ANNALES MATHÉMATIQUES BLAISE PASCAL

ANN VERDOODT

A method for constructing orthonormal bases for non-archimedean Banach spaces of continuous functions

Annales mathématiques Blaise Pascal, tome 7, nº 1 (2000), p. 87-99
<http://www.numdam.org/item?id=AMBP_2000__7_1_87_0>

Ann. Math. Blaise Pascal, Vol. 7, N° 1, 2000, pp 87-99

A Method for Constructing Orthonormal Bases for Non-Archimedean Banach Spaces of Continuous Functions

Ann Verdoodt

Klein-Antwerpenstraat 64
9280 Lebbeke, Belgium

Abstract

Let K be a local field, let M be a regular compact in K with diameter 1 and let $C(M \to K)$ be the Banach space of continuous functions from M to K. Our aim is to find orthonormal bases for $C(M \to K)$ by taking powers, products and infinite linear combinations of functions in $C(M \to K)$. To construct the orthonormal bases, we use very well distributed sequences in M.

1. Introduction

In this paper we construct orthonormal bases for non-Archimedean Banach spaces of continuous functions. The method used here generalises the results from reference [9]. For the convenience of the reader, we start by recalling some definitions and some previous results. All these results can be found in [1], chapter 1, sections 1 and 2, and chapter 2, sections 5 and 6. For additional information we refer the reader to [1]. We remark that the notations used in this section are sometimes different from the notations used in [1]. The notations used in this section can also be found in [3]. Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}_0$ is the set of natural numbers without zero.

Definition 1.1 A *countable projective system of finite sets* $(M_i, \varphi_{i,j})_{i \leq j \in \mathbb{N}}$ consists of

1) a sequence (M_n) of finite sets

2) mappings $\varphi_{k,n}$ of M_n in M_k defined for $k \leq n$ and such that for $k \leq n \leq m$ we have

$$\varphi_{k,m} = \varphi_{k,n} \circ \varphi_{n,m}$$

and $\varphi_{n,n}$ is the identity mapping on M_n.

We denote by $M = \varprojlim M_k$ the projective limit of this system. M is compact and ultrametric, since it is the projective limit of the sequence of sets M_n equipped with the discrete topology. $M = \{(x_i) \in \prod_{i \in \mathbb{N}} M_i | \forall i \leq j \in \mathbb{N} : x_i = \varphi_{i,j}(x_j)\}$. On $M \times M$ we define the function $v(x,y)$ as follows : $v(x,y) = \sup\{i \in \mathbb{N} | x_k = y_k \ \forall k \leq i\}$ if $x \neq y$; $v(x,x) = +\infty$. v satisfies the ultrametric inequality $v(x,z) \geq inf\{v(x,y), v(y,z)\}$ and this for all $x, y, z \in M$. The function $d(x,y) = \alpha^{v(x,y)}$ ($\alpha \in (0,1)$, α fixed) is an ultrametric on M, which induces the projective limit topology.

Definition 1.2

A countable projective system of finite sets $(M_i, \varphi_{i,j})_{i \leq j \in \mathbb{N}}$ is called a *regular projective system* if it satisfies

1) $\# M_0 = 1$,
2) for all $i \leq j \in \mathbb{N} : \varphi_{i,j}$ is surjective,
3) there exist $q_1, q_2, \ldots \in \mathbb{N} \setminus \{0,1\}$ such that for all $n \in \mathbb{N}_0$, for all $\omega \in M_{n-1}$: $\#\{\varphi_{n-1,n}^{-1}(\omega)\} = q_n$.

In this case we put $N_0 = \# M_0 = 1$, $N_n = \# M_n = q_1 q_2 \cdots q_n$.

From 3), definition 1.2, it immediately follows that, for $M = \varprojlim M_k$,

Each closed ball in M with radius α^n is the finite disjoint union of q_{n+1} closed balls with radius α^{n+1}.(*)

Let us now look at something more general.

Let M be a set with an integer valued function v defined on $M \times M$ which satisfies for all $x, y, z \in M$

$$v(x,y) = v(y,x)$$

$$v(x,y) = +\infty \Leftrightarrow x = y$$

$$v(x,z) \geq inf\{v(x,y), v(y,z)\}.$$

We say that M is valued by v. The function $d(x,y) = \alpha^{v(x,y)}$ ($\alpha \in (0,1)$, α fixed) is an ultrametric on M. If M is compact, then each closed ball with radius α^n is a finite disjoint union of balls with radius α^{n+1}. M is then called a valued compact.

Let M be a valued compact such that the closed balls satisfy (*). Let $B_b(r)$ denote the 'closed' ball with center b and radius r. We introduce

- the equivalence relation π_k on M, defined for $k \geq 0$ by

$$x \pi_k y \iff B_x(\alpha^k) = B_y(\alpha^k)$$

- *the quotient* M_k *of* M *by* π_k, *and the canonical projection* pr_k *of* M *on* M_k,

- *the mapping* $\varphi_{k,n}$ *of* M_n *on* M_k *defined for* $k < n$ *by*

$$\varphi_{k,n}(\omega) = pr_k(pr_n^{-1}(\omega)) \quad for \quad \omega \in M_n$$

and $\varphi_{n,n}$ is the identity mapping on M_n.

The system $(M_i, \varphi_{i,j})_{i \leq j \in \mathbb{N}}$ is then a regular projective system, and its projective limit is isomorphic to M. If the closed balls of a valued compact M satisfy condition (*), then we call M a *regular valued compact.* Let us consider the following examples :

Example 1.1 Let $A_1, \ldots, A_n, \ldots$ be non-empty finite sets and let $q_n \geq 2$ be the the cardinality of A_n. The products $M_n = A_1 \times \ldots \times A_n$, ($M_0$ consisting of one element), equipped with the canonical projections from M_n on M_k defined by $(x_1, \ldots, x_n) \to (x_1, \ldots, x_k)$, $k \leq n$, $x_i \in A_i$, form a regular projective system.
Conversely, every regular projective system is isomorphic to such a system.

Example 1.2 Let p be a prime number. $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is a regular valued compact, with $q_n = p$, $N_n = p^n$.

Example 1.3 Let p be an odd prime number. The unit circle $\{x \in \mathbb{Z}_p \| x| = 1\}$, where $|.|$ denotes the p-adic valuation, is also a regular valued compact, with $q_1 = p - 1$, $q_n = p$ if $n \geq 2$. To see this put in example 1.1 $A_1 = \{1, 2, \ldots, p-1\}$, $A_n = \{0, 1, \ldots, p-1\}$ if $n \geq 2$.

Now we give the definition of very well distributed sequences. Therefore, let A be a finite set, $\#A = N$, and $M = \varprojlim M_k$ a regular valued compact.

Definition 1.3

A sequence $u : \mathbb{N} \to A$ is *well distributed* if for all $n \in \mathbb{N}_0$, for all $a \in A$: $\#\{i < nN | u_i = a\} = n$.
A sequence $u : \mathbb{N} \to M$ is called *very well distributed* if for all $n \in \mathbb{N}$: $\varphi_n \circ u : \mathbb{N} \to M_n$ is well distributed, where $\varphi_n : M \to M_n$ is the canonical projection.

A very well distributed sequence u in M is always injective, and lays dense in M.

Example 1.4 If p is a prime number, then $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is a regular valued compact (example 1.2). The sequence (u_i), $u_i = i$ for all i, is a very well distributed sequence in $\mathbb{Z}_p$.

For more details concerning regular valued compacts and very well distributed sequences, we refer the reader to [1].

Let K be a local field (i.e. a locally compact, non-trivially, non-Archimedean valued field) with valuation $|.|$ and logarithmic valuation v and let k be the finite residue class field of K. We have $|x| = \alpha^{v(x)}$, where $\alpha = |\pi|$, $|\pi|$ the generator of the value group of K. K is an ultrametric space by putting $v(x,y) = v(x-y)$, $d(x,y) = |x-y|$. The closed unit ball of K is a regular valued compact.

Definition 1.4

A compact part M of the closed unit ball satisfying (*) is called a *regular compact* of K.

$C(M \to K)$ denotes the non-Archimedean Banach space of continuous functions from M to K, equipped with the supremum norm $\|\cdot\|_\infty$: $\|f\|_\infty = \sup\{|f(x)| \mid x \in M\}$. An orthonormal basis for $C(M \to K)$ is defined as follows :

Definition 1.5

A sequence $e_0, e_1, e_2, \ldots$ of elements of $C(M \to K)$ is called an *orthonormal basis* for $C(M \to K)$ if every element f of $C(M \to K)$ has a unique representation $f = \sum_{i=0}^{+\infty} x_i e_i$ where $x_i \in K$ and $|x_i| \to 0$ if $i \to \infty$, and if $\|f\|_\infty = \max_{0 \leq i}\{|x_i|\}$.

The aim of this paper is to construct orthonormal bases for $C(M \to K)$, by taking powers, products and infinite linear combinations of functions in $C(M \to K)$. To construct the orthonormal bases, we use very well distributed sequences in M. In section 2 we prove some preliminary lemmas, and in section 3 we prove the main theorem of this paper. We also give some examples and in particular we obtain some well-known orthonormal bases.

2. Preliminary Lemmas

Let K be a local field with valuation $|.|$ and finite residue class field k and let M be a regular compact in K. We will assume that the diameter of M equals 1. We put $\alpha = |\pi|$, π the generator of the value group of K. Throughout sections 2 and 3, (u_n) denotes an arbitrary but fixed very well distributed sequence in M. First we introduce some functions on M which are going to play an important role in this paper.

We define sequences (q_n) and (ϕ_n) on M as follows :

$$q_0(x) = \phi_0(x) = 1,$$

$$q_n(x) = \frac{(x-u_0)\ldots(x-u_{n-1})}{(u_n-u_0)\ldots(u_n-u_{n-1})} \quad n \geq 1,$$

$$\phi_n(x) = 1 \;\; if \;\; x \in B_{u_n}(r_n), \;\; where \;\; r_n = \alpha^{i+1} \;\; if \;\; N_i \leq n < N_{i+1},$$

$$otherwise \;\; \phi_n(x) = 0 \;\; n \geq 1.$$

The functions q_n and ϕ_n are clearly continuous. The functions q_n where introduced by Amice (see [1], sections 2.4 and 6). The sequence (q_n) forms an orthonormal basis for $C(M \to K)$ ([1], section 6.2). It is clear that $||\phi_n||_\infty = 1$ for all n. Since (q_n) forms an orthonormal basis we have that $||q_n||_\infty = 1$ for all n. It is easy to see that for all n

$$q_n(u_n) = \phi_n(u_n) = 1,$$

$$q_n(u_j) = \phi_n(u_j) = 0 \quad if \quad j < n.$$

We only have to prove $\phi_n(u_j) = 0$ if $j < n$. To see this, suppose that $N_i \leq n < N_{i+1}$. Then ϕ_n is the characteristic function of the ball $B_{u_n}(r_n)$ where $r_n = \alpha^{i+1}$. There are N_{i+1} disjoint balls with radius α^{i+1}, namely the balls with centers $u_0, u_1, \ldots u_j, \ldots, u_n, \ldots, u_{N_{i+1}-1}$. So $|u_j - u_n| > \alpha^{i+1}$ and we conclude that $\phi_n(u_j) = 0$ if $j < n$.

For the functions q_n and ϕ_n we can prove the following (see [9], lemmas 1 and 2)

Lemma 2.1

1) *If* $x, y \in M$, $|x - y| \leq \alpha^t$ *then* $|q_n(x) - q_n(y)| \leq \alpha$ *if* $0 \leq n < N_t$.

2) *If* $x, y \in M$, $|x - y| \leq \alpha^t$ *then* $\phi_n(x) = \phi_n(y)$ *if* $0 \leq n < N_t$.

Proof

1) This can be found in [1] (p. 135, lemma 4).
2) Let $x, y \in M$ such that $|x - y| \leq \alpha^t$ and let $0 \leq n < N_t$. Then $|x - y| \leq \alpha^t \leq r_n$. So the elements x and y either belong both to $B_{u_n}(r_n)$ and then $\phi_n(x) = \phi_n(y) = 1$, or none of them is in $B_{u_n}(r_n)$ and then $\phi_n(x) = \phi_n(y) = 0$. □

Let (ψ_i) be a sequence of functions where ψ_i can be equal to the function q_i or to the function ϕ_i (this can be different for every index i). For instance, $(q_0, \phi_1, q_2, \phi_3, \ldots, q_{2i}, \phi_{2i+1}, \ldots)$ and $(q_0, q_1, \phi_2, \ldots, q_{3i}, q_{3i+1}, \phi_{3i+2}, \ldots)$ are sequences of this type. We then have, for all n,

$$||\psi_n||_\infty = 1, \quad \psi_n(u_n) = 1, \quad \psi_n(u_k) = 0 \quad if \quad k < n.$$

Let us consider functions of the following form :

$$\sum_{i=0}^{\infty} a_i \psi_i, \quad a_i \in K \quad for \quad all \quad i, \quad |a_i| \to 0 \quad if \quad i \to \infty.$$

It is clear that these functions are continuous since $a_i \psi_i$ tends to zero uniformly.

For functions of this type we can prove the following lemmas :

Lemma 2.2

Let n_0 be a natural number and let f be the function

$$f = \sum_{i=0}^{\infty} a_i \psi_i, \quad a_i \in K \;\; for \;\; all \;\; i, \;\; |a_i| \to 0 \;\; if \;\; i \to \infty,$$

where the coefficients a_i satisfy

$$|a_{n_0}| = 1 \;\; and \;\; |a_k| < 1 \;\; for \;\; k \neq n_0.$$

Then we have that $|f(u_{n_0})| = 1$ and $|f(u_k)| < 1$ if $0 \leq k < n_0$. Furthermore, $||f||_\infty = 1$.

Proof

We clearly have $|a_i \psi_i(u_{n_0})| < 1$ if $i \neq n_0$, $|a_{n_0} \psi_{n_0}(u_{n_0})| = 1$ and so we have $|f(u_{n_0})| = \max_{i \geq 0}\{|a_i \psi_i(u_{n_0})|\} = 1$. In an analogous way we have, for $k < n_0$, $|a_i \psi_i(u_k)| < 1$ if $i \neq n_0$, $|a_{n_0} \psi_{n_0}(u_k)| = 0$ and so $|f(u_k)| \leq \max_{i \geq 0}\{|a_i \psi_i(u_k)|\} < 1$. Since $||\psi_i||_\infty = 1$ for all i it now immediately follows that $||f||_\infty = 1$. □

Lemma 2.3

Let the function f be as defined in lemma 2.2.

Let $x, y \in M$ such that $|x - y| \leq \alpha^t$. Assume $0 \leq n_0 < N_t$. Then for every $j \in \mathbb{N}$, we have

$$|f(x)^j - f(y)^j| \leq \alpha.$$

Proof

For j=1 we have $|f(x) - f(y)| \leq \max_{i \geq 0}\{|a_i||\psi_i(x) - \psi_i(y)|\} \leq \alpha$ by lemma 2.1 and the fact that $|a_i| \leq \alpha$ for $i \neq n_0$. The case $j = 0$ is trivial and for $j > 1$ we find

$$|f(x)^j - f(y)^j| = |f(x) - f(y)|.|\sum_{s=0}^{j-1} f(x)^s f(y)^{j-1-s}| \leq \alpha.$$

□

In lemmas 2.4 and 2.5, we introduce a function g which is an infinite linear combination of powers of functions of the previous type.

Lemma 2.4

Let (f_n) be a sequence of continuous functions, where for every n, f_n is of the form

$$f_n = \sum_{i=0}^{\infty} a_{n,i}\psi_i, \quad a_{n,i} \in K, \quad |a_{n,i}| \to 0 \quad if \quad i \to \infty,$$

$$with \quad |a_{n,n}| = 1 \quad and \quad |a_{n,i}| < 1 \quad if \quad i \neq n.$$

Let n_0 be a natural number and let g be a continuous function of the form

$$g = \sum_{i=0}^{\infty} c_i (f_i)^{m_i}, \quad c_i \in K, \quad |c_i| \to 0 \quad if \quad i \to \infty,$$

$$with \quad |c_{n_0}| = 1, \quad |c_i| < 1 \quad if \quad i \neq n_0 \quad and \quad m_i \in \mathbb{N}_0 \quad for \quad all \quad i.$$

Then $|g(u_{n_0})| = 1$, $|g(u_j)| < 1$ if $0 \leq j < n_0$. Furthermore, $||g||_\infty = 1$.

Proof

We remark that g is continuous. From lemma 2.2 it follows that, for all n, $|f_n(u_n)| = 1$, $||f_n||_\infty = 1$ and since $|c_i| \leq \alpha$ for $i \neq n_0$ we have, for $i \neq n_0$, $|c_i f_i(u_{n_0})^{m_i}| < |c_{n_0} f_{n_0}(u_{n_0})^{m_{n_0}}| = 1$ so $|g(u_{n_0})| = \max_{i\geq 0}\{|c_i f_i(u_{n_0})^{m_i}|\} = 1$.
Further, if $j < n_0$, $|g(u_j)| \leq \max_{i\geq 0}\{|c_i f_i(u_j)^{m_i}|\} < 1$ since $|c_i| \leq \alpha$ for $i \neq n_0$ and $|f_{n_0}(u_j)| < 1$ (lemma 2.2). It now follows immediately that $||g||_\infty = 1$, since $||f_j||_\infty = 1$ for all j. □

Lemma 2.5

Let the function g be as defined in lemma 2.4.

Let $x, y \in M$ such that $|x - y| \leq \alpha^t$. Assume $0 \leq n_0 < N_t$. Then for every $j \in \mathbb{N}$, we have

$$|g(x)^j - g(y)^j| \leq \alpha.$$

Proof

Since $|c_i| \leq \alpha$ for $i \neq n_0$, we have $|g(x) - g(y)| \leq \max_{i\geq 0}\{|c_i||f_i(x)^{m_i} - f_i(y)^{m_i}|\} \leq \alpha$ by lemma 2.3. If $j > 1$ then

$$|g(x)^j - g(y)^j| = |g(x) - g(y)||\sum_{s=0}^{j-1} g(x)^s g(y)^{j-1-s}| \leq \alpha$$

since $||g||_\infty = 1$ (lemma 2.4). So $|g(x)^j - g(y)^j| \leq \alpha$ for all $j \in \mathbb{N}$ (the case $j = 0$ is trivial).
□

3. Orthonormal bases for $C(M \to K)$

We are going to use the lemmas in section 2 to construct orthonormal bases for $C(M \to K)$ with the aid of the following result. Let k be the finite residue class field of K.

If f is an element of $C(M \to K)$ with $||f||_\infty \leq 1$, let $\overline{f}$ denote the canonical projection of f on $C(M \to k)$. Then we have the following ([6], lemme 1)

A sequence (e_i) of elements of $C(M \to K)$ forms an orthonormal basis for $C(M \to K)$ if and only if

1) $||e_i||_\infty \leq 1$ for all i

2) $(\overline{e}_i)$ forms an algebraic basis for the k-vectorspace $C(M \to k)$.

This allows us to prove the main result of this paper. Let $(\psi_{1,i})$ and $(\psi_{2,i})$ be sequences of functions where $\psi_{l,i}$ ($l = 1$ or 2) is equal to the function q_i or to the function ϕ_i.

Remark that this can be different for every index i, just as in section 2. As an example of such a sequence we have $(\phi_0, \phi_1, q_2, \phi_3, \phi_4, q_5, \ldots, \phi_{3i}, \phi_{3i+1}, q_{3i+2}, \ldots)$

Theorem

Let $(f_{1,n})$, $(f_{2,n})$, $(g_{1,n})$ and $(g_{2,n})$ be sequences of continuous functions of the following form :

for $l = 1$ or $l = 2$, for every $n \geq 0$, $f_{l,n}$ is of the form

$$f_{l,n} = \sum_{i=0}^{\infty} a_{l,n,i}\psi_{l,i}, \quad a_{l,n,i} \in K, \quad |a_{l,n,i}| \to 0 \ \ if \ \ i \to \infty,$$
$$with \ \ |a_{l,n,n}| = 1 \ \ and \ \ |a_{l,n,i}| < 1 \ \ if \ \ i \neq n,$$

and for $l = 1$ or $l = 2$, for every $n \geq 0$, $g_{l,n}$ is of the form

$$g_{l,n} = \sum_{i=0}^{\infty} c_{l,n,i}(f_{l,i})^{m_{l,n,i}}, \quad c_{l,n,i} \in K, \quad |c_{l,n,i}| \to 0 \ \ if \ \ i \to \infty,$$

with

$$|c_{l,n,n}| = 1, \quad |c_{l,n,i}| < 1 \ \ if \ \ i \neq n,$$

and

$$m_{l,n,i} \in \mathbb{N}_0 \ \ for \ \ all \ \ i.$$

If (k_n) is a sequence in $\mathbb{N}$ and if (j_n) is a sequence in $\mathbb{N}_0$, then the sequence $((g_{1,n})^{k_n}(g_{2,n})^{j_n})$ forms an orthonormal basis for $C(M \to K)$.

Proof

The functions $f_{1,n}$, $f_{2,n}$, $g_{1,n}$ and $g_{2,n}$ are clearly continuous. We remark that, by lemma 2.4, $|g_{1,n}(u_n)| = |g_{2,n}(u_n)| = 1$, $|g_{1,n}(u_i)| < 1$ and $|g_{2,n}(u_i)| < 1$ for all n in $\mathbb{N}$ and for all i, $i < n$, and $||(g_{1,n})^{k_n}(g_{2,n})^{j_n}||_\infty = 1$. By the remark above, it suffices to prove that $(\overline{(g_{1,n})^{k_n}(g_{2,n})^{j_n}})$ forms an algebraic basis for $C(M \to k)$. Let C_t be the subspace of $C(M \to k)$ of the functions constant on closed balls with radius α^t. Since $C(M \to k) = \bigcup_{t \geq 0} C_t$ it suffices to prove that $(\overline{(g_{1,n})^{k_n}(g_{2,n})^{j_n}}|n < N_t)$ forms a basis for C_t. M is the union of N_t disjoint balls with centers u_n, $0 \leq n < N_t$, radius α^t. Let χ_i denote the characteristic function of the ball with center u_i.

For $x, y \in M$ satisfying $|x - y| \leq \alpha^t$, for $0 \leq n < N_t$ we have

$$|g_{1,n}(x)^{k_n} g_{2,n}(x)^{j_n} - g_{1,n}(y)^{k_n} g_{2,n}(y)^{j_n}| \leq \alpha.$$

This can be seen as follows :

$|g_{1,n}(x)^{k_n} g_{2,n}(x)^{j_n} - g_{1,n}(y)^{k_n} g_{2,n}(y)^{j_n}|$
$\leq \max\{|g_{1,n}(x)^{k_n} g_{2,n}(x)^{j_n} - g_{1,n}(x)^{k_n} g_{2,n}(y)^{j_n}|, |g_{1,n}(x)^{k_n} g_{2,n}(y)^{j_n} - g_{1,n}(y)^{k_n} g_{2,n}(y)^{j_n}|\}$
$\leq \max\{|g_{1,n}(x)^{k_n}||g_{2,n}(x)^{j_n} - g_{2,n}(y)^{j_n}|, |g_{2,n}(y)^{j_n}||g_{1,n}(x)^{k_n} - g_{1,n}(y)^{k_n}|\} \leq \alpha$ by lemma 2.5.

It now follows that

$$\begin{aligned}\overline{(g_{1,n}(x))^{k_n}(g_{2,n}(x))^{j_n}} &= \sum_{i=0}^{N_t-1} \chi_i(x)\overline{(g_{1,n}(u_i))^{k_n}(g_{2,n}(u_i))^{j_n}} \\ &= \sum_{i=n}^{N_t-1} \chi_i(x)\overline{(g_{1,n}(u_i))^{k_n}(g_{2,n}(u_i))^{j_n}}\end{aligned}$$

since $|(g_{1,n}(u_i))^{k_n}(g_{2,n}(u_i))^{j_n}| < 1$ if $i < n$ (lemma 2.4) and hence the transition matrix from $(\chi_n|n < N_t)$ to $(\overline{(g_{1,n})^{k_n}(g_{2,n})^{j_n}}|n < N_t)$ is triangular. Since $|(g_{1,n}(u_n))^{k_n}(g_{2,n}(u_n))^{j_n}| = 1$ (lemma 2.4), $(\overline{(g_{1,n})^{k_n}(g_{2,n})^{j_n}}|\ n < N_t)$ forms a basis for C_t. □

In the next corollary (ψ_i) is, as before, a sequence of functions where ψ_i is equal to the function q_i or to the function ϕ_i (this can be different for every index i).

Corollary

Let (f_n) and (g_n) be sequences of functions defined as follows : for all n,

$$f_n = \sum_{i=0}^{\infty} a_{n,i}\psi_i, \quad a_{n,i} \in K, \quad |a_{n,i}| \to 0 \ \ if \ \ i \to \infty,$$

$$\text{with } |a_{n,n}| = 1 \text{ and } |a_{n,i}| < 1 \text{ if } i \neq n.$$

$$g_n = \sum_{i=0}^{\infty} c_{n,i}(f_i)^{m_{n,i}}, \quad c_{n,i} \in K, \quad |c_{n,i}| \to 0 \text{ if } i \to \infty,$$

$$\text{with } |c_{n,n}| = 1, \quad |c_{n,i}| < 1 \text{ if } i \neq n \text{ and } m_{n,i} \in \mathbb{N}_0 \text{ for all } i.$$

If (j_n) is a sequence in $\mathbb{N}_0$, then

1) the sequence $((g_n)^{j_n})$ forms an orthonormal basis for $C(M \to K)$,

2) the sequence $((f_n)^{j_n})$ forms an orthonormal basis for $C(M \to K)$.

In particular, the sequences (g_n) and (f_n) form orthonormal bases for $C(M \to K)$.

Proof

1) Apply the theorem above with $k_n = 0$ for all n.
2) Put in 1) for all n $m_{n,n} = c_{n,n} = 1$ and $c_{n,i} = 0$ if $i \neq n$.
If we put $j_n = 1$ for all n, it follows that (g_n) and (f_n) form orthonormal bases for $C(M \to K)$. □

Remark 3.1 If we look at the proof of the theorem and at the proofs of the lemmas in section 2, it is not difficult to extend the theorem and to find more orthonormal bases for $C(M \to K)$. However, the notations get more and more complicated.

Remark 3.2 If we put in the theorem $a_{2,0,0} = 1$, $a_{2,0,i} = 0$ for $i \neq 0$, then $f_{2,0} = 1$ so for $m_{2,n,0} = 0$, $n = 0, 1, \ldots$ the theorem also holds. If we then put $c_{2,0,0} = 1$, $c_{2,0,i} = 0$ for $i \neq 0$, then $g_{2,0} = 1$ and so the theorem holds for $j_0 = 0$. In analogous way we see that the corollary is also valid for $m_{n,0} = 0$, $n = 0, 1, \ldots$ and $j_0 = 0$.

Let us now look at some examples. Examples 3.1, 3.2 and 3.3 can also be found with the results of [9].

Example 3.1 The sequence $((p_n)^{j_n})$ $(j_n \in \mathbb{N}_0)$ forms an orthonormal basis for $C(M \to K)$, where for all n, p_n is a function defined as follows : $p_n = \sum_{i=0}^{n} a_{n,i}\phi_i$, with $|a_{n,n}| = 1$ and $|a_{n,i}| < 1$ if $0 \leq i < n$, $a_{n,i} \in K$. To see this, apply the corollary above for $f_n = \sum_{i=0}^{n} a_{n,i}\psi_i$ with $\psi_i = \phi_i$. In particular, the sequence (ϕ_n) forms an orthonormal basis.

Example 3.2 Let for all n, p_n be a polynomial of degree n defined as follows : $p_n = \sum_{i=0}^{n} a_{n,i}q_i$

with $|a_{n,n}| = 1$ and with $|a_{n,i}| < 1$ if $0 \leq i < n$ ($a_{n,i} \in K$). If we apply the corollary above for $f_n = \sum_{i=0}^{n} a_{n,i}\psi_i$ with $\psi_i = q_i$ for all i, then the sequence $((p_n)^{j_n})$ ($j_n \in \mathbb{N}_0$) forms an orthonormal basis for $C(M \to K)$. In particular, the sequence $((q_n)^{j_n})$ forms an orthonormal basis. If we put $j_n = 1$ for all n then the sequence (q_n) also forms an orthonormal basis. (q_n) is known as Amice's basis (see also [1], p. 143, theorem 1).

Example 3.3 Put $K = \mathbb{Q}_p$, the field of the p-adic numbers and $M = \mathbb{Z}_p$, the ring of the p-adic integers, and let $|\cdot|$ be the p-adic valuation on $\mathbb{Q}_p$. From examples 1.2 and 1.4 we know that $\mathbb{Z}_p$ is a regular compact and that $(u_n), u_n = n$ for all n , is a very well distibuted sequence in $\mathbb{Z}_p$.

From example 3.1 it follows that the sequence (ϕ_n) defined on M by

$$\phi_0(x) = 1 \quad \text{for all} \quad x \text{ in } M$$

$$\phi_n(x) = 1 \text{ if and only if } x \in B_{u_n}(r_n)\ (n \geq 1), \text{ where } r_n = \alpha^{i+1} \text{ if } p^i \leq n < p^{i+1}$$

forms an orthonormal basis for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$. Then, (ϕ_n) is known as van der Put's basis ([4] , example 7.2).

Define the sequence of polynomials $\left(\binom{x}{k}\right)$ by $\binom{x}{0} = 1$, $\binom{x}{k} = \dfrac{x(x-1)\ldots(x-k+1)}{k!}$ if $k \geq 1$. From example 3.2 it follows that the sequences $\left(\binom{x}{k}\right)$ (Mahler's basis, [5]), and $\left(\binom{x}{k}^s\right)$ (Caenepeel, [2]) form orthonormal bases for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$ $\quad (s \in \mathbb{N}_0)$.

Example 3.4 Put $K = \mathbb{Q}_p$, $M = \mathbb{Z}_p$ and let $|.|$ be the p-adic valuation on $\mathbb{Q}_p$. V_q is the closure of the set $\{aq^n | n = 0, 1, 2, \ldots\}$ where a and q are two units of $\mathbb{Z}_p$, q not a root of unity.

Let m be the smallest integer such that $q^m \equiv 1$ (mod p) $(1 \leq m \leq p-1)$. There exists a k_0 such that $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$. If $(p, k_0) = (2, 1)$, i.e. $q \equiv 3 \pmod 4$, then there exists a natural number N such that $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$, $\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$. Then we have ([7], lemmas 4 and 5)

1) Let $q^m \equiv 1 \pmod{p^{k_0}}$, $q^m \not\equiv 1 \pmod{p^{k_0+1}}$ with $(p, k_0) \neq (2, 1)$.
Then $V_q = \bigcup_{0 \leq r \leq m-1} \{x \in \mathbb{Z}_p || x - aq^r| \leq p^{-k_0}\}$.

2) Let $q \equiv 3 \pmod 4$, $q = 1 + 2 + 2^2\varepsilon$, $\varepsilon = \varepsilon_0 + \varepsilon_1 2 + \varepsilon_2 2^2 + \ldots$,
$\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{N-1} = 1$, $\varepsilon_N = 0$.

Then $V_q = \{x \in \mathbb{Z}_2 \| x-a| \leq 2^{-(N+3)}\} \bigcup \{x \in \mathbb{Z}_2 \| x - aq| \leq 2^{-(N+3)}\}$.

The method used in this paper can be applied to construct orthonormal bases for the Banach space $C(V_q \to \mathbb{Q}_p)$ of continuous functions from V_q to $\mathbb{Q}_p$. A theorem concerning this item can be found in [8].

Example 3.5 Put $K = \mathbb{Q}_p$, $M = \mathbb{Z}_p$ and let $|.|$ be the p-adic valuation on $\mathbb{Q}_p$. With the theorem and the corollary above it is not so difficult to find orthonormal bases for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$. Let $\binom{x}{n}$ denote Mahler's basis (example 3.3). Put $f_n = \sum_{i=0}^{\infty} p^i \binom{x}{i+n} = \sum_{k=n}^{\infty} p^{k-n} \binom{x}{k}$. Then the sequences (f_n), $((f_n)^n)$ and $(f_0, f_1, (f_2)^2, \ldots, (f_n)^n, \ldots)$ form orthonormal bases for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$. Put $g_n = \sum_{i=0}^{\infty} p^i \binom{x}{i+n}^n = \sum_{k=n}^{\infty} p^{k-n} \binom{x}{k}^n$. Then (g_n), $((g_n)^n)$ and $(g_0, g_1, (g_2)^2, \ldots, (g_n)^n, \ldots)$ are orthonormal bases for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$. Let (ϕ_n) be van der Put's basis for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$ (example 3.3). Put $f_n = \phi_n + \sum_{i \geq 0, i \neq n} p^{2^i} \phi_i$, then (f_n) and $((f_n)^{2^n})$ are orthonormal bases for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$.

Another orthonormal basis for $C(\mathbb{Z}_p \to \mathbb{Q}_p)$ forms the sequence (f_n), where $f_n = \psi_n + \sum_{\substack{i \geq 0 \\ i \neq n}} (p)^{p^i} \psi_i$, with $\psi_{2i}(x) = \phi_{2i}(x), \psi_{2i+1}(x) = \binom{x}{2i+1}$ for all $n, i \in \mathbb{N}$.

References

[1] Y. Amice, "Interpolation p-adique", Bull. Soc. math. France, 92, 1964, p. 117-180.

[2] S. Caenepeel, "About p-adic Interpolation of Continuous and Differentiable Functions". Groupe d' étude d'analyse ultramétrique (Y. Amice, G. Christol, P. Robba), 9e année, 1981/82, no. 25, 8 p.

[3] S. Caenepeel, "p-adic Interpolation of Continuous and Analytic functions", Seminar on p-adic Analysis, 1981-82, Vrije Universiteit Brussel.

[4] L. Gruson and M. van der Put, "Banach Spaces", Table Ronde d' Analyse non Archimédienne (1972 Paris), Bulletin de la Société Mathématique de France, Mémoire 39-40, 1974, p. 55 - 100.

[5] K. Mahler, "An Interpolation Series for Continuous Functions of a p-adic Variable", Journal für reine und angewandte Mathematik, vol. 199, 1958, p. 23-34.

[6] J.P. Serre, Endomorphismes Complètement Continus des Espaces Banach p-adiques,

Presses Universitaires de France, Paris, 1962, Institut Hautes Etudes Scientifiques, Publications Mathématiques, 12, p. 69-85.

[7] A. Verdoodt, "Normal Bases for Non-Archimedean Spaces of Continuous Functions", Publicacions Matemàtiques, Vol. 37, 1993, p. 403-427.

[8], A. Verdoodt, "Orthonormal Bases for Spaces of Continuous and Continuously Differentiable Functions defined on a Subset of $\mathbb{Z}_p$, Revista Matemática de la Universidad Complutense de Madrid, Vol. 9, no. 2, 1996, p.295-307.

[9] A. Verdoodt, "Orthonormal Bases for Non-Archimedean Banach Spaces of Continuous Functions", in "p-Adic Functional Analysis", editors J. Kakol, N. De Grande-De Kimpe, C. Pérez-Garcia, 1999, p. 323-331, chapter 21.