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Convolution of Nörlund methods in non-archimedean fields

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Abstract

In the present paper we obtain a few inclusion theorems for the convolution of Nörlund methods in the form $(N, r_n) \subseteq (N, p_n) * (N, q_n)$ in complete, non-trivially valued, non-archimedean fields.

Throughout the present paper K denotes a complete, non-trivially valued, non-archimedean field. Infinite matrices and sequences, which are considered in the sequel, have entries in K. If $A = (a_{nk})$, $a_{nk} \in K$, $n, k = 0, 1, 2, \dots$ is an infinite matrix, the A-transform $Ax = \{(Ax)_n\}$ of the sequence $x = \{x_k\}$, $x_k \in K$, $k = 0, 1, 2, \dots$ is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k, \quad n = 0, 1, 2, \dots,$$

where it is assumed that the series on the right converge. If $\lim_{n\to\infty} (Ax)_n = s$, we say that $\{x_k\}$ is A-summable to s, written as $x_k \to s(A)$ or A-lim $x_k = s$. If $\lim_{n\to\infty} (Ax)_n = s$ whenever $\lim_{k\to\infty} x_k = s$, we say that A is regular. The following result is well-known (see [2], [4]).

Theorem 1 $A = (a_{nk})$ *is regular if and only if*

$$\sup_{n,k} |a_{nk}| < \infty; \tag{1}$$

$$\lim_{n\to\infty} a_{nk} = 0, \quad \textit{for every fixed } k; \tag{2}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = 1. \tag{3}$$

Any matrix A for which (1) holds is called a K_r-matrix. If A and B are two infinite matrices such that $x_k \to s(A)$ implies $x_k \to s(B)$, we say that A is included in B, written as $A \subseteq B$. A is said to be row-finite if for $n = 0, 1, 2, \ldots$, there exists a positive integer k_n such that $a_{nk} = 0$, $k > k_n$.

Given two infinite matrices A and B, their convolution is defined as the matrix $C = (c_{nk})$, where

$$c_{nk} = \sum_{i=0}^{k} a_{ni} b_{n,k-i}, \quad n, k = 0, 1, 2, \ldots . \tag{4}$$

In such a case we write $C = A * B$.

The following properties of the convolution can be easily proved.

1. *If A and B are both row-finite or both K_r, then their convolution C is row-finite or K_r respectively and their row sums satisfy*

$$\sum_{k=0}^{\infty} c_{nk} = \left(\sum_{k=0}^{\infty} a_{nk}\right)\left(\sum_{k=0}^{\infty} b_{nk}\right), \quad n = 0, 1, 2, \ldots . \tag{5}$$

2. *If A, B are both regular, then C is regular too.*

The Nörlund method of summability i.e., (N, p_n) method in K is defined as follows (see [5]): (N, p_n) is defined by the infinite matrix (a_{nk}) where

$$a_{nk} = \frac{p_{n-k}}{P_n}, \quad k \leq n;$$
$$= 0, \quad k > n,$$

where $p_0 \neq 0$, $|p_0| > |p_j|$, $j = 1, 2, \ldots$ and $P_n = \sum_{k=0}^{n} p_k$, $n = 0, 1, 2, \ldots$. It is to be noted that $|P_n| = |p_0| \neq 0$ so that $P_n \neq 0$, $n = 0, 1, 2, \ldots$.
The following result is very useful in the sequel.

Theorem 2 (See [5], Theorem 1.) (N, p_n) *is regular if and only if*

$$p_n \to 0, \ n \to \infty. \tag{6}$$

The purpose of the present paper is to prove a few inclusion theorems for the convolution of Nörlund methods in the form $(N, r_n) \subseteq (N, p_n) * (N, q_n)$.

We need to define $\{\overline{p}_n\}$ by

$$p_0\overline{p}_0 = 1,\ p_0\overline{p}_n + p_1\overline{p}_{n-1} + \cdots + p_n\overline{p}_0 = 0, \quad n \geq 1 \tag{7}$$

i.e., $\overline{p}(x) = \sum\limits_{n=0}^{\infty} \overline{p}_n x^n = \dfrac{1}{\sum\limits_{n=0}^{\infty} p_n x^n} = \dfrac{1}{p(x)}$, assuming that these series converge.

The following result is an easy consequence of Kojima-Schur theorem (see [2], [4]).

Lemma 1 *Let $A = (a_{nk})$ be a row-finite matrix and (N, p_n) be a regular Nörlund method. Then A-$\lim x_k$ exists whenever (N, p_n)-$\lim x_k$ exists if and only if*

$$\sup_{0\leq\gamma\leq k_n} |P_\gamma \sum_{k=\gamma}^{k_n} a_{nk}\overline{p}_{k-\gamma}| = O(1), \quad n \to \infty; \tag{8}$$

$$\lim_{n\to\infty} \sum_{k=\gamma}^{k_n} a_{nk}\overline{p}_{k-\gamma} = \delta_\gamma, \quad \textit{for every fixed } \gamma; \tag{9}$$

and

$$\lim_{n\to\infty} \sum_{k=0}^{k_n} a_{nk} = \delta. \tag{10}$$

Corollary 1 *$(N, p_n) \subseteq A$ if and only if (8), (9) and (10) hold with $\delta_\gamma = 0$, $\gamma = 0, 1, 2, \dots$ and $\delta = 1$.*

Corollary 2 *If (N, p_n) and (N, q_n) are regular Nörlund methods, then $(N, p_n) \subseteq (N, q_n)$ if and only if $h_n \to 0$, $n \to \infty$ where*

$$h(x) = \sum_{n=0}^{\infty} h_n x^n = \frac{\sum\limits_{n=0}^{\infty} q_n x^n}{\sum\limits_{n=0}^{\infty} p_n x^n} = \frac{q(x)}{p(x)}$$

(see [5]).

Let (N, p_n), (N, q_n), (N, r_n) be regular Nörlund methods. Let $p_n(x) = \sum\limits_{i=n}^{\infty} p_i x^i$, $p_0(x) = p(x)$ with similar expressions for $q_n(x)$, $r_n(x)$. Let

$$\begin{aligned} \frac{p(x)q(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \theta_\gamma x^\gamma; \\ \frac{p_{n+1}(x)q(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^\gamma; \\ \frac{p(x)q_{n+1}(x)}{r(x)} &= \sum_{\gamma=0}^{\infty} \beta_{n\gamma} x^\gamma, \end{aligned} \tag{11}$$

and

$$\frac{1}{r(x)}\{p(x)q(x) - p_{n+1}(x)q(x) - p(x)q_{n+1}(x)\} = \sum_{\gamma=0}^{\infty} \varphi_{n\gamma} x^{\gamma}. \tag{12}$$

It now follows that

$$\varphi_{n\gamma} = \theta_{\gamma} - \alpha_{n\gamma} - \beta_{n\gamma} \tag{13}$$

and

$$\alpha_{n\gamma} = \beta_{n\gamma} = 0, \quad 0 \le \gamma \le n. \tag{14}$$

Taking $C = (N, p_n) * (N, q_n)$, C is a row-finite matrix with

$$c_{nk} = \frac{1}{P_n Q_n} \sum_{i=\max(0,k-n)}^{\min(k,n)} p_{n-i} q_{n-k+i} \quad \text{with } k_n = 2n. \tag{15}$$

We write

$$f_{n\gamma} = \sum_{k=\max(0,2n-\gamma)}^{2n} c_{nk} \overline{r}_{k+\gamma-2n}, \quad n, \gamma \ge 0. \tag{16}$$

Lemma 2

$$P_n Q_n f_{n\gamma} = \varphi_{n\gamma}, \quad 0 \le \gamma \le 2n+1. \tag{17}$$

Proof. The result follows as in [6].

Theorem 3 $(N, p_n) * (N, q_n)$-$\lim x_k$ *exists whenever* (N, r_n)-$\lim x_k$ *exists if and only if*

$$\sup_{0 \le \gamma \le 2n} |R_{2n-\gamma} \varphi_{n\gamma}| = O(1), \quad n \to \infty; \tag{18}$$

and

$$\lim_{n\to\infty} \frac{\varphi_{n,2n-\gamma}}{P_n Q_n} = \delta_{\gamma}, \quad \textit{for every fixed } \gamma. \tag{19}$$

Proof. Let $(N, p_n) * (N, q_n)$-$\lim x_k$ exist whenever (N, r_n)-$\lim x_k$ exists. Applying Lemma 1 with $(N, p_n) = (N, r_n)$ and $A = (N, p_n) * (N, q_n) = (c_{nk})$, we have,

$$\sup_{0 \le \gamma \le 2n} |R_{\gamma} \sum_{k=\gamma}^{2n} c_{nk} \overline{r}_{k-\gamma}| = O(1), \quad n \to \infty \tag{20}$$

and

$$\lim_{n\to\infty} \sum_{k=\gamma}^{2n} c_{nk} \overline{r}_{k-\gamma} = \delta_{\gamma}, \quad \text{for every fixed } \gamma. \tag{21}$$

Using (16) and (20), we get

$$\sup_{0 \le \gamma \le 2n} |R_{2n-\gamma} f_{n\gamma}| = O(1), \quad n \to \infty. \tag{22}$$

Using (17), we note that $|f_{n\gamma}| = \dfrac{|\varphi_{n\gamma}|}{|p_0|\,|q_0|}$ since $|P_n| = |p_0|$ and $|Q_n| = |q_0|$. Consequently, in view of (22), we get

$$\sup_{0\le\gamma\le 2n} |R_{2n-\gamma}\varphi_{n\gamma}| = O(1), \quad n \to \infty.$$

In view of (16) and (21), we have

$$\lim_{n\to\infty} f_{n,2n-\gamma} = \delta_\gamma, \quad \text{for every fixed } \gamma.$$

Now, using (17), we get

$$\lim_{n\to\infty} \frac{\varphi_{n,2n-\gamma}}{P_n Q_n} = \delta_\gamma, \quad \text{for every fixed } \gamma.$$

Thus (18) and (19) hold. Conversely (18) and (19) imply (20) and (21) respectively. Using (5), we have, $\lim_{n\to\infty} \sum_{k=0}^{2n} c_{nk} = 1$. Using Lemma 1, the result follows, completing the proof of the theorem.

Corollary 3 *$(N, r_n) \subseteq (N, p_n) * (N, q_n)$ if and only if (18) and (19) hold with $\delta_\gamma = 0$.*

Corollary 4 *If $\lim_{n\to\infty} \bar{r}_n = 0$, then $(N, r_n) \subseteq (N, p_n) * (N, q_n)$ if and only if (18) holds.*

Proof. The result follows using (9) and the fact that $(N, p_n) * (N, q_n)$ is regular.

Theorem 4 *If*

$$\varphi_{n,2n-\gamma} = o(1), \quad n \to \infty, \quad \textit{for every fixed } \gamma, \tag{23}$$

and either

$$\varphi_{n\gamma} = O(1), \quad n, \gamma \to \infty, \tag{24}$$

or

$$\theta_\gamma, \alpha_{n\gamma}, \beta_{n\gamma} = O(1), \quad n, \gamma \to \infty, \tag{25}$$

then

$$(N, r_n) \subseteq (N, p_n) * (N, q_n).$$

Proof. Using (23), (19) follows with $\delta_\gamma = 0$ since $|P_n| = |p_0|$ and $|Q_n| = |q_0|$. Because of (13) and (25), (24) holds. So if (24) or (25) holds, (18) holds since $R_n = O(1)$, $n \to \infty$, (N, r_n) being a regular method. The result now follows from Corollary 3.

We shall now take up an application of Theorem 4.

Theorem 5 *Let* $\overline{p}_n, \overline{q}_n \to 0$, $n \to \infty$ *and* $t_n = p_0q_n + p_1q_{n-1} + \cdots + p_nq_0$, $n = 0, 1, 2, \ldots$. *Then*

$$(N, t_n) \subseteq (N, p_n) * (N, q_n)$$

and

$$(N, p_n) \subseteq (N, t_n), \ \ (N, q_n) \subseteq (N, t_n).$$

Proof. We apply Theorem 4 with $r_n = t_n$. With the usual notation we have $t(x) = p(x)q(x)$ and $\overline{t}(x) = \overline{p}(x)\overline{q}(x)$. Since $\overline{p}_n, \overline{q}_n \to 0$, $n \to \infty$, $\overline{t}_n \to 0$, $n \to \infty$ (see [3], Theorem 1). Consequently (23) follows using (9). In view of (11), we have,

$$\sum_{\gamma=0}^{\infty} \theta_\gamma x^\gamma = \frac{p(x)q(x)}{t(x)} = 1,$$

so that

$$\theta_0 = 1 \text{ and } \theta_\gamma = 0, \ \gamma \geq 1;$$

$$\sum_{\gamma=0}^{\infty} \alpha_{n\gamma} x^\gamma = \frac{p_{n+1}(x)q(x)}{t(x)} = p_{n+1}(x)\overline{p}(x),$$

so that

$$\begin{aligned} \alpha_{n\gamma} &= \sum_{\lambda=0}^{\gamma-(n+1)} \overline{p}_\lambda p_{\gamma-\lambda}, \ \ \gamma \geq n+1; \\ &= 0, \ \ 0 \leq \gamma \leq n. \end{aligned}$$

Consequently $\alpha_{n\gamma} = O(1)$, $n, \gamma \to \infty$. Similarly $\beta_{n\gamma} = O(1)$, $n, \gamma \to \infty$. In view of Theorem 4, $(N, t_n) \subseteq (N, p_n) * (N, q_n)$. Now $\dfrac{t(x)}{p(x)} = q(x)$ and $q_n \to 0$, $n \to \infty$, (N, q_n) being regular, by (6). So by Corollary 2, $(N, p_n) \subseteq (N, t_n)$. Similarly $(N, q_n) \subseteq (N, t_n)$. The proof of the theorem is now complete.

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