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# BÖCHER'S THEOREM IN A SPACE OF DIMENSION ONE

**Premalatha and T. Sowmya**

**Abstract:**

In this paper we express a harmonic function $h$ defined outside a compact set in a B.H. space $\Omega$ as an integral with respect to a signed measure in $\Omega$ assuming $\Omega$ satisfies the axiom of local proportionality. If in particular $h$ is positive and $\Omega$ has harmonic dimension one then this expression leads to an analogue of Böcher's theorem in a space of dimension one.

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## §1. Introduction

We consider a harmonic function $h$ defined outside a compact set in a B.H. space $\Omega$. This can be written as the difference of two superharmonic functions in $\Omega$ where both functions have the same compact support in $\Omega$. If we assume the axiom of local proportionality this leads to an integral representation for $h$ with respect to a signed measure which looks like the Riesz representation. This is of interest because the Riesz representation does not give an integral for a harmonic function as the measure associated with a harmonic function is zero. This theorem gives an analogue of Böcher's theorem in a B.H. space of harmonic dimension one if we assume $h$ is positive.

## §2. Preliminaries

Let $\Omega$ be a harmonic space satisfying the axioms 1,2,3 of M.Brelot. We assume that constants are harmonic in $\Omega$ in which case $\Omega$ is referred to as a B.H.space. $\Omega$ is called a B.P. or B.S. space according as there exists a positive potential or not in $\Omega$. For a nonlocally polar outer regular compact set $k \subset \Omega$ and a continuous function $f$ on $\partial k$, as in [1], the notation $B_k f$ stands for the Dirichlet solution in $\Omega - k$ with values $f$ or $\partial k$ and 0 at the point at infinity.

In the case of a B.S. space $\Omega$, we fix an outer regular compact set $K$ and a regular domain $\omega$, $K \subset \omega$ with respect to which flux is defined (for definition see [1]). We also fix a harmonic function $H > 0$ in $\Omega - K$ tending to 0 on $\partial K$ with flux at infinity one.

We recall the definition of a B.H. potential in a B.S. space $\Omega$: Let $\{\Omega_i\}$ be a fixed regular exhaustion of $\Omega$. Fix an ultrafilter $e$ finer than the filter of sections of $\{\Omega_i\}$. Let $\mathcal{D}(u)$ be the limit of $\overline{\mathcal{H}}_u^{\Omega_i}$ according to the ultrafilter $e$. An admissible superharmonic function $u$ in a B.S. space $\Omega$ with flux at infinity $\alpha$ is said to be a B.H. - potential if $\mathcal{D}(u - \alpha H) = 0$.

It can be easily seen that a superharmonic function $u$ with compact support in a B.P. (respectively B.S.) space can be written uniquely as the sum of a potential (respectively B.H. potential) and a harmonic function.

Let $\Omega$ be a B.H. space satisfying the axiom of local proportionality.

**Case (i).** Let $\Omega$ be a B.P. space. If $\delta$ is a regular domain and $z$ a fixed point in $\delta$, then for any $y$ there exists a unique potential $q_y(x)$ with support $y$ such that $\int q_y(x) d\rho_z^\delta(x) = 1$ where $d\rho_z^\delta$ is the harmonic measure of $\delta$ with respect to $z$.

If $u$ is a potential with compact support $A$ then there exists a unique Radon measure $\mu \geq 0$ supported by $A$ such that $u(x) = \int q_y(x) d\mu(y)$; and conversely if $\mu \geq 0$ is a Radon measure with compact support then $\int q_y(x) d\mu(y)$ is a potential.

**Case (ii):** Let $\Omega$ be a B.S. space. In this case, for any $y$, there exists a unique B.H. potential $q_y(x)$ with support $y$ and flux $q_y$ at infinity $-1$. Then if $u(x)$ is a B.H. potential with compact support $A$, there exists a unique Radon measure $\mu \geq 0$ supported by $A$ such that $u(x) = \int q_y(x) d\mu(y)$; and conversely, if $\mu \geq 0$ is a Radon measure with compact support, then $u(x) = \int q_y(x) d\mu(y)$ is a B.H. potential with flux $u$ at infinity $= -\int d\mu$.

## § 3. BÖCHER'S THEOREM IN A SPACE OF DIMENSION ONE

**Theorem 1.**

*Let $h$ be a harmonic function defined outside a compact set $X$ in a B.H. space $\Omega$ and $\omega_0$ be any regular domain such that $X \subset \omega_0$. Assume that $\Omega$ has a countable base and satisfies the axiom of local proportionality. Then* *there exists a signed measure $\mu$ with support contained in $\partial\omega_0$ and a uniquely determined harmonic function $u$ in $\Omega$ such that $h(x) = \int q_y(x)d\mu(y) + u(x)$ in $\Omega \sim \overline{\omega_0}$.*

*Here $q_y(x)$ is the potential (respectively B.H. potential) that we fix in $\Omega$ as explained in §2, if $\Omega$ is a B.P. (respectively B.S.) space. Moreover if the harmonic dimension at infinity of $\Omega$ is 1, $u$ is a constant if and only if $h$ is bounded on one side near the point at infinity $\mathcal{A}$.*

**Proof.**

Let $x_0 \in X$ and $s_{x_0}$ be a superharmonic function in $\Omega$ with point support $x_0$.

Choose an outer regular compact set $K_1$ such that $X \subset K_1^0 \subset K_1 \subset \omega_0$.

Without loss of generality we can assume that $h$ is harmonic in $\omega_0 \sim K_1$ and continuous in $\overline{\omega_0 \sim K_1}$. For a continuous function $f$ on $\partial\omega_0$ let $Df = H_f^{\omega_0}$ denote the Dirichlet solution in $\omega_0$ with boundary value $f$.

Since $Ds_{x_0} < s_{x_0}$ in $\omega_0$ we have $\inf_{\partial K_1}(s_{x_0} - Ds_{x_0}) > 0$.

Choose $\alpha > 0$ such that

$$\alpha(s_{x_0} - Ds_{x_0}) > Dh - h \text{ on } \partial K_1.$$

Then $h + \alpha s_{x_0} > D(h + \alpha s_{x_0})$ on $\partial K_1$.

Since $h + \alpha s_{x_0} = D(h + \alpha s_{x_0})$ on $\partial\omega_0$, by minimum principle of harmonic functions we get

$$h + \alpha s_{x_0} > D(h + \alpha s_{x_0}) \text{ in } \omega_0 \sim K_1.$$

Define $$h_1 = \begin{cases} h + \alpha s_{x_0} & \text{in } \Omega \sim \omega_0 \\ D(h + \alpha s_{x_0}) & \text{in } \omega_0 \end{cases}$$

and $$h_2 = \begin{cases} \alpha s_{x_0} & \text{on } \Omega \sim \omega_0 \\ D(\alpha s_{x_0}) & \text{on } \omega_0. \end{cases}$$

Then $h_1$ and $h_2$ are finite, continuous, superharmonic functions in $\Omega$ with compact support in $\partial\omega_0$ such that

$$h = h_1 - h_2 \text{ on } \Omega \sim \overline{\omega}_0.$$

Now, $h_i = p_i + u_i \quad i = 1,2$ where $p_i$ is a potential (respectively B.H. potential) with support in $\partial\omega_0$ if $\Omega$ is a B.P. (respectively B.S.) space and $u_i$ is harmonic in $\Omega$.

Hence $h = p_1 - p_2 + u$ where $u = u_1 - u_2$ is harmonic in $\Omega$. But $p_i(x) = \int q_y(x) d\mu_i(y)$, $i = 1,2$ where $\mu_i, i = 1,2$ is a Radon measure with support contained in $\partial\omega_0$.

Hence $h(x) = \int q_y(x) d\mu(y) + u(x)$ where $\mu = \mu_1 - \mu_2$ is a signed measure with support contained in $\partial\omega_0$.

We shall complete the proof by considering the two cases of a B.P. space and a B.S. space separately.

**Case (i).** Let $\Omega$ be a B.P. space.

Suppose $h(x) = \int q_y(x) d\mu'(y) + u'(x)$ where $\mu'$ is also a signed measure with support contained in $\partial\omega_0$ and $u'$ is harmonic in $\Omega$.

Then $h$ can be written as

$$h = p_1 - p_2 + u = q_1 - q_2 + u' \text{ in } \Omega \sim \overline{\omega}_0$$

where $q_i$, $i = 1,2$ are potentials in $\Omega$ with compact support.

$$\begin{aligned} \text{Then } \mathcal{D}(p_1) &= \mathcal{D}(p_2) = \mathcal{D}(q_1) = \mathcal{D}(q_2) = 0 \text{ gives} \\ \mathcal{D}(u) &= u = u' = \mathcal{D}(u'). \end{aligned}$$

Thus $u$ is uniquely determined in $\Omega$.

Now $h = p_1 - p_2 + u$ on $\Omega \sim \overline{\omega}_0$.

Since $p_1$ and $p_2$ are potentials with compact support, they are bounded outside a compact set in $\Omega$.

Hence if $h$ is bounded on one side near $\mathcal{A}$ so is $u$.

Therefore if $\Omega$ is of harmonic dimension one, we see that $u$ reduces to a constant [2].

If $u$ is a constant then clearly $h$ is bounded on one side near $\mathcal{A}$.

**Case (ii):** Let $\Omega$ be a B.S. space.

Let flux $p_1 = \alpha_1$ and flux $p_2 = \alpha_2$.

Then $h - (\alpha_1 - \alpha_2)H = (p_1 - \alpha_1 H) - (p_2 - \alpha_2 H) + u$

gives $\mathcal{D}(h - (\alpha_1 - \alpha_2)H) = u$ by definition of a B.H. potential.

Since $\alpha_1 - \alpha_2 =$ flux $h$, we see that given $h, u$ is uniquely determined in $\Omega$.

Now since $D(p_i - \alpha_i H) = 0$ we get $p_i - \alpha_i H$, $i = 1, 2$ are bounded outside a compact set.

Hence $h = u + (\alpha_1 - \alpha_2)H +$ a bounded harmonic function outside a compact set.

If $h$ is bounded on one side near $A$, then $u + (\alpha_1 - \alpha_2)H$ is bounded on one side near $A$.

If $\Omega$ has harmonic dimension one this implies that $u$ is a constant [2].

If $u$ is a constant, $h$ is obviously bounded on one side near $A$.

This completes the proof of the theorem.

Now if we take the function $h$ in the above theorem to be $\geq 0$ we can deduce the analogue of the inverted version of Böcher's theorem, which may be stated as follows, in a space of harmonic dimension one.

Böcher's theorem: (Inverted version). Let $u$ be positive and harmonic in $\mathbf{R}^n - \bar{B}, n \geq 2$ where $B$ is the unit ball about the origin. Then

$$u(x) = \begin{cases} \alpha \log |x| + b(x) & \text{if } n = 2 \\ \alpha + b(x) & \text{if } n \geq 3 \end{cases}$$

where $b(x)$ is a bounded harmonic function in $\mathbf{R}^n - \bar{B}$ and $\alpha \geq 0$ is a constant. If $n \geq 3$, $b(x)$ is actually bounded by a bounded potential.

This can be proved by applying the Kelvin's transform to the standard form of Böcher's theorem [3].

**Theorem 2.**

*Let $\Omega$ be a B.H. space of harmonic dimension one and $h$ be a positive harmonic function defined outside a compact set $X$. If $\Omega$ is a B.P. space then $h = \alpha + b$ where $\alpha$ is a constant and $b$ is a harmonic function bounded by a bounded potential outside a compact set.*

*If $\Omega$ is B.S., then $h = \alpha H + b$ outside a compact set where $\alpha$ is a constant and $b$ is a bounded harmonic function outside a compact set.*

**Proof.**

**Case (i).** Let $\Omega$ be a B.P. space.

Take $\omega_0, p_1, p_2$ as in Theorem 1.

Since $h \geq 0$, $u$ is a constant say $\alpha$.

Let $K'$ be an outer regular compact set such that $(K')^0 \supset \partial\omega_0$.

Then for $i = 1,2$, $v_i = \begin{cases} 0 & \text{on } K' \\ p_i - B_{K'}p_i & \text{on } \Omega \sim K' \end{cases}$
is a subharmonic function on $\Omega$ such that $0 \leq v_i \leq p_i$.

Since $p_i$ is a potential this implies that $v_i \equiv 0$ or $p_i = B_{K'}p_i$ outside the compact set $K'$.

If $p_i \leq \lambda$ on $\partial K'$, then $B_{K'}p_i \leq \lambda B_{K'}1$.
Hence $h = p_1 - p_2 + \alpha = \alpha + b$ where $b = p_1 - p_2$ is such that $|b| \leq 2\lambda B_{K'}1$, a bounded potential outside a compact set.

**Case (ii).** Let $\Omega$ be a B.S. space.

Then as in the proof of the above theorem since $u$ is a constant we get

$$\begin{aligned} h &= (\alpha_1 - \alpha_2)H + \text{a bounded harmonic function} \\ &= \alpha H + b \qquad \text{outside a compact set.} \end{aligned}$$

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