian structure with the usual signature, that there is a system of fundamental particles, and that there is *local* symmetry about each of these particles. Then there is a field of time-like vectors, the *fundamental vectors*, which are the tangent vectors to the fundamental particles' world-lines ; these vectors can be taken to have unit length. Also each point x has a neighbourhood in which the space-time is symmetric about the fundamental vector at x, i.e. admits the group O(3) of motions leaving x and the fundamental vector at x invariant. This neighbourhood is assumed to be so small that cubes of distances from x can be neglected, since this is sufficient to enable us to calculate the effect of symmetry on the curvature tensor at x.

Referring to local coordinates in the neighbourhood of any point x, and writing g_{ij}, R_{hijk}, λ^i for the components of the metric tensor, curvature tensor and fundamental vector at x, we find(1) that the assumption of local symmetry about the fundamental vector at x implies that the fundamental world-line through x has zero curvature at x, and that, at x,

$$R_{hijk} = p(g_{hk}\lambda_i\lambda_j + g_{ij}\lambda_h\lambda_k - g_{hj}\lambda_i\lambda_k - g_{ik}\lambda_h\lambda_j) + q(g_{hk}g_{ij} - g_{hj}g_{ik})$$

for some scalars p, q, where $\lambda_i = g_{ij}\lambda^j$. From this it follows that the Ricci tensor (contracted curvature tensor) is given by an expression of the form :

$$R_{ij} = \alpha\lambda_i\lambda_j + \beta g_{ij}$$

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(1) The calculations referred to in this discussion on the effect of local symmetry are to be found in the *Quart. Journ. Math.*, 6 (1935), 81-93.

for some scalars α, β, and substituting in the expression for the curvature tensor we find :

$$C_{hijk} = 0$$

where C_{hijk} are the components of the conformal tensor.

These results apply to each point of space-time, and from the assumption of local symmetry we have therefore derived a local structure in the sense of differential geometry. We have a Riemannian space-time with a unit time-like vector field (λ^i), the paths of this vector field are geodesics, the space-time is conformally flat, i. e. $C_{hijk} = 0$, and the Ricci tensor satisfies $R_{ij} = \alpha\lambda_i\lambda_j + \beta g_{ij}$ for some scalars α, β. It is now a straightforward calculation to find all the Riemannian spaces satisfying these requirements and in particular to find canonical forms for their metrics.

It can be deduced that coordinates exist locally relative to which the metric of space-time is $dt^2 - R^2 d\sigma^2$ where R is a function of t, $d\sigma^2$ is the metric of a space C_3 of constant curvature, and the fundamental world-lines are orthogonal to C_3. From this we get the well known global models $T \times C_3$ where T is either the real number continuum (which is the more usual assumption) or a circle (giving curious but interesting models with closed fundamental world-lines), and C_3 is a complete space of constant curvature K. If K is positive C_3 is either a sphere or a projective space ; if K is negative C_3 is a hyperbolic space, and if K is zero C_3 is everywhere locally flat i. e. is either L^3 (euclidean space), $L^2 \times C_1$ or $L \times C_1^2$ (cylindrical), or C_1^3 (a 3-torus). In each of these cases C_3 is homogeneous and the Cosmological Principle is satisfied ; this homogeneity is thus a consequence of the assumption of local symmetry. We note however that some but not all of the models are globally symmetric about each fundamental particle, the exceptions being when K = 0 and C_3 is not euclidean. If we impose global symmetry then C_3 is either spherical, projective, hyperbolic or euclidean.

The value of K, or at least its sign if it is not zero, is obviously an important characteristic of a model but it has not so far been derived from any generally accepted assumptions, although various arguments have been put forward for one sign or another (or for zero curvature) in different cosmological theories. The detemination of this characteristic is also an outstanding problem in observational cosmology. The number K occurs in the theoretical formulae obtained when different observables for distant galaxies are correlated, and so might be determined when these correlations are made for actual observations. Unfortunately, however, the usual observables, such as redshift, distance, and number counts, are such that observations will need to be far more numerous and accurate than they are at present before their correlations will have the degree of accuracy necessary for the determination of the sign of K. What I want to describe now is the way Lie group theory may help in this problem by making it possible to include another observable, the *orientation* of galaxies, in the theoretical correlations. If this can be done it appears that the degree of accuracy necessary for the determination of the sign of K is not as high as with the earlier correlations ; and that the determination of this characteristic from observations will be practicable.

There is an "if" here because it is possible that the galaxies are oriented in a random manner, and if this turns out to be the case then there is no useful observable associated with orientation. The alternative is that the galaxies are oriented in some systematic way. If we assume this and apply the Cosmological Principle we can find all such systematic laws of orientation in each model ; in each case this leads to an observable which can be theoretically correlated with the other observables mentioned above.

In the previous discussions the galaxies were represented by points of C_3, but now we will assume that each galaxy has "shape", so that at each point of C_3 there is a preferred frame, or set of axes. For the Cosmological Principle to be satisfied these frames must be distributed over C_3 in such a way that the distribution appears the same from whichever point it is viewed. This implies that C_3 admits a group of motions, i. e. transformations into itself leaving the metric of C_3 invariant ; and since there is only one frame at each point it follows that the group is simply transitive, i. e. for any two points of C_3 there is just one transformation of the group which takes one point into the other. The group is therefore a three-dimensional Lie group and C_3 is the underlying manifold of the group ; our problem is now seen to be soluble because it is known that every complete 3-space of constant curvature does in fact admit such a group of motions.

Each space C_3 admits a 6-dimensional transitive group of motions, and what we want are all the 3-dimensional simply transitive subgroups. We then wish to determine how each of these subgroups distributes a frame (local set of axes) over C_3. These subgroups and frame distributions can be found by straightforward calculations(1), and the results can be summarised as follows.

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(1) See A.G. Walker, "Certain groups of motions in 3-space of constant curvature", *Quart. Journ. Math.*, 11, (1940) 81-94.

$K = 0$. In this case C_3 is euclidean with the euclidean metric, and the simply transitive group of motions is the group of translations. The frames over C_3 are therefore parallel.

$K > 0$. In this case C_3 is spherical or projective, and in each case there are precisely two simply transitive groups of motions. Spherical and projective 3-spaces are well known as Lie groups, and in each case the two simply transitive groups of motions appear as the left and right translation groups.

$K < 0$. In this case C_3 is euclidean with a hyperbolic metric and is found to admit many simply transitive groups of motions. Each such group is determined by (i) an arbitrary unit vector a at one point, which can be taken as the first frame vector at this point, and (ii) an arbitrary parameter τ. The distribution over C_3 of the first frame vector depends upon a but not τ, and if we assume that each galaxy is discoid, so that its orientation is determined by its axis which is taken to be the first frame vector, then the distribution of orientations over C_3 does not involve τ and is uniquely determined by the orientation at one point.

It thus appears that when $K \leqslant 0$ there is a unique "law of orientation" but that when $K > 0$ there are two essentially different possible laws, a fact that may perhaps provide an argument against models with positive curvature. In terms of observables the orientation of a galaxy, or rather of its directed axis, is described by two angles ϑ and Φ, where ϑ measures rotation about the line of sight from the observer to the galaxy and Φ measures rotation towards the line of sight in the plane of this line and the observer's galactic axis(1). The various cases are now found to be sharply distinguished as follows :

$$K = 0 \quad : \quad \vartheta = \Phi = 0 \; ;$$

$$K > 0 \quad : \quad \Phi = 0, \; \vartheta = \pm \alpha r \; ;$$

$$K < 0 \quad : \quad \vartheta = 0, \; \Phi = \beta \cos \lambda . \, r$$

where r is the distance of the galaxy under observation (calculated e.g. from apparent brightness or red-shift), λ is the galactic latitude of the line of sight, and α, β are positive constants. The formulae for ϑ when $K > 0$ and Φ when $K < 0$ are calculated to the first order of approximation, i.e. for galaxies not too distant. The sign in the formula for ϑ when $K > 0$ depends upon which of the two possible laws in this case is being considered, and we see that the two laws differ only in the sense of rotation of the nebular axis about the line of sight.

These results suggest that observations on those galaxies which are seen to be discoid, and so for which ϑ and Φ can be measured, should soon enable us to determine (i) whether or not the orientations are random, and (ii) if the orientations are not random, whether $K = 0$, $K > 0$ or $K < 0$ in the appropriate model of the universe.

DISCUSSION

M. LICHNEROWICZ - Le point de vue fort intéressant, développé par Mr. A.G. Walker présente certains rapports avec le point de vue développé par Mr. Cattaneo en ce qui concerne l'interprétation physique, en termes d'espace et de temps relatifs à un référentiel, des formules fondamentales de la relativité générale.

M. MERCIER - L'adoption du principe cosmologique dans la discussion de la situation expérimentale impose-t-elle une restriction sur le système d'axiomes à la base de votre théorie ?

M. WALKER - No. The cosmological principle is one of homogeneity and so is a consequence of the assumption of symmetry. This symmetry need only be assumed to exist locally, i.e. in the neighbourhood of each event.

M. MERCIER - Si donc le principe cosmologique est une conséquence de vos axiomes et en particulier de l'axiome de la symétrie locale, considérez-vous la symétrie locale comme épistémologiquement plus importante (ou plus fondamentale peut-être) que le principe cosmologique ?

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(1) See A.G. Walker, "The orientation of the extra-galactic nebulae", *Monthly Notices Roy. Astronom. Soc.*, 100, (1940), 623-630.

M. WALKER - Yes, I consider a principle of symmetry to be more important than the cosmological principle because it is more easily expressible in terms of primitive observables and does not require the comparison of one region of the universe with another.

M. TAYLOR - How did the system of axioms provide for the time to be a *complete* set of real numbers ?

M. WALKER - It is not necessary to postulate that the totally ordered set of events at a particle is closed, i. e. that every bounded sequence of events has a limit. If the particle-sets are not closed, new events can be defined, e.g. by sequences or as sections, so that the sets become closed, and the primitive light-mappings can be extended so that the axioms are still satisfied.

Mme TONNELAT - Quand on désire utiliser une définition physique de la "distance" (celle qui va intervenir dans les mesures des astronomes) on a plusieurs possibilités et ces possibilités ne coincident que d'une manière approchée (cf. MacVittie). La plus usuelle est la définition par l'éclairement $\Sigma = \frac{k}{d^2}$. Les axiomes utilisés ici permettent-ils de lever cette ambiguité ? Et dans quelle mesure se raccordent-ils avec ces difficultés ?

M. WALKER - The "distance" referred to in my lecture is a conventional measure between fundamental particles and is represented by the metric of C_3 as a space of constant curvature. The interpretation of distance as measured by astronomers, from either apparent brightness or apparent size, is a straightforward geometrical problem involving the study of thin cones of null geodesics in space-time. This problem was considered in detail in earlier papers of mine(1), where the relations between various definitions of distance were given, and I do not believe that there is any ambiguity provided it is remembered that the measure of "distance" depends upon its definition.

M. LANCZOS - Where does your axiomatic system demand that the experiments must be made by *light* signals and could not possibly be particles moving geodesically with constant velocity ? How then does the Minkowskian type of metric come about, when in the other case the same experiments could also be fitted by a positive definite metric ?

M. WALKER - The adoption of an indefinite metric for the space $T \times C_3$ is made as a matter of convenience, to simplify as much as possible the relation between physics and geometry. With this metric the light cone at an event is simply the null cone, and light paths are easily recognisable as null geodesics. It would be possible to make the experiments with material particles instead of light signals, but they would need to be carefully selected in order to satisfy the axioms and this would make the system highly artificial.

M. CAYREL - La suggestion de déterminer la signe de la courbure de R_3 en étudiant l'orientation des axes des galaxies lointaines est intéressante. Pour le moment, les observations astronomiques qui permettraient le mieux d'avoir une information sur ce signe sont la détermination du décalage vers le rouge et de la magnitude de galaxies très lointaines. L'observation de Minkowski de la source du Bouvier (v/c = 0,46) semble donner plus de vraisemblance à un signe + qu'à un signe - ; mais le résultat n'est peut être pas définitif.

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(1) See for example *Spatial distance in General Relativity* (*Quart. Journ. Math.*, 4 (1933), 71-80), and *Distance in an expanding universe* (*Monthly Notices Roy. Astron. Soc.* 94 (1934) 159-167).