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STRONG CONVERGENCE AND DINI THEOREMS FOR NON-UNIFORM SPACES

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ABSTRACT. We use the strong convergence to obtain some new insight concerning Dini theorem. A new concept generalizing the concepts of upper and lower semicontinuity is introduced and used to obtain some results for functions ant multifunctions at the same time. A version of Dini theorem for functions with non-uniform range space is proved, and some older results are improved. It is shown, that even classical assumptions of the Dini theorem imply a far stronger convergence than the mere uniform one.

RÉSUMÉ. Nous démontrons une version renforcée du théorème de Dini à partir de la notion de convergence forte. Nous introduisons de nouvelles notions de semi-continuité inférieure et supérieure, ainsi q'une notion de convergence monotone, qui généralisent les définitions classiques. Avec ces nouvelles définitions, nous obtenons une généralisation du théorème de Dini aux cas où l'espace d'arrivée n'est pas un espace uniforme. Celle-ci s'applique aussi bien aux fonctions qu'aux multi-fonctions. De plus, même sous les hypothèses classiques, le mode de convergence est strictement plus fort que la convergence uniforme.

Introduction. The methodical approach of this article was influenced by an idea of Ivan Kupka, which was developed when he worked on some problems concerning selections and fixed points in very general spaces (see [6] and [7]). This idea preaches "to replace ε by ε-covers and then the ε-covers by all covers". This enables us to work in topological spaces with "uniform-like" convergences. The new concept of the space of semicontinuity introduced in Definition 1 and the related concept of lower and upper semicontinuity from Definition 3 have the roots in the cited papers [6] and [7], too.

Using the open covers we define the strong convergence of the nets of functions with values in a topological space. If the range space is uniformisable, then the strong convergence is stronger than the uniform one but in general they are not equivalent as an example of [9] shows.

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The strong convergence does not need a uniform structure but still preserves some properties of functions which are not preserved by the uniform convergence, for exemple the fixed point property. The following theorem can be proved:

Let $(f_\gamma\colon X \to Y)_{\gamma\in\Gamma}, (g_\gamma\colon X \to Y)_{\gamma\in\Gamma}$ be nets of functions converging strongly to the functions f and g respectively. If for each $\gamma \in \Gamma$ exist a point $x_\gamma \in X$ such that $f_\gamma(x_\gamma) = g_\gamma(x_\gamma)$, then under very general conditions on the space X, Y there exists such a point x that $f(x) = g(x)$ ([8]).

In what follows we show that using strong convergence we can prove some generalizations of Dini type theorems in two directions. The range space suffices to be only a topological one and also the convergence is stronger than the uniform one. Moreover, there is no order on the range space and we obtain results for functions and multifunctions at the same time.

For example in [3] Beer has given a sufficient condition which ensures a uniform convergence of a sequence of multifunctions $(F_n\colon X \to Y)_{n\geq 1}$ to its "intersection" F satisfying $F(x) = \bigcap_{k\geq 1} F_k(x)$ for each x in X. Beer supposed the target space to be a metric one. The new approach presented here permits us to suppose Y to be an arbitrary topological space. We prove an assertion for nets of multifunctions, which is both a generalization and strengthening of the Beer's result.

Strong convergence and the Dini type theorem. To make this paper selfcontained we recall the definition of the strong convergence introduced in [9].

Definition 0. *Let X be an arbitrary set and $(Y, \mathcal{T})$ a topological space. Let us denote $\mathcal{P}$ the family of all open covers of the space Y. If $\wp \in \mathcal{P}$, we say that a net of functions $(f_\gamma\colon X \to Y)_{\gamma\in\Gamma}$ converges $\wp$-uniformly to a function $f\colon X \to Y$ if*

$$\exists\gamma_0 \in \Gamma\ \forall\gamma \geq \gamma_0\ \forall x \in X\ \exists O \in \wp\colon\ f_\gamma(x), f(x) \in O$$

We say that the net (f_γ) converges strongly to the function f if it converges $\wp$-uniformly for each $\wp \in \mathcal{P}$

The interested reader can find more details about this type of convergence in [9].

Definition 1. *An ordered quadruple $(Y, \mathcal{T}, \mathcal{U}, \mathcal{L})$ will be called the space of semicontinuity if $(Y, \mathcal{T})$ is a topological space, $\mathcal{U} \subset \mathcal{T}$, $\mathcal{L} \subset \mathcal{T}$ and the collection of sets $\mathcal{U} \sqcap \mathcal{L} := \{A \cap B;\ A \in \mathcal{U}$ and $B \in \mathcal{L}\}$ forms a base of the topology $\mathcal{T}$.*

Definition 2. *We say, that a net $(y_\gamma\colon \gamma \in \Gamma)$ of points of a space Y of semicontinuity converges monotonically from above to a point $y \in Y$ if the following three conditions are fulfilled:*

(l) $\forall L \in \mathcal{L}$ if $y \in L$ then $\forall\gamma \in \Gamma\colon\ y_\gamma \in L$;

(u) $\forall U \in \mathcal{U}$: 1. if there exists $\gamma \in \Gamma$ such that $y_\gamma \in U$ then $y \in U$;

2. if $\beta \in \Gamma$ and $y_\beta \in U$ then $\forall\delta \geq \beta\colon\ y_\delta \in U$ holds.

(c) The net $(y_\gamma\colon \gamma \in \Gamma)$ converges to y in the topology $\mathcal{T}$.

Definition 3. *Let (X, T) be a topological space, let $(Y, \mathcal{T}, \mathcal{U}, \mathcal{L})$ be a space of semicontinuity. A function $f\colon X \to Y$ is said to be lower semicontinuous (upper semicontinuous) if $\forall L \in \mathcal{L}\colon f^{-1}(L) \in T \quad (\forall U \in \mathcal{U}\colon f^{-1}(U) \in T)$ holds.*

Remark 1. A natural model of a space of semicontinuity is the space $(\mathbb{R}, \mathcal{T}, \mathcal{U}, \mathcal{L})$, where $(\mathbb{R}, \mathcal{T})$ is the set of real numbers with its natural topology, $\mathcal{U} := \{(-\infty, a);\ a \in \mathbb{R}\}$ and $\mathcal{L} := \{(b, +\infty);\ b \in \mathbb{R}\}$. In this case our definition of lower (upper) semicontinuity coincides with the classical one. It is well known that having a net of u.s.c. functions (f_γ) defined on a compact topological space X and with the range $\mathbb{R}$ which converges pointwise monotonically to a l.s.c. function $f\colon X \to \mathbb{R}$ from above, we can conclude that f is continuous. It is easy to see that this assertion remains true when we replace $\mathbb{R}$ with a space of semicontinuity and consider the lower (upper) semicontinuity in the sense of Definition 3. To realize this, it suffices to carefully examine the proof of the next theorem.

Theorem 1. *Let (X, T) be a compact topological space and $(Y, \mathcal{T}, \mathcal{U}, \mathcal{L})$ be a space of semicontinuity. Let $f\colon X \to Y$ be a lower semicontinuous function. Let $(f_\gamma\colon \gamma \in \Gamma)$ be a net of upper semicontinuous functions from X into Y. Let the net $(f_\gamma\colon \gamma \in \Gamma)$ converge to the function f pointwise monotonically from above, i.e. $\forall x \in X$, $f_\gamma(x)$ converges monotonically to $f(x)$ from above. Then the net $(f_\gamma\colon \gamma \in \Gamma)$ converges to the function f strongly.*

Proof. Let $\wp \subset \mathcal{T}$ be an open cover of Y. We have to show that the functions f_γ converge to f $\wp$-uniformly in the sense of the Definition 0. We will prove more by proving:

$$(*) \qquad \exists\, \alpha \in \Gamma\ \forall x \in X\ \exists\, O_x \in \wp\ \forall \gamma \geq \alpha\colon\ f(x) \in O_x \text{ and } f_\gamma(x) \in O_x.$$

So let $x \in X$. Then there exists an open set $O_x \in \wp$, such that $f(x) \in O_x$ and there exist sets $U_x \in \mathcal{U}$, $L_x \in \mathcal{L}$ such, that $f(x) \in U_x \cap L_x \subset O_x$. From the pointwise convergence of the functions f_γ to f it follows that there exists a $\gamma_x \in \Gamma$ such that $f_{\gamma_x}(x) \in U_x \cap L_x$. Let us denote $V_x := f_{\gamma_x}^{-1}(U_x) \cap f^{-1}(L_x)$, which is an open set. Let $z \in V_x$. Then $f_{\gamma_x}(z) \in U_x$ and therefore, since the net $(f_\gamma(z)\colon \gamma \in \Gamma)$ converges monotonically to $f(z)$ from above, $\forall \gamma \geq \gamma_x\colon f_\gamma(z) \in U_x$ and $f(z) \in U_x$ holds. The same argument gives us $\forall \gamma \in \Gamma\colon f_\gamma(z) \in L_x$ because of $z \in f^{-1}(L_x)$. Therefore

$$(1) \qquad \forall z \in V_x\colon\ f(z) \in U_x \cap L_x \subset O_x$$

$$(2) \qquad \forall \gamma \geq \gamma_x\colon\ f_\gamma(z) \in U_x \cap L_x \subset O_x.$$

Since $\{V_x;\ x \in X\}$ is an open cover of the compact space X, we can extract a finite subcover $\{V_{x_1}, \dots, V_{x_n}\}$. Let us choose $\alpha \in \Gamma$ such that $\alpha \geq \gamma_{x_i}$ for each $i \in \{1, 2, \dots, n\}$. Then from (1) and (2) we have (*). Q.e.d.

To illustrate the generality of Theorem 1, two corollaries are in order. First, considering what Remark 1 says, we can see that the following strengthened version of the "classical" Dini theorem holds:

Corollary 1. *Let $(f_k\colon\ k \in \mathbb{N})$ be a sequence of u.s.c. real valued functions defined on a compact topological space X. Suppose that f is a l.s.c. function defined on X and that $(f_k(x)\colon\ k \in \mathbb{N})$ converges monotonically to $f(x)$ from above at each point $x \in X$. Then $(f_k\colon\ k \in \mathbb{N})$ converges to f strongly (and therefore uniformly).*

To present a result concerning multifunctions which can be deduced from Theorem 1 we need to recall some notions and basic results concerning multifunctions and hyperspaces. They can be found in many books or papers, e.g. [1], [2], [5], sometimes in slightly modified form than presented here. In what follows, X and Y are topological spaces, $2^Y := \{A \subset Y \mid A \textit{ is nonempty and closed}\}$ and $C(Y) := \{A \subset Y \mid A \textit{ is nonempty and compact}\}$. By a multifunction from X to Y we mean a function F from X to 2^Y and we shall denote it by $F:\ X \longrightarrow\!\!\!\!\prec Y$. A multifunction $F:\ X \longrightarrow\!\!\!\!\prec Y$ is said to be upper semicontinuous - u.s.c. (lower semicontinuous - l.s.c) if for every open $V \subset Y$ the set $F^+(V) := \{z \in X \mid F(z) \subset V\}$ $\ (F^-(V) := \{z \in X \mid F(z) \cap V \neq \emptyset\})$ is open. It is said to be continuous if it is l.s.c. and u.s.c..

Three topologies can be defined on 2^Y and in the same way on $C(Y)$. In both cases we shall denote them by the same symbols and they are defined as follows:

1. So-called "upper-Vietoris" topology $\mathcal{U}$. Its base consists of all sets of the form $< U > := \{A \in 2^Y \mid A \subset U\}$, where U is an arbitrary open subset of Y.
2. So-called "lower-Vietoris" topology $\mathcal{L}$. Its base consists of all sets of the form $(V_1, \ldots, V_n) := \{A \in 2^Y \mid A \cap V_i \neq \emptyset,\ i = 1, 2, \ldots, n\}$, where $n \in \mathbb{N}$ and $V_1, \ldots, V_n$ are open.
3. Vietoris topology $\mathcal{V}$, whose base consists of all sets of the form $[U; V_1, \ldots, V_n] := \{A \in 2^Y \mid A \subset U$ and $A \cap V_i \neq \emptyset,\ i = 1, 2, \ldots, n\}$, where n is a positive integer and the sets $U, V_1, \ldots, V_n$ are open.

A closed-valued multifunction $F\colon X \longrightarrow\!\!\!\!\prec Y$ is u.s.c. (l.s.c., continuous) iff the function $F\colon X \to (2^Y, \mathcal{U})$ $\ (F\colon X \to (2^Y, \mathcal{L}),\ F\colon X \to (2^Y, \mathcal{V}))$ is continuous.

If the space Y is moreover a metric space with a metric d, the so-called Hausdorff metric can be defined for $C(Y)$ as follows: if $A,\ B \in C(Y)$ then

$$H(A, B) = \max\{\sup\{d(x, B)\colon\ x \in A\};\ \sup\{d(y, A)\colon\ y \in B\}\}.$$

If we consider the set $C(Y)$, then the Vietoris topology for $C(Y)$ coincides with the topology induced on this set by the Hausdorff metric.

Remark 2. Concerning the topologies $\mathcal{V}, \mathcal{U}, \mathcal{L}$ on 2^Y mentioned above it is well known that $\mathcal{U} \cup \mathcal{L}$ is a subbase for $\mathcal{V}$, so the quadruple $(2^Y, \mathcal{V}, \mathcal{U}, \mathcal{L})$ is a space of semicontinuity. Moreover, the lower (upper) semicontinuity and the continuity for multifunctions $F\colon X \to$

$(2^Y, \mathcal{V}, \mathcal{U}, \mathcal{L})$ in the sense of Definition 3 coincides with the classical ones introduced above. The same holds for the space $(C(Y), \mathcal{V}, \mathcal{U}, \mathcal{L})$.

Corollary 2. *Let X be a compact topological space. Let (Y,T) be a topological space and let $(C(Y), \mathcal{V}, \mathcal{U}, \mathcal{L})$ be the space of semicontinuity related to (Y,T). Let $(F_\gamma\colon \gamma \in \Gamma)$ be a net of u.s.c. multifunctions from X into Y, let $F\colon X \multimap Y$ be a l.s.c. multifunction. Let all F_γ and F have compact values. Let $\forall x \in X$, the following holds:*

(1) if $\alpha, \beta \in \Gamma$, $\alpha \geq \beta$ then $F_\beta(x) \supset F_\alpha(x) \supseteq F(x)$;

(2) $\bigcap_{\gamma \in \Gamma} F_\gamma(x) = F(x)$.

Then the net $(F_\gamma\colon \gamma \in \Gamma)$ converges to F strongly in Vietoris topology. Moreover, F is - according to Remark 2 - u.s.c. and is therefore continuous.

Proof. It sufficies to show that functions $F_\gamma\colon X \to (C(Y), \mathcal{V}, \mathcal{U}, \mathcal{L})$ converge pointwise monotonically to $F\colon X \to (C(Y), \mathcal{V}, \mathcal{U}, \mathcal{L})$ from above. But it is true, because for each $x \in X$ the following holds:

(l) $\forall V \in T$: if $F(x) \cap V \neq \emptyset$ then $\forall \gamma \in \Gamma$: $F_\gamma(x) \cap V \neq \emptyset$ (which is equivalent to $\forall L \in \mathcal{L}$: if $F(x) \in L$, then $\forall \gamma \in \Gamma$: $F_\gamma(x) \in L$);

(u) $\forall U \in T$; 1. if there exists $\gamma \in \Gamma$ such that $F_\gamma(x) \subset U$, then $F(x) \subset U$;
2. if $\beta \in \Gamma$ and $F_\beta(x) \subset U$ then $\forall \delta \geq \beta$: $F_\delta(x) \subset U$ holds.

(c) The net $(F_\gamma(x)\colon \gamma \in \Gamma)$ converges to $F(x)$ with respect to the Vietoris topology.

(The easy reformulation of (u) and (c) into the language of Definition 2 is analogous to (l) and is left to the reader. Compactness of the values is essential for (c).) Q.e.d.

The following Dini theorem is due to G. Beer.

Corollary 3. (Theorem 3 of [3].)*Let X be a compact topological space and let Y be a metric space. Suppose that $F\colon X \multimap Y$ is a l.s.c. compact valued multifunction. For $k = 1, 2, 3, \dots$ let $F_k\colon X \multimap Y$ be an u.s.c. compact valued multifunction such that for each $x \in X$ and each integer k we have $F_k(x) \supset F_{k+1}(x) \supset F(x)$. If $\bigcap_{k=1}^{\infty} F_k(x) = F(x)$ for each x, then $(F_k)_{k \geq 1}$ converges to F uniformly in the Hausdorff metric.*

Proof. According to Corollary 2 the sequence $(F_k\colon k \in \mathbb{N})$ converges to F strongly with respect to the Vietoris topology. But the values of our multifunctions are compact and the Vietoris topology on $C(Y)$ is equal to the topology derived from the Hausdorff metric on $C(Y)$. Hence $(F_k\colon k \in \mathbb{N})$ converges to $F\colon X \to C(Y)$ strongly also with respect to this topology. And strong convergence implies the uniform one, the fact which is proved among others in [9]. Q.e.d.

An interesting generalization of Corollary 3 for uniform spaces is in [10]. This result and our Corollary 2 don't imply each other.

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