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Sergey V. Ludkovsky

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# Quasi-invariant measures on non-Archimedean groups and semigroups of loops and paths, their representations. I.

Sergey V. Ludkovsky

permanent address: Theoretical Department, Institute of General Physics, Str. Vavilov 38, Moscow, 117942, Russia.

**Abstract**

Loop and path groups $G$ and semigroups $S$ as families of mappings of one non-Archimedean Banach manifold $M$ into another $N$ with marked points over the same locally compact field $\mathbf{K}$ of characteristic $char(\mathbf{K}) = 0$ are considered. Quasi-invariant measures on them are constructed. Then measures are used to investigate irreducible representations of such groups.

## 1 Introduction.

Loop and path groups are very important in differential geometry, algebraic topology and theoretical physics [2, 5, 19, 25]. Moreover, quasi-invariant measures are helpful for an investigation of the group itself. In the case of real manifolds Gaussian quasi-invariant measures on loop groups and semigroups were constructed and then applied for the investigation of unitary representations in [17].

In [12, 14, 16] quasi-invariant measures on non-Archimedean Banach spaces $X$ and diffeomorphism groups were investigated.

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During the recent time non-Archimedean functional analysis and quantum mechanics develop intensively [21, 27]. One of the reason for this is in the divergence of some important integrals and series in the real or complex cases and their convergence in the non-Archimedean case. Therefore, it is important to consider non-Archimedean loop semigroups and groups, that are new objects. There are many principal differences between classical functional analysis (over the fields $\mathbf{R}$ or $\mathbf{C}$) and non-Archimedean [21, 22, 24, 28]. Then the names of loop groups and semigroups in the non-Archimedean case are used here in analogy with the case of manifolds over the real field $\mathbf{R}$, but their meaning is quite different, because non-Archimedean manifolds $M$ are totally disconnected with the small inductive dimension $ind(M) = 0$ (see §6.2 and Ch. 7 in [8]) and real manifolds are locally connected with $ind(M) \ge 1$. In the real case loop groups $G$ are locally connected for $dim_{\mathbf{R}} M < dim_{\mathbf{R}} N$, but in the non-Archimedean case they are zero-dimensional with $ind(G) = 0$, where $1 \le dim_{\mathbf{R}} N \le \infty$ is the dimension of the tangent Banach space $T_x N$ over $\mathbf{R}$ for $x \in N$. Shortly the non-Archimedean loop semigroups were considered in [15].

In this article loop groups and semigroups are considered. The loop semigroups are quotients of families of mappings $f$ from one non-Archimedean manifold $M$ into another $N$ with $\lim_{x \to s_0} \bar{\Phi}^v f(x) = 0$ for $0 \le v \le t$ by the corresponding equivalence relations, where $s_0$ and $y_0 = 0$ are marked points in $\bar{M}$ and $N$ respectively, $M = \bar{M} \setminus \{s_0\}$, $\bar{\Phi}^v f$ are continuous extensions of the partial difference quotients $\Phi^v f$. Besides locally compact manifolds also non-locally compact Banach manifolds $M$ and $N$ are considered. This work presents results for manifolds $M$ and $N$ modelled on Banach spaces $X$ and $Y$ over locally compact fields $\mathbf{K}$ such that $\mathbf{Q_p} \subset \mathbf{K} \subset \mathbf{C_p}$, where $\mathbf{Q_p}$ is the field of $p$-adic numbers, $\mathbf{C_p}$ is the field of complex numbers with the corresponding non-Archimedean norm, that is, $\mathbf{K}$ are finite algebraic extensions of $\mathbf{Q_p}$.

More interesting are groups constructed with the help of A. Grothendieck procedure of an Abelian group from an Abelian monoid. This produces the non-Archimedean loop group. Also semigroups and groups of paths are considered, but it is only formal terminology. Both in the real and non-Archimedean cases compositions of pathes are defined not for all elements, but satisfying the additional condition. Since the non-Archimedean field $\mathbf{K}$ is not directed (apart from $\mathbf{R}$) this condition is another in the non-Archimedean case than in the real case. On the other hand, semigroups with units (that is, monoids) and groups of loops have indeed the algebraic structure of monoids and groups respectively. Quasi-invariant measures on these semigroups and groups are constructed in §3 of Part I and §2 of Part II. Then such measures are used for the investigation of irreducible unitary representations of loop groups in §3 of Part II.

To construct real-valued and also $\mathbf{Q_q}$-valued (for $q \neq p$) quasi-invariant measures specific antiderivations and isomorphisms of non-Archimedean Banach spaces are considered. Apart from the real-valued measures the notion of quasi-invariance for $\mathbf{Q_q}$-valued measures is quite different and is based on the results from [24]. For this a Banach space $L(\mu)$ of integrable functions defined for a tight measure $\mu$ on an algebra $Bco(X)$ of clopen subsets of a Hausdorff space $X$ with $ind(X) = 0$ is used. To construct measures we start from measures equivalent to Haar measures on $\mathbf{K}$. The real-valued nonnegative Haar measure $v$ on $\mathbf{K}$ as the additive group is characterised by the equation $v(x + A) = v(A)$ for each $x \in \mathbf{K}$ and $A \in Bf(\mathbf{K})$, where $Bf(X)$ denotes the Borel $\sigma$-field of $X$ (see Chapter VII in [4]). Each bounded nonnegative Borel measure on a clopen compact subset of $\mathbf{K}$ may contain only countable number of atoms, but for it each atom may be only a singleton. Therefore, the Haar measure $v$ certainly has not any atom. The $\mathbf{Q_q}$-valued Haar measure $w$ on $\mathbf{K}$ is characterised by $w(x + A) = w(A)$ for each $x \in \mathbf{K}$ and each $A \in Bco(\mathbf{K})$ (see Chapter 8 in [21]). In view of Monna-Springer Theorem 8.4 [21] a non-zero $\mathbf{Q_q}$-valued invariant measure $w$ on $Bco(\mathbf{K})$ exists for each $q \neq p$, but does not exist for $q = p$.

Pseudo-differentiability of measures with values in $\mathbf{R}$ and $\mathbf{Q_q}$ also is considered, because in the non-Archimedean case there is not any non-trivial differentiable function $f : \mathbf{K} \to \mathbf{R}$ or $f : \mathbf{K} \to \mathbf{Q_q}$ for $q \neq p$. This notion of pseudo-differentiability of real-valued measures is based on Vladimirov operator on the corresponding space of functions $f : \mathbf{K} \to \mathbf{R}$ [26, 27].

Semigroups and groups of loops and paths are investigated in §2 and Part II respectively. Here real-valued and also $\mathbf{Q_q}$-valued measures are considered (for $q \neq p$). Unitary representations of loop groups are given in Part II.

The loop groups are neither Banach-Lie nor locally compact and have a structure of a non-Archimedean Banach manifold (see Theorem II.2.3). A. Weil theorem states that, if there exists a non-trivial non-negative quasi-invariant measure $\mu$ on a topological group $G$ relative to left shifts $L_g$ for all $g \in G$, then $G$ is locally compact, where $L_g h = gh$ for each $g$ and $h \in G$ (see also Corollaries III.12.4,5 [9]). Therefore, the loop group and the loop semigroup has not any non-zero Haar measure. In Part I manifolds with disjoint atlases modelled on Banach spaces are considered. This is sufficient for many purposes. Moreover, in §II.3.4 it is shown, that arbitrary atlases of the corresponding class of smoothness of the same manifolds preserve loop groups and semigroups up to algebraic topological isomorphisms. In Part II loop and path groups for manifolds modelled on locally K-convex spaces also are discussed.

The notation is summarized in §II.6.

## 2 Loop semigroups.

To avoid misunderstandings we first give our definitions and notations. They are quite necessary, but a reader wishing to get main results quickly can begin to read from §2.6 and then to find appearing notions and notations in §§2.1-5.

**2.1. Notation.** Let K be a local field, that is, a finite algebraic extension of the $p$-adic field $\mathbf{Q_p}$ for the corresponding prime number $p$ [28]. For $b \in \mathbf{R}$, $0 < b < 1$, we consider the following mapping:

$$(1)\ j_b(\zeta) := p^{b \times ord_p(\zeta)} \in \Lambda_\mathbf{p}$$

for $\zeta \neq 0$, $j_b(0) := 0$, such that $j_b(*) : \mathbf{K} \to \Lambda_\mathbf{p}$, where $\mathbf{K} \subset \mathbf{C_p}$, $\mathbf{C_p}$ denotes the field of complex numbers with the non-Archimedean valuation extending that of $\mathbf{Q_p}$, $p^{-ord_p(\zeta)} := |\zeta|_\mathbf{K}$, $\Lambda_\mathbf{p}$ is a spherically complete field with a valuation group $\{|x| : 0 \neq x \in \Lambda_\mathbf{p}\} = (0, \infty) \subset \mathbf{R}$ such that $\mathbf{C_p} \subset \Lambda_\mathbf{p}$ [6, 21, 22, 28]. Then we denote $j_1(x) := x$ for each $x \in \mathbf{K}$.

**2.2. Note.** Each continuous function $f : M \to \mathbf{K}$ has the following decomposition

$$(1)\ f(x) = \sum_{m \in \mathbf{N_o^n}} a(m, f)\bar{Q}_m(x),$$

where $M = B(\mathbf{K^n}, 0, 1)$ is the unit ball in $\mathbf{K^n}$, $\bar{Q}_m(x)$ are basic Amice polynomials, $a(m, f) \in \mathbf{K}$ are expansion coefficients (see also §2.2 [13] and [1, 3]). Here $B(X, x, r) := \{y \in X : d(x, y) \leq r\}$, $B(X, x, r^-) := \{y \in X : d(x, y) < r\}$ are balls for a space $X$ with a metric $d$, $x \in X$, $r > 0$, $\mathbf{N} := \{1, 2, 3, ...\}$, $\mathbf{N_o} := \{0, 1, 2, 3, ...\}$.

**2.3. Definitions and Notes.** Let us consider Banach spaces $X$ and $Y$ over K. Suppose $F : U \to Y$ is a mapping, where $U \subset X$ is an open bounded subset. The mapping $F$ is called differentiable if for each $\zeta \in \mathbf{K}$, $x \in U$ and

$h \in X$ with $x + \zeta h \in U$ there exists a differential such that

$$(1)\ DF(x,h) := dF(x+\zeta h)/d\zeta\ |_{\zeta=0} := \lim_{\zeta \to 0}\{F(x+\zeta h) - F(x)\}/\zeta$$

and $DF(x,h)$ is linear by $h$, that is, $DF(x,h) =: F'(x)h$, where $F'(x)$ is a bounded linear operator (a derivative). Let

$$(2)\ \Phi^1 F(x;h;\zeta) := \{F(x+\zeta h) - F(x)\}/\zeta$$

be a partial difference quotient of order 1 for each $x + \zeta h \in U$, $\zeta h \neq 0$. If $\Phi^1 F(x;h;\zeta)$ has a bounded continuous extension $\bar{\Phi}^1 F$ onto $U \times V \times S$, where $U$ and $V$ are open neighbourhoods of $x$ and 0 in $X$, $U + V \subset U$, $S = B(\mathbf{K}, 0, 1)$, then

$$(3)\ \|\bar{\Phi}^1 F(x;h;\zeta)\| := \sup_{(x\in U, h\in V, \zeta\in S)} \|\bar{\Phi}^1 F(x;h;\zeta)\|_Y < \infty$$

and $\bar{\Phi}^1 F(x;h;0) = F'(x)h$. Such $F$ is called continuously differentiable on $U$. The space of such $F$ is denoted $C(1, U \to Y)$. Let

$$(4)\ \Phi^b F(x;h;\zeta) := (F(x+\zeta h) - F(x))/j_b(\zeta) \in Y_{\Lambda_p}$$

be partial difference quotients of order $b$ for $0 < b < 1$, $x + \zeta h \in U$, $\zeta h \neq 0$, $\Phi^0 F := F$, where $Y_{\Lambda_p}$ is a Banach space obtained from $Y$ by extension of a scalar field from $\mathbf{K}$ to $\mathbf{\Lambda_p}$. By induction using Formulas $(1-4)$ we define partial difference quotients of orders $n+1$ and $n+b$:

$$(5)\ \Phi^{n+1} F(x;h_1,...,h_{n+1};\zeta_1,...,\zeta_{n+1}) :=$$

$$\{\Phi^n F(x+\zeta_{n+1}h_{n+1};h_1,..,h_n;\zeta_1,...,\zeta_n) - \Phi^n F(x;h_1,...,h_n;$$
$$\zeta_1,...,\zeta_n)\}/\zeta_{n+1} \text{ and } (\Phi^{n+b}F) = \Phi^b(\Phi^n F)$$

and derivatives $F^{(n)} = (F^{(n-1)})'$. Then $C(t, U \to Y)$ is a space of functions $F : U \to Y$ for which there exist bounded continuous extensions $\bar{\Phi}^v F$ for each $x$ and $x + \zeta_i h_i \in U$ and each $0 \le v \le t$, such that each derivative $F^{(k)}(x) : X^k \to Y$ is a continuous $k$-linear operator for each $x \in U$ and $0 < k \le [t]$, where $0 \le t < \infty$, $h_i \in V$ and $\zeta_i \in S$, $[t] = n \le t$ and $\{t\} = b$

are the integral and the fractional parts of $t = n + b$ respectively. The norm in the Banach space $C(t, U \to Y)$ is the following:

$$(6)\ \|F\|_{C(t,U\to Y)} := sup_{(x,x+\zeta_i h_i\in U;h_i\in V;\zeta_i\in S;i=1,\dots,s=[v]+sign\{v\};0\le v\le t)}$$

$$\|(\bar{\Phi}^v F)(x; h_1, .., h_s; \zeta_1, ..., \zeta_s)\|_{Y_{\Lambda_p}},$$

where $0 \le t \in \mathbf{R}$, $sign(y) = -1$ for $y < 0$, $sign(y) = 0$ for $y = 0$ and $sign(y) = 1$ for $y > 0$.

Then the locally K-convex space $C(\infty, U \to Y) := \bigcap_{n=1}^{\infty} C(n, U \to Y)$ is supplied with the ultrauniformity given by the family of ultranorms $\| * \|_{C(n,U\to Y)}$.

**2.4. Definitions and Notes. 1.** Let $X$ be a Banach space over $\mathbf{K}$. Suppose $M$ is an analytic manifold modelled on $X$ with an atlas $At(M)$ consisting of disjoint clopen charts $(U_j, \phi_j)$, $j \in \Lambda_M$, $\Lambda_M \subset \mathbf{N}$ [18]. That is, $U_j$ and $\phi_j(U_j)$ are clopen in $M$ and $X$ respectively, $\phi_j : U_j \to \phi_j(U_j)$ are homeomorphisms, $\phi_j(U_j)$ are bounded in $X$. Let $X = c_0(\alpha, \mathbf{K})$, where

$$(1)\ c_0(\alpha, \mathbf{K}) := \{x = (x^i : i \in \alpha) | x^i \in \mathbf{K}, \text{ and for each } \epsilon > 0 \text{ the set}$$

$$(i : |x^i| > \epsilon) \text{ is finite } \} \text{ with}$$

$$(2)\ \|x\| := \sup_i |x^i| < \infty$$

and the standard orthonormal base $(e_i : i \in \alpha)$ [21], $\alpha$ is an ordinal, $\alpha \ge 1$ [11]. Its cardinality is called a dimension $card(\alpha) =: dim_{\mathbf{K}} c_0(\alpha, \mathbf{K})$ over $\mathbf{K}$.

Then $C(t, M \to Y)$ for $M$ with a finite atlas $At(M)$, $card(\Lambda_M) < \aleph_0$, denotes a Banach space of functions $f : M \to Y$ with an ultranorm

$$(3)\ \|f\|_t = \sup_{j\in\Lambda_M} \|f|_{U_j}\|_{C(t,U_j\to Y)} < \infty,$$

where $Y := c_0(\beta, \mathbf{K})$ is the Banach space over $\mathbf{K}$, $0 \le t \in \mathbf{R}$, their restrictions $f|_{U_j}$ are in $C(t, U_j \to Y)$ for each $j$, $\beta \ge 1$.

**2.4.2.** Let $X, Y$ and $M$ be the same as in §2.4.1 for a local field $\mathbf{K}$. When $X$ or $Y$ are infinite-dimensional over $\mathbf{K}$, then the Banach space $C(t, M \to Y)$ is in general of non-separable type over $\mathbf{K}$ for $0 \le t \in \mathbf{R}$. For constructions of quasi-invariant measures it is necessary to have spaces of separable type. Therefore, subspaces of type $C_0$ are defined below. Their construction is analogous to that of $c_0$ from $l^\infty$ by imposing additional conditions (see for comparison [21]).

We denote by $C_0(t, M \to Y)$ a completion of a subspace of cylindrical functions restrictions of which on each chart $f|_{U_l}$ are finite K-linear combinations of functions $\{\bar{Q}_m(x_m)q_i|_{U_l} : i \in \beta, m\}$ relative to the following norm:

$$(1)\ \|f\|_{C_0(t,M\to Y)} := \sup_{i,m,l} |a(m, f^i|_{U_l})| J_l(t,m),$$

where multipliers $J_l(t,m)$ are defined as follows:

$$(2)\ J_l(t,m) := \|\bar{Q}_m|_{U_l}\|_{C(t,\phi_l(U_l)\cap \mathbf{K}^n \to \mathbf{K})},$$

$m \in c_0(\alpha, \mathbf{Q_p})$ with components $m_i \in \mathbf{N_o}$, non-zero componets of $m$ are $m_{i_1}, ..., m_{i_n}$ with $n \in \mathbf{N}$, $\bar{m} := (m_{i_1}, ..., m_{i_n})$ for each $m \neq 0$, $x_m := (x^{i_1}, ..., x^{i_n}) \in \mathbf{K}^n \hookrightarrow X$, $\bar{Q}_0 := 1$ (see aslo §2.2).

**Lemma.** *If $f \in C_0(t, M \to Y)$, then*

$$(3)\ (f|_{U_j})(x) = \sum_{i,m} a(m, f^i|_{U_j})\bar{Q}_m(x_m)q_i|_{U_j}$$

*for each $j \in \Lambda_M$, where $a(m, f^i|_{U_j}) \in \mathbf{K}$ are expansion coefficients such that for each $\epsilon > 0$ a set*

$$(4)\ \{(i,m,j) :\ |\, a(m, f^i|_{U_j})\,|\, J(t,m) > \epsilon\}\ \textit{is finite}.$$

**Proof.** This follows immediately from the definition, since

$$(5)\ f(x) = \sum_{i\in\beta} f^i(x)q_i,$$

where $f^i(*) \in C_0(t, M \to \mathbf{K})$.

In view of Formulas $(1-5)$ the space $C_0(t, M \to Y)$ is of separable type over K, when $card(\alpha \times \beta \times \Lambda_M) \leq \aleph_0$. Evidently, for compact $M$ the spaces $C_0(t, M \to Y)$ and $C_0(t, M \to Y)$ are isomorphic.

**2.4.3.a.** Now we define uniform spaces of the corresponding mappings from one manifold into another, which are necessary for the subsequent definitions of loop semigroups and groups.

Let $N$ be an analytic manifold modelled on $Y$ with an atlas

$$(1)\ At(N) = \{(V_k, \psi_k) :\ k \in \Lambda_N\},\ \text{such that } \psi_k : V_k \to \psi_k(V_k) \subset Y$$

are homeomorphisms, $card(\Lambda_N) \le \aleph_0$ and $\theta : M \to N$ be a $C(t')$-mapping, also $card(\Lambda_M) < \aleph_0$, where $V_k$ are clopen in $N$, $t' \ge \max(1,t)$ is the index of a class of smoothness, that is, for each admissible $(i,j)$:

$$(2)\ \theta_{i,j} \in C_*(t', U_{i,j} \to Y)$$

with $*$ empty or an index $*$ taking value 0 respectively,

$$(3)\ \theta_{i,j} := \psi_i \circ \theta|_{U_{i,j}},$$

where $U_{i,j} := [U_j \cap \theta^{-1}(V_i)]$ are non-void clopen subsets. We denote by $C^\theta_*(\xi, M \to N)$ for $\xi = t$ with $0 \le t \le \infty$ a space of mappings $f : M \to N$ such that

$$(4)\ f_{i,j} - \theta_{i,j} \in C_*(\xi, U_{i,j} \to Y).$$

In view of Formulas $(1-4)$ we supply it with an ultrametric

$$(5)\ \rho^\xi_*(f,g) = \sup_{i,j} \|f_{i,j} - g_{i,j}\|_{C_*(\xi, U_j \to Y)}$$

for each $0 \le \xi < \infty$.

**2.4.3.b.** For a construction of quasi-invariant measures particular types of function spaces are necessary, which are obtained by imposing simple relations on vectors $h_i$ for partial difference quotients. Let $M$ and $N$ be two analytic manifolds with finite atlases, $dim_{\mathbf{K}} M = n \in \mathbf{N}$, $\theta_{i,j} \in C(\infty, U_j \to Y)$ for each $i,j$.

We denote by $C^\theta_0((t,s), M \to N)$ a completion of a locally $\mathbf{K}$-convex space

$$(1)\ \{f \in C^\theta_0(t+sn, M \to N) :\ \rho_0^{(t,s)}(f,\theta) < \infty$$

$$\text{and for each } \epsilon > 0 \text{ a set } \{(k,m) :\ \sum_{i,j} |a(m, f^k_{i,j} - \theta^k_{i,j})| J((t,s),m) > \epsilon\} \text{ is finite }\}$$

relative to an ultrametric

$$(2)\ \rho_0^{(t,s)}(f,g) := \sup_{i,j,m,k} |a(m, f^k_{i,j} - g^k_{i,j})| J_j((t,s),m),$$

where $s \in \mathbf{N_o}$, $0 \le t < \infty$,

$$(3)\ J_j((t,s),m) := \max_{(v \le [t]+sign(t)+sn)} \|(\bar{\Phi}^v \bar{Q}_m|_{U_j})(x;$$

$$h_1,...,h_v;\zeta_1,...,\zeta_v)\|_{C_0(0,U_j\to Y)} \text{ with}$$

$$(4)\ h_1 = ... = h_\gamma = e_1, ..., h_{(n-1)\gamma+1} = ... = h_{n\gamma} = e_n$$

for each integer $\gamma$ such that $1 \le \gamma \le s$ and for each $v \in \{[t]+\gamma n, t+\gamma n\}$.

In view of Formulas $(1-3)$ this space is separable, when $N$ is separable, since $M$ is locally compact.

**2.4.4.** For infinite atlases we use the traditional procedure of inductive limits of spaces. For $M$ with the infinite atlas, $card(\Lambda_M) = \aleph_0$, and the Banach space $Y$ over $\mathbf{K}$ we denote by $C^\theta_*(\xi, M \to Y)$ for $\xi = t$ with $0 \le t \le \infty$ or for $\xi = (t,s)$ a locally $\mathbf{K}$-convex space, which is the strict inductive limit

$$(1)\ C^\theta_*(\xi, M \to Y) := str - ind\{C^\theta_*(\xi,(U^E \to Y), \pi^F_E, \Sigma\},$$

where $E \in \Sigma$, $\Sigma$ is the family of all finite subsets of $\Lambda_M$ directed by the inclusion $E < F$ if $E \subset F$, $U^E := \bigcup_{j\in E} U_j$ (see also §2.4 [13]).

For mappings from one manifold into another $f : M \to N$ we therefore get the corresponding uniform spaces. Then as in §2.4.4(b) [13] we denote them by $C^\theta_*(\xi, M \to N)$.

We introduce notations

$$(2)\ G(\xi, M) := C^\theta_0(\xi, M \to M) \cap Hom(M),$$

$$(3)\ Diff(\xi, M) = C^\theta(\xi, M \to M) \cap Hom(M),$$

that are called groups of diffeomorphisms (and homeomorphisms for $0 \le t < 1$ and $s = 0$), $\theta = id$, $id(x) = x$ for each $x \in M$, where $Hom(M) := \{f : f \in C(0, M \to M)$, $f$ is bijective , $f(M) = M$, $f$ and $f^{-1} \in C(0, M \to M)\}$ denotes the usual homeomorphism group. For $s = 0$ we may omit it from the notation, which is always accomplished for $M$ infinite-dimensional over $\mathbf{K}$.

**2.5. Notes.** Henceforth, ultrametrizable separable complete manifolds $\bar{M}$ and $N$ are considered. Since a large inductive dimension $Ind(\bar{M}) = 0$ (see Theorem 7.3.3 [8]), hence $\bar{M}$ has not boundaries in the usual sense. Therefore,

$$(1)\ At(\bar{M}) = \{(\bar{U}_j, \bar{\phi}_j) :\ j \in \Lambda_{\bar{M}}\}$$

has a refinement $At'(\bar{M})$ which is countable and its charts $(\bar{U}'_j, \bar{\phi}'_j)$ are clopen and disjoint and homeomorphic with the corresponding balls $B(X, y_j, \bar{r}'_j)$, where

$$(2)\ \bar{\phi}'_j : \bar{U}'_j \to B(X, y'_j, \bar{r}'_j) \text{ for each } j \in \Lambda'_{\bar{M}}$$

are homeomorphisms (see [8, 18]). For $\bar{M}$ we fix such $At'(\bar{M})$.

We define topologies of groups $G(\xi, \bar{M})$ and locally K-convex spaces $C_*(\xi, \bar{M} \to Y)$ relative to $At'(\bar{M})$, where $Y$ is the Banach space over K. Therefore, we suppose also that $\bar{M}$ and $N$ are clopen subsets of the Banach spaces $X$ and $Y$ respectively. Up to the isomorphism of loop semigroups (see below their definition) we can suppose that $s_0 = 0 \in \bar{M}$ and $y_0 = 0 \in N$.

For $M = \bar{M} \setminus \{0\}$ let $At(M)$ consists of charts $(U_j, \phi_j)$, $j \in \Lambda_M$, while $At'(M)$ consists of charts $(U'_j, \phi'_j)$, $j \in \Lambda'_M$, where due to Formulas $(1, 2)$ we define

$$(3)\ U_1 = \bar{U}_1 \setminus \{0\},\ \phi_1 = \bar{\phi}_1|_{U_1};\ U_j = \bar{U}_j \text{ and } \phi_j = \bar{\phi}_j \text{ for each } j > 1,$$

$$0 \in \bar{U}_1,\ \Lambda_M = \Lambda_{\bar{M}},\ U'_1 = \bar{U}'_1 \setminus \{0\},\ \phi'_1 = \bar{\phi}'_1|_{U'_1},\ U'_j = \bar{U}'_j \text{ and } \phi'_j = \bar{\phi}'_j$$
$$\text{for each } j > 1,\ j \in \Lambda'_M = \Lambda'_{\bar{M}},\ \bar{U}'_1 \ni 0.$$

**2.6. Definitions and Notes. 1.** Let the spaces be the same as in §2.4.4 (see Formulas 2.4.4.(1-3)) with the atlas of $M$ defined by Conditions 2.5.(3). Then we consider their subspaces of mappings preserving marked points:

$$(1)\ C^\theta_0(\xi, (M, s_0) \to (N, y_0)) := \{f \in C^\theta_0(\xi, \bar{M} \to N) : \lim_{|\zeta_1| + \dots + |\zeta_k| \to 0} \bar{\Phi}^v(f - \theta)(s_0; h_1, \dots, h_k; \zeta_1, \dots, \zeta_k) = 0 \text{ for each } v \in \{0, 1, \dots, [t], t\},\ k = [v] + sign\{v\}\},$$

where for $s > 0$ and $\xi = (t, s)$ in addition Condition 2.4.3.b.(4) is satisfied for each $1 \le \gamma \le s$ and for each $v \in \{[t] + n\gamma, t + n\gamma\}$,
and the following subgroup:

$$(2)\ G_0(\xi, M) := \{f \in G(\xi, \bar{M}) : f(s_0) = s_0\}$$

of the diffeomorphism group, where $s \in \mathbf{N}_o$ for $dim_{\mathbf{K}} M < \aleph_0$ and $s = 0$ for $dim_{\mathbf{K}} M = \aleph_0$.

With the help of them we define the following equvalence relations $K_\xi$: $f K_\xi g$ if and only if there exist sequences

$$\{\psi_n \in G_0(\xi, M) :\ n \in \mathbf{N}\},$$

$$\{f_n \in C_0^\theta(\xi, M \to N) : \ n \in \mathbf{N}\} \text{ and}$$
$$\{g_n \in C_0^\theta(\xi, M \to N) : n \in \mathbf{N}\} \text{ such that}$$

(3) $f_n(x) = g_n(\psi_n(x))$ for each $x \in M$ and $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$.

Due to Condition (3) these equivalence classes are closed, since $(g(\psi(x))' = g'(\psi(x))\psi'(x)$, $\psi(s_0) = s_0$, $g'(s_0) = 0$ for $t + s \geq 1$. We denote them by $< f >_{K,\xi}$. Then for $g \in < f >_{K,\xi}$ we write $gK_\xi f$ also. The quotient space $C_0^\theta(\xi, (M, s_0) \to (N, y_0))/K_\xi$ we denote by $\Omega_\xi(M, N)$, where $\theta(M) = \{y_0\}$.

**2.6.2.** Let as usually $A \vee B := A \times \{b_0\} \cup \{a_0\} \times B \subset A \times B$ be the wedge product of pointed spaces $(A, a_0)$ and $(B, b_0)$, where $A$ and $B$ are topological spaces with marked points $a_0 \in A$ and $b_0 \in B$. Then the composition $g \circ f$ of two elements $f, g \in C_0^\theta(\xi, (M, s_0) \to (N, y_0))$ is defined on the domain $\bar{M} \vee \bar{M} \setminus \{s_0 \times s_0\} =: M \vee M$.

Let $M = \bar{M} \setminus \{0\}$ be as in §2.5. We fix an infinite atlas $\tilde{At}'(M) := \{(\tilde{U}'_j, \phi'_j) : j \in \mathbf{N}\}$ such that $\phi'_j : \tilde{U}'_j \to B(X, y'_j, r'_j)$ are homeomorphisms,

$$\lim_{k\to\infty} r'_{j(k)} = 0 \text{ and } \lim_{k\to\infty} y'_{j(k)} = 0$$

for an infinte sequence $\{j(k) \in \mathbf{N} : k \in \mathbf{N}\}$ such that $cl_{\bar{M}}[\bigcup_{k=1}^\infty \tilde{U}'_{j(k)}]$ is a clopen neighbourhood of 0 in $\bar{M}$, where $cl_{\bar{M}}A$ denotes the closure of a subset $A$ in $\bar{M}$. In $M \vee M$ we choose the following atlas $\tilde{At}'(M \vee M) = \{(W_l, \xi_l) : l \in \mathbf{N}\}$ such that $\xi_l : W_l \to B(X, z_l, a_l)$ are homeomorphisms,

$$\lim_{k\to\infty} a_{l(k)} = 0 \text{ and } \lim_{k\to\infty} z_{l(k)} = 0$$

for an infinite sequence $\{l(k) \in \mathbf{N} : k \in \mathbf{N}\}$ such that $cl_{\bar{M}\vee\bar{M}}[\bigcup_{k=1}^\infty W_{l(k)}]$ is a clopen neighbourhood of $0 \times 0$ in $\bar{M} \vee \bar{M}$ and

$$card(\mathbf{N} \setminus \{l(k) : k \in \mathbf{N}\}) = card(\mathbf{N} \setminus \{j(k) : k \in \mathbf{N}\}).$$

Then we fix a $C(\infty)$-diffeomorphisms $\chi : M \vee M \to M$ such that

(1) $\chi(W_{l(k)}) = \tilde{U}'_{j(k)}$ for each $k \in \mathbf{N}$ and

(2) $\chi(W_l) = \tilde{U}'_{\kappa(l)}$ for each $l \in (\mathbf{N} \setminus \{l(k) : k \in \mathbf{N}\})$, where

(3) $\kappa : (\mathbf{N} \setminus \{l(k) : k \in \mathbf{N}\}) \to (\mathbf{N} \setminus \{j(k) : k \in \mathbf{N}\})$

is a bijective mapping for which

$$(4)\ p^{-1} \le a_{l(k)}/r'_{j(k)} \le p \text{ and } p^{-1} \le a_l/r'_{\kappa(l)} \le p.$$

This induces the continuous injective homomorphism

(5) $\chi^* : C^\theta_0(\xi, (M \vee M, s_0 \times s_0) \to (N, y_0)) \to C^\theta_0(\xi, (M, s_0) \to (N, y_0))$ such that

$$(6)\ \chi^*(g \vee f)(x) = (g \vee f)(\chi^{-1}(x))$$

for each $x \in M$, where $(g \vee f)(y) = f(y)$ for $y \in M_2$ and $(g \vee f)(y) = g(y)$ for $y \in M_1$, $M_1 \vee M_2 = M \vee M$, $M_i = M$ for $i = 1, 2$. Therefore

$$(7)\ g \circ f := \chi^*(g \vee f)$$

may be considered as defined on $M$ also, that is, to $g \circ f$ there corresponds the unique element in $C^\theta_0(\xi, (M, s_0) \to (N, y_0))$.

**2.6.3.** The composition in $\Omega_\xi(M, N)$ is defined due to the following inclusion $g \circ f \in C^\theta_0(\xi, (M, s_0) \to (N, y_0))$ (see Formulas 2.6.2.(1-7)) and then using the equivalence relations $K_\xi$ (see Condition 2.6.1.(3)).

It is shown below that $\Omega_\xi(M, N)$ is the monoid, which we call the loop monoid.

**2.7. Theorem.** *The space $\Omega_\xi(M, N)$ from §2.6 is the complete separable Abelian topological Hausdorff monoid. Moreover, it is non-discrete, topologically perfect and has the cardinality $\mathsf{c} := card(\mathbf{R})$.*

**Proof.** We have $f(\psi) \in C^\theta_0(\xi, (M, s_0) \to (N, y_0))$ for each $f \in C^\theta_0(\xi, (M, s_0) \to (N, y_0))$ and $\psi \in G_0(\xi, M)$ (see also [16, 18]). The diffeomorphism $\chi : M \vee M \to M$ is of class $C(\infty)$ and from Condition 2.6.1.(1) for $f_i \in C^0_0(\xi, (M, s_0) \to (N, y_0))$ it follows that for $f = \chi^*(f_1 \vee f_2)$ also Condition 2.6.1.(1) is satisfied, since $\chi$ fulfils Conditions 2.6.2.(1-4), where $i \in \{1, 2\}$, $x + \xi_j h_j \in M$ for each $j$, $n = [v] + sign\{v\}$, $h_j \in X$, $\xi_j \in \mathbf{K}$. Due to Condition 2.6.2(4) the composition in $C^\theta_0(\xi, (M, s_0) \to (N, y_0))$ is evidently continuous, since

$$\|f \circ g\|_{C^\theta_0(\xi, M \to Y)} \le p^d \times \max\{\|f\|_{C^\theta_0(\xi, M \to Y)}, \|g\|_{C^\theta_0(\xi, M \to Y)}\}$$

for finite $At(M)$ and using the strict inductive limit for infinte $At(M)$, when $t < \infty$ and $d = [t] + 1$ for $\xi = t$ with $dim_{\mathbf{K}} M \le \aleph_0$, $d = [t] + 1 + s\alpha$ for $\xi = (t, s)$ with $\alpha = dim_{\mathbf{K}} M < \aleph_0$, where $0 \le t \in \mathbf{R}$ and $s \in \mathbf{N_o}$. Due to

Formulas 2.6.1.(1-3) and 2.6.2.(5-7) $< f >_{K,\xi} \circ < g >_{K,\xi} = < f \circ g >_{K,\xi}$ for each $f$ and $g \in C^\theta_0(\xi, (M, s_0) \to (N, y_0))$, since if $\tilde{f}_n(x) = f_n(\eta_n(x))$ and $\tilde{g}_n(x) = g_n(\zeta_n(x))$ for each $x \in M$, then $(\tilde{f}_n \vee \tilde{g}_n)(x) = (f_n(\eta_n) \vee g_n(\zeta_n))(x)$, where $\eta_n$ and $\zeta_n \in G_0(\xi, M)$. Hence the composition is continuous for the quotient space.

In view of Formulas 2.6.2.(1-3) $M_1 \vee M_2$ and $M_2 \vee M_1$ are $C(\infty)$-diffeomorphic, hence these semigroups are Abelian. Evidently, this composition is associative, since $M \vee (M \vee M)$ is $C(\infty)$-diffeomorphic with $(M \vee M) \vee M$. In view of Conditions 2.6.1.(1-3) for each $f \in C^\theta_0(\xi, (M, s_0) \to (N, y_0))$ there exist sequences $\{\psi_n : n \in \mathbf{N}\}$, $\{\eta_n : n \in \mathbf{N}\}$ and $\{\zeta_n : n \in \mathbf{N}\}$ in $G_0(\xi, M)$, $\{f_n : n \in \mathbf{N}\}$, $\{w_{0,n} : n \in \mathbf{N}\}$ and $\{g_n : n \in \mathbf{N}\}$ in $C^\theta_0(\xi, (M, s_0) \to (N, y_0))$ such that $[w_{0,n} \vee f_n](\psi_n \vee \eta_n(x)) = g_n(\zeta_n(\chi(x)))$, where

$$(1) \lim_{n\to\infty} diam\{x \in M : \chi^*[w_{0,n} \vee f_n](\psi_n \vee \eta_n(x)) = y_0\} = 0,$$

$$(2) \lim_{n\to\infty} f_n = f, \ \lim_{n\to\infty} g_n = f,$$

$$(3) \lim_{n\to\infty} w_{0,n} = w_0, \ f_n(x) \ne y_0 \text{ for each } x \in M,$$

$$(4) \ diam(A) := \sup_{x_1, x_2 \in A} \|x_1 - x_2\|_X,$$

$A \subset M \subset X$. On the other hand, from $\lim_{n\to\infty}(f_n \vee g_n) = f \vee g$ it follows that $\lim_{n\to\infty} f_n = f$ and $\lim_{n\to\infty} g_n = g$. Using Formulas $(1-4)$ we get $< w_0 \circ f >_{K,\xi} = < f >_{K,\xi}$ and $< w_0 >_{K,\xi} = e$ is the unit element in $\Omega_\xi(M, N)$, since $< f >_{K,\xi} \circ < g >_{K,\xi} = < f \circ g >_{K,\xi}$ for each $f$ and $g \in C^\theta_0(\xi, (M, s_0) \to (N, y_0))$.

For each $f, h$ and $g \in C^\theta_0(\xi, (M, s_0) \to (N, y_0))$ from $f \vee g = f \vee h$ it follows that $g = h$. If there exists $\psi \in G_0(\xi, M)$ such that $\chi^*(f \vee g)(\psi(x)) = \chi^*(f \vee h)(x)$ for each $x \in M$, then there exists $\tilde{\psi} \in G_0(\xi, M \vee M)$ for which $(f \vee g)(\tilde{\psi}(x)) = (f \vee h)(x)$ for each $x \in M \vee M$ and $f(\tilde{\psi}|_{M_1})(x) = f(x)$ for each $x \in M_1$, $M_1 = M_2 = M$, hence $g(x) = h(\bar{\psi}(x))$ for each $x \in M$, where $\bar{\psi} \in G_0(\xi, M)$ corresponds to $\tilde{\psi}|_{M_2}$. Then

$$(5) \ < f >_{K,\xi} \circ < g >_{K,\xi} = < f >_{K,\xi} \circ < h >_{K,\xi} \text{ implies}$$

$$< g >_{K,\xi} = < h >_{K,\xi},$$

since it is true for representatives of these classes. Implication (5) is called the cancellation property. Therefore, the composition in $\Omega_\xi(M, N)$ is associative, commutative and there is the unit element $e$, consequently, $\Omega_\xi(M,N)$ is the monoid with the cancellation property.

To show that $\Omega_\xi(M,N)$ is the Hausdorff space it is sufficient to consider $M$ with a finite atlas, since $C^0_0(\xi,\bar{M}\to N)$ is defined with the help of inductive limit. In view of the monoid structure it is sufficient to consider two elements $g$ and $e$ with $g\ne e$. Let

$$\rho^\xi_\Omega(f,g):=\inf_{(\tilde{g}\in g,\tilde{f}\in f)}\rho^\xi_0(\tilde{g},\tilde{f})$$

be a pseudoultrametric in $\Omega_\xi(M,N)$, that is,

$$(6)\ \rho^\xi_\Omega(g,f)\ge 0,\ \rho^\xi_\Omega(f,f)=0,$$

$$(7)\ \rho^\xi_\Omega(g,f)=\rho^\xi_\Omega(f,g)\text{ and}$$

$$(8)\ \rho^\xi_\Omega(g,f)\le\max\{\rho^\xi_\Omega(g,h),\rho^\xi_\Omega(h,f)\}\text{ for each } f,g \text{ and } h\in\Omega_\xi(M,N),$$

where $\tilde{g}$ and $\tilde{f}\in C^0_0(\xi,(M,s_0)\to(N,y_0))$, $<\tilde{g}>_{K,\xi}=g$, $<\tilde{f}>_{K,\xi}=f$, $f$ and $g\in\Omega_\xi(M,N)$. Evidently, $\rho^\xi_\Omega(M,N)$ is continuous relative to the quotient topology (see §2.4 and §4.1 [8]). If $\rho^\xi_\Omega(g,e)=0$, then there exist $\phi_n\in G_0(\xi,M)$ and $\tilde{g}_n\in g$ such that

$$(9)\ \lim_{n\to\infty}\{\sup_{j\in\Lambda_N}\|\psi_j\circ\tilde{g}_n\circ\phi_n\|_{C_0(\xi,\tilde{M}\to Y)}\}=0$$

(see Formulas 2.4.2.(3-5) and 2.4.3.a.(1)). In view of Conditions 2.6.1(1), 2.6.2(4) and Formula (9) for each $f\in\Omega_\xi(M,N)$ there are $\tilde{f}_n\in f$, $\tilde{\phi}_n$ and $\bar{\phi}_n\in G_0(\xi,M)$ such that

$$\lim_{n\to\infty}\{\sup_j\|\psi_j\circ(\chi^*(\tilde{f}_n(\tilde{\phi}_n)\vee\tilde{g}_n(\bar{\phi}_n))-\psi_j\circ\tilde{f}_n(\tilde{\phi}_n)\|_{C_0(\xi,\tilde{M}\to Y)}\}=0,$$

consequently, $f\circ g=f=g\circ f$ for each $f\in\Omega_\xi(M,N)$, hence $g=e$. This contradicts the assumption $g\ne e$, consequently, $\epsilon:=\rho^\xi_\Omega(g,e)>0$ and $W_g\cap W_e=\emptyset$ for $W_f:=\{h\in\Omega_\xi(M,N):\rho^\xi_\Omega(h,f)<\epsilon/p\}$, where $W_f$ are open subsets of $\Omega_\xi(M,N)$. Then $\Omega_\xi(M,N)$ is Hausdorff, since

$$(10)\ \rho^\xi_\Omega(g,f)>0\text{ for each } g\ne f$$

and $\rho^\xi_\Omega(g,f)$ satisfying $(6-8,10)$ is the ultrametric.

The space $C_0^\theta(\xi,(M,s_0)\to(N,y_0))$ is separable and complete (see §§2.4 and 2.6) such that for each Cauchy sequence in the loop monoid there exists a Cauchy sequence in $C_0^\theta(\xi,(M,s_0)\to(N,y_0))$, hence this monoid is complete.

For each pair of elements $f$ and $g\in C_0^\theta(\xi,(M,s_0)\to(N,y_0))$ with different images $f(M)\ne g(M)$ we have $<f>_{K,\xi}\ne<g>_{K,\xi}$. Since $N$ is embedded into the Banach space $Y$ and $N$ is clopen in $Y$ it is possible to consider shifts along the basic vectors and retractions of images $f(M)$ within $N$ of the corresponding class of smoothness, hence this monoid is non-discrete. The manifolds $M$ and $N$ and the space $C_0^\theta(\xi,(M,s_0)\to(N,y_0))$ are separable, hence $\mathsf{c}=card(N)\le card(\Omega_\xi(M,N))\le card(C_0^\theta(\xi,(M,s_0)\to(N,y_0))=\mathsf{c}$. For each $g\in\Omega_\xi(M,N)$ there exists a Cauchy sequence $\{g_i : i\in\mathbf{N}\}\subset\Omega_\xi(M,N)\setminus\{g\}$ such that $\lim_{i\to\infty}g_i=g$, since this is true for $C_0^\theta(\xi,M\to N)$ and choosing representatives in classes. Hence $\Omega_\xi(M,N)$ is dense in itself and perfect as the topological space.

**2.8. Note.** For each chart $(V_i,\psi_i)$ of $At(N)$ (see Equality 2.4.3.a.(1)) there are local normal coordinates $y=(y^j : j\in\beta)\in B(Y,a_i,r_i)$, $Y=c_0(\beta,\mathbf{K})$. Moreover, $TV_i=V_i\times Y$, consequently, $TN$ has the disjoint atlas $At(TN)=\{(V_i\times X,\psi_i\times I) : i\in\Lambda_N\}$, where $I_Y : Y\to Y$ is the unit mapping, $\Lambda_N\subset\mathbf{N}$, $TN$ is the tanget vector bundle over $N$.

Suppose $V$ is an analytic vector field on $N$ (that is, by definition $V|_{V_i}$ are analytic for each chart and $V\circ\psi_i^{-1}$ has the natural extension from $\psi_i(V_i)$ on the balls $B(X,a_i,r_i)$). Then by analogy with the classical case we can define the following mapping

$$\bar{exp}_y(zV)=y+zV(y) \text{ for which}$$

$$\partial^2\bar{exp}_y(zV(y))/\partial z^2=0$$

(this is the analog of the geodesic), where $\|V(y)\|_Y|z|\le r_i$ for $y\in V_i$ and $\psi_i(y)$ is also denoted by $y$, $z\in\mathbf{K}$, $V(y)\in Y$. Moreover, there exists a refinement $At"(N)=\{(V"_i,\psi"_i) : i\in\Lambda"_N\}$ of $At(N)$. This $At"(N)$ is embedded into $At(N)$ by charts such that it is also disjoint and analytic and $\psi"_i(V"_i)$ are K-convex in $Y$. The latter means that $\lambda x+(1-\lambda)y\in\psi"_i(V"_i)$ for each $x,y\in\psi"_i(V"_i)$ and each $\lambda\in B(\mathbf{K},0,1)$. Evidently, we can consider $\bar{exp}_y$ injective on $V"_i$, $y\in V"_i$. The atlas $At"(N)$ can be chosen such that

$$(\bar{exp}_y|_{V"_i}) : V"_i\times B(Y,0,\tilde{r}_i)\to V"_i$$

to be the analytic homeomorphism for each $i \in \Lambda"_M$, where $\infty > \tilde{r}_i > 0$, $y \in V"_i$,

$$\bar{exp}_y : (\{y\} \times B(Y, 0, \tilde{r}_i)) \to V"_i$$

is the isomorphism. Therefore, $\bar{exp}$ is the locally analytic mapping, $\bar{exp} : \tilde{T}N \to N$, where $\tilde{T}N$ is the corresponding neighbourhood of $N$ in $TN$.

Then

$$(1)\ T_f C^\theta_*(\xi, M \to N) = \{g \in C^{(\theta,0)}_*(\xi, M \to TN) :\ \pi_N \circ g = f\},$$

consequently,

$$(2)\ C^\theta_*(\xi, M \to TN) = \bigcup_{f \in C^\theta_*(\xi, M \to N)} T_f C^\theta_*(\xi, M \to N) = TC^\theta_*(\xi, M \to N),$$

where $\pi_N : TN \to N$ is the natural projection, $* = 0$ or $* = \emptyset$ ($\emptyset$ is omitted). Therefore, the following mapping

$$(3)\ \omega_{\bar{exp}} :\ T_f C^\theta_*(\xi, M \to N) \to C^\theta_*(\xi, M \to N)$$

is defined by the formula given below

$$(4)\ \omega_{\bar{exp}}(g(x)) = \bar{exp}_{f(x)} \circ g(x),$$

that gives charts on $C^\theta_*(\xi, M \to N)$ induced by charts on $C^\theta_*(\xi, M \to TN)$.

**2.9. Definition and Note.** In view of Equalities 2.8.(1,2) the space $C^\theta_0(\xi, M \to N)$ is isomorphic with $C^\theta_0(\xi, (M, s_0) \to (N, y_0)) \times N^\xi$, where $y_0 = 0$ is the marked point of $N$. Here

$$(1)\ N^\xi := N \otimes (\bigotimes_{j=1}^{d} \tilde{L}_\xi(X^j \to Y)) \text{ for } t \in \mathbf{N_0} \text{ with } t + s > 0;$$

$$(2)\ N^\xi = N \text{ for } t + s = 0;$$

$$(3)\ N^\xi = N \otimes (\bigotimes_{j=1}^{d} \tilde{L}_\xi(X^j \to Y)) \otimes C^0_0(0, M^k \to Y_\lambda) \text{ for } t \in \mathbf{R} \setminus \mathbf{N},$$

where $N^\xi$ is with the product topology, $d = [t]$ for $\xi = t$, $d = [t] + n\alpha$ for $\xi = (t, s)$ with $\alpha = dim_{\mathbf{K}} M < \aleph_0$, when $s > 0$, $k = d + sign\{t\}$, $Y_\lambda := c_0(\beta, \lambda)$, $\lambda$ is the least subfield of $\mathbf{\Lambda_p}$ such that $\lambda \supset \mathbf{K} \cup j_{\{t\}}(\mathbf{K})$

(see Equation 2.1.(1)). Then $\tilde{L}_\xi(X^j \to Y)$ denotes the Banach space of continuous $j$-linear operators $f_j : X^j \to Y$ with

$$(4)\ \|f_j\|_{\tilde{L}_\xi(X^j \to Y)} := \sup_{i,m} \|f_j^i\|_m \text{ and}$$

$$(5)\ \lim_{i+|m|+k\to\infty} \|f_j^i\|_m = 0, \text{ where}$$

$$(6)\ \|f_j^i\|_m := \sup_{0\ne h_l\in \mathbf{K}^k, l=1,...,j} \|f_j^i(h_1,...,h_j)\|_Y J'(\xi,m)/(\|h_1\|_X...\|h_j\|_X),$$

$\mathbf{K}^k := sp_{\mathbf{K}}(e_1,...,e_k) \hookrightarrow X$ is a $\mathbf{K}$-linear span of the standard basic vectors, $m = (m_1,...,m_k)$, $|m| = m_1 + ... + m_k$, $k \in \mathbf{N}$; $h_1 = ... = h_{m_1},...,h_{m_{k-1}+1} = ... = h_{m_k}$ for $s = 0$; in addition Condition 2.4.3.b.(4) is satisfied for each $0 < \gamma \le s$, when $s > 0$; $f = (f_0, f_1, ..., f_j, ...) \in N^\xi$, $\sum_i f_j^i q_i = f_j$, $f_j^i : X^j \to \mathbf{K}$,

$$J'(\xi,m) := |\bar{\partial}^m \bar{Q}_m(x)|_{x=0}|_{\mathbf{K}}$$

(see §2.2 and Equations 2.4.2.(1-5), 2.4.3.b.(1-3)).

**2.10. Theorem.** *Let $G = \Omega_\xi(M,N)$ be the same monoid as in §2.6.*

*If $1 \le t+s$, $0 \le t \in \mathbf{R}$ and $\xi = (t,s)$, $s \in \mathbf{N}_o$, then*

*(1) $G$ is an analytic manifold and for it the mapping $\tilde{E} : \tilde{T}G \to G$ is defined, where $\tilde{T}G$ is the neighbourhood of $G$ in $TG$ such that $\tilde{E}_\eta(V) = \bar{exp}_{\eta(x)} \circ V_\eta$ from some neighbourhood $\tilde{V}_\eta$ of the zero section in $T_\eta G \subset TG$ onto some neighbourhood $W_\eta \ni \eta \in G$, $\tilde{V}_\eta = \tilde{V}_e \circ \eta$, $W_\eta = W_e \circ \eta$, $\eta \in G$ and $\tilde{E}$ belongs to the class $C(\infty)$ by $V$, $\tilde{E}$ is the uniform isomorphism of uniform spaces $\tilde{V}$ and $W$;*

*(2) if $At(M)$ is finite, then there are $\tilde{A}t(TG)$ and $\tilde{A}t(G)$ for which $\tilde{E}$ is locally analytic. Moreover, $G$ is not locally compact for each $0 \le t$.*

**Proof.** (A.) Let at first $M$ be with a finite atlas $At(M)$.

Let $V_\eta \in T_\eta G$ for each $\eta \in G$, $V \in C_0(\xi, G \to TG)$, suppose also that $\tilde{\pi} \circ V_\eta = \eta$ be the natural projection such that $\tilde{\pi} : TG \to G$, then $V$ is a vector field on $G$ of class $C_0(\xi)$. Indeed, let $\tilde{V} = \{g \in C_0(\xi, (M, s_0) \to (N, y_0)) : \rho_0^\xi(g, w_0) \le 1/p\}$. Then $C_0(\xi, (M, s_0) \to (N, y_0))$ and $C_0(\xi, (M, s_0) \to (TN, y_0 \times 0))$ have disjoint atlases with clopen charts, since there are neighbourhoods $P \ni w_0$ in $C_0(\xi, M \to N)$ and $\tilde{P} \ni (w_0 \otimes 0)$ in $C_0(\xi, M \to TN)$ homeomorphic to clopen subsets in $C_0(\xi, M \to Y)$ and $C_0(\xi, M \to Y \times Y)$, where $P$ and $\tilde{P}$ are such that they may be embedded into $A := \bigoplus_i C_0(\xi, U_i \to Y)$ and $B := \bigoplus_i C_0(\xi, U_i \to Y \times Y)$ respectively. Moreover, there are the natural embeddings $A \hookrightarrow C_0(\xi, M \to Y)$ and $B \hookrightarrow C_0(\xi, M \to Y \times Y)$.

The disjoint and analytic atlases $At(C_0(\xi, M \to N))$ and $At(C_0(\xi, M \to TN)$ induce disjoint clopen atlases in $G$ and $TG$ with the help of the corresponding equivalence relations, since the metrics in these quotient spaces satisfy Inequality 2.7.(8). These atlases are countable, since $G$ and $TG$ are separable.

Let us suppose, that $G$ has a compact clopen neighbourhood $W_e$ of $e$. Since $ind(G) = 0$, then there exists $W_e$, which is the submonoid (see §7.7 and §9 [10]). There exists $< f >_{K,\xi} \in W_e$ for which $f$ is locally linear, that is, $f(x) = a + b(x - \tilde{x}_1)$ for each $x \in B(X, \tilde{x}_1, \tilde{r}_1)$, $f(x) = 0$ for each $x \in M \setminus B(X, \tilde{x}_1, \tilde{r}_1)$, where $B(X, \tilde{x}_1, \tilde{r}_1) \subset M$, $a$ and $b \in \mathbf{K}$, $0 \notin B(X, \tilde{x}_1, \tilde{r}_1)$, $|a| = p\tilde{r}_1$, $|b| \ge p^2$, $\infty > \tilde{r}_1 > 0$, $B(Y, 0, |b|\tilde{r}_1) \subset N$. Then $f$ has compositions $f^n := f \circ f^{n-1}$ for each $n > 1$, $n \in \mathbf{N}$, where $f^1 := f$ and $f^0 := w_0$, for which $< f^n >_{K,\xi} \in W_e$ for each $n$, hence the sequence $\{< f^n >_{K,\xi}: n\}$ has a convergent subsequence $\{< f^{n_i} >_{K,\xi}: i \in \mathbf{N}\}$ in $W_e$, since $G$ is Lindelöf (see Theorems 3.10.1 and 3.10.31 [8]). But

$$\|f^n(x) - f^l(\psi(x))\|_{C^0_0(\xi, \tilde{M} \to Y)} \ge \delta \|f\|_{C^0_0(\xi, \tilde{M} \to Y)} =: \epsilon > 0$$

for each $n \ne l$, $n$ and $l \in \mathbf{N}$, $\psi \in G_0(\xi, M)$, consequently, due to Equations 2.6.2.(1-4)

$$\rho^\xi_0(< f^n >_{K,\xi}, < f^l >_{K,\xi}) \ge \epsilon$$

for each $n \ne l \in \mathbf{N}$, where $\infty > \delta > 0$, since

$$0 < \|f\|_{C_0(0, \tilde{M} \to Y)} \le \|f^n\|_{C_0(\xi, \tilde{M} \to Y)}.$$

This contradiction means that $G$ is not locally compact. In view of §2.7 the space $T_\eta G$ is not locally compact, hence it is infinite-dimensional over $\mathbf{K}$, since $dim_{\mathbf{K}} C^0_0(\xi, B(\mathbf{K}, 0, 1) \to \mathbf{K}) = \aleph_0$.

Due to Equations 2.6.2.(1-7) multiplications

$$(1)\ R_f : G \to G,\ g \mapsto g \circ f =: R_f(g) \text{ and}$$

$$(2)\ \alpha_h : C^0_0(\xi, (M, s_0) \to (N, y_0)) \to C^0_0(\xi, (M, s_0) \to (N, y_0)),\ \alpha_h(v) := v \circ h$$

for $f, g \in G$ and $h, v \in C^0_0(\xi, (M, s_0) \to (N, y_0))$ belong to the class $C(\infty)$. Let $\tilde{V}$ be a collection of all $g \in C^0_0(\xi, (M, s_0) \to (N, 0))$ for which $g = w_0 + Z$ with $\|\psi_i(Z|_{U_j})\|_{C^0_0(\xi, U_j \to Y)} \le 1/p$ for each $i$ and $j$, since $N$ is clopen in $Y$. Hence $\tilde{g}_z = w_0 + zZ \in \tilde{V}$ for each $z \in B(\mathbf{K}, 0, 1)$ and $(\partial \alpha_h(\tilde{g}_z)/\partial z)|_{z=0} =$

$\alpha_h(Z)$. Then $Z(x) = p_2(v(g(x)))$ for each $x \in M$, where $v$ is a vector field on $N$ such that $\pi_N v(y) = y$, $\pi_N : TN \to N$ is the natural projection and $v(y) \in T_yN$ for each $y \in N$, $p_2 : TN \to Y$, $p_2(y,a) = a$ for each $(y,a) \in T_yN$. Due to the equivalence relations there are a neighbourhood $W$ of $e$ in $G$ and a curve $\bar{g}_z \subset W$ corresponding to $\tilde{g}_z$ such that $(\partial R_f \bar{g}_z/\partial z)|_{z=0} = R_f \bar{Z}$, where $\bar{Z}$ is a vector field on $G$, that is a section of the vector bundle $\tilde{\pi} : TG \to G$, $\tilde{\pi}(\bar{Z}_\eta) = \eta$ for each $\eta \in G$, $\bar{Z}_\eta \in T_\eta G$ (compare with the classical case in [7]). From this it follows that each vector field $V$ of class $C_0(\xi)$ on $G$ is invariant, since $G$ is Abelian. This means that $R_f V_\eta = V_{\eta\circ f}$ for each $f$ and $\eta \in G$.

Therefore, the vector field $V$ on $G$ of class $C_0(\xi)$ has the form

$$(3)\ V_{\eta(x)} = v(\eta(x)),$$

where $v$ is a vector field on $N$ of the class $C_0(\xi)$, $\eta \in G$, $v(< f >_{K,\xi} (x)) := \{v(g(x)) : g \in < f >_{K,\xi}\}$. Since $\tilde{e}xp : \tilde{T}N \to N$ is analytic on the corresponding charts, then $\tilde{E}(V) = \tilde{e}xp \circ V$ has the necessary properties (see Equations 2.8.(3,4)).

(B.) Let now $M$ be with an infinite countable atlas $At(M)$. Let us take a subset

$$(4)\ \tilde{V}' := \{V \in C^0_0(\xi,(M,s_0) \to (\tilde{T}N, 0\times 0)) : \ supp(V) \subset U^{E(V)},\ E(V) \in \Sigma$$

$$\text{such that } \|V|_{U^{E(V)}}\|_{C_0(\xi,U^{E(V)}\to Y\times Y)} \le 1/p$$

$$\text{and } \pi_N \circ V_{\eta(x)} = \eta(x) \text{ for each } \eta \in C^0_0(\xi,(M,s_0) \to (N,0)) \text{ and } x \in M\},$$

where $E(V) \ni 1$ (see §2.4.4). In view of Equations 2.8.(1,2) if $\tilde{V} \in TG$, then there exists $\tilde{v} \in TN$ such that $\tilde{V}_\eta = \tilde{v}(\eta)$ for each $\eta \in G$. If $F(t)$ is a $C(\infty)$-mapping from $B(\mathbf{K},0,1)$ into $C^0_0(\xi,(M,s_0) \to (N,y_0))$, then

$$\lim_{x\to s_0} \lim_{|\zeta_1|+\ldots+|\zeta_n|\to\infty} \bar{\Phi}^v(\partial F(t)/\partial t)(x;h_1,\ldots,h_n;\zeta_1,\ldots,\zeta_n) \in Y^\xi$$

as a mapping by $h_1,\ldots,h_n$, hence

$$(5)\ TC^0_0(\xi,(M,s_0) \to (N,y_0)) = C^0_0(\xi,(M,s_0) \to (TN,0\times 0)) \times Y^\xi,$$

since $T(N^\xi)$ is isomorphic with $N^\xi \times Y^\xi$ (see Equations 2.9.(1-3)).

Therefore, using the equivalence relation $K_\xi$ from §2.6.1 we get the uniform isomorphism $\tilde{E}_\eta : \bar{V}_\eta \to W_\eta$, where $\bar{V} \subset \Theta(\tilde{V}')$, $\bar{V}_\eta = \bar{V} \cap T_\eta G$,

$$(6)\ \Theta : C^0_0(\xi,(M,s_0) \to (TN,0\times 0)) \to$$

$$[C_0^0(\xi,(M,s_0)\to(TN,0\times 0))\times Y^\xi]/(K_\xi\times I_Y)=TG$$

is the quotient mapping, $I_Y$ is the identity operator in $Y^\xi$; $(f,v)(K_\xi \times I_Y)(g,w)$ if and only if $v_{f_n(\psi_n(x))}=w_{g_n(x)}$ and $f_n(\psi_n(x))=g_n(x)$ for each $x\in M$, where $\psi_n$, $f_n$ and $g_n$ satisfy Condition 2.6.1.(3), $W=\tilde E(\bar V)$, $W_\eta=\tilde E(\bar V_\eta)$, where $K_\xi$ is for $C_0^0(\xi,(M,s_0)\to(TN,0\times 0))$. In view of Equations $(3-6)$ we have a mapping $\tilde E_\eta: T_\eta G\to G$ such that $\tilde E$ is of class $C(\infty)$.

**2.11. Definition.** Let $f(x)$ be in $C((t,s-1),X'\to \mathbf{K})$ (see §2.4.3.b), then an antiderivation $P(l,s)$ is defined by the formula:

$$(1)\ P(l,s)f(x):=\sum\{(\partial^j f(x_m))(x_{m+\bar u}-x_m)^{(j+\bar u)}/(j+\bar u)!:\ m\in \mathbf{N}_o^n,$$
$$j=j'+s'\bar u,\ s'\in\{0,1,...,s-1\},\ \mid j'\mid=0,...,l-1\},$$

where $\partial^j=\partial_1^{j(1)}...\partial_n^{j(n)}$, $j=(j(1),...,j(n))$, $\bar u=(1,...,1)\in\mathbf{N}^n$, $x_m=\sigma_m(x)$,

$$\{\sigma_m:\ m\in \mathbf{No}^n\} \text{ is an approximation of the identity in } X',$$

$X'$ is a clopen subset in $B(\mathbf{K}^n,0,R)$, $\infty>R>0$, $1\le t\in\mathbf{R}$, $l=[t]+1$, $n\in\mathbf{N}$.

**2.12. Lemma.** *Let $f\in C((t,s-1),X'\to\mathbf{K})$, $t=l+b-1$, $0<b<1$ and $\partial^{\bar u}f(x)\in C((t,s-1),X'\to\mathbf{K})$. Suppose in addition that*

$$(1)\ f(x)=f(y)+\sum_{(1\le|j'|<l,\ j=j'+s'\bar u,\ s'\in\{0,1,...,s\})}(\partial^j f(y))(x-y)^j/j!$$
$$+\sum_{(|j'|=l-1,j=j'+s\bar u)}(x-y)^jR(n,j;x,y),$$

*where $R(n,j;x,y)\in C(b,X'\times X'\to\mathbf{K})$ and they are zero on the diagonal $((x,y)\in X'\times X':\ x=y)$. Then for each $z\in X'$:*

$$(2)\ \lim_{x,y\to z;|\zeta_1|+...+|\zeta_n|\to 0}J_j^{k,q}f(x,y;\zeta_1,...,\zeta_{qn})=\partial^{q\bar u}f^{(k)}(z)/(k+qn)!,$$

*for each $k=1,...,l$ and*

$$(3)\ \lim_{x,y\to z}J_j^{l-1,q}f(x,y;\zeta_1,...,\zeta_{qn})(x-y)^{l-1}/j_b(\zeta)=\partial^{q\bar u}(\bar\Phi^t f)(z,...,z)/(qn)!,$$

*where*

$$(4)\ (J_j^{k,q}f(x,y;\zeta_1,...,\zeta_{qn}))(x-y)^k:=(\bar\Phi^{k+qn}f)\ (y;x-y,...,x-y,h_1,...,h_{qn};$$

$$0,...,0,1,...,1,\zeta_1,...,\zeta_{qn})$$

*has j zeros in (...), the function* $j_b(\zeta)$ *was defined in §2.1,* $x = y + \zeta e_i$ *in Formula (3),* $h_1 = h_2 = ... = h_q = e_1$, $h_{q+1} = ... = h_{2q} = e_2, ...$ $h_{q(n-1)+1} = ... = h_{qn} = e_n$, $\zeta_i \in \mathrm{K}$, $q \in \{0,1,...,s\}$, $x, y \in X'$, $y + \zeta_i h_i \in X'$.

**Proof.** By the Taylor formula

$$f(x) = f(y) + \sum_{1 \le |j| \le l-1} (\partial^j f(y))(x-y)^j / j!$$

$$+(\bar{\Phi}^l f)(y; x-y, ..., x-y; 1, ..., 1, 1) - f^{(l)}(y)(x-y)^l / l!.$$

Then using §78.3 by induction by each variable and §78.A [22] and Formulas 2.11.(1), 2.12.(1,4) we get formulas (2, 3).

**2.13. Lemma.** *Let* $f \in C((t, s-1), B \to \mathrm{K})$, $B = B(\mathrm{K}^n, 0, 1)$ *and* $S = B(\mathrm{K}, 0, r)$ *be balls in* $\mathrm{K}^n$ *and* K *respectively,* $t = l + b$, $0 \le b < 1$, $l \in \mathbf{N}$. *Suppose that* $J_j^{k,q} f(x, y; \zeta_1, ..., \zeta_{qn})(x-y)^k \in S$ *for each* $x, y \in B$, $\zeta_1, .., \zeta_{qn} \in B(\mathrm{K}, 0, 1)$ *and for each* $0 < j < n+1$, $k < l+1$. *Then* $(\bar{\Phi}^{k+qn} f)(y; x_1, ..., x_k, h_1, ..., h_{qn}; \zeta_1, ..., \zeta_{k+qn}) \in S$ *for each* $x_i \in B$ *and* $|\zeta_i| \le 1$, *where* $h_i$ *are the same as in §2.12.*

**Proof.** Applying §81.2 [22] by induction by each variable we get the statement of the lemma, since $f^{(k)}(y)(x-y)^k / k! = \bar{\Phi}^k(y; x-y, ..., x-y; 0, ..., 0)$.

**2.14. Lemma.** *Let* $f \in C((t, s-1), X' \to \mathrm{K})$ *and* $\partial^{\mathbf{u}} f \in C((t, s-1), X' \to \mathrm{K})$ *and*

$$(1)\ f(x) = f(y) + \sum_{(1 \le |j'| \le l,\ j = j' + s'\mathbf{u},\ s' \in \{0,1,...,s\})} (\partial^j f(y))(x-y)^j / j!$$

$$+ \sum_{(|v'| = l,\ v = v' + s\mathbf{u})} (x-y)^v \times R(n, v; x, y),$$

*where* $R(n, v; x, y) / j_b(\zeta)$ *are continuous functions zero on the diagonal for* $x - y = \zeta e_i$ *with* $\zeta \in \mathrm{K}$. *Then* $f \in C((t, s), X' \to \mathrm{K})$.

**Proof.** For $s = 1$ by assumption $\partial^{\mathbf{u}} f \in C((t, 0), X' \to \mathrm{K})$, hence $f \in C((t, 1), X' \to \mathrm{K})$. Then by induction applying Lemmas 2.12 and 2.13 we get the statement of this lemma for each $s \in \mathbf{N}$.

**2.15. Theorem.** *Let* $f \in C((t, s-1), X' \to \mathrm{K})$. *Then*

$$(1)\ P(l, s) f(x) - P(l, s) f(y) = \sum_{(j = j' + s'\mathbf{u},\ 0 \le |j'| < l,\ s' \in \{1, ..., s\})} (\partial^{j'} f(y))(x-y)^j / j!$$

$$+\sum_{(v=v'+s\bar{u},\ |v'|=l-1)} (x-y)^v R(n,v;x,y),$$

*where* $R(n,v;x,y)$ *and* $R(n,v;x,y)/j_b(\zeta)$ *(with* $x-y=\zeta e_i$, $\zeta\in\mathbf{K}$ *for* $i=1,...,n$ *in the latter case) are continuous functions equal to zero on the diagonal.*

**Proof.** Applying §78.2 [22] by each variable and using Lemma 2.14 we get $\bar{\partial}^j f\in C((t-|j|,s-1),X'\to\mathbf{K})$. In view of Formula 2.14.(1) there are continuous functions $A(j,v;*)$ together with $A(j,v;x,y)/j_b(\zeta)$ (for $x-y=\zeta e_i$, $i=1,...,n$ in the latter case), such that

$$(2)\ \bar{\partial}^j f(x_m)=\sum_{(|q|=0,...,l-|j|-1)} (\bar{\partial}^{j+q} f(y))(x_m-y)^q/q!$$

$$+\sum_{(|v|=l-|j|-1)} (x_m-y)^v A(j,v;x_m,y) \text{ and}$$

$$(3)\ P(l,s)f(x)=\sum_{(m\in\mathbf{N}_o^n,\ |j'|=0,...,l-1,\ j=j'+s'\bar{u},\ s'\in\{0,1,...,s-1\})} [\sum_{(|q|=0,...,l-|j|-1)} (\bar{\partial}^{j+q} f(y))(x_m-y)^q/q!+\sum_{|v|=l-|j|-1} (x_m-y)^v A(j,v;x_m,y)(x_{m+\bar{u}}-x_m)^{j+\bar{u}}/(j+\bar{u})!]$$

$$=\sum_{(\ |j'|=0,...,l-1,\ j=j'+s'\bar{u},\ s'\in\{0,1,...,s-1\})} \{(\bar{\partial}^j f(y))/(j+\bar{u})![(x-y)^{j+\bar{u}}+(-1)^n\times(x_0-y)^{j+\bar{u}}]$$

$$+\sum_{(m\in\mathbf{N}_o^n,|v|=l-|j|-1)} (x_m-y)^v(x_{m+\bar{u}}-x_m)^{j+\bar{u}}A(j,v;x_m,y)/(j+\bar{u})!\},$$

analogously for $P(l,s)f(y)$. From Formulas (2,3) and 2.14.(1) we get

$$\sum_{(|v'|=l-1,\ v=v'+s\bar{u})} (x-y)^v R(l,v;x,y):$$

$$=\sum_{(m\in\mathbf{N}_o^n,\ |j|=0,...,l-1;\ |v'|=l-|j|-1,\ v=v'+s\bar{u})} [(x_m-y)^v (x_{m+\bar{u}}-x_m)^{j+\bar{u}}A(j,v;x_m,y)-(y_m-y)^v(y_{m+\bar{u}}-y_m)^{j+\bar{u}}A(j,v;y_m,y)]/(j+\bar{u})!.$$

We finish the proof as in Theorem 80.3 [22].

**2.16. Corollary.** *Let* $1\le t\in\mathbf{R}$. *Then each* $f\in C((t,s-1),X'\to\mathbf{K})$ *has a* $C((t,s),X'\to\mathbf{K})$*-antiderivative:*

$$(1)\ \bar{\partial}^{\bar{u}}(P(l,s)f)(x)=f(x) \textit{ for each } x\in X',$$

*moreover, for each $j = (j(1), ..., j(n))$ with $j(i) < 2$ for each $i = 1, ..., n$, and each $x = (x^1, ..., x^n)$ the following equation is fulfilled:*

$$(2)\ \partial^j P(l,s)f(x)\,|_{\{\text{ there is } x^i = x^i_0\}} = 0.$$

**Proof.** This follows immediately from §29.12 [22] and Formulas 2.11.(1), 2.15.(1).

**2.17. Lemma.** *Let $G = \Omega_\xi(M, N)$ be the same monoid as in §2.6. If $At'(\tilde{M})$ has $card(\Lambda'_{\tilde{M}}) \ge 2$, then $G$ is isomorphic with $G_1 = \Omega_\xi(\tilde{M}, N)$, where $\tilde{M} = U'_1 \cup U'_2$ (see §2.5). Moreover, $T_\eta G$ is the Banach space for each $\eta \in G$ and $G$ is ultrametrizable.*

**Proof.** Let $\bar{\chi}_i$ be the characteristic function of $\bar{U}_i$, then $f = \sum_{i\in\Lambda_{\bar{M}}} \bar{\chi}_i f$ for each $f \in C^0_0(\xi, M \to Y)$, where $(\bar{\chi}_i f) \in C^0_0(\xi, \bar{U}_i \to Y)$. The spaces $C^0_0(\xi, \bar{M} \to Y)$ and $\bigoplus_{i\in\Lambda_{\bar{M}}} C^0_0(\xi, \bar{U}_i \to Y)$ are isomorphic, since $\bar{U}_i$ are clopen in $\bar{M}$ (see §12.1 and (12.2.2) [20]). In view of Formulas 2.5.(1-3) each $\bar{U}'_i$ for $At'(\bar{M} \setminus \bar{U}_1)$ is $C(\infty)$-diffeomorphic with $\bar{U}'_j$, when $i$ and $j > 1$. Hence $< f >_{K,\xi}$ is completely defined by the restriction $(f|_{\tilde{M}})$. Therefore, $G$ and $G_1$ are isomorphic, consequently, $TG$ and $TG_1$ are isomorphic.

The space $TG_1$ is isomorphic with $[C^0_0(\xi, (\tilde{M}, s_0) \to (TN, 0 \times 0)) \times Y^\xi]/(K_\xi \times I_Y)$ (see Formula 2.10.(6)). Let $\tilde{\rho}^\xi_0$ be the norm in $C^0_0(\xi, \tilde{M} \to Y \times Y)$, it is also the norm in its complete subset $C^0_0(\xi, (\tilde{M}, s_0) \to (TN, 0\times 0))$, where $TN$ is isomorphic with $N \times Y$ and $N$ is locally K-convex. In view of Theorem 2.7 $G_1$ and hence $TG_1$ are Hausdorff spaces. Then $\rho^\xi_0$ induces an ultrametric

$$(1)\ \rho_{TG}(f,h) := \inf_{(\tilde{f}\in f, \tilde{h}\in h)} \tilde{\rho}^\xi_0(\tilde{f} - \tilde{h}),$$

where $f, h \in TG_1$ and $\tilde{f}, \tilde{h} \in TC^0_0(\xi, (\tilde{M}, s_0) \to (N, 0))$. In view of Formula (1) the ball $B(T_gG, g \times 0, 1/p)$ is K-convex and it is contained in $T_gG$ for each $g \in G$.

The tangent space $T_{\eta_1}G_1$ is complete, Hausdorff and has a K-convex bounded neihgbourhood of 0, consequently, $T_{\eta_1}G_1$ is the Banach space over K for each $\eta_1 \in G_1$, since $At(\tilde{M})$ is finite and $C^0_0(\xi, \tilde{M} \to Y)$ is the Banach space over K (see §(7.2.1) and Exer. 7.119 [20]). Hence $G_1$ is metrizable by an ultrametric $\rho$ together with $G$ such that

$$(2)\ \rho(< f_1 >_{K,\xi}, < f_2 >_{K,\xi}) = \inf_{(g_i\in< f_i >_{K,\xi}, i=1,2)} \|g_1 - g_2\|_{C^0_0(\xi,\tilde{M}\to Y)},$$

since it satisfies Conditions 2.7.(6-8,10).

# 3 Quasi-invariant and pseudo-differentiable measures on loop semigroups.

**3.1. Definition.** Let $G$ denote the Hausdorff totally disconnected group. A function $f : \mathbf{K} \to Y$ is called pseudo-differentiable of order $b$, if there exists the following integral:

$$(1)\ PD(b, f(x)) := \int_{\mathbf{K}} [(f(x) - f(y)) \times g(x, y, b)] v(dy),$$

where $g(x, y, b) := | x - y |^{-1-b}$ for $Y = \mathbf{C}$ and $g(x, y, b) := q^{(-1-b)\times ord_p(x-y)}$ for $Y = \mathbf{\Lambda_q}$ with the corresponding Haar $Y$-valued measure $v$ and $b \in \mathbf{C}$ (see also §2.1). We introduce the following notation $PD_c(b, f(x))$ for such integral by $B(\mathbf{K}, 0, 1)$ instead of the entire $\mathbf{K}$.

**3.2. Remarks. 1.** For a Hausdorff topological space $X'$ with a small inductive dimension $ind(X') = 0$ [8] the Borel $\sigma$-field is denoted $Bf(X')$. Henceforth, measures $\mu$ are given on a measurable space $(X', E)$, where $E$ is a $\sigma$-algebra such that $E \supset Bf(X')$ and $\mu$ has values in $\mathbf{R}$ or in the local field $\mathbf{K_q} \supset \mathbf{Q_q}$. The completion of $Bf(X')$ relative to $\mu$ is denoted by $Af(X', \mu)$. The total variation of $\mu$ with values in $\mathbf{R}$ is denoted by $|\mu|(A)$ for $A \in Af(X', \mu)$. If $\mu$ is non-negative and $\mu(X') = 1$, then it is called a probability measure.

We recall that a mapping $\mu : E \to \mathbf{K_q}$ is called a measure, if the following conditions are accomplished:

(1) $\mu$ is additive and $\mu(\emptyset) = 0$,

(2) for each $A \in E$ there exists the following norm

$\|A\|_\mu := \sup\{|\mu(B)|_{\mathbf{K_q}} : B \subset A, B \in E\} < \infty$,

(3) if there is a shrinking family $F$, that is, for each $A, B \in F$

there exist $F \ni C \subset (A \cap B)$ and $\cap\{A : A \in F\} = \emptyset$, then

$$\lim_{A \in F} \mu(A) = 0$$

(see chapter 7 [21] and also about the completion $Af(X', \mu)$ of the $\sigma$-field $Bf(X')$ by the measure $\mu$). A measure with values in $\mathbf{K_q}$ is called a probability measure if $\|X'\|_\mu = 1$ and $\mu(X') = 1$. For functions $f : X' \to \mathbf{K_q}$ and

$\phi : X' \to [0, \infty)$ we consider the prenorm

$$(4)\ \|f\|_\phi := \sup_{x \in X'} (|f(x)|\phi(x))$$

and define the function

$$(5)\ N_\mu(x) := \inf_{(U \in Bco(X'),\ x \in X')} \|U\|_\mu,$$

where $Bco(X')$ is a field of closed and at the same time open (clopen) subsets in $X'$.

Tight measures (that is, measures defined on an algebra $E$ such that $E \supset Bco(X')$) compose a Banach space $M(X')$ with a norm

$$(6)\ \|\mu\| := \|X'\|_\mu.$$

For $\mathbf{K} \supset \mathbf{Q_p}$ let $\mu$ take the values in $\mathbf{K_q}$, where $q \neq p$, if another is not specified. Everywhere below there are considered $\sigma$-additive measures for which $|\mu|(X') < \infty$ and $\|X'\|_\mu < \infty$ for $\mu$ with values in $\mathbf{R}$ and $\mathbf{K_q}$ respectively, if it is not specified another. Then $L(X', \mu, \mathbf{K_q}) = L(\mu)$ denotes a space of $\mu$-measurable functions $f : X' \to \mathbf{K_q}$ for which

$$(7)\ \|f\|_{L(\mu)} := \|f\|_{N_\mu} < \infty.$$

**3.2.** Let on a completely regular space $X'$ with $ind(X') = 0$ two non-zero real-valued (or $\mathbf{K_q}$-valued) measures $\mu$ and $\nu$ are given. Then $\nu$ is called absolutely continuous relative to $\mu$ if $\nu(A) = 0$ for each $A \in Bf(X')$ with $\mu(A) = 0$ (or there exists $f \in L(\mu)$ such that

$$(8)\ \nu(A) = \int_A f(x)\mu(dx)$$

for each $A \in Bco(X')$, respectively) and it is denoted by $\nu \ll \mu$. Measures $\nu$ and $\mu$ are singular to each other if there is $F \in Bf(X')$ with $|\mu|(X' \setminus F) = 0$ and $|\nu|(F) = 0$ (or $F \in Bco(X')$ for which $\|X' \setminus F\|_\mu = 0$ and $\|F\|_\nu = 0$) and it is denoted by $\nu \perp \mu$.

If $\nu \ll \mu$ and $\mu \ll \nu$ then they are called equivalent, which is denoted by $\nu \sim \mu$.

**3.3. Remark.** Let $G$ be a topological Hausdorff semigroup and $(\mathsf{M}, \mathbf{F})$ be a space $\mathsf{M}$ of measures on $(G, Bf(G))$ with values in either $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{K_q}$.

Let also $G'$ and $G''$ be dense subsemigroups in $G$ such that $G'' \subset G'$ and a topology T on M is compatible with $G'$, that is, $\mu \mapsto \mu_h$ is the homomorphism of (M, F) into itself for each $h \in G'$, where $\mu_h(A) := \mu(h \circ A)$ for each $A \in Bf(G)$. Let T be the topology of convergence for each $E \in Bf(G)$. If $\mu \in (\mathsf{M}, \mathsf{F})$ and $\mu_h \sim \mu$ for each $h \in G'$ then $\mu$ is called quasi-invariant on $G$ relative to $G'$. We shall consider $\mu$ with the continuous quasi-invariance factor

$$(1)\ \rho_\mu(h, g) := \mu_h(dg)/\mu(dg).$$

If $G$ is a group, then we use the traditional definition of $\mu_h$ such that $\mu_h(A) := \mu(h^{-1} \circ A)$.

**3.4. Definition.** Let $S(r, f) = g(r, f)$ be a curve on the subsemigroup $G''$, such that $S(0, f) = f$ and there exists $\partial S(r, f)/\partial r \in TG''$ and $\partial S(r, f)/\partial r|_{r=0} =: A_f \in T_f G''$, where $r \in B(\mathbf{K}, 0, R)$, $\infty > R \geq 1$. Then a measure $\mu$ on $G$ is called pseudo-differentiable of order $b$ relative to $S$ if there exists $PD_c(b, \bar{S}(r, \mu)(B))$ by $r \in B(\mathbf{K}, 0, 1)$ for each $B \in Bf(G)$ (see §3.1), where $\bar{S}(r, \mu)(B) := \mu(S(-r, B))$ for each $B \in Bf(G)$. A measure $\mu$ is called pseudo-differentiable of order $b$ if there exists a dense subsemigroup $G''$ of $G$ such that $\mu$ is pseudo-differentiable of order $b$ for each curve $S(r, f)$ on $G''$ described above, where $b \in \mathbf{C}$.

**3.5. Note.** Now let us describe dense loop submonoids which are necessary for the investigation of quasi-invariant measures on the entire monoid. For finite $At(M)$ and $\xi = (t, s)$ let $C^\theta_{0,\{k\}}(\xi, M \to Y)$ be a subspace of $C^\theta_0(\xi, M \to Y)$ consisting of mappings $f$ for which

$$(1)\ \|f - \theta\|_{C^\theta_{0,\{k\}}(\xi, M \to Y)} := \sup_{i,m,j} |a(m, f^i|_{U_j})|_{\mathbf{K}} J_j(\xi, m) p^{k(i,m)} < \infty \text{ and}$$

$$(2)\ \lim_{i+|m|+Ord(m)\to\infty} \sup_j |a(m, f^i|_{U_j})|_{\mathbf{K}} J_j(\xi, m) p^{k(i,m)} = 0,$$

where $k(i, m) := c' \times i + c \times (|m| + Ord(m))$, $c'$ and $c$ are non-negative constants, $|m| := \sum_i m_i$,

$$Ord(m) := \max\{i : m_i > 0 \text{ and } m_l = 0 \text{ for each } l > i\}$$

(see also Formulas 2.4.2.(2) and 2.4.3.b.(3)).

For finite-dimensional $M$ over $\mathbf{K}$ this space is isomorphic with $C^\theta_{0,\{k'\}}(\xi, M \to Y)$, where $k'(i, m) = c' \times i + c \times |m|$. For finite-dimensional $Y$ over $\mathbf{K}$ the space $C^\theta_{0,\{k\}}(\xi, M \to Y)$ is isomorphic with $C^\theta_{0,\{k''\}}(\xi, M \to Y)$, where

$k"(i,m) = c \times (|m| + Ord(m))$. For $c' = c = 0$ this space coincides with $C_0^\theta(\xi, M \to Y)$ and we omit $\{k\}$.

Then as in §2.6 we define spaces $C^\theta_{0,\{k\}}(\xi,(M,s_0) \to (N,0))$, groups

$$(3)\ G^{\{k\}}(\xi,M) := C^{id}_{0,\{k\}}(\xi, M \to M) \cap Hom(M),$$

$$(4)\ G_0^{\{k\}}(\xi,M) := \{\psi \in G^{\{k\}}(\xi,M) : \psi(s_0) = s_0\}$$

and the equivalence relation $K_{\xi,\{k\}}$ in it for each $M$ and $N$ from §2.4 and §2.5. Therefore,

$$(5)\ G' := \Omega_\xi^{\{k\}}(M,N) =: C^0_{0,\{k\}}(\xi,(M,s_0) \to (N,0))/K_{\xi,\{k\}}$$

is the dense submonoid in $\Omega_\xi(M,N)$.

**3.6. Theorem.** *On the monoid $G = \Omega_\xi(M,N)$ from §2.6 and each $b \in$ $\mathbf{C}$ there exist probability quasi-invariant and pseudo-differentiable of order $b$ measures $\mu$ with values in $\mathbf{R}$ and $\mathbf{K_q}$ for each prime number $q$ such that $q \ne p$ relative to the dense submonoid $G'$ from §3.5 with $c > 0$ and $c' > 0$.*

**Proof.** In view of Lemma 2.17 it is sufficient to consider the case of $M$ with the finite atlas $At'(M)$. Let $(x^1,...,x^m,...) =: x$ be the natural coordinates in $\bar M$, since $\bar M$ is embedded into the Banach space $X$, where $x^j \in \mathbf{K}$ for each $j \in \alpha$. The space $C_0^0(\xi, \bar M \to N)$ is complete, hence it is closed in the complete space $C_0^0(\xi, M \to Y)$, since $0 \in N \subset Y$. From the definition of the topology in $C_0^0(\xi, \bar M \to Y)$ it follows that $C_0^0(\xi, \bar M \to N)$ is the clopen neighbourhood of zero in $C_0^0(\xi, \bar M \to Y)$.

Then there exists a continuous mapping

$$A_a :\ C_0^0(\xi, \bar M_a \to N) \to C_0^0(\xi', \bar M_a \to Y)$$

given by the following formula:

$$(1)\ A_a(F)(x_a) := ((P_a(l,s+1)F^1)(x_a), ..., (P_a(l,s+1)F^k)(x_a), ...),$$

where $\bar M_a := \bar M \cap \mathbf{K}^a$ for each $a \in \mathbf{N}$, $\mathbf{K}^a = sp_{\mathbf{K}}(e_1,...,e_a) \hookrightarrow X$, $\xi' = (t, s+1)$ for $\xi = (t,s)$, $F(x_a) = (F^1(x_a),...,F^k(x_a),...) \in Y$ for each $x_a \in \bar M_a$, $x_a :=$ $(x^1,...,x^a)$, $F \in C_0^0(\xi, \bar M_a \to N)$, $P_a(l,s+1)$ is the antiderivation by $x_a$ defined on the space $C_0^0((t,s), \bar M_a \to \mathbf{K})$ as in §2.11, since $\bar M_a$ is with the finite atlas $At'(\bar M_a)$ consisting of bounded charts.

Let $\tilde{A}_a$ be defined on the tangent spaces to these with the help of the local diffeomorphism , $\omega_{Exp} : V_{f,a} \to U_{f,a}$, where $V_{f,a}$ is a neighbourhood of the zero section in $T_f C^0_0(\xi, \bar{M}_a \to N)$ and $U_{f,a}$ is a neighbourhood of $f$ in $C^0_0(\xi, \bar{M}_a \to N)$, for example, $f = w_0$ (see Formulas 2.8.(1-4)). Then it is continuously strongly differentiable such that

$$(2)\ (D\tilde{A}_a(F)(x_a))(\xi) = \tilde{A}_a(\xi)(x_a),$$

where $F, \xi \in U_{N,a} \subset \tilde{T}C^0_0(\zeta, M_a \to N)$, $U_{N,a}$ is the corresponding neighbourhood of the zero section.

In view of Corollary 2.16 this mapping $A_a$ is injective, hence $\tilde{A}_a$ is injective on $U_{N,a}$. Moreover, the restriction of $A_a$ on $C^0_0(\xi, (\tilde{M}_a, s_0) \to (N, 0))$ has the image in $C^0_0(\xi', (M_a, s_0) \to (Y, 0))$. For $M_a = M$ let $\tilde{A} = \tilde{A}_a$, for $M$ modelled on $c_0(\omega_0, \mathbf{K})$ let

$$(3)\ \tilde{A} = \sum_{a \in \mathbf{N}} \kappa_a \{\tilde{A}_a(f|_{M_a}) - \tilde{A}_{a-1}(f|_{M_{a-1}})\} w_a, \text{ consequently}$$

$$(4)\ \tilde{A}:\ T_0 C^0_0(\xi, (M, s_0) \to (N, 0)) \to \tilde{Z}$$

is injective and continuous for suitable $\kappa_a \in \mathbf{K}$ with $p^{-1} \le |\kappa_a| \times \|\tilde{A}_a\| \le 1$, where

$$(5)\ \tilde{Z} := c_0(\{H_a : a \in \mathbf{N}\})$$

is the following Banach space with elements $z = (z^a : z^a \in H_a, a \in \mathbf{N})$ having the norm

$$(6)\ \|z\| := \sup_a \|z^a\|_{H_a} < \infty \text{ and}$$

$$(7)\ \lim_a \|z^a\|_{H_a} = 0,$$

$$(8)\ \|L\| := \sup_{x \ne 0} \|Lx\|_{Y'} / \|x\|_{X'}$$

for a bounded linear operator $L : X' \to Y'$ and Banach spaces $X'$ and $Y'$ over $\mathbf{K}$, $w_a \in \theta_a(H_a)$, $\theta_a : H_a \hookrightarrow c_0(\{H_a : a \in \mathbf{N}\})$ are the natural embeddings, $\|w_a\|_{H_a} = 1$ for each $a \in \mathbf{N}$; $\tilde{A}_0 := 0$. We choose $H_a = T_0 C^0_0(\xi', M_a \to Y)$. In view of the definition of the space $C^0_0(\xi, M \to Y)$ this mapping $\tilde{A}$ is the isomorphism of $T_0 C^0_0(\xi, (M, s_0) \to (N, 0))$ onto the Banach subspace of $\tilde{Z}$. Hence $\tilde{A}$ is defined on a neighbourhood of the zero section in $TC^0_0(\xi, (M, s_0) \to (N, 0))$ into a neighbourhood of the zero section either in $TC^0_0(\xi', (M, s_0) \to (Y, 0))$ for finite-dimensional $M$ over $\mathbf{K}$, or into

$c_0(\{TC^0_0(\xi',(M_a,s_0) \to (Y,0)) : a \in \mathbf{N}\}) =: \bar{Z}$ for $dim_{\mathbf{K}}M = \aleph_0$, where charts in the manifold $\bar{Z}$ are induced by $c_0(\{T_{(f|_{M_a})}C^0_0(\xi',(M_a,s_0) \to (Y,0)) : a \in \mathbf{N}\})$ for $f \in C^0_0(\xi',(M,s_0) \to (Y,0))$,

$$(9)\ c_0(\{TC^0_0(\xi',(M_a,s_0) \to (Y,0)) :\ a \in \mathbf{N}\}) =$$

$$\bigcup_{\{f \in C^0_0(\xi',(M,s_0)\to(Y,0))\}} c_0(\{T_{(f|_{M_a})}C^0_0(\xi',(M_a,s_0) \to (Y,0)) :\ a \in \mathbf{N}\}) =$$

$$\bigcup_{\{f \in C^0_0(\xi',(M,s_0)\to(Y,0))\}} T_f c_0(\{C^0_0(\xi',(M_a,s_0) \to (Y,0)) :\ a \in \mathbf{N}\}),$$

$g \in T_f c_0(\{C^0_0(\xi',(M_a,s_0) \to (Y,0)) :\ a \in \mathbf{N}\})$ means by the definition that $\pi_2 \circ g \in c_0(\{C^0_0(\xi',M_a \to Y) : a \in \mathbf{N}\})$ and $\pi_1(g) = f$, $\pi_i : Y_1 \times Y_2 \to Y_i$ are projections, $Y_1 = Y_2 = Y$, $i \in \{1,2\}$.

Let the equivalence relation $\tilde{K}_\xi$ acts in $TC^0_0(\xi,(M,s_0 \to (N,0))$ as $K_\xi \times I_Y$ in Formula 2.10.(6). Therefore, $\tilde{A}$ is continuous and injective and from $f \circ \psi = g$ it follows $\tilde{A}(f \circ \psi) = \tilde{A}(g)$ and $\tilde{A} < f >_{\tilde{K},\xi}$ is a closed subset in the corresponding manifold $TC^0_0(\xi',(M,s_0) \to (Y,0))$ or $\bar{Z}$ (see Theorem (14.4.5) and Exer. 14.110 in [20]). Then the factorization by the equivalence relation $\tilde{K}_\xi$ is correct due to Definition 2.11 and Corollary 2.16, which produces the mapping $\tilde{\Upsilon}$ from the corresponding neighbourhood of the zero section in $T\Omega_\xi(M,N)$ into a neighbourhood of the zero section in $T\Omega_{\xi'}(M,Y)$ for $dim_{\mathbf{K}}M < \infty$ or into $c_0(\{T\Omega_{\xi'}(M_a,Y) : a \in \mathbf{N}\})$ for $dim_{\mathbf{K}}M = \aleph_0$.

Therefore they are continuously strongly differentiable with $(D\tilde{\Upsilon}(f))(v) = \tilde{\Upsilon}(f)(v)$, where $f$ and $v \in V_N \subset T_e\Omega_\xi(M,N)$, $V_N$ is the corresponding neighbourhoods of zero sections for the unit element $e =< \omega_0 >_{K,\zeta}$. In view of the existence of the mapping $\tilde{E}$ (see Theorem 2.10 and Formula 2.10.(3)) and Formulas (1-9) for $\tilde{T}G$ there exists the local diffeomorphism

$$(10)\ \Upsilon : W_e \to V'_0$$

induced by $\tilde{E}$ and $\tilde{\Upsilon}$, where $W_e$ is a neighbourhood of $e$ in $G$, $V'_0$ is a K-convex neighbourhood of zero either in the Banach subspace $\tilde{H}$ of $T_e\Omega_{\xi'}(M,Y)$ for $dim_{\mathbf{K}}M < \infty$ or in the Banach subspace $\tilde{H}$ of $c_0(\{T_e\Omega_{\xi'}(M_a,Y) : a \in \mathbf{N}\})$ for $dim_{\mathbf{K}}M = \aleph_0$. In view of Formulas 2.6.2.(5-7) there exists the homomorphism

$$T\chi^* : TC^0_0(\xi,(M \vee M,s_0)) \to (N,0)) \to TC^0_0(\xi,(M,s_0)) \to (N,0))$$

such that $\chi^*$ is in the class of smoothness $C(\infty)$. Therefore, there is the linear mapping (differential)

$$(11)\ D\chi^*(h) : T_hC^0_0(\xi, (M \vee M, s_0)) \to (N, 0)) \to \mathbf{F}$$

for each $h \in C^0_0(\xi, (M \vee M, s_0)) \to (N, 0))$, where $\mathbf{F}$ is the Banach space such that $T_zC^0_0(\xi, (M, s_0)) \to (N, 0)) = \{z\} \times \mathbf{F}$ for each $z \in C^0_0(\xi, (M \vee M, s_0)) \to (N, 0))$, in particular for $z = \chi^*(h)$.

Let now $W'_e$ be a neighbourhood of $e$ in $G'$ such that $W'_eW_e = W_e$. It is possible, since the topology in $G$ and $G'$ is given by the corresponding ultrametrics (see Formulas 2.7.(6-8,10) and Lemma 2.17) and there exists $W_e$ with $W_eW_e = W_e$, hence it is sufficient to take $W'_e \subset W_e$. For $g \in W_e$, $v = \tilde{E}^{-1}(g)$, $\phi \in W'_e$ the following operator

$$(12)\ S_\phi(v) := \Upsilon \circ L_\phi \circ \Upsilon^{-1}(v) - v$$

is defined for each $(\phi, v) \in W'_e \times V'_0$, where $L_\phi(g) := \phi \circ g$. Then $S_\phi(v) \in V"_0 \subset V'_0$, where $V"_0$ is an open neighbourhood of the zero section either in the Banach subspace of $T_eG'$ for $dim_{\mathbf{K}}M < \infty$ or in the Banach subspace of $c_0(\{T_eG'_a : a \in \mathbf{N}\})$ for $dim_{\mathbf{K}}M = \aleph_0$, where $G'_a = \Omega^{\{k\}}_\xi(M_a, N)$. Moreover, $S_\phi(v)$ is the $C(\infty)$-mapping by $\phi$ and $v$, since $\Upsilon$ and $L_\phi$ are $C(\infty)$-mappings.

In view of Theorems 5.13 and 5.16 [21] the Banach space $H$ is isomorphic with $c_0(\omega_0, \mathbf{K})$. The Borel $\sigma$-algebras of $c_0(\omega_0, \mathbf{K})$ relative to the norm and weak topologies coincide. Suppose that there exists a sequence of finite-dimensional over $\mathbf{K}$ distributions $\nu_{L_n}$ on $c_0(\omega_0, \mathbf{K})$, which means by the definition, that $L_n := sp_{\mathbf{K}}(e_1, ..., e_n)$ is a sequence of finite-dimensional over $\mathbf{K}$ subspaces such that $L_n \subset L_m$ for each $n \le m$, $\bigcup_n L_n$ is dense in $c_0$, $\nu_{L_n}$ is a family of measures on $Bf(L_n)$ all with values in one chosen field among either $\mathbf{R}$ or $\mathbf{K}_q$ satisfying the following condition

$$(13)\ \nu_{L_m}((\pi_n^{-1}(A)) \cap L_m) = \nu_{L_n}(A)$$

for each $A \in Bf(L_n)$ and each $n \le m$, where $\pi_n : c_0 \to L_n$ are projections such that $\pi_n(x) = x_n$, $x = \sum_{j=1}^\infty x^je_j \in c_0$, $x_n = \sum_{j=1}^n x^je_j$ and $x^j \in \mathbf{K}$. When the sequence of finite-dimensional over $\mathbf{K}$ distributions

$$(14)\ \nu_{L_n}(dx_n) = \otimes_{j=1}^n \nu_j(dx^j)$$

generates a measure $\nu$ on $c_0$ we write

$$(15)\ \nu(dx) := \otimes_{j=1}^\infty \nu_j(dx^j),$$

where $\nu_j(dx^j)$ are measures on $\mathbf{K}e_j$. There exist the following $\sigma$-additive measures $\nu$ with values in $[0,\infty)$ and $\mathbf{K_q}$:

$$(16)\ \nu(dx) = \bigotimes_{j=1}^{\infty} \nu_{l(j)}(dx^j),$$

where $\nu_j(\mathbf{K}) = 1$ for each $j \in \mathbf{N}$,

$$(17)\ \nu_j(dx^j) = f_j(x^j)w(dx^j),$$

$w$ is the $\sigma$-finite Haar measure on $\mathbf{K}$ with values in $[0,\infty]$ or $\mathbf{K_q}$ with $w(B(\mathbf{K},0,1)) = 1$, $f_j \in L^1(\mathbf{K}, w, \mathbf{R})$ for real-valued $w$ and $f_j \in L(\mathbf{K}, w, \mathbf{K_q})$ for $\mathbf{K_q}$-valued $w$ (see §3.2). It is possible to take, for example,

$$f_j(x^j)|_{S(j,n)} = a(j,n),$$

where

$$S(j,n) := B(\mathbf{K},0,p^{-j}) \setminus B(\mathbf{K},0,p^{-j-1})$$

for $j \in \mathbf{Z}$ with $j < n$,

$$S(n,n) := B(\mathbf{K},0,p^{-n}),$$

$$a(j,n) = r^{n(j-n)}(1 - r^{-n})(1 - 1/p)p^{-n}$$

for $j < n$ and

$$a(n,n) = (1 - r^{-n})p^{-n}$$

with $1 < r$ for the real-valued case;

$$a(j,n) = (1 - q)(1 - 1/p)q^{2n-1-j}p^{-n}$$

for $j < n$ and

$$a(n,n) = (1 - q^n)p^{-n}$$

for the $\mathbf{K_q}$-valued case.

Let

$$(18)\ l(j) \le l(j+1)$$

for each $j \in \mathbf{N}$,

$$(19)\ \lim_{j\to\infty} l(j) = \infty \text{ and}$$

$$(20)\ \lim_{j\to\infty} p^{(l(j)-k(i_j,m_j))} = 0$$

(these limits are taken relative to the usual metric in $\mathbf{R}$), where $\tilde{\psi} : \mathbf{N} \to \mathsf{L}$ is a bijection,

$$\mathsf{L} := \{(i, m) : \text{indices corresponding to different classes } < \bar{Q}_m q_i >_{K,(\xi,\{k\})} \text{ with } |m| > 0\}$$

such that to one class there corresponds one index, $\tilde{\psi}(j) =: (i_j, m_j)$, $\bar{Q}_m$ are considered on $\tilde{M}$ (see Formulas 2.4.2.(3-5), 3.5.(1-4) and Lemma 2.17). We can take $\tilde{M} \subset U_1$ (see Formulas 2.5.(2,3)). When either $i \neq i'$ or $|m| \neq |m'|$ then $< \bar{Q}_m q_i >_{K,(\xi,\{k\})} \neq < \bar{Q}_{m'} q_{i'} >_{K,(\xi,\{k\})}$, since $\bar{Q}_m$ are completely defined by its values on $\tilde{M}_{Ord(m)}$ and have $|m|$ pairwise distinct zeros, where $\tilde{M}_n := \tilde{M} \cap sp_{\mathbf{K}}(e_1, ..., e_n)$.

In view of Prohorov Theorem §IX.4.2 [4] for real measures or its analog for measures with values in $\mathbf{K_q}$ 7.6(ii) [21] $\nu$ has the countably-additive extension on $Bf(\tilde{H})$. The restriction of $\nu$ on $Bf(V_0')$ is non-trivial.

Then $S_\phi$ are compact operators [23] for each $\phi \in W_e'$. Let

$$(21)\ U_\phi = I + S_\phi \text{ and } \nu_\phi(J) := \nu(U_\phi^{-1}(J))$$

for each $J \in Bf(V_0')$. In view of Formulas (10-20) there exists the quasi-invariance factor

$$(22)\ \nu_\phi(dx)/\nu(dx) = |det\ U'(U^{-1}(x))|_{\mathbf{K}} \rho_\nu(x - U^{-1}(x), x),$$

where $U = U_\phi$ for a given $\phi \in W_e'$, $U'(y) = dU(y)/dy$,

$$\rho_\nu(z, x) := \nu^z(dx)/\nu(dx),\ \nu^z(J) := \nu(J - z),\ z = x - U^{-1}(x),$$

$$\text{either } \nu_\phi(dx)/\nu(dx) \in L^1(V_0', \nu, \mathbf{R}) \text{ or } \nu_\phi(dx)/\nu(dx) \in L(V_0', \nu, \mathbf{K_q})$$

in the corresponding cases, since $V_0'$ is $\mathbf{K}$-convex and $U_\phi(V_0') \subset V_0'$ for each $\phi \in W_e'$. This $\nu$ on $Bf(V_0')$ is also pseudo-differentable realtive to $W_e'$. Moreover, $\nu_\phi(dx)/\nu(dx)$ is continuous by $(\phi, x) \in W_e' \times V_0'$. If $f : B(\mathbf{K}, 0, 1) \to G'$ is a continuous mapping, then $f(B(\mathbf{K}, 0, 1))$ is a compact subset in $G'$, hence for each neighbourhood $W"$ of $e$ in $G'$ there are $h_1, ..., h_k \in G'$ and $k \in \mathbf{N}$ such that $f(B(\mathbf{K}, 0, 1)) \subset \bigcup_{j=1}^k h_j W"$. This measure is pseudo-differentiable of order $b$, since $V_0'$ is bounded in $\tilde{H}$ and there exists a neighbourhood $\tilde{W}_e'$ of $e$ in $G'$ and local coordinates in $\tilde{W}_e'$ such that $\nu_\phi(dx)/\nu(dx)$ depends on finite number of local coordinates.

More general classes of quasi-invariant and pseudo-differentiable of order $b$ measures $\nu$ with values in $[0,\infty)$ or in $\mathbf{K_q}$ exist in view of Theorems **3.23**, **3.28** and **4.3** [14] on $V'_0$ relative to the action of $\phi \in W'_e$ such that $(\phi, v) \mapsto v + S_\phi(v)$, where $v \in V'_0$.

This measure induces a measure $\tilde{\mu}$ on $W_e$ with the help of $\Upsilon$ such that

$$(23)\ \tilde{\mu}(A) = \nu(\Upsilon(A)) \text{ for each } A \in Bf(W_e),$$

since $\|\nu\|(V'_0) > 0$. The monoids $G$ and $G'$ are separable and metrizable, hence there are locally finite coverings $\{\phi_i \circ W_i : i \in \mathbf{N}\}$ and $\{\phi_i \circ W'_i : i \in \mathbf{N}\}$ of $G$ and $G'$ with $\phi_i \in G'$ such that $W_i$ are open subsets in $W_e$, $W'_i$ are open subsets in $W'_e$, $\phi_1 = e$, $W_1 = W_e$ and $W'_1 = W'_e$ [8], that is,

$$\bigcup_{i\in\mathbf{N}} \phi_i \circ W_i = G \text{ and } \bigcup_{i=1}^{\infty} \phi_i \circ W'_i = G'.$$

Then $\tilde{\mu}$ can be extended onto $G$ by the following formula

$$(24)\ \mu(A) := (\sum_{i=1}^{\infty} \tilde{\mu}((\phi_i^{-1} \circ A) \cap W_i) r^i)/(\sum_{i=1}^{\infty} \tilde{\mu}(W_i) r^i)$$

for each $A \in Bf(G)$, where $0 < r < 1$ for real $\tilde{\mu}$ or $r = q$ for $\tilde{\mu}$ with values in $\mathbf{K_q}$. This $\mu$ is the desired measure, which is quasi-invariant and pseudo-differentiable of order $b$ relative to the submonoid $G" = G'$ (see §§**3.3**, **3.4**).

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