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(*t*-)orthogonality in *p*-adic Banach spaces

(t-)ORTHOGONALITY IN P-ADIC BANACH SPACES

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Abstract. Let $(E, \| \cdot \|)$ be a Banach space over $\mathbf{K}$. We shall prove that orthogonality in E can be expressed in terms of distances to convex hulls of points in E instead of distances to the linear hulls of these same points, which gives in fact a new characterization of orthogonality. Also, for an absolutely convex, weakly c'-compact subset of a Banach space having the Hahn-Banach extension property, an equivalent condition is given for the compactoidness of the set.

INTRODUCTION

(For unexplained terms, see below). Since 1986, when it was proved in [8] that each weakly c'-compact set is c-compact (and hence compactoid) in case E is a Banach space with a base, the following question kept popping up :

Is every weakly c'-compact set in a strongly polar K-Banach space necessarily c'-compact? Progress came about in 1989 ([1], theorem 5.2.13) when this could be proved for an arbitrary Banach space over a spherically complete K and even more in 1992, when it was proved that this statement is also true assuming the continuum hypothesis and assuming K to have the cardinality of the continuum ([5], theorem 2.5). The general question for K not spherically complete however, remains open. In order to pursue the answer still further, I studied the technique used for the proof in [1]. Kernel of this technique is the relationship between a closed, absolutely convex subset A of E and $A/\overline{A^i}$ and more particularly, the relationship between (t-)orthogonality in E and in $A/\overline{A^i}$. This led to a new characterization of orthogonality in E (theorem 2.4) and also sheds some new light on the open problem stated above, in case E has the Hahn-Banach extension property. (Paragraph 3). As an important result, an equivalent condition for the compactoidness of a set is given.

NOTATIONS AND PRELIMINARIES

Throughout the text, K is a complete non-trivially valued field with valuation $|.|$ and k is its residue class field. We denote by $|K^*|$ the valuegroup of K. We put $B(0,1) = \{\lambda \in K \mid |\lambda| \leq 1\}$ and $B(0,1^-) = \{\lambda \in K \mid |\lambda| < 1\}$. If the valuation on K is discrete, there exists $\rho \in B(0,1^-)$ such that $|K^*| = \{|\rho|^n \mid n \in \mathbb{Z}\}$ and $B(0,1^-) = B(0,|\rho|)$. We denote by $\pi_1 : B(0,1) \to k : \lambda \to \overline{\lambda} = \lambda + B(0,1^-)$ the canonical surjection.

Throughout E is a K-Banach space with norm $||.||$. We write $(E, ||.||)$.

For $S \subset E$, we denote by coS the absolutely convex (a.c.) hull of S, by $\overline{co}S$ the closure of coS and by $[S]$ the linear hull of S.

For an absolutely convex subset A of E, we put $A^i = \{\lambda.a \mid \lambda \in B(0,1^-), a \in A\}$ and by $\partial A = A \backslash \overline{A^i}$. Note that A^i and $\overline{A^i}$ are absolutely convex.

For $A \subset E$ absolutely convex and closed, we denote by $\pi : A \to A/\overline{A^i} : x \to x + \overline{A^i}$ the canonical surjection. The quotient $V_A = A/\overline{A^i}$ is, in a natural way, a k-vector space. The formula $||\pi(x)|| = inf\{||x - a|| \mid a \in \overline{A^i}\}$ defines a norm on V_A for which renders it a k-Banach space ([3], proposition 3.2). If $A = \overline{co}X$, then $V_A = \overline{[\pi(X)]}$. ([3], proposition 3.4).

1. THE DISTANCE FUNCTION

1.1.LEMMA : Let $x \in E$ and $\emptyset \neq B \subset E$. Then $dist(x,B) = dist(x,\overline{B})$.

1.2.LEMMA : Let $B \subset E$ be a.c. Then $\overline{B^i} = \overline{(\overline{B})^i}$.
Proof : Straightforward.

1.3.LEMMA : Let $(a_i)_{i \in I}$ be a family in E. Then $z \in co(a_i \mid i \in I)^i$ if and only if there exists $J \subset I$ finite and $(\lambda_j)_{j \in J} \subset B(0,1^-)$ such that $z = \sum_{j \in J} \lambda_j.a_j$.

Proof : Straightforward.

1.4.PROPOSITION : Let $(a_i)_{i \in I}$ be a family in $E \backslash \{o\}$ and put $A = \overline{co}(a_i \mid i \in I)$. Then for each $i \in I$, $dist(a_i, \overline{A^i}) \geq dist(a_i, \overline{co}(a_j \mid j \neq i))$.
Proof : According to lemma 1.1, it suffices to prove that for each $i \in I$, $dist(a_i, A^i) \geq dist(a_i, co(a_j \mid j \neq i))$. Now, as $\overline{co(a_j \mid j \in I)^i} = \overline{A^i}$ (lemma 1.2), it follows by lemma 1.1 that for $x \in E$, $dist(x, \overline{A^i}) = dist(x, co(a_j \mid j \in I)^i)$.

Choose $i \in I$ and let $z \in co(a_j \mid j \in I)^i$. We are going to calculate $||a_i - z||$. According to lemma 1.3, $z = \sum_{j \in J} \lambda_j.a_j$ where $J \subset I$ is finite and $(\lambda_j)_{j \in J} \subset B(0,1^-)$.

There are two cases to consider :
i) $i \notin J$. Then $||a_i - z|| \geq dist(a_i, co(a_j \mid j \neq i))$, since $z \in co(a_j \mid j \neq i)$.

ii) $i \in J$. Then $\|a_i - z\| = \|(1 - \lambda_i)a_i - \sum_{j \in J \setminus \{i\}} \lambda_j . a_j\| = \|a_i - \sum_{j \in J \setminus \{i\}} \frac{\lambda_j}{1 - \lambda_i} a_j\|$, since $|1 - \lambda_i| = 1$.

Now, $\sum_{j \in J \setminus \{i\}} \frac{\lambda_j}{1 - \lambda_i} a_j \in co(a_j \mid j \neq i)$ and thus $\|a_i - z\| \geq dist(a_i, co(a_j \mid j \neq i))$.

Hence, for each $z \in co(a_j \mid j \in I)^i, \|a_i - z\| \geq dist(a_i, co(a_j \mid j \neq i))$ and thus $dist(a_i, \overline{A^i}) \geq dist(a_i, co(a_j \mid j \neq i))$, which completes the proof.

1.5.COROLLARY : Let $t \in (0, 1)$, let $(a_i)_{i \in I}$ be a (t-)orthogonal family in $(E, \|.\|)$ and put $A = \overline{co}(a_i \mid i \in I)$. Then for each $i \in I$, $\|\pi(a_i)\| \geq dist(a_i, co(a_j \mid j \neq i)) \geq (t).\|a_i\|$.

Proof : According to proposition 1.4, for each $i \in I$, $dist(a_i, \overline{A^i}) \geq dist(a_i, \overline{co}(a_j \mid j \neq i))$. On the other hand, for each $i \in I$, $dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq dist(a_i, \overline{[\{a_j \mid j \neq i\}]}) \geq (t).\|a_i\|$ and the statement follows, since $\|\pi(a_i)\| = dist(a_i, \overline{A^i})$. (See preliminaries).

1.6.LEMMA : Let $(a_i)_{i \in I}$ be a family in E such that for each $i \in I, dist(a_i, co(a_j \mid j \neq i)) > 0$. Then $(a_i)_{i \in I}$ is a linearly independent family in E.

Proof : Let $J \subset I$ be finite and let $(\lambda_j)_{j \in J} \subset \mathbf{K}^*$. It is no restriction to assume that $\max_{j \in J} |\lambda_j| = 1$. Choose $j_o \in J$ such that $|\lambda_{j_o}| = 1$.

Then $\|\sum_{j \in J} \lambda_j . a_j\| = \|a_{j_o} + \sum_{j \in J} \frac{\lambda_j}{\lambda_{j_o}} a_j\| \geq dist(a_{j_o}, co(a_j \mid j \neq j_o)) > 0$.

Hence, $(a_i)_{i \in I}$ is a linearly independent family in E.

2. THE IMAGE BY π OF A FAMILY IN E

2.1.THEOREM : Let $(a_i)_{i \in I}$ be a family in E and put $A = \overline{co}(a_i \mid i \in I)$. Then we have the following :

i) If for each $i \in I$, $dist(a_i, \overline{co}(a_j \mid j \neq i)) > 0$, then $(\pi(a_i))_{i \in I}$ is a linearly independent family in V_A.

ii) If there exists a $t \in (0, 1)$ such that for each $i \in I$, $dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq (t).\|\pi(a_i)\| > 0$, then $(\pi(a_i))_{i \in I}$ is a (t-)orthogonal base of $(V_A, \|.\|)$.

Proof : First note that for each $i \in I, a_i \neq o$. Now, let $J \subset I$ be finite and let $(\lambda_j)_{j \in J} \subset k^*$. For each $j \in J$, we choose $\mu_j \in B(0, 1)$ such that $\pi_1(\mu_j) = \lambda_j$. Both for i) as for ii), we have to calculate $\|\sum_{j \in J} \lambda_j . \pi(a_j)\| = dist(\sum_{j \in J} \mu_j . a_j, \overline{A^i})$ or equivalently, $dist(\sum_{j \in J} \mu_j . a_j, co(a_i \mid i \in I)^i)$. (Lemma 1.2). (Note also that by lemma 1.6, $\sum_{j \in J} \mu_j . a_j \neq o$).

We proceed with the proof of both statements :

Therefore, we consider $z = \sum_{m \in M} \nu_m . a_m \in co(a_i \mid i \in I)^i$ (by lemma 1.3, this means that

$M \subset I$ is finite and that $\nu_m \in B(0, 1^-)$ for each $m \in M$) and we calculate $\| \sum_{j \in J} \mu_j . a_j - z \|$.

i) Let $j_o \in J$. Since $|\mu_{j_o}| = 1$, we have the following :

$$\| \sum_{j \in J} \mu_j . a_j - z \| = \| a_{j_o} + \sum_{j \in J \setminus \{j_o\}} \frac{\mu_j}{\mu_{j_o} - \nu_{j_o}} a_j - \sum_{m \in M \setminus \{j_o\}} \frac{\nu_m}{\mu_{j_o} - \nu_{j_o}} a_m \| \quad \text{if } j_o \in M.$$

$$\| \sum_{j \in J} \mu_j . a_j - z \| = \| a_{j_o} + \sum_{j \in J \setminus \{j_o\}} \frac{\mu_j}{\mu_{j_o}} a_j - \sum_{m \in M \setminus \{j_o\}} \frac{\nu_m}{\mu_{j_o}} a_m \| \quad \text{if } j_o \notin M.$$

It follows that $\| \sum_{j \in J} \mu_j . a_j - z \| \geq dist(a_{j_o}, co(a_i \mid i \neq j_o))$.

Consequently, $dist(\sum_{j \in J} \mu_j . a_j, co(a_i \mid i \in I)^i) \geq dist(a_{j_o}, co(a_i \mid i \neq j_o)) > 0$,

and thus $\| \sum_{j \in J} \lambda_j . \pi(a_j) \| > 0$.

ii) First note that $(\pi(a_i))_{i \in I} \subset V_A \setminus \{o\}$. On the other hand, it suffices to prove that $(\pi(a_i))_{i \in I}$ is a (t-)orthogonal family in $(V_A, \|.\|)$ since $V_A = \overline{[\{\pi(a_i) \mid i \in I\}]}$. (See preliminaries). Put $\alpha = \max_{j \in J} \|\pi(a_j)\|$ and choose $j_o \in J$ such that $\|\pi(a_{j_o})\| = \alpha$. In the same way as above, $\| \sum_{j \in J} \mu_j . a_j - z \| \geq dist(a_{j_o}, co(a_i \mid i \neq j_o))$ and thus $\| \sum_{j \in J} \mu_j . a_j - z \| \geq (t). \|\pi(a_{j_o})\|$

$= (t).\alpha = (t). \max_{j \in J} \|\pi(a_j)\|$. It follows immediately that $\| \sum_{j \in J} \lambda_j . \pi(a_j) \| \geq (t). \max_{j \in J} \|\pi(a_j)\|$.

2.2.COROLLARY : Let $t \in (0, 1)$, let $(a_i)_{i \in I}$ be a (t-)orthogonal family in $(E, \|.\|)$ and put $A = \overline{co}(a_i \mid i \in I)$. Then $(\pi(a_i))_{i \in I}$ is a (t-)orthogonal base of $(V_A, \|.\|)$.
Proof : According to theorem 2.1, it suffices to prove that for each $i \in I, dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq (t).\|\pi(a_i)\| > 0$. So, let $i \in I$. We deduce from corollary 1.5 that $\|\pi(a_i)\| > 0$. On the other hand, $dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq dist(a_i, \overline{[\{a_j \mid j \neq i\}]}) \geq (t).\|a_i\| \geq (t).\|\pi(a_i)\|$ and we are done.

2.3.PROPOSITION : Let $A \subset E$ be a.c. and closed and let $(s_i)_{i \in I}$ be an orthogonal family in $(V_A, \|.\|)$. If for each $i \in I$, we can choose $a_i \in \partial A$ such that $\pi(a_i) = s_i$ and $\|a_i\| = \|\pi(a_i)\|$, then $(a_i)_{i \in I}$ is an orthogonal family in $(E, \|.\|)$.
Proof: Let $J \subset I$ be finite and let $(\lambda_j)_{j \in J} \subset K^*$. Put $\alpha = \max_{j \in J} \|\lambda_j . a_j\|$ and put $J_1 = \{j \in J \mid \|\lambda_j . a_j\| = \alpha\}$. Choose $\lambda \in K$ such that $|\lambda| = \max_{j \in J} |\lambda_j|$ and put $J_2 = \{j \in J \mid |\lambda_j| = |\lambda|\}$.

Then $\| \sum_{j \in J_1} \lambda_j . a_j \| = |\lambda| . \| \sum_{j \in J_1} \frac{\lambda_j}{\lambda} a_j \| \geq |\lambda| . \max_{j \in J_1} |\frac{\overline{\lambda_j}}{\lambda}| . \|\pi(a_j)\| = |\lambda| . \max_{j \in J_2} \|\pi(a_j)\| = |\lambda| . \max_{j \in J_2} \|a_j\| = \max_{j \in J_2} \|\lambda_j . a_j\| = \alpha$.

Hence, $\| \sum_{j \in J} \lambda_j . a_j \| = \| \sum_{j \in J_1} \lambda_j . a_j \| = \alpha = \max_{j \in J} \| \lambda_j . a_j \|.$

2.4.THEOREM : Let $(a_i)_{i \in I}$ be a family in $E \backslash \{o\}$ and put $A = \overline{co}(a_i \mid i \in I)$. Then the following are equivalent :
i) $(a_i)_{i \in I}$ is an orthogonal family in $(E, \|.\|)$;
ii) For each $i \in I, dist(a_i, \overline{co}(a_j \mid j \neq i)) = \|a_i\|$.
Proof :
i) $\Rightarrow$ ii) : Since for each $i \in I, dist(a_i, \overline{[\{a_j \mid j \neq i\}]} = \|a_i\|$, the statement follows easily.
ii) $\Rightarrow$ i) : From proposition 1.4, we deduce that $\|\pi(a_i)\| = \|a_i\|$. According to proposition 2.3, it suffices to prove that $(\pi(a_i))_{i \in I}$ is an orthogonal family in $(V_A, \|.\|)$. This, however follows immediately from the second part of theorem 2.1.

2.5.REMARK : Regarding theorem 2.4, one could ask if the following more general question is true as well :
Let $(a_i)_{i \in I}$ be a family in $E \backslash \{o\}$ and put $A = \overline{co}(a_i \mid i \in I)$. Are the following statements necessarily equivalent ?
i) There exists $t \in (0, 1]$ such that $(a_i)_{i \in I}$ is a t-orthogonal family in $(E, \|.\|)$;
ii) For each $i \in I, dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq t.\|a_i\|$.
The implication i) $\Rightarrow$ ii) is stated in corollary 1.5. The other implication is true in case $t = 1$. For $t \in (0, 1)$ however, the answer is no in general. To see this, consider the following example, obtained by C. Perez-Garcia through informal communication :
Put $\mathbf{K} = Q_2$ and $t = \frac{1}{2}$. Let $a = (2, 8, \frac{1}{4})$; $b = (1, 8, \frac{1}{2})$ and $c = (2, 4, 2)$.
Then $dist(a, co(b, c)) \geq \frac{1}{2}.\|a\|$, $dist(b, co(a, c)) \geq \frac{1}{2}.\|b\|$ and $dist(c, co(a, b)) \geq \frac{1}{2}.\|c\|$.
On the other hand, $\|2a + b + \frac{1}{2}.c\| < \frac{1}{2}.max\{\|2a\|, \|b\|, \|\frac{1}{2}.c\|\} = 1$.
However, we do have the following :

2.6.PROPOSITION : Let $(a_i)_{i \in I}$ and A be as in remark 2.5 and suppose there exists $t \in (0, 1)$ such that for each $i \in I, dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq t.\|a_i\|$.
Then, for each $i \in I, dist(a_i, \overline{[\{a_j \mid j \in J_i\}]} \geq t.\|a_i\|$ where $J_i = \{j \in I \backslash \{i\} \mid \|a_j\| \geq \|a_i\|\}$.
Proof : Let $i \in I$ and let $L \subset J_i$ be finite. For $(\lambda_j)_{j \in L} \subset \mathbf{K}^*$, we have to calculate $\|a_i - \sum_{j \in L} \lambda_j . a_j\|$. (Lemma 1.1).
Put $\lambda_i = -1, M = L \cup \{i\}$ and choose $j_o \in M$ such that $|\lambda_{j_o}| = \max_{j \in M} |\lambda_j|$.

Then, $\|a_i - \sum_{j \in L} \lambda_j . a_j\| = \| \sum_{j \in M} \lambda_j . a_j\| = |\lambda_{j_o}|. \| \sum_{j \in M} \frac{\lambda_j}{\lambda_{j_o}}. a_j\| \geq |\lambda_{j_o}|.dist(a_{j_o}, \overline{co}(a_j \mid j \neq i))$
$\geq t.|\lambda_{j_o}|.\|a_{j_o}\| \geq t.|\lambda_{j_o}|.\|a_i\| \geq t.\|a_i\|.$

2.7.REMARK : We can formulate a kind of converse for proposition 2.6. In fact, taking any closed a.c. subset A of E, we can construct a family in A satisfying similar properties as stated in proposition 2.6 (see proposition 2.10).

2.8.CONSTRUCTION : Let $(\alpha_n)_{n \in \mathbb{Z}}$ be a sequence in $\mathbb{R}_+^*$ with the following properties :

i) $\alpha_o = 1$;

ii) For each $n \in \mathbb{Z}, \alpha_{n+1} < \alpha_n$;

iii) $\lim\limits_{n \to +\infty} \alpha_n = 0$ and $\lim\limits_{n \to -\infty} \alpha_n = +\infty$;

iv) $\lim\limits_{n \to +\infty} \dfrac{\alpha_{n+1}}{\alpha_n} > 0$ and $\lim\limits_{n \to -\infty} \dfrac{\alpha_{n+1}}{\alpha_n} > 0$.

As an example of such a sequence, we can choose $\alpha_n = t^n \, (n \in \mathbb{Z})$ where $t \in (0,1)$ or $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}_o$ and $\alpha_n = -n$ for $n \in \mathbb{Z} \backslash \mathbb{N}$.

2.9.PROPOSITION : Let $A \subset E$ be a.c. and closed and let $(\alpha_n)_{n \in \mathbb{Z}}$ be a sequence with the properties stated in 2.8. For $x \in V_A$, we define $p(x)$ as follows :

i) If $x = 0$, we put $p(x) = 0$;

ii) If $x \neq 0$, there exists $n \in \mathbb{Z}$ such that $\alpha_{n+1} \leq \|x\| < \alpha_n$. We put $p(x) = \alpha_{n+1}$. Then p is a norm on V_A equivalent to $\|.\|$.

Proof : We leave it to the reader to prove that p is a norm on V_A. To prove the equivalence of both norms, we proceed as follows :

a) It is clear that $p(x) \leq \|x\|$, for each $x \in V_A \backslash \{o\}$.

b) Now, for $x \in V_A \backslash \{o\}$, there exists $n \in \mathbb{Z}$ such that $\alpha_{n+1} \leq \|x\| < \alpha_n$. Obviously, $p(x) = \alpha_{n+1} \geq \frac{\alpha_{n+1}}{\alpha_n}.\alpha_n > \frac{\alpha_{n+1}}{\alpha_n}.\|x\|$. On the other hand, there exists $\varepsilon > 0$ such that for each $n \in \mathbb{Z}, \frac{\alpha_{n+1}}{\alpha_n} \geq \varepsilon$. It follows that $p(x) \geq \varepsilon.\|x\|$.

2.10.PROPOSITION : Let $A, (\alpha_n)_{n \in \mathbb{Z}}$ and p be as in proposition 2.9 and let $(s_i)_{i \in I}$ be an orthogonal base of (V_A, p). ([4], theorem 2.5). For each $i \in I$, let n_i be such that $p(s_i) = \alpha_{n_i+1}$ and for each $i \in I$, choose $a_i \in A$ such that $\pi(a_i) = s_i$ and $\|s_i\| \leq \|a_i\| < \alpha_{n_i}$. Then we have the following :

i) For each $i \in I, dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq \frac{\alpha_{n_i+1}}{\alpha_{n_i}}.\|a_i\|$

ii) For each $i \in I, dist(a_i, \overline{[\{a_j \mid j \in J_i\}]}) \geq \frac{\alpha_{n_i+1}}{\alpha_{n_i}}.\|a_i\|$ where $J_i = \{j \in I \backslash \{i\} \mid \|a_j\| \geq \|a_i\|\}$

iii) There exists $t \in (0,1)$ such that $(\pi(a_i))_{i \in I}$ is a t-orthogonal base of $(V_A, \|.\|)$.

Proof :

i) Let $i \in I$ and let $L \subset J_i$ be finite. For $(\lambda_j)_{j \in L} \subset B(0,1)$, we have to calculate $\|a_i - \sum\limits_{j \in L} \lambda_j.a_j\|$. (Lemma 1.1).

Now, $\|a_i - \sum\limits_{j \in L} \lambda_j.a_j\| \geq p(s_i - \sum\limits_{j \in L} \overline{\lambda_j}.s_j) \geq p(s_i) = \alpha_{n_i+1} \geq \frac{\alpha_{n_i+1}}{\alpha_{n_i}}.\|a_i\|$.

ii) Let $i \in I$ and let $L \subset J_i$ be finite. For $(\lambda_j)_{j \in L} \subset K^*$, we have to calculate $\|a_i - \sum\limits_{j \in L} \lambda_j.a_j\|$. (Lemma 1.1).

Put $\lambda_i = -1, M = L \cup \{i\}$ and choose $j_o \in M$ such that $|\lambda_{j_o}| = \max\limits_{j \in M} |\lambda_j|$.

Then, $\left\| a_i - \sum_{j \in L} \lambda_j . a_j \right\| = |\lambda_{j_o}| . \left\| \sum_{j \in M} \dfrac{\lambda_j}{\lambda_{j_o}} . a_j \right\| \geq |\lambda_{j_o}| . p(s_{j_o}) = |\lambda_{j_o}| . \alpha_{n_{j_o}+1} \geq |\lambda_{j_o}| . \alpha_{n_i+1} >$

$|\lambda_{j_o}| . \dfrac{\alpha_{n_i+1}}{\alpha_{n_i}} . \|a_i\| \geq \dfrac{\alpha_{n_i+1}}{\alpha_{n_i}} . \|a_i\|$ since obviously, $|\lambda_{j_o}| \geq 1$.

Hence, $dist(a_i, \overline{[\{a_j \mid j \in J_i\}]}) \geq \frac{\alpha_{n_i+1}}{\alpha_{n_i}} . \|a_i\|$.

iii) There exists $\varepsilon > 0$ such that for each $x \in V_A$, $p(x) \geq \varepsilon . \|x\|$. Now, for $J \subset I$ finite and for $(\lambda_j)_{j \in J} \subset \mathbf{k}^*$, $\left\| \sum_{j \in J} \lambda_j . \pi(a_j) \right\| \geq p(\sum_{j \in J} \lambda_j . \pi(a_j)) = \max_{j \in J} |\lambda_j| . p(\pi(a_j)) \geq \varepsilon . \max_{j \in J} |\lambda_j| . \|\pi(a_j)\|$ and we are done.

2.11.COROLLARY : Let $A \subset E$ be a.c and closed, let $t \in (0,1)$ and for each $n \in \mathbf{Z}$, put $\alpha_n = t^n$. Suppose $(a_i)_{i \in I}$ is a family in A with the properties stated in proposition 2.10. Then $(a_i)_{i \in I}$ is a p-adic t-frame in E.

Proof : According to [7], definition 2.1 we have to prove that for $n \in \mathbf{N}_o$ and $\{i_1, ..., i_n\} \subset I$, $Vol(a_{i_1}, ..., a_{i_n}) \geq t^{n-1} . \|a_{i_1}\| . ..... \|a_{i_n}\|$. (Recall that for $n \in \mathbf{N}_o$ and $x_1, ..., x_n \in E$, $Vol(x_1, ..., x_n) = \|x_1\| . dist(x_2, [x_1]) . .... . dist(x_n, [x_1, ..., x_{n-1}])$ and that this Volume Function is symmetrical. ([11], theorem 1.3). So, let $n \in \mathbf{N}_o$ and $\{i_1, ..., i_n\} \subset I$. It is no restriction to assume that $\|a_{i_1}\| \geq \|a_{i_2}\| \geq ... \geq \|a_{i_n}\|$. According to proposition 2.10, for each $k \in \{2, ..., n-1\}$ we have that $dist(a_{i_k}, [a_{i_1}, ..., a_{i_{k-1}}]) \geq t . \|a_{i_k}\|$. It follows immediately that $Vol(a_{i_1}, ..., a_{i_n}) \geq t^{n-1} . \|a_{i_1}\| . .... . \|a_{i_n}\|$.

3. BANACH SPACES WITH THE HAHN-BANACH EXTENSION PROPERTY

3.1.REMARK : Throughout the text, $(E, \|.\|)$ is a Banach space with the Hahn-Banach extension property (HBEP). (i.e. for any linear subspace D of E and for any $f \in D'$, there exists $\overline{f} \in E'$ such that $\overline{f} = f$ on D).

3.2.LEMMA : Let $A \subset E$ be a.c., closed and weakly c'-compact. Let $T \subset A$ such that $\overline{[\pi(T)]} = V_A$. Then $\overline{[A]} = \overline{[T]}$.

Proof : It is not hard to see that $A = \overline{\overline{A^i} + coT}$. Now, if $\overline{[A]} \neq \overline{[T]}$, then there exists $x \in A \backslash \overline{[T]}$. Since $(E, \|.\|)$ has the HBEP, $\overline{[T]}$ is weakly closed and thus there exists $f \in E'$ such that $f(\overline{[T]}) = \{o\}$ and $f(x) \neq o$. ([8], corollary 4.8).
Put $\alpha = \max_{y \in A} |f(y)|$. Note that $\alpha > 0$.

Now, for each $y \in \overline{A^i} + coT$, $|f(y)| < \alpha$ since $f \equiv 0$ on coT, whereas $|f| < \alpha$ on $\overline{A^i}$. ([2], lemma 1.2. (The proof is also valid for $|f|$)).

Hence, for each $y \in A$, $|f(y)| < \alpha$ since $A = \overline{\overline{A^i} + coT}$. But this is of course a contradiction since A is weakly c'-compact.

3.3.COROLLARY : Let $A \subset E$ and $T \subset A$ be as in lemma 3.2. If $\overline{[A]} \neq \overline{[T]}$ then A cannot

be weakly c'-compact.

Proof: Apply lemma 3.2.

3.4.PROPOSITION : Let $A \subset E$ be a.c., closed and weakly c'-compact. Then the following are equivalent :

i) A is c'-compact ;

ii) $(V_A, \|.\|)$ is of countable type.

Proof:

i) $\Rightarrow$ ii) : There exists a t-orthogonal sequence $(x_n)_{n \in \mathbf{N}_o}$ in $(E, \|.\|)$ such that $A = \overline{co}(x_n \mid n \in \mathbf{N}_o)$. ([10], corollary 2.8). It follows that $V_A = \overline{[\{x_n \mid n \in \mathbf{N}_o\}]}$.

ii) $\Rightarrow$ i) : Let $(s_n)_{n \in \mathbf{N}_o} \subset V_A$ such that $V_A = \overline{[\{s_n \mid n \in \mathbf{N}_o\}]}$. For each $n \in \mathbf{N}_o$, choose $a_n \in A$ such that $\pi(a_n) = s_n$. According to lemma 3.2, $\overline{[A]} = \overline{[\{a_n \mid n \in \mathbf{N}_o\}]}$. Now apply [8], lemma 3.1 and observe that a weakly c'-compact compactoid is c'-compact. (Apply [9], theorem 5.12)

3.5.LEMMA : Suppose the valuation on **K** is dense, let $A \subset E$ be a.c., closed, weakly c'-compact and infinite-dimensional and let $(a_i)_{i \in I}$ be a family in A such that $(\pi(a_i))_{i \in I}$ is a t-orthogonal base of $(V_A, \|.\|) \, (t \in (0,1))$. Then $\inf_{i \in I} \|\pi(a_i)\| = 0$.

Proof: First note that since $\overline{[A]} = \overline{[\{a_i \mid i \in I\}]}$ (lemma 3.2), I is infinite. Now, suppose $\inf_{i \in I} \|\pi(a_i)\| = \alpha > 0$. Put $\beta = \sup_{x \in A} \|x\|$. (According to [9], corollary 7.7, A is bounded, since $(E, \|.\|)$ is a polar space ([6], proposition 5.1)). It follows by [3], lemma 3.11 that $(a_i)_{i \in I}$ is a t'-orthogonal family in $(E, \|.\|)$ where $t' = \frac{t.\alpha}{\beta}$.

Let $(i_k)_{k \in \mathbf{N}_o} \subset I$ and choose $(\xi_k)_{k \in \mathbf{N}_o} \subset \mathbf{K}$ such that $|\xi_1| < |\xi_2| < ... < 1$ and $\lim_{k \to \infty} |\xi_k| = 1$.

Put $f : \overline{[\{a_i \mid i \in I\}]} \to \mathbf{K} : \sum_{i \in I} \lambda_i.a_i \to \sum_{k \in \mathbf{N}_o} \lambda_{i_k}.\xi_k$. f is well defined and for $x \in \overline{[\{a_i \mid i \in I\}]}$,

$|f(x)| < \max_{k \in \mathbf{N}_o} |\lambda_{i_k}| \le \frac{1}{\alpha} . \max_{k \in \mathbf{N}_o} |\lambda_{i_k}|.\|a_{i_k}\| \le \frac{1}{t'.\alpha}.\|x\|$, hence f is continuous.

Now, for $x \in co(a_i \mid i \in I), |f(x)| = |f(\sum_{i \in I} \lambda_i.a_i)| \le \sup_{k \in \mathbf{N}_o} |\lambda_{i_k}|.|\xi_k| < \sup_{k \in \mathbf{N}_o} |\lambda_{i_k}| \le 1$. But

$\sup_{x \in co(a_i \mid i \in I)} |f(x)| = \sup_{k \in \mathbf{N}_o} |f(a_{i_k})| = 1$.

Hence, there is no $x \in co(a_i \mid i \in I)$ such that $|f(x)| = 1$. But according to [3], proposition 4.3 (the condition that A is polar is not needed in the proof), $co(a_i \mid i \in I)$ is weakly c'-compact and this contradicts the previous.

3.6.PROPOSITION : Suppose the valuation on **K** is dense and that **K** is not spherically complete. Let $A \subset E$ be a.c., weakly c'-compact and infinite-dimensional, let $t \in (0,1)$ and let $(a_i)_{i \in I}$ be a family in $A \backslash \{o\}$ such that $(\pi(a_i))_{i \in I}$ is a t-orthogonal base of $(V_A, \|.\|)$

and such that $(a_i)_{i \in I}$ satisfies the conditions of corollary 2.11. Then the following are equivalent :

i) $\overline{[A]}$ has a base ;

ii) $\lim\limits_{i \in I} \|a_i\| = 0$.

Proof :

ii) $\Rightarrow$ i) : Suppose $\overline{[A]}$ has a base. Then A is c'-compact ([8], theorem 2.7, $(\alpha) \Rightarrow (\delta)$) and thus $\overline{[A]}$ is of countable type. Now, as $(a_i)_{i \in I}$ is a p-adic t-frame (corollary 2.11), it follows that I is countable. ([7], theorem 2.5). But then $\lim\limits_{i \in I} \|\pi(a_i)\| = 0$ (lemma 3.5), and thus, by construction $\lim\limits_{i \in I} \|a_i\| = 0$.

i) $\Rightarrow$ ii) : Suppose $\lim\limits_{i \in I} \|a_i\| = 0$. As for each $i \in I, \|a_i\| \neq 0$, it follows that I is countable. But then V_A is of countable type and thus A is c'-compact. (Proposition 3.4). It follows that $\overline{[A]}$ is of countable type ([10], proposition 2.3), hence $\overline{[A]}$ has a base ([12], theorem 3.16).

3.7.PROPOSITION : Suppose the valuation on **K** is dense and that **K** is not spherically complete. Let $A \subset E$ be a.c., polar, weakly c'-compact and infinite-dimensional, let $t \in (0, 1)$ and let $(a_i)_{i \in I}$ be a family in A such that for each $i \in I, dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq t.\|a_i\| > 0$. (Note that A is closed ([9], proposition 3.4), hence, according to proposition 2.10, such a family exists). Now, if $\lim\limits_{i \in I} \|a_i\| \neq 0$, then $(a_i)_{i \in I}$ cannot be an α-orthogonal family in $(E, \|.\|)$, for each $\alpha \in (0, 1]$.

Proof : First note that in Banach spaces with the HBEP, A is weakly c'-compact in E if and only if A is weakly c'-compact in $\overline{[A]}$. Note also that the condition $dist(a_i, \overline{co}(a_j \mid j \neq i)) \geq t.\|a_i\| > 0$ implies that $(a_i)_{i \in I}$ is a t-orthogonal base of $(V_A, \|.\|)$ (theorem 2.1) and thus I is infinite. (Same reasoning as in the proof of lemma 3.5).

Suppose there exists $\alpha \in (0, 1]$ such that $(a_i)_{i \in I}$ is an α-orthogonal family in $(E, \|.\|)$. According to lemma 3.2, $\overline{[A]} = \overline{[\{a_i \mid i \in I\}]}$. It follows that $(a_i)_{i \in I}$ is an α-orthogonal base of $\overline{[A]}$. Let $(i_k)_{k \in \mathbf{N}_o} \subset I$ such that for each $k \in \mathbf{N}_o, \|a_{i_k}\| \geq \varepsilon > 0$ and let $(\xi_k)_{k \in \mathbf{N}_o} \subset \mathbf{K}$ such that $|\xi_1| < |\xi_2| < ... < 1$ and $\lim\limits_{k \to \infty} |\xi_k| = 1$.

Put $f : \overline{[A]} \to \mathbf{K} : \sum\limits_{i \in I} \lambda_i.a_i \to \sum\limits_{k \in \mathbf{N}_o} \lambda_{i_k}.\xi_k$. f is well defined and for $x \in \overline{[A]}, |f(x)| <$

$\max\limits_{k \in \mathbf{N}_o} |\lambda_{i_k}| \leq \dfrac{1}{\varepsilon}.\max\limits_{k \in \mathbf{N}_o} |\lambda_{i_k}|.\|a_{i_k}\| \leq \dfrac{1}{\varepsilon.\alpha}.\|x\|$, hence f is continuous.

Now, for $x \in co(a_i \mid i \in I), |f(x)| = |f(\sum\limits_{i \in I} \lambda_i.a_i)| \leq \sup\limits_{k \in \mathbf{N}_o} |\lambda_{i_k}|.|\xi_k| < \sup\limits_{k \in \mathbf{N}_o} |\lambda_{i_k}| \leq 1$.

Since $A = \overline{co}^\sigma(a_i \mid i \in I)$ ([3], corollary 4.7. (The proof is valid in polar Banach spaces, hence in Banach spaces with HBEP. ([6], proposition 5.1)), it follows that for each $x \in A, |f(x)| < 1$. But, $\sup\limits_{x \in A} |f(x)| = \sup\limits_{k \in \mathbf{N}_o} |f(a_{i_k})| = 1$, which contradicts the fact

that A is weakly c'-compact.

3.8.REMARK : I don't know whether the following is true :
Suppose the valuation on **K** is dense and that **K** is not spherically complete. Let $A \subset E$ be a.c., polar, weakly c'-compact and infinite-dimensional, let $t \in (0,1)$ and let $(a_i)_{i \in I}$ be a family in $A \backslash \{o\}$ such that $(\pi(a_i))_{i \in I}$ is a t-orthogonal base of $(V_A, \|.\|)$ and such that $(a_i)_{i \in I}$ satisfies the conditions of corollary 2.11.
Are the following equivalent ?

i) $\overline{[A]}$ has a base ;

ii) $\lim\limits_{i \in I} \|a_i\| = 0$;

iii) There exists $\alpha \in (0,1]$ such that $(a_i)_{i \in I}$ is an α-orthogonal family in $(E, \|.\|)$.
As we have seen in proposition 3.6, i) $\Leftrightarrow$ ii), whereas iii) $\Rightarrow$ i) and iii) $\Rightarrow$ ii) (proposition 3.7). The problem is to prove one of the other implications. These implications would be easily proved if a p-adic t-frame in a Banach space with a base is an α-orthogonal family for some $\alpha \in (0,1]$.
I don't know if the equivalence of i), ii) and iii) above would bring us closer to solving the (still?) open problem that in strongly polar Banach spaces (or perhaps in Banach spaces with HBEP) each weakly c'-compact subset is c'-compact, but it might perhaps help in answering a weaker version of the problem stated in remark 2.5.

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