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ARENS ALGEBRAS, ASSOCIATED WITH COMMUTATIVE VON NEUMANN ALGEBRAS

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1. Introduction. Let (Ω, Σ, μ) be a measurable space with a finite measure, $L^p(\mu) = L^p(\Omega, \Sigma, \mu)$ the Banach space of all μ-measurable complex functions on Ω, integrable with the degree, $p \in [1, +\infty)$. R. Arens [1] introduced and studied the set $L^\omega(\mu) = \bigcap\limits_{1 \leq p < \infty} L^p(\mu)$. He showed, in particular, that $L^\omega(\mu)$ is a complete locally-convex metrizable algebra with respect to "t" topology generated by the system of norms $\|f\|_p = \left(\int\limits_\Omega |f|^p \, d\mu\right)^{1/p}$, $p \geq 1$. Later G.R. Allan [2] observed that $(L^\omega(\mu), t)$ is a GB^*-algebra with the unit ball $B_0 = \{f \in L^\infty : \|f\|_p \leq 1\}$. Further investigation of properties of the Arens algebra $L^\omega(\mu)$ was made by S.J. Bhaft [3,4]. He described the ideals of the algebrá $L^\omega(\mu)$ and considered some classes of homomorphisim of this algebra. B.S. Zakirov [5] showed that $L^\omega(\mu)$ is an EW^*-algebra and gave an example of two measures, μ and ν, on an atomic Boolean algebra, for which the algebras $L^\omega(\mu)$ and $L^\omega(\nu)$ are not isomorphic. It is clear that the problem of complete classification of the Arens algebras arises. Speaking more preciesly, what conditions should be imposed on measures μ and ν for the corresponding Arens algebras to be isomorphic? It is natural to solve this problem in the class of equivalent measures. Therefore instead of a measurable space with a measure, one should consider a commutative von Neumann algebra M with faithful normal finite traces μ and ν on M and study the problem of $*$-isomorphism of EW^*- algebras $L^\omega(M;\mu) = \bigcap\limits_{1 \leq p < \infty} L^p(M;\mu)$ and $L^\omega(M, \nu)$

The present article gives the complete solution of the mentioned problem, a classification of the normalized Boolean algebras from the book by D.A. Vladimirov [6] being considerably used. All neccessary notations and results from the theory of von Neumann algebras are taken from [7] and the theory of integration on von Neumann algebras is from [8].

2. Preliminaries. Let M be an arbitrary von Neumann algebra, μ a faithful normal finite trace on M, $P(M)$ the lattice of all projections of M. Let $K(M,\mu)$ be the $*$-algebra of all μ-measurable operators affiliated with M [8].

In the commutative case, when $M = L^\infty(\Omega,\Sigma,\mu)$ and $\mu(x) = \int\limits_\Omega x\,d\mu$, where (Ω,Σ,μ) is a measurable space, the algebra $K(M,\mu)$ coincides with the algebra of all measurable complex functions on (Ω,Σ,μ).

For every set $A \subset K(M,\mu)$ we shall denote by A_h (respectively, by A_+) the set of all self-adjoint (respectively, positive self-adjoint) operators from A. The partial order in $K_h(M,\mu)$ generated by the positive cone $K_+(M,\mu)$ will be denoted by $x \le y$.

Put $M(x) = \sup\{\mu(y) | 0 \le y \le x, y \in M\}$ for every $x \in K_+(M,\mu)$. Let $p \in [1,\infty)$ and $L^p(M,\mu) = \{x \in K(M,\mu) | \mu(|x|^p) < \infty\}$, where $|x| = (x^*x)^{1/2}$. The set $L^p(M,\mu)$ is a subspace in $K(M,\mu)$ and the function $\|x\|_p = \mu(|x|^p)^{1/p}$ is a Banach norm on $L^p(M,\mu)$ [9]. Moreover,

1. $\|x\|_p = \|x^*\|_p = \|xu\|_p$ for all $x \in L^p(M,\mu)$ and a unitary element $u \in M$;

2. If $|x| \le |y|$, $x \in K(M,\mu)$, $y \in L^p(M,\mu)$, then $x \in L^p(M,\mu)$ and $\|x\|_p \le \|y\|_p$;

3. If $x \in L^p(M,\mu), y \in L^q(M,\mu)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $1 < p,q,r < \infty$, then $xy \in L^r(M,\mu)$ and $\|xy\|_r \le \|x\|_p \|y\|_q$.

From these properties of the norm $\|\cdot\|_p$ it follows that the set $L^\omega(M,\mu) = \bigcap\limits_{1\le p<\infty} L_p(M,\mu)$ is a $*$-subalgebra in $K(M,\mu)$, and $M \subset L^\omega(M,\mu)$. It was shown in [5] that $M = L^\omega(M,\mu)$ if and only if $\dim M < \infty$. Furthermore, since $L^\omega(M,\mu)$ is a solid $*$-subalgebra in $K(M,\mu)$ (e.g. the inequality $|x| \le |y|, x \in K(M,\mu), y \in L^\omega(M,\mu)$ implies $x \in L^\omega(M,\mu)$), $L^\omega(M,\mu)$ is an EW^*-algebra, the bounded part of which coincides with M [10].

Now we cite from [6] some information which will be used in the seqnel.

Let X be an arbitrary complete Boolean algebra, $e \in X$, $X_e = [0,e] = \{g \in X | \; g \le e\}$. The minimal cardinality of the set which is dense in X_e in the (o)-topology will be denoted $\tau(X_e)$. An infinite complete Boolean algebra X is called homogeneous, if $\tau(X_e) = \tau(X_g)$ for any non-zero $e,g \in X$. The cardinality of $\tau(X) = \tau(X_{\mathbf{1}})$ where $\mathbf{1}$ – is the unit of the Boolean algebra X is called a weight of a homogeneous Boolean algebra X.

Let μ be a strictly positive countably additive measure on X. If $\mu(\mathbf{1}) = 1$, then the pair (X, μ) is called a normalized Boolean algebra. It was shown in [6] that for any cardinal number τ there existed a complete homogeneous normalized Boolean algebra X with the weight $\tau(X) = \tau$. The next theorem gives a criterion of isomorphism of two homogeneous normalized Boolean algebras.

Theorem ([6]). *Let (X, μ) and (Y, ν) be homogeneous normalized Boolean algebras. The following conditions are equivalent:*

1) $\tau(X) = \tau(Y)$;
2) *There exists an isomorphism $\varphi : X \to Y$ for which $\nu(\varphi(x)) = \mu(x)$ for all $x \in X$.*

This theorem enables us to describe the class of von Neumann algebras for which the existence of $*$-isomorphism between the Arens algebras $L^w(M, \mu)$ and $L^\omega(N, \nu)$ is equivalent to isomorphism between M and N.

Proposition 1. *Let M and N be commutative von Neumann algebras, the Boolean algebras $P(M)$ and $P(N)$ of which are homogeneous, and let μ and ν be faithful normal finite traces on M and N, respectively. The following conditions are equivalent:*

1) *The Arens algebras $L^\omega(M, \mu)$ and $L^\omega(N, \nu)$ are $*$-isomorphic;*
2) *The von Neumann algebras M and N – are $*$-isomorphic;*
3) $\tau(P(M)) = \tau(P(N))$.

Proof. Sience $L^\omega(M, \mu)$ and $L^\omega(N, \nu)$ are EW^*-algebras the bounded parts of which coincide with M and N respectively, restriction on M of any $*$-isomorphism from $L^\omega(M, \mu)$ on $L^\omega(N, \nu)$ is a $*$-isomorphism from M on N. On the other hand if the von Neumann algebras M and N are $*$-isomorphic, then their Boolean algebras of projectors are also isomorphic and therefore, in this case, $\tau(P(M)) = \tau(P(N))$.

Now suppose that $\tau(P(M)) = \tau(P(N))$ and assume $\mu'(x) = \mu(x)/\mu(\mathbf{1})$, $\nu'(y) = \nu(y)/\nu(\mathbf{1})$, $x \in M$, $y \in N$. According to the theorem 1, there exists an isomorphism of Boolean algebras $\varphi : X \to Y$ for which $\nu(\varphi(x)) = \mu'(x)$ for all $x \in X$. This isomorphism extends to a $*$-isomorphism $\Phi : K(M, \mu) \to K(N, \nu)$ (See [11]): At the same time $\mu'(x) = \nu'(\Phi(x))$ for all $x \in L^1(M, \mu')$. Since $\mu'(|x|^p) = \nu'(\Phi(|x|^p)) = \nu'(|\Phi(x)|^p)$ we have $\Phi(L^p(M, \mu)) = \Phi(L^p(M, \mu')) = L^p(N, \nu') = L^p(N, \nu)$ for all $p \geq 1$. Hence $\Phi(L^\omega(M, \mu)) = L^\omega(N, \nu)$.

Corolary. *Let M and N be non-atomic commutative von Neumann algebras on separable Hilbert spaces, μ and ν faithful normal finite traces on M and N, respectively. Then the Arens algebras $L^\omega(M,\mu)$ and $L^\omega(N,\nu)$ are $*$-isomorphic.*

Proof. At first, show that if M acts on a separable Hilbert space H, then the Banach space $(L^r(M,\mu), \|\cdot\|_r)$ is also separable. To start one should note that in this case the strong topology is metrizable on the unit ball M_1 of the algebra M ([12] p.24). In addition, the convergence $x_\alpha \xrightarrow{so} 0$ in the strong topology in M_1 is equivalent to the convergence $\mu(x_\alpha^* x_\alpha) \to 0$ ([12] p.130).

Thus, for any sequence of $\{x_n\} \subset M$ and $x \in M$ the convergence $x_n \xrightarrow{so} x$ implies $\sup \|x_n\|_M < \infty$ and $\|x_n - x\|_2 \to 0$, where $\|\cdot\|_M$ is a C^*-norm in M. Hence, on any ball $M_n = \{x \in M | \|x\|_M \le n\}$ the strong topology coincides with the topology induced from $L_2(M,\mu)$. Since H is separable, there exists a countable set $X_n \subset M$ which is dense in M_n in the strong topology ([13], p.568). Hence the countable set $X = \bigcup_{n=1}^{\infty} X_n$ is dense in M in the topology induced from $L_2(M,\mu)$. Since M is dense in $(L_2(M,\mu), \|\cdot\|_2)$, $(L_2(M,\mu), \|\cdot\|_2)$ is separable.

There is one thing left to say: the (o)-topology in $(P(M),\mu)$ coincides with the topology induced from $(L^2(M,\mu), \|\cdot\|_2)$. Therefore, the $P(M)$ is a non-atomic Boolean algebra which is separable in the (o)-topology. Hence it is homogeneous [6]. Similarly, $P(N)$ is a non-atomic Boolean algebra and $\tau(P(M)) = \tau(P(N))$. According to the proposition 1, the Arens algebras $L^\omega(M,\mu)$ and $L^\omega(N,\nu)$ – are $*$-isomorphic.

Let (X,μ) be an arbitrary complete non-atomic normalized Boolean algebra. It was shown in [6] that there is a sequence $\{e_n\}$ of non-zero pairwise disjoint elements for which the Boolean algebras $[0,e_n]$ are homogeneous and $\tau_n = \tau([0,e_n]) < \tau_{n+1}$, $n = 1,2,\ldots$ This collection is determined uniquely and the matrix

$$\begin{pmatrix} \tau_1 & \tau_2 & \cdots \\ \mu(e_1) & \mu(e_2) & \cdots \end{pmatrix}$$

is called the passport of the Boolean algebra (X,μ)

The following theorem will be used for investigation of isomorphisms of Arens algebra.

Theorem 2 [6]. *Let (X, μ) and (Y, ν) be complete non-atomic normalized Boolean algebras. The following conditions are equivalent.*

1. *There exists an isomorphism $\varphi : X \to Y$ for which $\nu(\varphi(x)) = \mu(x)$ for all $x \in X$.*
2. *The passports of the Boolean algebras (X, μ) and (Y, ν) coincide.*

3. Main results. A von Neuman algebra M is called σ-finite if it admits at most countable family of orthogonal projections. On any σ-finite von Neumann algebra M, there exists a normal state, in particular, if M is commutative, then its Boolean algebra of projections $P(M)$ is a normed one. The next theorem discribes the class of commutative σ-finite von Neumann algebras M for which the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are $*$-isomorphic for any faithful normal finite traces of μ and ν on M.

Theorem 3. *For a commutative σ-finite von Neumann algebra M the following conditions are equivalent:*

1. *The Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are $*$-isomorphic for any faithful normal finite traces μ and ν on M.*
2. *$M = M_0 + \sum_{i=1}^n M_i$, where M_0 is a finite-dimentional commutative von Neumann algebra, M_i is an infinite-dimensional commutative von Neumann algebra in which the lattice of projections $P(M_i)$ is a homogeneous Boolean algebra and $\tau_i = \tau(P(M_i) < \tau_{i+1}$, $i = 1, \ldots, n-1$ (the summand M_0 are $\sum_{i=1}^n M_i$ may be absent).*

Proof. 1) $\to$ 2). Let Δ be the set of all atoms in $P(M)$ and $e = \sup \Delta$. Suppose that Δ is a countable set. Then $M_0 = eM$ coincides with the algebra ℓ_∞ of all bounded sequences of complex numbers. Denote the atoms in $P(\ell_\infty)$ by $q_u = (0, \ldots, 0, 1, 0, \ldots)$. Consider two faithful normal finite traces μ and ν on M, for which $\mu(q_n) = n^{-2}$, $\nu(q_n) = e^{-2n}$ and $\mu(x) = \nu(x)$ for all $x \in (\mathbf{1} - e)M$. Suppose, that a $*$-isomorphism Φ from $L^\omega(M, \nu)$ on $L^\omega(M, \mu)$ exists. Since $\Phi(M_0) = M_0$, we have $\Phi(L^\omega(M_0, \nu) = L^\omega(M_0, \mu)$. Choose $x \in K(M_0, \nu)$ such that $xq_n = 2^n$. The series

$$\sum_{n=1}^{\infty} \frac{2^{pn}}{e^{2^n}} = \nu(|x|^p)$$

converges for all $p \geq 1$. Therefore $x \in L^\omega(M_0, \nu)$ and, so $\Phi(x) \in L^\omega(M_0, \nu)$. Since $M_0 = l_\infty$, the $*$-isomorphism Φ is generated by some bijection π of the set of natural numbers. It means that $\Phi(x) = \Phi(\{2^n\}) = \{2^{\pi(n)}\} = y \in L^\omega(M_0, \mu)$. In particular,

$$\nu(|y|) = \sum_{n=1}^{\infty} 2^{\pi(n)} n^{-2} < \infty$$

which is wrong. Hence, a set Δ is either finite or empty.

Now suppose that in the Boolean algebra $P((\mathbf{1}-e)M)$ there is a countable set $\{e_n\}$ of disjoint elements, for which the algebras $X_n = P(e_n M)$ are homogeneous and $\tau_n = \tau(X_n) < \tau_{n+1}$. Choose two faithful normal finite traces μ and ν on M such that $\mu(e_n) = n^{-2}$, $\nu(e_n) = e^{-2^n}$ and $\mu(x) = \nu(x)$ for all $x \in M_0$. Let Φ be a $*$-isomorphisms from $L^\omega(M, \nu)$ on $L^\omega(M, \mu)$. Then $\Phi((\mathbf{1} - e)M) = (\mathbf{1} - e)M$ and, since weights τ_n are different, $\Phi(e_n M) = e_n(M)$ (See [6]). Choose $x \in K((\mathbf{1}-e)M, \nu)$ such that $xe_n = 2^n e_n$. Then $x \in L^\omega((\mathbf{1} - e)M, \nu)$, $\Phi(x) = x$ and

$$\mu(|\Phi(x)|) = \sum_{n=1}^{\infty} 2^n n^{-2} = \infty,$$

i.e. $\Phi(x)$ does not belong to $L^\omega(M, \nu)$.

The obtained contradiction implies that the set $\{e_n\}$ is at most countable.

2) $\to$ 1). Let $M = M_0 + \sum_{i=1}^{n} M_i$, where M_0 is finite-dimensional and M_i is infinite dimensional commutative von Neumann algebra, the Boolean algebra $P(M_i)$ being homogeneous, $\tau_i < \tau_{i+1}$, $i = 1, \ldots, n-1$.

Take arbitrary faitful normal traces μ and ν on M. As $\dim M_0 < \infty$, $L^\omega(M_0, \mu) = M_0 = L^\omega(M_0, \nu)$. According to the proposition 1 a $*$-isomorphism Φ_i from $L^\omega(M, \mu)$ on $L^\omega(M_i, \nu)$ exists. Each element x from $L^\omega(M, \mu)$ is represented as $x = x_0 + \sum_{i=1}^{n} x_i$, where $x_0 \in M_0 = L^\omega(M_0, \mu)$, $x_i \in L^\omega(M_i, \mu)$, $i = 1, \ldots, n$. It is obvious that $\Phi(x) = x_0 + \sum_{i=1}^{n} \Phi_i(x_i)$ is a $*$-isomorphism from $L^\omega(M, \mu)$ on $L^\omega(M, \nu)$. The theorem is proved.

Using theorem 3, it is easy to construct an example of a non-atomic commutative von Neumann algebra M with traces μ and ν, such that the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are isomorphic, while there is no $*$-isomorhism φ from M on M, for which $\nu \circ \varphi = \mu$. Indeed, assume that

$M = M_1 + M_2$, where M_1, M_2 are non-atomic commutative σ-finite von Neumann algebras in which the lattice of projections form homogeneous Boolean algebras and $\tau(P(M_1)) < \tau(P(M_2))$. Identify M_1 with the subalgebra $e_1 M_1$ and M_2 with $(\mathbf{1} - e_1)M_1$, $e_1 \in P(M)$. Let μ be an arbitrary faithful normal finite trace on M, $\mu(\mathbf{1}) = 1$. Assume that

$$\nu(x) = p(\mu(e_1)^{-1}\mu(xe_1) + q(\mu(\mathbf{1} - e_1))^{-1}\mu(x(\mathbf{1} - e_1)),$$

$x \in M$, $p, q > 0$, $p + q = 1$. It is evident that ν is a faithful normal finite trace on M. Choose p and q such that $\mu(e_1) \neq \nu(e_1) = p$, $\mu(\mathbf{1} - e_1) \neq \nu(\mathbf{1} - e_1) = q$. According to the theorem 2, there is no $*$-isomorphism $\varphi : M \to M$ for which $\nu \circ \varphi = \mu$. At the same time, according to the theorem 3, the Arens algebras $L^\omega(M, \mu)$ and $L^\omega(M, \nu)$ are $*$-isomorphic.

Now, let us find out when the Arens algebras coincide for different traces. Let μ and ν be two faithful normal finite traces on a commutative von Neumann algebra M. Denote by $h = \frac{d\mu}{d\nu}$ the Radon-Nikodim derivate of the trace μ relative ν, i.e. h is the element from $L^1_+(M, \nu)$ for which $\mu(x) = \nu(hx)$ for all $x \in M$.

It is clear that the element x from $K(M, \mu)$ belongs to $L^1(M, \mu)$ if and only if $hx \in L^1(M, \nu)$. In this case the equality $\mu(x) = \nu(hx)$ holds.

Proposition 2. *$L^\omega(M, \nu) \subset L^\omega(M, \mu)$ if only if*

$$h \in \bigcup_{1<p\leq\infty} L^p(M, \nu),$$

where $L^\infty(M, \nu)$ is identified with M.

Proof. Let $L^\omega(M, \nu) \subset L^\omega(M, \mu) \subset L^1(M, \mu)$. Then $\mu(x) = \nu(hx)$ for all $x \in L^\omega(M, \nu)$, and μ is a positive linear functional on $L^\omega(M, \nu)$. Since $L^\omega(M, \nu)$ is a complete metrizable locally-convex algebra with respect to the t-topology generated by the system of norms $\left\{\|x\|_p = \left(\nu(|x|^p)\right)^{1/p}\right\}_{p\geq 1}$ (see[3]) and involution in $L^\omega(M, \nu)$ is continuous in this topology, μ is continuous [14]. It was shown in [3] that the dual space of $(L^\omega(M, \nu)t)$ may be identified with $\bigcup_{1<p\leq\infty} L^p(M, \nu)$. Hence one can find such $y \in L^p(M, \nu)$ for some $p \in (1, \infty]$ that $\nu(hx) = \mu(x) = \nu(yx)$ for all $x \in L^\omega(M, \nu)$. It means that $h = y$ and $h \in \bigcup_{1<p\leq\infty} L^p(M, \nu)$.

Conversely, if $h \in L^p(M,\nu)$ for some $p \in (1,\infty]$, then $\nu(hx)$ is a t-continuous linear functional on $L^\omega(M,\nu)$(See[3]) and therefore $\mu(|x|^q) = \nu(h|x|^q) < \infty$ for any $x \in L^p(M,\nu)$ and $q \geq 1$; we recall that $|x|^q \in L^\omega(M,\nu)$ for all $x \in L^\omega(M,\nu)$ and $q \geq 1$. Thus,

$$L^\omega(M,\nu) \subset \bigcap_{q\geq 1} L^q(M,\mu) = L^\omega(M,\mu)$$

The following criterion of coincidence of the algebras $L^\omega(M,\mu)$ and $L^\omega(M,\nu)$ arises from the proposition 2.

Theorem 4. *Let μ, ν be faithful normal finite traces on a commutative von Neumann algebra M. Then $L^\omega(M,\mu) = L^\omega(M,\nu)$ if only if*

$$\frac{d\mu}{d\nu} \in \bigcup_{1<p\leq\infty} L^p(M,\nu) \text{ and } \frac{d\nu}{d\mu} \in \bigcup_{1<p\leq\infty} L^p(M,\mu).$$

Remarks.

1. In the example constructed after theorem 3 $L^\omega(M,\mu) = L^\omega(M,\nu)$ since

$$\frac{d\mu}{d\nu} = \mu(e_1)p^{-1}e_1 + \mu(1-e_1)q^{-1}(1-e_1)).$$

Now everything is ready to obtain the criterion of $*$-isomorphism of the Arens algebras $L^\omega(M,\mu)$ and $L^\omega(M,\nu)$. Let M be an arbitrary non-atomic commutative σ-finite von Neumann algebra. According to [6] the Boolean algebra $P(M)$ of projections M possesses uniquely determined collection $\{e_r\}$ of non-zero pairwise disjoint elements for which the Boolean algebras $X_n = \{e \in P(M) : e \leq e_n\}$ are homogeneous and $\tau(X_n) < \tau(X_{n+1})$. Assume that the collection $\{e_n\}$ is infinite otherwise all Arens algebras $L^\omega(M,\mu)$ are $*$-isomorphic (see theorem 3).

Theorem 5. *Let μ and ν be faithful normal finite traces on a non-atomic commutative σ-finite von Neumann algebra M. The following conditions are equivalent:*

1) *The Arens algebras $L^\omega(M,\mu)$ and $L^\omega(M,\nu)$ are $*$-isomorphic;*
2) *There are such $p, q \in (1,\infty]$ that*

$$\sum_{n=1}^{\infty} \mu_n^p \nu_n^{1-p} < \infty, \quad \sum_{n=1}^{\infty} \nu_n^q \mu_n^{1-q} < \infty$$

in the case $p \neq \infty$, $q \neq \infty$, *and* $\sup_{n\geq 1} |\mu_n \nu_n^{-1}| < \infty$ *if* $p = \infty$, $\sup_{n\geq 1} |\nu_n \mu_n^{-1}| < \infty$ *if* $q = \infty$.

Proof. 1) $\to$ 2). Let Φ be a $*$-isomorhism from $L^\omega(M,\mu)$ on $L^\omega(M,\nu)$. Since all $\tau(x_n)$ are different, $\Phi(e_n\mu) = e_n\mu$.

Denote by N the atomic von Neumann subalgebra of all elements x from M, for which $xe_n = \lambda_n$ for some complex numbers λ_n, $n = 1, \ldots$. It is evident that N is identified with the algebra l_∞ of all bounded sequences of complex numbers. Since $\Phi(e_n) = e_n$, $n = 1, 2, \ldots$,it follows that $\Phi(z) = z$ for all $z \in N$. If $z \in L^\omega(N,\mu) \bigcap K(N,\mu) = L^\omega(N,\mu)$, $z \geq 0$, then $z = \sup_{m\geq 1} z\sum_{n=1}^m e_n$, and $\left(z\sum_{n=1}^m e_n\right) \in N_+$. Therefore,

$$\Phi(z) = \sup_{m\geq 1} \Phi(z\sum_{n=1}^{m} e_n) = \sup_{m\geq 1} z\sum_{n=1}^{m} e_n = z.$$

Thus the restriction of Φ on $L^\omega(N,\mu)$ coincides with the identity mapping. It means that $L^\omega(N,\nu) = \Phi(L^\omega(N,\mu)) = L^\omega(N,\mu)$.

Therefore, according to the theorem 4 $h \in \bigcup_{1<p\leq\infty} L^p(N,\nu)$, and $h^{-1} \in \bigcup_{1<p\leq\infty} L^p(N,\mu)$, where h is the Radon-Nikodym's derivative of the trace μ relative the trace ν, being considered in N. So using the equality $he_n = \mu_n\nu_n^{-1}e_n$, $n = 1, 2, \ldots$, the required inequalities follow from the condition 2).

2) $\to$ 1). Let the inequalities from the condition 2) hold. Consider the faithful normal finite trace on M given by the equality

$$\lambda(x) = \sum_{n=1}^{\infty} \nu_n\mu_n^{-1}\mu(e_n x), \ \ x \in M.$$

Since x_n is a homogeneous Boolean algebra and $\lambda(e_n) = \nu_n = \nu(e_n)$, using the proof of proposition 1, construct a $*$-isomorphism $\Phi_n : K(e_nM,\nu) \to K(e_nM,\lambda)$ for which $\nu(y) = \lambda(\Phi_n(y))$ for all $y \in L^1(e_nM,\nu)$. For each $x \in K(M,\nu)$ denote by $\psi(\lambda)$ such an element from $K(M,\lambda)$ for which $e_n\psi(x) = \Phi_n(e_nx)$. It is evident that ψ is a $*$-isomorphism from $K(M,\nu)$ on $K(M,\lambda)$. At the same time, if $x \in L^1_+(M,\nu)$, then

$$\nu(x) = \sum_{n=1}^{\infty} \nu(e_nx) = \sum_{n=1}^{\infty} \lambda(\Phi_n(e_nx)) =$$

$$\sum_{n=1}^{\infty} \lambda(e_n\psi(x)) = \lambda(\psi(x)),$$

therefore $\psi(L^\omega(M,\nu)) = L^\omega(M,\lambda)$.

Let is show that $L^\omega(M,\lambda) = L^\omega(M,\mu)$. Let h be such an element from $K(M,\mu)$ that $he_n = \mu_n\nu_n^{-1}e_n$. For every $x \in M$ we have

$$\lambda(hx) = \sum_{n=1}^{\infty} \lambda(he_nx) = \sum_{n=1}^{\infty} \mu_n\nu_n^{-1}\lambda(e_nx) =$$
$$= \sum_{n=1}^{\infty} \mu(e_nx) = \mu(x),$$

therefore $h = \frac{d\mu}{d\lambda}$. According to the inequalities from the condition 2, we obtain that

$$h^{-1} \in \bigcup_{1<p\leq\infty} L^p(M,\mu).$$

If $\sup\limits_{n\geq 1}(\mu_n\nu_n^{-1}) < \infty$, then $h \in M$.

Suppose that $\sum_{n=1}^{\infty} \mu_n^p\nu_n^{1-p} < \infty$ for some $p \in (1,\infty)$. Then

$$\lambda(h^p) = \sum_{n=1}^{\infty} \nu_n\mu_n^{-1}\mu(e_nh^p) = \sum_{n=1}^{\infty} \mu_n^p\nu_n^{1-p} < \infty.$$

Thus,

$$h \in \bigcup_{1<p\leq\infty} L^p(M,\lambda)$$

and, using the theorem 4, we get $L^\omega(M,\lambda) = L^\omega(M,\mu)$.

Therefore $\psi(L^\omega(M,\nu)) = L^\omega(M,\mu)$.

Remarks 2. Repeating the argument from the proof of the theorem 5, it is easy to obtain the following criterion of $*$-isomorphism of the Arens algebras $L^\omega(l_\infty,\mu)$ and $L^\omega(l_\infty,\nu)$:

Let μ and ν be faithful normal finite traces on a infinite dimensional atomic commutative von Neumann algebra N, $\{q_n\}_{n=1}^{\infty}$ – the set of all atoms in $P(N)$, $\mu_n = \mu(q_n)$, $\nu_n = \nu(q_n)$, $n = 1,2,\ldots$. Then, the Arens algebras

$L^\omega(N,\mu)$ and $L^\omega(N,\nu)$ are $*$-isomorphic only in the case when there are such p, $q \in (1,\infty)$ and permutation π of a set of natural numbers, that

$$\sum_{n=1}^{\infty} \mu_n^p \nu_{\pi(n)}^{1-p} < \infty, \quad \sum_{n=1}^{\infty} \nu_{\pi(n)}^q \mu_n^{1-q} < \infty, \quad \text{in the case } \; p,q \in (1,\infty)$$

and $\sup\limits_{n\geq 1} |\mu_n \nu_n^{-1}| < \infty$ if $p=\infty$, $\sup\limits_{n\geq 1} |\nu_n \mu_n^{-1}| < \infty$ if $q=\infty$.

3. Any von Neumann algebra M is represented as $M = M_1 + M_2$, where M is an atomic von Neumann algebra and M_2 is a non-atomic von Neumann algebra. Moreover, if Φ is a $*$-automorphism of M, then $\Phi(M_1) = M_1$ and $\Phi(M_2) = M_2$. Therefore theorem 5 and Remark 2 give criterion of isomorphism of Arens algebras for arbitrary commutative σ-finite von Neumann algebras

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