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CLASSES OF MODELS AND SETS OF SENTENCES WITH THE INTERSECTION PROPERTY (*)

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INTRODUCTION.

The usual structures such as groups, rings, fields, and algebraically-closed fields, considered in Algebra have the property that the intersection of substructures of any of these structures is again a structure of the same kind. This *intersection property* is a fundamental feature of algebraic structures and is used, for example, whenever we want to prove existence of subsystem generated by a set of elements.

From the logician's point of view the classes of structures mentioned above have another special property. Each class is the class of all models of a sentence or set of sentences of first-order logic. In other words, these are elementary, or elementary in the wider sense, classes of structures.

It is thus natural to ask : what can be said in general about elementary classes of structures with intersection property. Alternatively we may wish to obtain general results concerning sets S of formal sentences (axioms) of first-order logic having the property that the class of all models of S has the intersection property.

A. Robinson was the first to study classes with i. p. in a systematic way, as part of his scheme of application of logic to algebra, in his book [5]. The important Theorem 2 is quoted from his work. The intersection property was introduced by Robinson under the name "convexity". We prefer the former name firstly because it seems to describe more directly the property in question, and secondly because "convexity" has meanwhile been expropriated by some logicians to name another, completely unrelated, concept.

The intersection property of a set S of sentences is a *semantical property* of S, i. e. it is defined by referring to models of S. Whenever dealing with a semantical property P of formal sentences (or sets of sentences), one of the natural and fundamental problems is to find a syntactical characterization of sentences having property P. We try to describe in syntactical (i. e. by reference only to the *form* of sentences) a set Σ of sentences such that every $\sigma \in \Sigma$ has property P and furthermore, if a sentence σ_1 has P, then there exists a sentence $\sigma_1 \in \Sigma$ such that $\sigma \longleftrightarrow \sigma_1$ is logically valid. Usually we also try to find such a set Σ which is recursive. Such characterization problems were solved for several semantical properties P.

Here we solve the problem of syntactical characterization for sentences with intersection property (Theorem 7) as well as sets of sentences with i. p. (Theorem 8). We distinguish between the case of a single sentence (or *finite* set of sentences) and the case of an (infinite) set of sentences for the following reason. The characterization for single sentences is of the kind described before (with a suitable recursive Σ). The characterization theorem for sets of sentences, however, is of a new kind and is different in form from characterizations found in the literature for other semantical properties. We explain this situation in Section 8 and prove that there does not exist a characterization of the usual kind.

The main tool for obtaining the syntactical characterization is a new concept introduced here

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(Section 3) under the name *relative intersection property*. We establish some basic properties of this notion and also give syntactical characterization of the property that a sentence σ has intersection property relative to a sentence τ (Theorem 6).

Given an elementary class K of structures with intersection property we may assume that the sentence σ defining K has the syntactical form given in the characterization theorem. In proving theorems about elementary classes with intersection property we may use this assumption without loss of generality. In fact the test for usefulness of such a characterization theorem is whether by using the characterization we can obtain results about classes with intersection property which are not obvious from the definition of intersection property. In Chapter III we apply the characterization theorems to obtain results about models of a set S with intersection property which are generated by a set of elements. We also prove that each element of such a model has just a finite number of conjugates by automorphisms of the model over the set of generators.

We conclude this paper by discussing a conjecture of C. C. Chang and indicating another characterization of classes with intersection property as algebras with respect to generalized multi-valued operations.

CHAPTER I

BASIC NOTIONS AND RESULTS

0 - NOTATIONS.

We shall employ first-order calculi (languages) with equality having the individual variables $v_0, v_1, \ldots$, and the usual logical constants $=, \wedge, \vee, \rightarrow, \sim$. Universal and existential quantification with respect to a variable v_i will be denoted by $\bigwedge v_i$ and $\bigvee v_i$ respectively.

In pratice we shall rarely write actual formulas of the formal calculus, instead we shall use meta-mathematical abbreviations to denote the formulas we have in mind. The notational conventions to be adopted are the following.

$x, y, z, x_1, y_1, z_1, \ldots$, (sometimes also with superscripts) will be used as meta-mathematical variables denoting the individual variables of the formal calculus.

Bold face letters will denote sequences. In particular, $\underline{s}$, $\underline{t}$, $\underline{u}$ will denote arbitrary sequences of positive integers ; $\underline{k}, \underline{n}, \underline{m}, \underline{p}, \underline{q}, \underline{n+k}$, etc. will denote the sequences $\langle 1, 2, \ldots, k\rangle, \ldots, \langle 1, 2, \ldots, n+k\rangle$ which are initial segments of the sequence of positive integers.

If $\underline{t} = \langle i_1, \ldots, i_q\rangle$ then by, definition,

$$|\underline{t}| = \max_{1 \leq j \leq q} i_j .$$

If $a_1, a_2, \ldots$, is a sequence of variables or elements of a set and $\underline{t}$ is as above then $a_{\underline{t}}$ will denote the sequence $a_{i_1}, \ldots, a_{i_q}$; thus $x_{\underline{n}}$ denotes $x_1, \ldots, x_n$.

$\bigwedge_{1 < i < n} F_i$ abbreviates the conjunction $F_1 \wedge \ldots \wedge F_n$; similarly for disjunction.

$\bigwedge x_{\underline{n}}$ abbreviates the string $\bigwedge x_1 \ldots \bigwedge x_n$ of n universal quantifiers ; similarly for existential quantifiers. Thus $\bigwedge x_{\underline{n}} \bigvee y_{\underline{m}} B(x_{\underline{n}}, y_{\underline{m}})$ denotes the formula

$$\bigwedge x_1 \ldots \bigwedge x_n \bigvee y_1 \ldots \bigvee y_m \; B(x_1, \ldots, x_n, y_1 \ldots, y_m).$$

Let A be a formula with q variable-places, $\bigvee_{|\underline{t}|<r} A(x_{\underline{t}})$ denotes the repeated disjunction

$$\bigvee_{1 \leq i_1 \leq r} \ldots \bigvee_{1 \leq i_q \leq r} A(x_{i_1}, \ldots, x_{i_q})$$

of r^q substitution instances of A.

Let $\underline{s} = \langle j_1, \ldots, j_p\rangle$, $\underline{t} = \langle i_1, \ldots, i_q\rangle$ the formula $x_{\underline{s}} \subseteq y_{\underline{t}}$ abbreviates

$$\bigwedge_{1 \leq m \leq p} \bigwedge_{1 \leq n \leq q} x_{j_m} = y_{i_n} .$$

In words : every x_{j_m} equals some y_{i_n}. Thus if $\xi_{\underline{p}} \subseteq \eta_{\underline{q}}$ (this does not denote a formula of the calculus, what is meant is that the sequences of elements $\xi_1, \ldots, \xi_p$, $\eta_1, \ldots, \eta_q$ *satisfy* the formula $x_{\underline{p}} \subseteq y_{\underline{q}}$), then $\{\xi_1, \ldots, \xi_p\} \subseteq \{\eta_1, \ldots, \eta_q\}$.

A *sentence* is a formula without free variables. Tne notation $S \vdash T$ means that every sentence in T is a logical consequence of the set S of sentences. The sets S and T will be called *equivalent* (notation : $S \equiv T$) if $S \vdash T$ and $T \vdash S$.

A sentence is said to be *universal-existential* if it is in prenex form and all universal quantifiers precede all existential quantifiers (e.g. $\bigwedge v_1 \bigvee v_2 \; P(v_1, v_2)$).

AE will denote the set of all universal-existential sentences.

1 - STRUCTURES AND MODELS.

Even though the results in this paper are general and apply to structures with an arbitrary fixed number of operations and relations (e.g. ordered fields) we shall restrict the discussion, for the sake of notational simplicity, to structures of type $\langle 2 \rangle$.

By a first-order *structure* (of type $\langle 2 \rangle$) we mean a system $\mathfrak{A} = \langle A, R \rangle$ where $A = D(\mathfrak{A})$ is a set called the *domain* of $\mathfrak{A}$, and $R \subseteq A \times A$ is a binary relation on A. The structure $\langle \emptyset, \emptyset \rangle$ with empty domain will be referred to as the *improper structure*.

Inasmuch as structures of types other than $\langle 2 \rangle$ will be used as examples or in proofs, the deviation from the restriction to type $\langle 2 \rangle$ will be obvious from the context.

The *restriction* $S|A$ of a relation $S \subseteq B \times B$ to a subset $A \subseteq B$ is defined by $S|A = S \cap (A \times A)$. A structure $\mathfrak{A} = \langle A, R \rangle$ is a *substructure* of a structure $\mathfrak{B} = \langle B, S \rangle$ (notation : $\mathfrak{A} \subseteq \mathfrak{B}$) if $A \subseteq B$ and $R = S|A$.

If $\mathfrak{A}_i = \langle A_i, R_i \rangle$, $i \in I$, is a set of structures then the *union* and *intersection* of these structures are defined by

$$\bigcup_{i \in I} \mathfrak{A}_i = \langle \bigcup_{i \in I} A_i , \bigcup_{i \in I} R_i \rangle, \quad \bigcap_{i \in I} \mathfrak{A}_i = \langle \bigcap_{i \in I} A_i , \bigcap_{i \in I} R_i \rangle.$$

Note that the intersection of a set of structures may be the improper structure.

With the class of all structures (of type $\langle 2 \rangle$) we associate a first-order calculus L having, besides the individual variables and logical constants, a binary predicate **P**.

We assume the notion of *satisfaction* of a formula $F(v_0, \ldots, v_n)$ of L by a sequence of elements $a_0, \ldots a_n \in D(\mathfrak{A})$ in the structure $\mathfrak{A}$ to be known. If the sequence satisfies the formula in $\mathfrak{A}$ we shall write

$$\mathfrak{A} \models F(a_0, \ldots, a_n)$$

and say that $F(a_0, \ldots, a_n)$ *holds* or is *true* in $\mathfrak{A}$.

A structure $\mathfrak{A}$ is called a *model of a set* S of *sentences* (notation : $\mathfrak{A} \models S$) if $\sigma \in S$ implies $\mathfrak{A} \models \sigma$. The improper structure is considered to be a model of every set of sentences. If $\mathfrak{B} \models S$ and S is held fixed in the discussion, then a substructure $\mathfrak{A} \subseteq \mathfrak{B}$ will be called a *submodel* if $\mathfrak{A} \models S$.

2 - THE INTERSECTION PROPERTY.

We turn now to the basic concept of this investigation namely the intersection property (i.p.). We start by defining both the i.p. and the finite i.p. for classes of structures.

DEFINITION 1. A class K of structures is said to have the *intersection property* if (i) K contains the improper structure, (ii) $\mathfrak{A} \in K$ and $\mathfrak{A}_i \subseteq \mathfrak{A}$, $\mathfrak{A}_i \in K$, for $i \in I$, imply $\bigcap_{i \in I} \mathfrak{A}_i \in K$. K is said to have the *finite i.p.* if (i) and (ii) hold with the sets I in (ii) restricted to finite sets.

The class of all rings (considered as structures of an appropriate type), which is assumed in this context to contain the improper ring having an empty domain, is an example of a class with i.p. Similarly fields, groups, boolean algebras, and in fact most of the systems usually considered in algebra clearly have i.p. Somewhat more sophisticated examples of classes with i.p. are algebraically closed fields and real closed fields (we shall omit the proofs of these two statements).

Not much can be said about general classes with i.p. All the above classes, however, are elementary classes (in the wider sense) of structures, i.e. classes of models of certain sets of sentences. For this more restricted case of elementary classes with i.p. we can in fact develop a detailed theory. We are thus led to the following.

DEFINITION 2. A (consistent) set S of sentences of L has the *i.p.* (*finite i.p.*) if the class K(S) of all models of S has the i.p. (finite i.p.). A sentence σ is said to have i.p. (finite i.p.) if $\{\sigma\}$ has i.p. (finite i.p.).

For general classes of structures the finite i.p. does not imply the i.p. All the above examples, however, satisfy both finite i.p. and i.p. This is not accidental in view of the following result which was announced without proof by Chang [1] and later rediscovered by the present author.

THEOREM 1. If S has the finite intersection property then S has the (unrestricted) intersection property.

PROOF. Let $\mathfrak{A} = \langle A, R\rangle$ be a model of S and $\mathfrak{A}_i$, $i \in I$, be a system of submodels. Denote the substructure $\bigcap_{i\in I} \mathfrak{A}_i \subseteq \mathfrak{A}$ by $\mathfrak{B}$. Augment the language L to a language L' by adding one individual constant ξ for each element ξ of $D(\mathfrak{A})$ (the same constant is used to denote the element in the metalanguage), and a one-place predicate B(x). If σ is a sentence of L then $\sigma^{B(x)}$ will denote sentence of L' obtained from σ by relativizing all quantifiers of σ to B(x)(*). $S^{B(x)}$ will be the set $\{\sigma^{B(x)} \mid \sigma \in S\}$.

Consider a finite subset $\{\xi_1, \dots, \xi_n, \xi_{n+1}, \dots, \xi_m\}$ of $D(\mathfrak{A})$ such that $\xi_i \notin D(\mathfrak{B})$, $1 \le j \le n$, $\xi_k \in D(\mathfrak{B})$, $n+1 \le k \le m$. For each $1 \le j \le n$ there exists an $i_j \in I$ such that $\xi_j \notin D(\mathfrak{A}_{i_j})$. Let $\mathfrak{B}_1 = \bigcap_{1\le j\le n} \mathfrak{A}_{i_j}$; $\mathfrak{B}_1$ is a submodel of $\mathfrak{A}$ (by finite i.p. of S). By interpreting B(x) as $x \in D(\mathfrak{B}_1)$ we see that the set of sentences

$$S \cup \underline{A} \cup S^{B(x)} \cup \{\sim B(\xi_j) \mid 1 \le j \le n\} \cup \{B(\xi_k) \mid n+1 \le k \le m\}$$

where $\underline{A}$ is the *diagram* [5, p.74] of $\mathfrak{A}$, is consistent. This implies the consistency of the set

$$S \cup \underline{A} \cup S^{B(x)} \cup \{\sim B(\xi) \mid \xi \in A,\ \xi \notin D(\mathfrak{B})\} \cup \{B(\xi) \mid \xi \in D(\mathfrak{B})\}.$$

This set, therefore, has a model $\mathfrak{C}' = \langle C, T, B_1, c_\xi\rangle_{\xi\in A}$ where T and B_1 are binary and unary relations corresponding to the predicates P and B(x) respectively and the elements $c_\xi \in C$ correspond to the constants ξ of L'. We may identify each c_ξ with the corresponding $\xi \in A$; thus $A \subseteq C$ and since $\mathfrak{C}' \models \underline{A}$ we have $T|A = R$. Let $\mathfrak{C} = \langle C, T\rangle$, clearly $\mathfrak{C} \models S$ and $\mathfrak{A} \subseteq \mathfrak{C}$. From $\mathfrak{C}' \models S^{B(x)}$ and $B_1 = \{\xi \mid \mathfrak{C}' \models B(\xi)\}$ it follows that $\mathfrak{B}_1 = \langle B_1, T|B_1\rangle$ is a model of S. Now, for $\xi \in A$ we have $\mathfrak{C}' \models B(\xi)$ if and only if $\xi \in D(\mathfrak{B})$ (B_1 may, however, contain elements which are not in A). Hence $A \cap B_1 = D(\mathfrak{B})$ and thus $\mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}_1$. But $\mathfrak{A}$ and $\mathfrak{B}_1$ are models of S which are submodels of the model $\mathfrak{C}$ (of S). Thus $\mathfrak{B}$ is a model of S by the finite i.p. of S. Since $\mathfrak{B}$ was an arbitrary intersection of submodels of $\mathfrak{A}$, S has the unrestricted i.p.

In view of Theorem 1 the notions of finite intersection property and intersection property are equivalent and hence mutually exchangeable throughout the following discussion.

Robinson proved in [5, p.117] the following fundamental theorem concerning sets with intersection property.

THEOREM 2. If S has i.p. then the union of every system $\mathfrak{A}_i$, $i \in I$, of models of S such that for every i, $j \in I$, $\mathfrak{A}_i \subseteq \mathfrak{A}_j$ or $\mathfrak{A}_j \subseteq \mathfrak{A}_i$, is a model of S.

The property of S established in the above theorem may be called *closure under unions of chains*. Closure under unions of chains implies by a theorem of Chang, Suzko and Łoś [2,3] that $S \equiv S'$ for a set $S' \subset AE$.

COROLLARY. If S has i.p. then $S \equiv S'$ for a set $S' \subseteq AE$. The set $\underline{S}'$ can be taken to consist of all sentences $\sigma \in AE$ for which $S \vdash \sigma$.

3 - RELATIVE INTERSECTION PROPERTY.

Our next aim is to express the intersection property of an arbitrary set S by properties of *finite* subsets of S. If S has i.p. and $\sigma \in S$ then σ need not have the i.p.; still for every model of S and every system of submodels, σ holds in the intersection of the submodels. We contend that this behavior of σ depends just on some finite subset of S. This is made precise by the following definition and theorem.

DEFINITION 3. The sentence σ is said to have intersection property *relative to the sentence* τ if $\mathfrak{A} \models \tau$ and $\mathfrak{A} \supseteq \mathfrak{A}_i \models \tau$, $i \in I$, imply $\bigcap_{i\in I} \mathfrak{A}_i \models \sigma$.

(*) If F(x, y) is a formula containing the free variable x and possibly also the free variable y and σ is any sentence, then σ^F - the result of relativizing the quantifiers of σ to F(x, y) considered as a predicate in x, is constructed as follows. Change alphabetically all the (bounded) occurrences of y in σ. Starting with the innermost quantifier of σ and proceeding outwards, replace each part $\bigwedge zB$ by $\bigwedge z\,[F(z, y) \longrightarrow B]$ and each par $\bigvee zB$ by $\bigvee z\,[F(z, y) \wedge B]$. Proceed until all the original quantifiers of σ have been dealt with.

A set S is called *(quasi) conjunctive* if for every $\tau_1, \ldots, \tau_n \in S$ there exists a $\tau \in S$ such that $\tau_1 \wedge \ldots \wedge \tau_n \equiv \tau$. Examples of conjunctive sets are sets $S \subseteq AE$ such that $\sigma \in AE$ and $S \vdash \sigma$ imply $\sigma \in S$. This follows from the fact that a conjunction of sentences in AE is logically equivalent to a sentence in AE.

THEOREM 3. A conjunctive set S has i.p. if and only if for every $\sigma \in S$ there exists a sentence $\tau \in S$ such that σ has i.p. relative to τ.

PROOF. The sufficiency part of the theorem is immediate.

To prove the necessity part assume by way of contradiction that S is conjunctive and has i.p. and for some $\sigma \in S$, σ does *not* have i.p. relative to any $\tau \in S$. Thus for every $\tau \in S$ there exists a model $\mathfrak{A}_0 = \langle A, R \rangle$ of τ and a system of submodels $\mathfrak{A}_0 \supseteq \mathfrak{A}_\eta \models \tau$, $\eta \in I$, such that $\mathfrak{B} = \bigcap_{\eta \in I} \mathfrak{A}_\eta$ is non-empty and $\mathfrak{B} \models \sim \sigma$.

We may assume that $0 \in I$ (i.e. $\mathfrak{A}_0$ is one of the $\mathfrak{A}_\eta$) and that $A \cap I = \emptyset$. Construct a new structure $\mathfrak{A}_1 = \langle A \cup I, R, I, E \rangle$ of type $\langle 2, 1, 2 \rangle$ where $E = \{\langle \xi, \eta \rangle \mid \eta \in I, \xi \in D(\mathfrak{A}_\eta)\}$. Add to L the predicates I(y) and E(x, y) and call the new calculus L_1.

For any sentence λ of L and any formula F(x) or F(x, y) of L_1, λ^F will denote the formula of L_1 obtained by relativizing the quantifiers of λ to F (see Footnote (*), p.43) where F is considered as a predicate in x. Note that $\lambda^{F(x,y)}$ may contain y as a free variable.

Let $A(x) = \bigvee y E(x,y)$ and $B(x) = \bigwedge y[I(y) \longrightarrow E(x,y)]$. It is clear that $\mathfrak{A}_1 \models A(\xi)$ if and only if $\xi \in A$, and $\mathfrak{A}_1 \models B(\xi)$ if and only if $\xi \in D(\mathfrak{B})$. Furthermore, the construction of $\mathfrak{A}_1$ implies that the sentences

$$\tau^{A(x)}, \quad \varphi(\tau) = \bigwedge y[I(y) \longrightarrow \tau^{E(x,y)}], \quad \sim \sigma^{B(x)}, \bigvee x\, B(x). \tag{1}$$

hold in $\mathfrak{A}_1$. These sentences are, therefore, consistent.

We notice now that

$$\tau \vdash \tau_1 \text{ implies } \tau^{A(x)} \longrightarrow \tau_1^{A(x)}, \quad \tau \vdash \tau_1 \text{ implies } \varphi(\tau) \longrightarrow \varphi(\tau_1). \tag{2}$$

Since S is conjunctive, the consistency of (1) for every $\tau \in S$ together with (2) imply that the set

$$\{\tau^{A(x)} \mid \tau \in S\} \cup \{\varphi(\tau) \mid \tau \in S\} \cup \{\sim \sigma^{B(x)}, \bigvee x\, B(x)\} \tag{3}$$

is consistent.

Thus (3) has a model $\mathfrak{A}' = \langle A', R', I', E' \rangle$. Define $A_1 = \{\xi \mid \mathfrak{A}' \models A(\xi)\}$, $A_\eta = \{\xi \mid \langle \xi, \eta \rangle \in E'\}$ for $\eta \in I'$, $B_1 = \{\xi \mid \mathfrak{A}' \models B(\xi)\}$. From the fact that $\mathfrak{A}'$ is a model of (3) it follows at once that $\langle A_1, R'|A_1 \rangle \models S$; $\langle A_\eta, R'|A_\eta \rangle \models S$ for $\eta \in I'$; $B_1 = \bigcap_{\eta \in I'} A_\eta$; $B_1 \neq \emptyset$; $\langle B_1, R'|B_1 \rangle \models \sim \sigma$. This contradicts the i.p. of S. Thus for some $\tau \in S$, σ has i.p. relative to τ.

We can now combine the Corollary of Theorem 2 with Theorem 3 to get the following.

THEOREM 4. S has i.p. if and only if there exists a set S_1 such that $S \equiv S_1$, $S_1 \subseteq AE$ and for every $\sigma \in S$ there exists a $\tau \in S_1$ such that σ has i.p. relative to τ. If S_1 is finite then S can be taken to consist of a single sentence.

PROOF. Let S_1 be the set of all AE consequences of S. By the Corollary of Theorem 2 $S \equiv S_1$ so that S_1 has i.p. By the remark following Definition 3 S_1 is conjunctive so that the result follows from the previous theorem.

In case S is finite $S_1 \equiv \{\sigma\} \subseteq AE$ for a suitable σ. Since σ has i.p. it trivially has i.p. relative to itself.

CHAPTER II

SEQUENCES OF SUCCESSORS, AND AND SYNTACTICAL CHARACTERIZATION THEOREMS

4 - A SIMPLE CASE.

Having Theorem 4 at our disposal, it is clear that we shall be able to characterize sets with i.p. if we could find a satisfactory necessary and sufficient condition for a sentence σ to have i.p. relative to a sentence τ, where both σ and τ are in AE. Let us consider the simplest case where $\sigma = \bigwedge x \bigvee y A(x, y)$, $\tau = \bigwedge x \bigvee y B(x, y)$.

Let $\mathfrak{A} = \langle A, R \rangle \models \tau$ and $\xi \in A$. We shall call $\underline{\eta} = \langle \eta_i \rangle_{i<\omega}$ a *sequence of successors* (of length ω) *of* ξ *by* τ if $\eta_0 = \xi$ and $\mathfrak{A} \models B(\eta_i, \eta_{i+1})$, $0 \leq i \leq \omega$.

The following two statements are easily verifiable. *Let* $\mathfrak{A} \models \tau$, *every* $\xi \in A$ *has at least one sequence of successors by* τ . *If* $\xi \in A$ *and* $\underline{\eta}$ *is sequence of successors of* ξ *by* τ *then the substucture* $\mathfrak{A}_{\underline{\eta}} \subseteq \mathfrak{A}$

$$\mathfrak{A}_{\underline{\eta}} = \langle \{ \eta_i \mid 0 \leq i < \omega \}, \ R | \{ \eta_i \mid 0 \leq i < \omega \} \rangle \qquad (4)$$

is a model of τ *containing* ξ.

Using the notion of sequence of successors and the above two statements we can easily give a *semantical* condition for σ to have i.p. relative to τ as follows.

THEOREM. Let σ and τ be as above. σ has i.p. relative to τ if and only if for every element ξ of an arbitrary model $\mathfrak{A}$ of τ, every sequence of successors $\underline{\eta}$ of ξ by τ contains an element η_i such that $\mathfrak{A} \models A(\xi, \eta_i)$ and η_i is contained in every other sequence $\underline{\eta}'$ of successors of ξ by τ.

PROOF. We start by proving necessity of the condition. Let σ have i.p. relative to τ and assume that the condition does not hold. Thus for a suitable model $\mathfrak{A}$ of τ, some $\xi \in D(\mathfrak{A})$, and a sequence $\underline{\eta}$ of successors of ξ, for every element η_j of $\underline{\eta}$ such that $\mathfrak{A} \models A(\xi, \eta_j)$ there would exist a sequence $\underline{\eta}^j$ of successors of ξ by τ *not* containing η_j.

Let $J = \{ j | j < \omega, \ \mathfrak{A} \models A(\xi, \eta_j) \}$. The substructure $\mathfrak{B} = \mathfrak{A}_{\eta} \cap \bigcap_{j \in J} \mathfrak{A}_{\eta^j}$ (cf. (4)) of $\mathfrak{A}$ is an intersection of models of τ each containing ξ, and for $\xi \in D(\mathfrak{B})$ there is no $\eta \in D(\mathfrak{B})$ such that $\mathfrak{B} \models A(\xi, \eta)$. Thus $\mathfrak{B} \models \sim \bigwedge x \bigvee y A(x, y)$ contradicting the assumption that σ has i.p. relative to τ.

To prove sufficiency, assume the condition to hold. Let $\mathfrak{A}$ and $\mathfrak{A}_\alpha \subseteq \mathfrak{A}$, $\alpha \in I$, be models of τ. Let $\mathfrak{B} = \bigcap_{\alpha \in I} \mathfrak{A}_\alpha$, we may assume that $D(\mathfrak{B}) \neq \emptyset$. Let $\xi \in D(\mathfrak{B})$. The element ξ has a sequence $\underline{\eta}$ of successors by τ in $\mathfrak{A}$ and, for each $\alpha \in I$, since $\xi \in D(\mathfrak{A}_\alpha)$ and $\mathfrak{A}_\alpha \models \tau$, ξ has a sequence $\underline{\eta}^\alpha$ of successors by τ in $\mathfrak{A}_\alpha$. By assumption, one of the elements η_i of $\underline{\eta}$ satisfies $\mathfrak{A} \models A(\xi, \eta_i)$ and appears in every sequence $\underline{\eta}^\alpha$, $\alpha \in I$. Thus $\eta_i \in D(\mathfrak{B})$. The sentence $\sigma = \bigwedge x \bigvee y A(x, y)$ is therefore true in $\mathfrak{B}$, and σ has i.p. relative to τ.

5 - GENERAL SEQUENCES OF SUCCESSORS.

We have seen that relative i.p. for sentences in AE involving two quantifiers can be characterized by a property of sequences of successors. To obtain a general result we have to generalize the concept of a sequence of successors to formulas with more than two variables.

DEFINITION 4. Let n, p, m be positive integers, a system of functions $f_1, \ldots, f_n$ from non-negative integers to positive integers will be called a (p, m) *selection system* if i. $f_j(i) \leq p + mi$, $1 \leq j \leq n$, $0 \leq i \leq \omega$, ii. $\langle f_1(i), \ldots, f_n(i) \rangle$, $0 \leq i < \omega$, runs through all n-tuples of positive integers.

In the following we assume that for every n, p, m a fixed (p, m) selection system of n functions $f_j^{(n,p,m)}$, $1 < j < n$, has been chosen and that this was done in such a way that $f_j^{(n,p,m)}(i)$ is recursive in all variables.

All subsequent definitions and theorems will refer to these fixed selection systems

We recall some of our notational conventions. $\underline{x}_n$ denotes the sequence $x_1, \ldots, x_n$ of n individual variables, thus $B(\underline{x}_n, \underline{y}_m)$ stands for $B(x_1, \ldots, x_n, y_1, \ldots, y_m)$; similarly $\underline{\xi}_n$, $\underline{\eta}_n$, etc. denote sequences of n elements $\xi_1, \ldots, \xi_n$ etc. $\bigwedge \underline{x}_n$ stands for the sequence $\bigwedge x_1 \ldots \bigwedge x_n$ of universal quantifiers, similarly for $\bigvee \underline{y}_m$.

DEFINITION 5. Let $\tau = \bigwedge \underline{x}_n \bigvee \underline{y}_m B(\underline{x}_n, \underline{y}_m)$. $S_{\tau,p,k}(\underline{x}_p, \underline{y}_{p+mk})$ will abbreviate the formula

$$\bigwedge_{1 \leq i \leq p} x_i = y_i \wedge \bigwedge_{0 \leq i < k} B(y_{f_1(i)}, \ldots, y_{f_n(i)}, y_{p+mi+1}, \ldots, y_{p+m(i+1)}). \tag{5}$$

If $\mathfrak{A} \models S_{\tau,p,k}(\underline{\xi}_p, \underline{\eta}_{p+mk})$ then $\underline{\eta}_{p+mk}$ will be called a *sequence of successors by* τ (of length p+mk) of $\xi_1, \ldots, \xi_p$.

We shall usually drop the index p and write $S_{\tau,k}(\underline{x}_p, \underline{y}_{p+mk})$ for the formula (5).

REMARK. For every $0 \leq k$ clearly

$$S_{\tau,k+1}(\underline{x}_p, \underline{y}_{p+m(k+1)}) \equiv S_{\tau,k}(\underline{x}_p, \underline{y}_{p+mk}) \wedge B(y_{f_1(k)}, \ldots, y_{f_n(k)}, y_{p+mk+1}, \ldots, y_{p+m(k+1)}). \tag{6}$$

LEMMA . If $\mathfrak{A} \models \tau$ and $\xi_1, \ldots, \xi_p \in D(\mathfrak{A})$ then there exists a sequence $\langle \eta_i \rangle_{1 \leq i < \omega}$ of elements of $D(\mathfrak{A})$ such that

$$\mathfrak{A} \models S_{\tau,k}(\xi_1, \ldots, \xi_p, \eta_1, \ldots, \eta_{p+mk}), \quad 1 \leq k < \omega. \tag{7}$$

If a sequence $\langle \eta_i \rangle_{1 \leq i < \omega}$ *satisfies (7) and if* $B = \{\eta_i \mid 1 \leq i < \omega\}$ *then the structure* $\mathfrak{B} = \langle B, B|R \rangle \subseteq \mathfrak{A}$ *is a model of* τ *containing* $\xi_1, \ldots, \xi_p$.

PROOF. Define $\eta_1 = \xi_1, \ldots, \eta_p = \xi_p$. Assume that we have constructed a sequence $\eta_1, \ldots, \eta_{p+mr}$ satisfying (7) for $1 \leq k \leq r$. By i. of Definition 4, $f_j(r) \leq p+mr$, $i \leq j \leq n$, thus the elements $\eta_{f_j(r)}$, $1 \leq j \leq n$, are already defined. Since $\mathfrak{A} \models \tau$ there exist elements $\vartheta_1, \ldots, \vartheta_m$ such that $\mathfrak{A} \models B(\eta_{f_1(r)}, \ldots, \eta_{f_n(r)}, \vartheta_1, \ldots, \vartheta_m)$. Define $\eta_{p+mr+1} = \vartheta_1, \ldots, \eta_{p+m(r+1)} = \vartheta_m$ then by (6) $\mathfrak{A} \models S_{\tau,r+1}(\xi_1, \ldots, \xi_p, \eta_1, \ldots, \eta_{p+m(r+1)})$ and the sequence $\eta_1, \ldots, \eta_{p+m(r+1)}$ satisfies (7) for $1 \leq k \leq r+1$. This process can be continued to construct the infinite sequence $\langle \eta_i \rangle_{1 \leq i < \omega}$.

To prove the second assertion, let $\eta_{j_1}, \ldots, \eta_{j_n}$, be arbitrary elements of $B = D(\mathfrak{B})$. By ii. of Definition 4 $< j_1, \ldots, j_n > = < f_1(k), \ldots, f_n(k) >$ for some k. It follows from (7) (for k+1) and (6) that

$$\mathfrak{A} \models B(\eta_{f_1(k)}, \ldots, \eta_{f_n(k)}, \eta_{p+mk+1}, \ldots, \eta_{p+m(k+1)}).$$

Now $\eta_{p+mk+i} \in B$, $1 \leq i \leq m$. Thus $\mathfrak{B} \models \bigwedge \underline{x}_n \bigvee \underline{y}_m B(\underline{x}_n, \underline{y}_m)$.

REMARK. Lemma 5 generalizes to the case of arbitrary formulas τ in AE, the two italicized statements of Section 4. From Lemma 5 it is now possible to derive a *semantical* characterization of relative i.p. which is completely analogous to the theorem given in Section 4. As the statement of the corresponding theorem and its proof are almost identical with what was done in Section 4, we skip the details.

6 - SYNTACTICAL CHARACTERIZATION OF RELATIVE i.p.

The desired syntactical characterization is now an almost direct consequence of Lemma 5 and the completeness theorem.

THEOREM 6. Let σ $\bigwedge \underline{x}_p \bigvee \underline{y}_q A(\underline{x}_p, \underline{y}_q)$, $\tau = \bigvee \underline{x}_n \bigwedge \underline{y}_m B(\underline{x}_n, \underline{y}_m)$. σ has i.p. relative to τ if and only if for some $1 \leq k$, $\vdash \tau \longrightarrow \tau_{\sigma,k}$ where

$$\tau_{\sigma,k} = \bigwedge \underline{x}_p \bigwedge \underline{y}_{p+mk} [S_{\tau,k}(\underline{x}_p, \underline{y}_{p+mk}) \longrightarrow \bigvee_{|t| \leq p+mk} [A(\underline{x}_p, \underline{y}_t) \bigwedge \underline{z}_{p+mk} [S_{\tau,k}(\underline{x}_p, \underline{z}_{p+mk}) \longrightarrow \underline{y}_t \subseteq \underline{z}_{p+mk}]]]^{(*)} \tag{8}$$

(*) For the notations used in this formula see Section 0.

REMARK. In semantical terms the above statement reads : *σ has i.p. relative to τ if and only if for some k, in every model $\mathfrak{A}$ of τ and for every $\xi_1,\ldots,\xi_p \in D(\mathfrak{A})$, every sequence $\underline{\eta}_{p+mk}$ of length* p+mk *of successors of $\underline{\xi}_p$ by τ contains* q *elements $\eta_{i_1},\ldots,\eta_{i_q}$ satisfying $\mathfrak{A} \models A(\underline{\xi}_p, \eta_{i_1},\ldots,\eta_{i_q})$ so that these* q *elements appear in every other sequence $\underline{\zeta}_{p+mk}$ of successors of $\underline{\xi}_p$ by τ.* Note that this statement runs completely parallel to the semantical condition given in Section 4 and its generalization which was mentioned at the end of Section 5. The only difference is that here we use sequences of successors of *finite* length and are therefore able to write a formal sentence (8).

PROOF. Assume $\vdash \tau \longrightarrow \tau_{\sigma,k}$ for some $1 \leq k$. Let $\mathfrak{A}$ and $\mathfrak{A}_i \subseteq \mathfrak{A}$, $i \in I$, be models of τ ; let $\mathfrak{B} = \bigcap_{i\in I} \mathfrak{A}_i$. If $D(\mathfrak{B}) = \emptyset$ then there is nothing to prove. Assume therefore that $\xi_1,\ldots,\xi_p$ are p elements in $D(\mathfrak{B})$. Since $\mathfrak{A} \models \tau$ and $\mathfrak{A}_i \models \tau$, $i \in I$, there is, by Lemma 5, a sequence $\eta_1,\ldots,\eta_{p+mk}$ of successors by τ of $\xi_1,\ldots,\xi_p$ in $\mathfrak{A}$ and, for $i \in I$, a sequence $\zeta_1^i,\ldots,\zeta_{p+mk}^i \in D(\mathfrak{A}_i)$ of successors of $\xi_1,\ldots,\xi_p$ in $\mathfrak{A}_i$.

The formula $B(\underline{x}_n, \underline{y}_m)$ is open and hence also $S_{\sigma,k}(\underline{x}_p, \underline{y}_{p+mk})$ is open. Consequently, since $\zeta_1^i,\ldots,\zeta_{p+mk}^i$ are successors of the $\xi_1,\ldots,\xi_p$ in $\mathfrak{A}_i$, they are also successors in $\mathfrak{A}$.

From the assumptions we have $\mathfrak{A} \models \tau_{\sigma,k}$ so that among $\eta_1,\ldots,\eta_{p+mk}$ there are q elements $\eta_{i_1},\ldots\eta_{i_q}$ which appear in every sequence $\zeta_1^i,\ldots,\zeta_{p+mk}^i$ and furthermore satisfy $\mathfrak{A} \models A(\underline{\xi}_p, \eta_{i_1},\ldots,\eta_{i_q})$. Thus $\eta_{i_1},\ldots,\eta_{i_q} \in D(\mathfrak{A}_i)$ for every $i \in I$ and hence $\eta_{i_1},\ldots,\eta_{i_q} \in D(\mathfrak{B})$. Thus $\mathfrak{B} \models A(\underline{\xi}_p, \eta_{i_1},\ldots,\eta_{i_p})$. The sentence $\sigma = \bigwedge \underline{x}_p \bigvee \underline{y}_q A(\underline{x}_p, \underline{y}_q)$ is therefore true in the intersection $\mathfrak{B}$ so that σ has i.p. relative to τ.

To prove necessity of the condition, let σ have i.p. relative to τ and assume by way of contradiction that for every $1 \leq k$ *not* $\vdash \tau \longrightarrow \tau_{\sigma,k}$.

By our assumption there exists, for every $1 \leq k$, a model $\mathfrak{A}$ of τ, a sequence $\xi_1,\ldots,\xi_p \in D(\mathfrak{A})$, a sequence $\eta_1,\ldots,\eta_{p+mk}$ of successors of $\underline{\xi}_p$ and, for every $\underline{t} = \langle i_1,\ldots,i_q\rangle$ such that $|\underline{t}| \leq p+mk$, a sequence $\zeta_1^t,\ldots,\zeta_{p+mk}^t$ of successors of $\underline{\xi}_p$ such that $\mathfrak{A} \models \sim A(\underline{\xi}_p, \underline{\eta}_t) \vee \sim [\underline{\eta}_t \subseteq \underline{\zeta}^t_{p+mk}]$.

Thus, putting $\alpha_{t,k} = \sim A(\underline{x}_p, \underline{y}_t) \vee \sim [\underline{y}_t \subseteq \underline{z}^t_{p+mk}]$ where $\underline{t} = \langle i_1,\ldots,i_q\rangle$ and $\underline{z}^t_{p+mk} = \langle z_1^t,\ldots,z_{p+mk}^t\rangle$, the set of formulas

$$H_k = \{\tau,\ S_{\tau,k}(\underline{x}_p,\ \underline{y}_{p+mk})\} \cup \{\alpha_{t,k}\ ,\ S(\underline{x}_p,\ \underline{z}^t_{p+mk}) \mid |\underline{t}| \leq p+mk\}, \tag{9}$$

is consistent.

It can be verified that if $k \leq h$ then $\vdash S_{\tau,h}(\underline{x}_p, \underline{y}_{p+mh}) \longrightarrow S_{\tau,k}(\underline{x}_p, \underline{y}_{p+mk})$

(compare the remark following Definition 5). Furthermore it follows from $\vdash [\underline{y}_t \subseteq \underline{z}_{p+mk} \longrightarrow [\underline{y}_t \subseteq \underline{z}_{p+mh}]$, for $k \leq h$, that $\vdash \alpha_{t,h} \longrightarrow \alpha_{t,k}$. Consequently, if $k \leq h$ then $H_h \vdash H_k$.

This implies that the set $H = \bigcup_{1\leq i<\omega} H_k$ is a consistent set of formulas and hence has a model $\mathfrak{A} = \langle A, R\rangle$. Let $\langle \xi_1,\ldots,\xi_p\rangle = \underline{\xi}_p$ be the elements of A corresponding to $x_1,\ldots,x_p$; $\langle \eta_i\rangle_{1\leq i<\omega} = \underline{\eta}$ and $\langle \zeta_i^t\rangle_{1\leq i<\omega} = \underline{\zeta}^t$, for $\underline{t} = \langle i_1,\ldots,i_q\rangle$, $1 \leq i_j < \omega$, be the sequences of elements of A corresponding to $\langle y_i\rangle_{1\leq i<\omega}$ and $\langle z_i^t\rangle_{1\leq i<\omega}$ respectively. By Lemma 5, since $\underline{\eta}$ and $\underline{\zeta}^t$ are sequences of successors of $\underline{\xi}_p$ by τ, the substructures $\mathfrak{A}_0$, $\mathfrak{A}_{\underline{t}}$

$$\mathfrak{A}_0 = \langle \{\xi_i\}_{1\leq i<\omega},\ R|\{\xi_i\}_{1\leq i<\omega}\rangle\ ,\ \mathfrak{A}_{\underline{t}} = \langle \{\xi_i^t\}_{1\leq i<\omega},\ R|\{\xi_i^t\}_{1\leq i<\omega}\rangle$$

are models of τ containing $\underline{\xi}_p$. Let $\mathfrak{B} = \mathfrak{A}_0 \bigcap_t \mathfrak{A}_{\underline{t}}$ then $\xi_1,\ldots,\xi_p \in D(\mathfrak{B})$.

By i.p. of σ relative to τ we must have $\mathfrak{B} \models \bigwedge \underline{x}_p \bigvee \underline{y}_q A(\underline{x}_p, \underline{y}_q)$. Assume that for some $\vartheta_1,\ldots,\vartheta_q \in D(\mathfrak{B})$, $\mathfrak{B} \models A(\underline{\xi}_p, \underline{\vartheta}_q)$. Since $\mathfrak{B} \subseteq \mathfrak{A}_0$ we have $\vartheta_1 = \eta_{i_1},\ldots,\vartheta_q = \eta_{i_q}$ for some $\underline{t} = \langle i_1,\ldots,i_q\rangle$. Now $\vartheta_1,\ldots,\vartheta_q \in \mathfrak{B} \subseteq \mathfrak{A}_{\underline{t}}$ and hence $\eta_{i_1} = \vartheta_1 = \zeta^t_{j_1},\ldots,\eta_{i_q} = \vartheta_q = \zeta^t_{j_q}$ for suitable $j_1,\ldots,j_q$. Let k be such that $j_e \leq p+mk$ for $1 \leq e \leq q$, thus $\underline{\eta}_t \subseteq \underline{\zeta}^t_{p+mk}$ holds. We have $\mathfrak{A} \models \alpha_{t,k}$ i.e. $\mathfrak{A} \models \sim A(\underline{\xi}_p, \underline{\eta}_t) \vee \sim [\eta_t \subseteq \xi^t_{p+mk}]$.

Since $A(\underline{\xi}_p, \underline{\eta}_t)$ is true in $\mathfrak{B}$ and hence, $A(\underline{x}_p, \underline{y}_q)$ being open, is true in $\mathfrak{A}$, it follows that $\mathfrak{A} \models \sim [\underline{\eta}_t \subseteq \underline{\zeta}^t_{p+mk}]$, a contradiction with preceding statements.

Thus for $\underline{\xi}_p$ there is no sequence $\vartheta_1,\ldots,\vartheta_q \in D(\mathfrak{B})$ such that $\mathfrak{B} \models A(\underline{\xi}_p, \underline{\vartheta}_q)$, contrary to $\mathfrak{B} \models \sigma$. We conclude that for some $1 \leq k$, $\vdash \tau \longrightarrow \tau_{\sigma,k}$.

REMARK. Note that in the sentence $\tau_{\sigma,k}$ (8) the universal quantifiers $\bigwedge \underline{z}_{p+mk}$ can actually be moved into the prefix so that $\tau_{\sigma,k}$ is (logically equivalent to) a *universal* sentence.

7 - SYNTACTICAL CHARACTERIZATION OF SETS WITH i.p.

We can now turn to the syntactical characterization of sets of sentences with i.p. Even though the result could be stated in a form which applies both to single sentences with i.p. and to (infinite) sets with i.p. we prefer to separate these two cases.

THEOREM 7. A sentence σ_1 has i.p. if and only if σ_1 is logically equivalent to a sentence of the form $\sigma \wedge \sigma_{\sigma,k}$ where $\sigma = \bigwedge \underline{x}_p \bigvee \underline{y}_q\ A(\underline{x}_p, \underline{y}_q)$ is a sentence in AE and $\sigma_{\sigma,k}$ is of the form (8) (with τ being σ).

PROOF. Assume that σ_1 has i.p. The Corollary to Theorem 2 yields that $\sigma_1 \equiv \sigma$ where σ is in AE. The sentence σ has i.p. relative to σ and therefore, by Theorem 6, $\vdash \sigma \longrightarrow \sigma_{\sigma,k}$ for some $1 \leq k$. Thus $\sigma \equiv \sigma \wedge \sigma_{\sigma,k}$ for this k and σ_1 is logically equivalent to a sentence in the desired from.

Assume now that for some k, $\sigma_1 \equiv \sigma \wedge \sigma_{\sigma,k}$ where σ is in AE. We shall prove that a sentence σ' of the form $\sigma \wedge \sigma_{\sigma,k}$ (if consistent) has i.p. Let $\mathfrak{A} \models \sigma'$ and $\mathfrak{A} \supseteq \mathfrak{A}_i \models \sigma'$, $i \in I$ be models of σ' and Let $\mathfrak{B} = \bigcap_i \mathfrak{A}_i$. Since $\sigma_{\sigma,k}$ is universal we have $\mathfrak{B} \models \sigma_{\sigma,k}$. Now $\mathfrak{A} \models \sigma, \sigma_{\sigma,k}$, and $\mathfrak{A}_i \models \sigma$, $i \in I$. This implies(see proof of sufficiency in Theorem 6) that $\mathfrak{B} \models \sigma$. Thus $\mathfrak{B} \models \sigma'$.

THEOREM 8. A set S_1 has i.p. if and only if there exists a set S of sentences such that

a) $S_1 \equiv S$,

b) the sentences in S are either universal or in form AE,

c) for every $\sigma \in S$ which is not universal there exists a $\tau \in S$ such that for some $1 \leq k$ also $\tau_{\sigma,k} \in S$.

PROOF. Let S_1 have i.p. By the Corollary of Theorem 2 $S_1 \equiv S'$ where S' is the set of all AE consequences of S_1. The set S' is conjunctive and has i.p., therefore for every $\sigma \in S'$ there exists a $\tau \in S'$ such that σ has i.p. relative to τ. By Theorem 6 $\vdash \tau \longrightarrow \tau_{\sigma,k}$ for some $1 \leq k$. Let

$$S = \{\tau_{\sigma,k} \mid \sigma, \tau \in S', S' \vdash \tau_{\sigma,k}\} \cup S',$$

then $S \equiv S'$ and satisfies a - c.

The proof that if a set S_1 satisfies a - c then it has i.p. is completely analogous to the corresponding proof for Theorem 7.

REMARK. Theorem 7 can actually be subsumed by Theorem 8 by putting $S = \{\sigma, \sigma_{\sigma,k}\}$. We stated it separately in order to emphasize the fact that in this case S can be taken to be *finite* .

8 - A COUNTER-EXAMPLE.

Most of the syntactical characterizations of sets of sentences with a given semantical property which are found in the literature have the following form : S_1 has the semantical property P if and only if S_1 is logically equivalent to a set of sentences S such that every $\sigma \in S$ has a certain fixed syntactical form. Our Theorem 7 has this form. Theorem 8, however, is a syntactical characterization of a new kind. Namely S_1 has the property P (in this case P is the i.p.) if and only if $S_1 \equiv S$ where S as a *set* of sentences has certain properties (a - c) which are expressed by referring both to the whole set S and to the syntactical form of elements of S.

Is it possible to give a syntactical characterization of sets of sentences with i.p. which is of the same kind as the usual characterizations mentioned above? We are able to *prove* that the answer is *negative*.

Assume that there exists a syntactical characterization of the form given above, for sets S_1 with semantical property P. From this we can conclude that a set S_1 has property P if and only if S_1 is logically equivalent to a set S of sentences such that every $\sigma \in S$ has property P. We shall construct an example of an (infinite) set S_1 of sentences with i.p. such that S_1 is not logically equivalent to any set S such that each sentence in S has i.p.

Let $P_1(x, y_1), \ldots, P_n(x, y_1, \ldots, y_n), \ldots,$ be a sequence of predicate constants. Define for $1 \leq n < \omega$,

$$\alpha_n = \bigwedge x \bigwedge \underline{y}_{n+1} [P_{n+1}(x, \underline{y}_{n+1}) \longrightarrow P_n(x, \underline{y}_n)].$$

$$\beta_n = \bigwedge x \bigwedge \underline{y}_{n+1} \bigwedge \underline{z}_{n+1} [P_{n+1}(x, \underline{y}_{n+1}) \wedge P_{n+1}(x, \underline{z}_{n+1}) \rightarrow \bigwedge_{1 \leq i \leq n} y_i = z_i].$$

$$\gamma_n = \bigwedge x \bigvee \underline{y}_n P_n(x, y_1, \ldots, y_n).$$

Let $U = \{\alpha_n, \beta_n \mid 1 \leq n < \omega\}$, $E = \{\gamma_n \mid 1 \leq n < \omega\}$ and $S_1 = U \cup E$.

LEMMA 9. S_1 has i.p.

PROOF. Let $\mathfrak{A} \models S_1$ and $\mathfrak{A} \supseteq \mathfrak{A}_i \models S_1$, $i \in I$, be a model of S_1 and a set of submodels, put $\mathfrak{B} = \bigcap_i \mathfrak{A}$. The sentences α_n, β_n are universal, hence $\mathfrak{B} \models U$. It remains to prove that $\mathfrak{B} \models \gamma_n$ for $1 \leq n < \omega$.

We may assume that $D(\mathfrak{B}) \neq \emptyset$. Let $\xi \in D(\mathfrak{B})$; we must show that there exist elements $\eta_1, \ldots, \eta_n \in D(\mathfrak{B})$ such that $\mathfrak{B} \models P_n(\xi, \underline{\eta_n})$. Now, $\mathfrak{A} \models \gamma_{n+1}$ and $\mathfrak{A}_i \models \gamma_{n+1}$, $i \in I$. There exist therefore elements $\eta_1, \ldots, \eta_{n+1} \in D(\mathfrak{A})$ and $\zeta_1^i, \ldots, \zeta_{n+1}^i \in D(\mathfrak{A}_i)$, for $i \in I$, such that $\mathfrak{A} \models P_{n+1}(\xi, \underline{\eta_{n+1}})$, $\mathfrak{A}_i \models P_{n+1}(\xi, \underline{\zeta_{n+1}^i})$. Thus $\mathfrak{A} \models P_{n+1}(\xi, \underline{\eta_{n+1}}) \wedge P_{n+1}(\xi, \underline{\zeta_{n+1}^i})$ for $i \in I$. Now, $\mathfrak{A} \models \beta_n$, hence $\eta_1 = \zeta_1^i, \ldots, \eta_n = \zeta_n^i$. From $\mathfrak{A} \models \alpha_n$ it follows that $\mathfrak{A} \models P_{n+1}(\xi, \underline{\eta_{n+1}}) \longrightarrow P_n(\xi, \underline{\eta_n})$ so that $P_n(\xi, \underline{\eta_n})$ holds in $\mathfrak{A}$ and hence also in $\mathfrak{B}$. Thus $\mathfrak{B} \models E$.

LEMMA 10. If $\mathfrak{A} \models U \cup \{\gamma_n\}$ then there exists a model $\mathfrak{B}$ *of* $U \cup \{\gamma_{n+1}\}$ and two submodels $\mathfrak{B}_1 \models U \cup \{\gamma_{n+1}\}$ and $\mathfrak{B}_2 \models U \cup \{\gamma_{n+1}\}$ *of* $\mathfrak{B}$ such that $\mathfrak{A} = \mathfrak{B}_1 \cap \mathfrak{B}_2$

PROOF. If $\mathfrak{A} \models \gamma_{n+1}$ there is nothing to prove. Let $\mathfrak{A} \models \sim \gamma_{n+1}$ and let

$$N = \{\xi \mid \xi \in D(\mathfrak{A}),\ \mathfrak{A} \models \sim \bigvee \underline{y_{n+1}}\ P_{n+1}(\xi, \underline{y_{n+1}}).$$

For each $\xi \in N$ define two sequences $\varphi(\xi, n), \Psi(\xi, n)$, $1 \leq n < \omega$, of elements which are pairwise distinct and are assumed not to belong to $D(\mathfrak{A})$.

Let now $\mathfrak{A} = \langle A, R_n \rangle_{1 \leq n < \omega}$ where R_n is the n+1-ary relation corresponding to P_n. Since $\mathfrak{A} \models \gamma_n$ we can pick for each $\xi \in N$ a fixed sequence $\eta_1^\xi, \ldots, \eta_n^\xi$ such that $\mathfrak{A} \models P_n(\xi, \underline{\eta_n^\xi})$.

Define $B_1 = A \cup \{\varphi(\xi, n) \mid \xi \in N,\ 1 \leq n < \omega\}$, $B_2 = A \cup \{\Psi(\xi, n) \mid \xi \in N,\ 1 \leq n < \omega\}$, and $B = B_1 \cup B_2$. Furthermore, define, for $1 \leq k \leq n+1$,

$$Q_k = \{ < \varphi(\xi, m), \ldots, \varphi(\xi, m+k)>,\ < \Psi(\xi, m), \ldots, \Psi(\xi, m+k) > \mid \xi \in N, 1 \leq m < \omega\},$$

$$\bar{R}_k = R_k \cup Q_k \quad 1 \leq k \leq n,$$

$$\bar{R}_{n+1} = R_{n+1} \cup Q_{n+1} \cup \{ < \xi, \eta_1^\xi, \ldots, \eta_n^\xi, \varphi(\xi, 1) >,\ < \xi, \eta_1^\xi, \ldots, \eta_n^\xi ; \Psi(\xi, 1) > \mid \xi \in N\},$$

and for $n+2 \leq k$ define $\bar{R}_k = R_k$.

Let now $\mathfrak{B} = < B, \bar{R}_i >_{1 \leq i < \omega}$, $\mathfrak{B}_j = < B_j, \bar{R}_i \mid B_j >_{1 \leq i < \omega}$, $j = 1, 2$. We have $\mathfrak{B}_1 \subseteq \mathfrak{B}$, $\mathfrak{B}_2 \subseteq \mathfrak{B}$ and $\mathfrak{A} = \mathfrak{B}_1 \cap \mathfrak{B}_2$. It can be verified that $\mathfrak{B}$, $\mathfrak{B}_1$, and $\mathfrak{B}_2$, are models of $U \cup \{\gamma_{n+1}\}$.

THEOREM 11. There does not exist a set S of sentences such that $S_1 \equiv S$ and each $\sigma \in S$ has i.p.

PROOF. Assume by way of contradiction that such a set S does exist. We have $S \vdash \gamma_2$ and hence there are $\sigma_1, \ldots, \sigma_m \in S$ such that for $\sigma = \sigma_1 \wedge \ldots \wedge \sigma_m$, $\vdash \sigma \longrightarrow \gamma_2$. On the other hand, $S_1 \vdash \sigma_i$, $1 \leq i \leq m$, so that $S_1 \vdash \sigma$.

It can be shown that $U \cup \{\gamma_n\} \vdash \gamma_k$ for $1 \leq k \leq n < \omega$ but *not* $U \cup \{\gamma_n\} \vdash \gamma_k$ for $n < k$; the last statement is proved by constructing an appropriate model.

Thus it is impossible that $U \vdash \sigma$, for then we would have $U \vdash \gamma_2$. Let n+1 be the smallest integer such that $U \cup \{\gamma_{n+1}\} \vdash \sigma$ and *not* $U \cup \{\gamma_n\} \vdash \sigma$; since $\vdash \sigma \longrightarrow \gamma_2$ we must have $2 \leq n+1$.

Let $\mathfrak{A}$ be an arbitrary model of $U \cup \{\gamma_n\}$. By Lemma 10 there exist models $\mathfrak{B}$, $\mathfrak{B}_1 \subseteq \mathfrak{B}$, and $\mathfrak{B}_2 \subseteq \mathfrak{B}$ of $U \cup \{\gamma_{n+1}\}$ such that $\mathfrak{A} = \mathfrak{B}_1 \cap \mathfrak{B}_2$. We have $\mathfrak{B} \models \sigma$, $\mathfrak{B}_j \models \sigma$, $j = 1, 2$ and since together with $\sigma_i \in S$, $1 \leq i \leq m$, also σ has i.p., we conclude $\mathfrak{A} \models \sigma$. Thus every model of $U \cup \{\gamma_n\}$ is a model of σ and hence $U \cup \{\gamma_n\} \vdash \sigma$, a contradiction.

CHAPTER III

GENERATED MODELS AND THEIR AUTOMORPHISMES

9 - MODELS AND GENERATORS.

In this chapter we shall apply the characterization given in Theorem 8 to the study of models of a set S with i.p. which are generated by a set of elements. In particular, we consider automorphisms of such models and prove a result which is a generalization of a basic fact of Galois theory for fields, namely that every element of a model $\mathfrak{A}$ generated by a set X has just a finite number of conjugates under the group of automorphisms of $\mathfrak{A}$ over X. We shall also obtain information on the structure of models generated by a set of elements.

DEFINITION 6. Let S be a set of sentences, $\mathfrak{A}$ be a model of S, and $X \subseteq D(\mathfrak{A})$ a subset of the domain of $\mathfrak{A}$. We shall say that $\mathfrak{A}$ *is generated by* X if the only submodel $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B}$ is a model of S and $X \subseteq D(\mathfrak{B})$ is $\mathfrak{A}$ itself. If $\mathfrak{A}$ is generated by $X \subseteq D(\mathfrak{A})$ then we shall also say that $\mathfrak{A}$ is a *minimal model* containing X.

If S is an arbitrary (consistent) set of sentences then a subset $X \subseteq D(\mathfrak{B})$, where $\mathfrak{B}$ is a model of S, need not generate a submodel $\mathfrak{A}$ of $\mathfrak{B}$.

For sets S with i.p., however, we have the following.

LEMMA 12. Let S be a set of sentences with i.p. If $\mathfrak{A}$ is a model of S and $X \subset D(\mathfrak{A})$ is a subset of the domain of $\mathfrak{A}$ then there exist a unique submodel $\mathfrak{A}(X) \subseteq \mathfrak{A}$ which is a model of S generated by X.

$\mathfrak{A}(X)$ clearly is the intersection of all models $\mathfrak{A}_1$ of S which satisfy $X \subseteq D(\mathfrak{A}_1)$, $\mathfrak{A}_1 \subseteq \mathfrak{A}$. The above observation is taken from Robinson's [5].

10 - CLOSURES WITH RESPECT TO RELATIONS AND THEIR AUTOMORPHISMS.

The following concepts and results are motivated by viewing a n-ary relation as a many-valued function of the first n-1 variables.

If $R \subseteq A^n$ is an n-ary relation on a set A then a subset $B \subseteq A$ is said to be *closed with respect to* R if $b_1, \dots, b_{n-1} \in B$ and $\langle b_1, \dots, b_{n-1}, b_n \rangle \in R$ imply that $b_n \in B$. In particular, if n = 1 (i.e. R is a subset of A) then closure of B with respect to R means $R \subseteq B$.

If Ω is a set of relations (of arbitrary orders) on A then a subset $B \subseteq A$ is said to be *closed with respect to* Ω if B is closed with respect to every relation $R \in \Omega$.

The *closure* C(X) of a subset $X \subseteq A$ with respect to a set Ω of relations is the intersection of all sets $B \subseteq A$ which contain X and are closed with respect to Ω.

The closure C(X) is clearly the smallest set B such that $X \subseteq B \subseteq A$ and B is closed with respect to Ω.

A 1-1 mapping φ of A onto A is called an *automorphism* of $\langle A, \Omega \rangle$ if for all relations $R \in \Omega$ $\langle \varphi(a_1), \dots, \varphi(a_n) \rangle \in R$ if and only if $\langle a_1, \dots, a_n \rangle \in R$.

An automorphism of $\langle A, \Omega \rangle$ is called an *automorphism over* X if for $\xi \in X$, $\varphi(\xi) = \xi$.

Two elements $a, b \in A$ are called *conjugate over* X if there exists an automorphism φ of $\langle A, \Omega \rangle$ over X such that $\varphi(a) = b$.

It can be verified that automorphisms of $\langle A, \Omega \rangle$ over X form a group (under composition of mappings) and, consequently, the relation of conjugacy over X is an equivalence relation.

A relation $R \subseteq A^n$ is *finite-valued* if for every $a_1, \dots, a_{n-1}$ there exists just a finite (possibly zero) number of elements $a_n \in A$ such that $\langle a_1, \dots, a_{n-1}, a_n \rangle \in R$.

THEOREM 13. Let Ω be a set of finite-valued relations over A, X a subset of A and C(X) the closure of X with respect to Ω. Every element of C(X) has just a finite number of conjugates over X.

PROOF. Let $\Omega|C(x) = \{R|C(X) \mid R \in \Omega\}$, we have to show that every $\xi \in C(X)$ is carried into just a finite number of elements by automorphisms of $\langle C(X), \Omega|C(X) \rangle$ over X.

Let $B \subseteq C(X)$ be the set of all elements $\xi \in C(X)$ with this finiteness property. By definition, $X \subseteq B \subseteq C(X)$. We contend that B is closed with respect to Ω, from this it would follow at once that $B = C(X)$.

To prove that B is closed, let $R \in \Omega$ be a relation of rank n ; $a_1 \in B, \ldots a_{n-1} \in B$, and $\langle a, \ldots, a_{n-1}, \xi \rangle \in R$. From the fact that C(X) is closed with respect to R it follows that $\xi \in C(X)$. Since each a_i, $1 \leq i \leq n-1$ has just a finite number of conjugates over X, there exist automorphisms $\varphi_1, \ldots \varphi_m$ of $< C(X), \Omega | C(X) >$ over X such that for any automorphism φ over X for some $1 \leq i \leq m$, $\langle \varphi(a_1), \ldots, \varphi(a_{n-1}) \rangle = < \varphi_i(a_1), \ldots, \varphi_i(a_{n-1}) >$. It follows now from $\langle a_1, \ldots, a_{n-1}, \xi \rangle \in R$ and the fact that φ is an automorphism of $\langle C(X), \Omega | C(X) \rangle$ that $\langle \varphi_i(a_1), \ldots, \varphi_i(a_{n-1}), \varphi(\xi) \rangle \in R$ for the above i. Since R is finite-valued there exist for each $1 \leq i \leq m$ just a finite number of elements $b \in A$ such that $\langle \varphi_i(a_1), \ldots, \varphi_i(a_{n-1}), b \rangle \in R$, hence there is just a finite number of possible values for $\varphi(\xi)$. The element ξ, therefore, belongs to B.

Thus $B = C(X)$. Our definition of B implies now that every element of C(X) has just a finite number of conjugates over X.

REMARK. Every automorphism Ψ of $\langle A, \Omega \rangle$ over X induces an automorphism $\Psi | C(X)$ of $\langle C(X), \Omega | C(X) \rangle$. In general, however, it may happen that the latter system has some automorphisms over X which are not restrictions of automorphisms of $\langle A, \Omega \rangle$. Note that Theorem 13 was proved for this, possibly larger, group of automorphisms.

11 - STRUCTURE OF MINIMAL MODELS.

We wish to represent the submodel $\mathfrak{A}(X) \subseteq \mathfrak{A}$ generated by a set $X \subseteq D(\mathfrak{A})$ as a closure of X with respect to certain finite-valued relations.

Let S have the i.p. We may assume, by Theorem 8, that S has the form given in that theorem.

Define for each $\tau \in S$ which is in form AE, $\tau = \bigwedge \underline{x}_n \bigvee \underline{y}_m B(\underline{x}_n, \underline{y}_m)$, and for every pair p, k of positive integers a formula $\bar{R}_{\tau,p,k}(\underline{x}_p, y)$ containing p+1 free variables

$$\bar{R}_{\tau,p,k}(\underline{x}_p, y) = \bigwedge \underline{y}_{p+mk} [S_{\tau,k}(\underline{x}_p, \underline{y}_{p+mk}) \longrightarrow \bigvee_{1 \leq i \leq p+mk} y = y_i], \tag{10}$$

(cf. Definition 5). In words, $\bar{R}_{\tau,p,k}(\underline{x}_p, y)$ asserts that y is an element of every sequence of successors of length p+mk of $\underline{x}_p$ by τ.

Let $\mathfrak{A}$ be a fixed model of S. We shall denote by $R_{\tau,p,k}$ the p+1-ary relation on $D(\mathfrak{A})$ defined by :

$$< \xi_1, \ldots, \xi_p, \eta > \in R_{\tau,p,k} \text{ if and only if } \mathfrak{A} \models \bar{R}_{\tau,p,k}(\xi_1, \ldots, \xi_p, \eta).$$

LEMMA 14. Let $\mathfrak{A}$ be a model of S and $\tau \in S$ be as above. Every relation $R_{\tau,p,k}$ is finite-valued.

PROOF. Let $\xi_1, \ldots, \xi_p$ be elements of $D(\mathfrak{A})$ and let $\eta_1, \ldots, \eta_{p+mk}$ be some fixed sequence of successors of $\underline{\xi}_p$ by τ. It follows from the definition of $R_{\tau,p,k}$ that if $\langle \xi_1, \ldots, \xi_p, \eta \rangle \in R_{\tau,p,k}$ then $\eta \in \{\eta_1, \ldots, \eta_{p+mk}\}$. The relation $R_{\tau,p,k}$ is thus finite-valued.

THEOREM 15. Let $\mathfrak{A} = \langle A, R \rangle$ be a model of S, where S has i.p. and is in the form of Theorem 8. The domain of the submodel $\mathfrak{A}(X)$ generated by a subset $X \subseteq D(\mathfrak{A})$ coincides with the closure C(X) of X with respect of the set Ω of relations

$$\Omega = \{R_{\tau,p,k} | \tau \in S \cap AE, \ 1 \leq p,k < \omega\}. \tag{11}$$

PROOF. We shall first show that $D(\mathfrak{A}(X))$ is closed with respect to the relations in Ω and infer that $C(X) \subseteq D(\mathfrak{A}(X))$.

Let $\tau = \bigwedge \underline{x}_n \bigvee \underline{y}_m B(\underline{x}_n, \underline{y}_m) \in S$. $\mathfrak{A}(X)$ is a model of S, hence $\mathfrak{A}(X) \models \tau$. It follows from Lemma 5 that for every $\xi_1, \ldots, \xi_p \in D(\mathfrak{A}(X))$ there exist $\eta_1, \ldots, \eta_{p+mk} \in D(\mathfrak{A}(X))$ such that

$$\mathfrak{A}(X) \models S_{\tau,k}(\underline{\xi}_p, \underline{\eta}_{p+mk}). \tag{12}$$

Assume that for these $\xi_1, \ldots, \xi_p \in D(\mathfrak{A}(X))$, $\eta \in D(\mathfrak{A})$ is any element such that $\langle \xi_1, \ldots, \xi_p, \eta \rangle \in R_{\tau,p,k}$. The element η belongs to every sequence of lenth p+mk of successors of $\underline{\xi}$, hence $\eta \in \{\eta_1, \ldots, \eta_{p+mk}\} \subseteq D(\mathfrak{A}(X))$. Thus $D(\mathfrak{A}(X))$ is closed with respect to every $R_{\tau,p,k} \in \Omega$ which proves the assertion.

Next we shall prove that $\mathfrak{C} = \langle C(X), R|C(X)\rangle$ is a model of S and infer that $D(\mathfrak{A}(X)) \subseteq C(X)$. Combined with the previous result this will entail $D(\mathfrak{A})(X)) = C(X)$.

Being a substructure of $\mathfrak{A}$ the structure $\mathfrak{C}$ satisfies every universal sentence which is an element of S. Let $\sigma = \bigwedge \underline{x}_p \bigvee \underline{y}_q A(\underline{x}_p, \underline{y}_q) \in S$ be an arbitrary sentence of S which is in form AE. The set S contains a sentence $\tau = \bigwedge \underline{x}_n \bigvee \underline{y}_m B(\underline{x}_n, \underline{y}_m)$ such that for some k also $\tau_{\sigma,k} \in S$ (cf. (8) and Thorem 8). We have to show that $\mathfrak{C} \models \sigma$. Let $\xi_1, \ldots, \xi_p \in C(x)$. Since C(X) is closed with respect to $R_{\tau,k}$ it contains all the elements $\eta_1, \ldots, \eta_r$ which are common to all sequences of length p+mk of successors of $\underline{\xi}_p$ by τ. The sentence $\tau_{\sigma,k}$ asserts that among the elements common to all sequences of length p+mk of successors of $\underline{\xi}_p$ by τ there exists a sequence $\eta_{i_1}, \ldots, \eta_{i_q}$, $1 \leq i_j \leq r$, such that $A(\underline{\xi}_p, \eta_{i_1}, \ldots \eta_{i_q})$ holds in $\mathfrak{A}$ (hence also in $\mathfrak{C}$). Thus $\mathfrak{C} \models \sigma$ and hence, σ being arbitrary, $\mathfrak{C} \models S$ which completes our proof.

12 - AUTOMORPHISMS OF MODELS GENERATED BY A SET OF ELEMENTS.

THEOREM 16. If the model $\mathfrak{A} = \langle A, R \rangle$ of the set S with i.p. is generated by a subset $X \subseteq A$ then every element $\xi \in A$ has just a finite number of conjugates under automorphisms of $\mathfrak{A}$ over X.

PROOF. We again assume that S the form given in Theorem 8, and that $R_{\tau,p,k}$, Ω (11), and C(X) retain their meaning from Theorem 15.

Since $\mathfrak{A}$ is generated by X we have $\mathfrak{A} = \mathfrak{A}(X)$ — the submodel generated by X. By Theorem 15, $D(\mathfrak{A}(X)) = C(X)$ — the closure of X with respect to Ω.

The finite-valued (Lemma 14) relations $R_{\tau,p,k} \in \Omega$ are *elementarily defined* from R by (10). Thus an automorphism of $\mathfrak{A}$ is also an automorphism with respect to every $R_{\tau,p,k} \in \Omega$. The automorphisms of $\mathfrak{A}$ over X are, therefore, also automorphisms of $\langle C(X), \Omega\rangle$ (i.e. $\langle A, \Omega\rangle$) over X. But by Theorem 13 (with A = C(X)) every element $\xi \in C(X)$ has just a finite number of conjugates by automorphisms of $\langle C(X), \Omega\rangle$ over X. Thus every $\xi \in A$ certainly has just a finite number of conjugates by automorphisms of $\mathfrak{A}$ over X.

REMARK. The conclusion of Theorem 16 applies also to a submodel $\mathfrak{A}(X) \subseteq \mathfrak{A}$ of a model $\mathfrak{A}$ of S, generated by a subset $X \subseteq D(\mathfrak{A})$. This is trivial because the model $\mathfrak{A}(X)$ (considered by itself) is now the model generated by X and Theorem 16 applies.

EXAMPLE. Let K be the class of algebraically closed fields. K has the i.p. and is an elementary class (in the wider sense). Thus our results apply to K. In particular if F is algebraically closed and $X \subseteq F$ is a subset of F, then there exist a minimal algebraically closed subfield $F(X) \subseteq F$ containing X, and every element of F(X) has just a finite number of conjugates by automorphisms of F(X) over X.

This result can, of course, be obtained also by methods of algebra. It seems interesting, however, that the result concerning finiteness of number of conjugates is a consequence of just the intersection property.

13 - CONCLUDING REMARKS.

C.C. Chang proposed in [2] the following characterization of (single) sentences with i.p. A sentence σ_1 has i.p. if and only if σ_1 is logically equivalent to a sentence

$$\sigma = \bigwedge x_1 \ldots \bigwedge x_p \bigvee! y_1 \ldots \bigvee! y_q \; A(x_1, \ldots, x_p, y_1, \ldots, y_q) \tag{13}$$

($\bigvee!$ is an abbreviation for : there exists exactly one). Every sentence σ of this form clearly has i.p. There exist, however, sentences σ_1 with i.p. which are not equivalent to any sentence σ of form (13).

We observe that if $\mathfrak{A}$ is a model of σ generated by a set $X \subseteq D(\mathfrak{A})$ then there is just one automorphism of over X (this can be proved by the methods of Section 10).

Let α be the axioms for a field and τ the sentence $\bigwedge x \bigvee y [y^2 = x]$, let $\sigma_1 = \sigma \wedge \tau$. Thus every model of σ_1 is a field every element of which has a square root. The class K of all such fields has i.p.

Let F be the subfield of the field of complex numbers obtained from the rationals by closing the field of rationals with respect to adjunction of square-roots. Clearly F is a model of σ_1 and in fact a minimal model (i.e. no proper subfield of F is a model of σ_1). If σ_1 were logically equivalent to a sentence (13) then F would have no non-trivial automorphism. But the complex conjugation mapping sending $a + ib$ into $a - ib$ (where $\underline{i}^2 = -1$) induces an automorphism of F. Thus σ_1 is not equivalent to any sentence of form (13), which disproves Chang's conjecture.

It can be easily verified that if $A(\underline{x}_p, y)$ is a quantifier-free formula then the sentence σ asserting that for every $x_1, \dots, x_p$ there exist *exactly* k different elements $y_1, \dots, y_k$ such that $A(\underline{x}_p, y_i)$, $1 \leq i \leq k$, has i.p. This raises the question whether one can give a syntactical characterization of sentences (or sets of sentences) with i.p. by using sentences of a form similar to that of σ_k.

A result along these lines was announced in [4]. A derivation of this result (and some improved versions) from the characterization given here will be presented in a subsequent paper.

BIBLIOGRAPHY

[1] C.C. CHANG - *A remark on convex classes*, Abstract 579, Bull. Amer. Math. Soc. 66 (1954), p. 396.

[2] C.C. CHANG - *Unions of chains of models and direct products of models*, Summer Institute for Symbolic Logic, Cornell University (1957), pp. 141-143.

[3] J. LOŚ and R. SUSZKO - *On the extending of models* IV, Fund. Math. 44 (1957), pp. 52-60.

[4] M.O. RABIN - *Characterization of convex systems of axioms*, Abstract 571-65, Notices Amer. Math. Soc. 7 (1960), p. 503.

[5] A. ROBINSON - *On the Metamathematics of Algebra*, North-Holland Publishing Co., Amsterdam, 1951.

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