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p-ADIC ANALYTIC INTERPOLATION

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Abstract. Let K be a complete ultrametric algebraically closed field. We study the Kernel of infinite van der Monde Matrices and show close connections with the zeroes of analytic functions. We study when such a matrix is invertible. Finally we use these results to obtain interpolation processes for analytic functions. They are more accurate if K is spherically complete.

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1. NOTATIONS, DEFINITIONS AND THEOREMS

K denotes an algebraically closed field complete for an ultrametric absolute value. Given $a \in \mathbf{K}$, $r > 0$, we denote by $d(a,r)$ (resp. $d(a,r^-)$) the disk $\{x \in \mathbf{K} : |x-a| \leq r\}$ (resp. $\{x \in \mathbf{K} : |x-a| < r\}$). Given $r > 0$ we denote by $C(0,r)$ the circle $d(0,r)\backslash d(0,r^-)$. Given $r_1, r_2 \in \mathbb{R}_+$ such that $0 < r_1 < r_2$, we denote by $\Gamma(0,r_1,r_2)$ the set $d(0,r_2^-)\backslash d(0,r_1)$.

Given $r > 0$, we denote by $A(d(0,r^-))$ the algebra of the power series $\sum_{n=0}^{\infty} b_n x^n$ converging for $|x| < r$.

Given **K**-vector spaces E, F, $\mathcal{L}(E,F)$ will denote the space of the **K**-linear mappings from E into F.

$\mathcal{E}$ will denote the **K**-vector space of the sequences in **K**, and $\mathcal{E}_0$ will denote the subspace of the bounded sequences. The identically zero sequence will be denoted by (0).

$\mathcal{E}_1$ will denote the set of the sequences (a_n) such that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq 1$. So $\mathcal{E}_1$ is seen to be a subspace of $\mathcal{E}$ isomorphic to the space $A(d(0,1^-))$, and obviously contains $\mathcal{E}_0$.

Let $\mathbf{M}_\infty$ be the set of the infinite matrices $(\lambda_{i,j})$ with coefficients in **K**.

$\delta_{i,j}$ will denote the Kronecker symbol. I_∞ will denote the infinite identical matrix defined as $\lambda_{i,j} = \delta_{i,j}$.

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In this paper, (a_n) will denote an injective sequence in $d(0, 1^-)$ such that $a_n \neq 0$ for every $n > 0$. and we will denote by $\mathcal{M}(a_n)$ the infinite matrix $M = (\lambda_{i,j})$ defined as $\lambda_{i,j} = (a_i)^j$, $(i, j) \in \mathbb{N} \times \mathbb{N}$.

A matrix $M = (\lambda_{i,j}) \in \mathbf{M}_\infty$ will be said to be *bounded* if there exists $A \in \mathbf{R}_+$ such that $|\lambda_{i,j}| \leq A$ whenever $(i, j) \in \mathbb{N} \times \mathbb{N}$.

M will be said to be *line-vanishing* if for each $i \in \mathbb{N}$, we have $\lim\limits_{j \to \infty} \lambda_{i,j} = 0$.

A line-vanishing matrix M is seen to define a $\mathbf{K}$-linear mapping ψ_M from $\mathcal{E}_0$ into $\mathcal{E}$.

So the matrix $M = \mathcal{M}(a_n)$ clearly defines a K-linear mapping ϕ_M from $\mathcal{E}_1$ into $\mathcal{E}$, because given a sequence $(b_n) \in \mathcal{E}_1$, the series $\sum\limits_{n=0}^{\infty} b_n (a_j)^n$ is obviously convergent.

Lemmas 1 and 2 are immediate :

Lemma 1 : *Let $M \in \mathbf{M}_\infty$ be line vanishing.*
The three following statements are equivalent :
ψ_M is continuous
ψ_M is an endomorphism of $\mathcal{E}_0$
M is bounded .

In particular, Lemma 1 applies to matrices of the form $\mathcal{M}(a_n)$.

Lemma 2 : *Let $M = \mathcal{M}(a_n)$ and let $(b_n) \in \mathcal{E}_1$. Then (b_n) belongs to* $\text{Ker}\phi_M$ *if and only if the analytic function $f(t) = \sum\limits_{n=0}^{\infty} b_n t^n$ admits each point a_j for zero.*

Theorem 1 : *Let $M = \mathcal{M}(a_n)$. Then* $\text{Ker}\phi_M \neq \{(0)\}$ *if and only if* $\lim\limits_{n \to \infty} |a_n| = 1$. *Besides* $\text{Ker}\psi_M \neq \{(0)\}$ *if and only if* $\prod\limits_{n=0}^{\infty} |a_n| > 0$.

Theorem 2 : *Let $b = (b_n) \in \mathcal{E}_0$. There exists an injective sequence (α_n) in $d(0, 1^-)$ such that $b \in$* $\text{Ker}\psi_{\mathcal{M}(\alpha_n)}$ *if and only if b satisfies $|b_j| < \sup\limits_{n \in \mathbb{N}} |b_n|$ for all $j \in \mathbb{N}$.*

Definitions and notations : An injective sequence (a_n) in $d(0, 1^-)$ will be called *a regular sequence* if $\inf\limits_{n \neq m} |a_n - a_m| > 0$ and $\lim\limits_{n \to \infty} |a_n| = 1$.

Let (a_n) be a regular sequence and let $\rho = \inf\limits_{n \neq m} |a_n - a_m|$. For every $r \in]0, 1[$, we will denote by $\Omega((a_n), r)$ the set $d(0, 1^-) \setminus (\bigcup\limits_{n \in \mathbb{N}} d(a_n, r^-))$, and by $\Omega(a_n)$ the set $d(0, 1^-) \setminus (\bigcup\limits_{n \in \mathbb{N}} d(a_n, \rho^-))$.

Let $\mathbf{a} = (a_n)$ and $\mathbf{b} = (b_n)$ be two sequences in K. We will denote by $\mathbf{a} * \mathbf{b}$ the convolution product (c_n) defined as $c_n = \sum_{j=0}^{n} a_j b_{n-j}$.

Theorem 3 : *Let (α_n) be a regular sequence of $d(0,1^-)$ such that there exists $g \in A(d(0,1^-))$ satisfying*

(i) α_n is a zero of order 1 of g for all $n \in \mathbb{N}$.

(ii) $g(x) \neq 0$ whenever $x \in d(0,1^-) \setminus \{\alpha_n : n \in \mathbb{N}\}$.

(iii) $\lim_{\substack{|x| \to 1^- \\ x \in \Omega(\alpha_n)}} |g(x)| = +\infty$.

Let $M = \mathcal{M}(\alpha_n)$. Then ψ_M is injective but its image does not contain $\mathcal{E}_0$. Also there exists $P = (\lambda_{i,j}) \in \mathcal{M}_\infty$ (not unique) satisfying

(1) P is line-vanishing.

(2) $\lim_{n \to \infty} \lambda_{n,j} \alpha_h^n = 0$ for all $(j,h) \in \mathbb{N} \times \mathbb{N}$.

(3) $\sum_{n=0}^{\infty} \lambda_{n,j} \alpha_h^n = \delta_{j,h}$ for all $(j,h) \in \mathbb{N} \times \mathbb{N}$.

(4) $MP = PM = I_\infty$.

(5) $P(\mathbf{b}) \in \mathcal{E}_1$ for all $\mathbf{b} \in \mathcal{E}_0$.

(6) $MP(\mathbf{b}) = \mathbf{b}$ for all $\mathbf{b} \in \mathcal{E}_0$.

(7) ψ_P is injective.

Let (ν_n) be a sequence in $\mathbf{K}$ such that $|\nu_0| \geq |\nu_n|$ for every $n > 0$. For every $j \in \mathbb{N}$, let $(\mu_{n,j})_{n \in \mathbb{N}}$ denote the sequence $\left(\frac{1}{\sum_{m=0}^{\infty} \nu_m \alpha_j^m}\right)((\lambda_{n,j}) * (\nu_n))$. Then the matrix $Q = (\mu_{i,j})$ also satisfies properties $(1)-(7)$ and is not equal to P for infinitely many sequences (ν_n).

Remarks. 1. Mainly, the proof of Theorem 3 takes inspiration from that of Lemma 3 in [7]. However, in this lemma, the considered matrix, roughly, was P. Here the matrix we consider is a van der Monde matrix M and we look for P.

2. Given M, the matrix P depends on g and therefore is not unique satisfying $(1)-(7)$. Indeed $\mathcal{M}_\infty$ is not a ring because the multiplication of matrices is not always defined and even when it is defined, is not always associative. As a consequence, if P, P' satisfy $MP = MP' = PM = P'M = I_\infty$, we cannot conclude $P' = P$.

Actually we can consider $\phi_M \circ \psi_P \in \mathcal{L}(\mathcal{E}_0, \mathcal{E})$ and then this is the identity in $\mathcal{E}_0$. Next we can consider $\psi_{P'} \circ \psi_M \in \mathcal{L}(\mathcal{E}_0, \mathcal{E}_1)$ and this is the identity in $\mathcal{E}_0$. But we cannot consider $\psi_{P'} \circ (\phi_M \circ \psi_P)$ because $\psi_{P'}$ is not defined in $\mathcal{E}_1$. In the same way, we cannot consider $(\psi_{P'} \circ \psi_M) \circ \psi_P$ because $\psi_{P'} \circ \psi_M$ is only defined in $\mathcal{E}_0$.

We consider the matrix P and look for "inverses" M such that $MP = PM = I_\infty$. Suppose that there exists a bounded matrix $M' \neq M$ such that $PM' = M'P = I_\infty$. Now we can consider $\phi_{M'} \circ (\psi_P \circ \psi_M) \in \mathcal{L}(\mathcal{E}_0, \mathcal{E})$. Since $\psi_P \circ \psi_M$ is the identity in $\mathcal{E}_0$, then $\phi_{M'} \circ (\psi_P \circ \psi_M)$ is equal to $\psi_{M'}$. Next we can consider $(\phi_{M'} \circ \psi_P) \circ \psi_M \in \mathcal{L}(\mathcal{E}_0, \mathcal{E})$. Since

$\phi_{M'} \circ \psi_P$ is the identity on $\mathcal{E}_0$, we have $(\phi_{M'} \circ \psi_P) \circ \psi_M = \psi_M$ and therefore $\psi_M = \psi_{M'}$ hence $M = M'$.

3. Let $P, Q \in \mathcal{M}_\infty$ satisfy $(1) - (7)$. Let $\mathcal{E}' = \psi_P(\mathcal{E}_0)$, let $\mathcal{E}'' = \psi_Q(\mathcal{E}_0)$. Then the restriction of ϕ_M to $\mathcal{E}'$ (resp. $\mathcal{E}''$) is just the reciprocal of ψ_P (resp. ψ_Q).

Conjecture. Under the hypothesis of Theorem 1, every matrix satisfying properties $(1) - (7)$ is of the form

$$\mu_{n,j} = \left(\frac{1}{\sum_{m=0}^{\infty} \nu_m \alpha_j^m}\right)((\lambda_{n,j}) * (\nu_n)).$$

Theorem 4 : *Let* $\mathbf{K}$ *be spherically complete, and let* (α_n) *be a sequence in* $d(0,1^-)$ *satisfying* $|\alpha_n - \alpha_m| \geq \min(|\alpha_n|, |\alpha_m|)$ *whenever* $n \neq m$, $\lim_{n\to\infty} |\alpha_n| = 1$, *and* $\prod_{n=0}^{\infty} |\alpha_n| = 0$.

Then $\mathcal{M}(\alpha_n)$ *admits inverses* P *such that, for every bounded sequence* $\mathbf{b} := (b_n)$ *in* $\mathbf{K}$, *the sequence* $\mathbf{a} := (a_n) = P(\mathbf{b})$ *defines a function* $f(x) = \sum_{n=0}^{\infty} a_n x^n \in A(d(0,1^-))$ *satisfying* $f(\alpha_n) = b_n$.

Theorem 5 : *Let* (α_n) *be a regular sequence in* $d(0,1^-)$. *There exists a regular sequence* (γ_n) *in* $d(0,1^-)$ *such that* (α_n) *is a subsequence of* (γ_n) *satisfying : for every inverse matrix* P *of* $\mathcal{M}(\gamma_n)$ *and for every bounded sequence* $\mathbf{b} = (b_n)$ *of* $\mathbf{K}$, *the sequence* $\mathbf{a} = P(\mathbf{b}) := (a_n)$ *defines an analytic function* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *such that* $f(\gamma_j) = b_j$ *whenever* $j \in \mathbb{N}$.

2. PROVING THEOREMS 1 AND 2.

For each set D in $\mathbf{K}$, we denote by $H(D)$ the set of the analytic elements in D (i. e., the completion of the set of the rational functions with no pole in D).

Given $f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-))$, one defines the valuation function $v(f,\mu)$ in the interval $]0, +\infty[$ as $v(f,\mu) = \inf_{n\in\mathbb{N}} (v(b_n) + n\mu)$.

Lemma 3 and 4 gather the main properties of the function $v(f,\mu)$ ([1],[4]).

Lemma 3 *Let* $f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-))$. *For every* $\mu > 0$, f *satisfies* $v(f,\mu) = \lim_{v(t)\to\mu, v(t)\neq\mu} v(f(t))$. *For every* $x \in d(0,1^-)$, f *satisfies* $v(f(x)) \geq v(f, v(x))$. *For every* $r \in]0,1[$, f *satisfies* $-\log \|f\|_{d(0,r)} = v(f, -\log r)$.

Besides f *is bounded in* $d(0,1^-)$ *if and only if the sequence* (b_n) *belongs to* $\mathcal{E}_0$. *If* f *is bounded in* $d(0,1^-)$, *then* $\|f\|_{d(0,1^-)} = \sup_{n\in\mathbb{N}} |b_n|$ *and* $-\log \|f\|_{d(0,1^-)} = \lim_{\mu\to 0} v(f,\mu)$.

Lemma 4 : *Let $f(t) \in A(d(0,1^-))$ and let $r_1, r_2 \in (0,1)$ satisfy $r_1 < r_2$. If f admits q zeros in $d(0,r_1)$ (taking multiplicities into account) and t distinct zeros $\alpha_1, \dots, \alpha_t$, of multiplicity order ζ_j $(1 \le j \le t)$ respectively in $\Gamma(0, r_1, r_2)$, then f satisfies*

$$v(f, -\log r_2) - v(f, -\log r_1) = -\sum_{j=1}^{t} \zeta_j (v(a_j) + \log r_2) - q(\log r_2 - \log r_1).$$

Proof of Theorem 1. Let $\mathbf{b} = (b_n) \in \mathcal{E}_1 \backslash \{(0)\}$ and let $f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-))$.

First we suppose $\mathrm{Ker}\phi_M \neq \{(0)\}$ and therefore we can assume $\mathbf{b} \in \mathrm{Ker}\phi_M$. Then, by Lemma 2, f satisfies $f(a_j) = 0$ for every $j \in \mathrm{N}$. But for every $r \in]0,1[$, we know that f belongs to $H(d(0,r))$ and has finitely many zeros in $d(0,r)$. Hence we have $\lim_{n\to\infty} |a_n| = 1$.

Reciprocally, let the sequence (a_n) satisfy $\lim_{n\to\infty} |a_n| = 1$. By Proposition 5 in [4], we know that there exists a not identically zero analytic function $f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-))$ which admits each a_j as a zero. Hence we have $\sum_{n=0}^{\infty} b_n a_j^n = 0$, and of course the sequence (b_n) belongs to $\mathcal{E}_1$, hence to $\mathrm{Ker}\phi_M$.

Now we suppose that $\mathrm{Ker}\psi_M \neq (0)$ and we assume that the sequence (b_n) belongs to $\mathrm{Ker}\psi_M$. In particular $\mathrm{Ker}\phi_M \neq (0)$ and therefore $\lim_{n\to\infty} |a_n| = 1$. Without loss of generality we may clearly assume $|a_n| \le |a_{n+1}|$ for all $n \in \mathbb{N}$. Besides, by definition we have $|a_1| > 0$. By Lemma 3 we know that $\inf_{n\in\mathbb{N}} v(b_n) = \lim_{\mu\to 0^+} v(f,\mu) = \lim_{|x|\to 1, x\in D} v(f(x)) = -\log \|f\|_{d(0,1^-)}$. Now for each $\mu > 0$, let $q(\mu)$ be the unique integer such that $v(a_n) \ge \mu$ for every $n \le q(\mu)$ and $v(a_n) < \mu$ for every $n > q(\mu)$. By Lemma 4, we check

$$v(f,\mu) - v(f, v(a_1)) \le \sum_{j=2}^{q(\mu)} \mu - v(a_j) + 2(\mu - v(a_1)).$$

Since $v(f,\mu)$ is bounded when μ approaches 0, by (1) it is seen that $\sum_{j=1}^{\infty} v(a_j)$ must be bounded and therefore we have $\prod_{n=1}^{\infty} |a_n| > 0$.

Reciprocally we suppose $\prod_{n=1}^{\infty} |a_n| > 0$. We can easily check that $\lim_{n\to\infty} |a_n| = 1$, and then we can assume $|a_n| \le |a_{n+1}|$ for all $n \in \mathbb{N}$ without loss of generality. For each

$j \in \mathbb{N}$ we put $P_j(x) = \prod_{m=1}^{j}(1 - x/a_m)$. By Theorem 1 in [2], we can check that there exists $f \in A(d(0,1^-))$ (f not identically zero) satisfying

(3) $f(a_m) = 0$ for all $m \in \mathbb{N}$, and

(4) $v(f,\mu) \geq v(P_{q(\mu)},\mu) - 1$ for all $\mu > 0$.

Now we notice that if $\mu_1 > \mu_2 > 0$ then we have $v(P_{q(\mu_1)},\mu_1) = v(P_{q(\mu_2)},\mu_1)$ and then we see that $\lim_{\mu\to 0^+} v(P_{q(\mu)},\mu) = \sum_{j=1}^{\infty} v(a_j)$. But by (2) we have $\sum_{j=1}^{\infty} v(a_j) < +\infty$ and therefore by (4), $v(f,\mu)$ is bounded in $]0,+\infty[$. Let $f(t) = \sum_{n=0}^{\infty} b_n t^n$. By Lemma 3 the sequence (b_n) is bounded and by (3) it clearly belongs to Ker.ψ_M. This finishes the proof of Theorem 1.

Lemma 5 : *Let $f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-))$ and let $r \in (0,1)$. Then f admits at least one zero in $C(0,r)$ if and only if there exist $k,l \in \mathbb{N}$ ($k < l$) such that $|b_k| r^k = |b_l| r^l$.*

Proof of Theorem 2. As a consequence of Lemma 5, a function $f(t) = \sum_{n=0}^{\infty} b_n t^n \in A(d(0,1^-))$ admits infinitely many zeros in $d(0,1^-)$ if and only if $|b_j| < \sup_{n\in\mathbb{N}} |b_n|$ for every $j \in \mathbb{N}$. Then the conclusion comes from Lemma 2.

3. PROVING THEOREM 3.

As an application of Corollary (of Theorem 5) in [8], we have this lemma.

Lemma 6 : *Let $f \in A(d(0,1^-))$ have a regular sequence of zeros (b_n) and satisfy $\lim_{\substack{|x|\to 1^- \\ x\in\Omega(b_n)}} |f(x)| = +\infty$. Then $1/f$ belongs to $H(\Omega(b_n))$.*

Proof of Theorem 3. We may obviously assume $|\alpha_n| \leq |\alpha_{n+1}|$ and therefore $\alpha_n \neq 0$ whenever $n > 0$. Since g is not bounded in $d(0,1^-)$, by Lemma 3 we have $\lim_{\mu\to 0^+} v(g,\mu) = -\infty$, and by Lemma 4 the sequence of the zeros (α_n) satisfies $\prod_{n=1}^{\infty} |\alpha_n| = 0$, hence ψ_M is injective. Now we look for P. Since g admits each α_j as a simple zero, it factorizes in $A(d(0,1^-))$ in the form $\psi_j(x)(1 - x/\alpha_j)$ and we have $\psi_j(\alpha_j) \neq 0$. We put $g_j(x) = \dfrac{\psi_j(x)}{\psi_j(\alpha_j)}$. Then g_j belongs to $A(d(0,1^-))$ and may be written as $\sum_{n=0}^{\infty} \lambda_{n,j} x^n$. We denote by P the matrix

$$\begin{pmatrix} \lambda_{00} & \lambda_{01} & \dots & \lambda_{0n} & \dots \\ \lambda_{10} & \lambda_{11} & \dots & \lambda_{1n} & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ \lambda_{j0} & \lambda_{j1} & \dots & \lambda_{jn} & \dots \\ \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

and we will show this satisfies Properties (1) – (7).

For convenience, we put $D = \Omega(\alpha_n)$. Since $\lim_{\substack{|x|\to 1^- \\ x\in D}} |g(x)| = +\infty$, by Lemma 6, we know that $1/g$ belongs to $H(D)$. For each $n \in \mathbb{N}$, we put $u_n = x^n/g$. Then in $H(D)$, u_n has a Mittag-Leffler series ([3], [5]) of the form $\sum_{j=0}^{\infty} \frac{\beta_{j,n}}{1 - x/\alpha_j}$. Now we put $\theta_j = \psi_j(\alpha_j)$ and we have $g(x) = \theta_j g_j(x)(1-x/\alpha_j)$. We will compute the $\beta_{j,n}$. Let $v_{j,n} = (1-x/\alpha_j)u_n$. Then we have $v_{j,n}(\alpha_i) = \frac{\alpha_j^n}{g_j(\alpha_j)\theta_j}$. But since $g_j(\alpha_j) = 1$ whenever $j \in \mathbb{N}$, we see that $\beta_{j,n} = \alpha_j^n/\theta_j$, hence $x^n g(x) = \sum_{i=0}^{n} \frac{\alpha_j^n}{\theta_j(1 - x/\alpha_j)}$. We notice that $\left\| \frac{\alpha_j^n}{1 - x/\alpha_j} \right\|_D = \frac{|\alpha_j|^{n+1}}{\rho}$ and then we have $\lim_{j\to\infty} |\theta_j| = +\infty$, because the sequence of the terms $x^n/g(x)$ must tend to 0. Now we have $x^n = \sum_{j=0}^{n} \frac{\alpha_j^n g(x)}{\theta_j(1 - x/\alpha_j)}$, while $g_j(x) = \frac{g(x)}{\theta_j(1 - x/\alpha_j)}$. Since $g_j(x) = \sum_{n=0}^{\infty} \lambda_{n,j} x^n$, we obtain

$$(8) \quad x^n = \sum_{j=0}^{\infty} \alpha_j^n (\sum_{h=0}^{\infty} \lambda_{h,j} x^h).$$

In particular, (8) holds in every disk $d(0,r)$ with $r \in]0,1[$. But then we know that $\|g_j\|_{d(0,r)} = \sup_{h\in\mathbb{N}} |\lambda_{j,h}| r^h \le \frac{\|\psi_j\|_{d(0,r)}}{|\theta_j|}$. Now, we have $\|\phi_j\|_{d(0,r)} \le \|g\|_{d(0,r)}$ as soon as $|\alpha_i| > r$ because then $\|1/(1 - x/\alpha_j)\|_{d(0,r)} = 1$ and therefore the sequence $(\|\phi_j\|_{d(0,r)})_{j\in\mathbb{N}}$ is bounded. Then the family $(|\lambda_{h,j}| r^h)_{j,h\in\mathbb{N}}$ tends to zero when j tends to $+\infty$, uniformly with respect to h. In particular, P is line-vanishing. For each $h \in \mathbb{N}$, we put $s_h = \sup_{j\in\mathbb{N}} |\lambda_{h,j}|$. We will show

$$(9) \quad \limsup_{h\to+\infty} s_h^{1/h} \le 1.$$

Indeed this is equivalent to show that for every $r \in]0,1[$, we have

$$(10) \quad \lim_{h\to\infty} s_h r^h = 0.$$

Let $r \in]0,1[$ and let $\epsilon > 0$. Since the family $(|\lambda_{h,j}| r^h)_{j,h\in\mathbb{N}}$ tends to zero uniformly with respect to h when j tends to $+\infty$, there clearly exists N such that $|\lambda_{h,j}| r^h < \epsilon$ whenever $j > N$, whenever $h \in \mathbb{N}$, hence for every $h \in \mathbb{N}$, we have $s_h r^h \leq \max_{1\leq j\leq N} |\lambda_{h,j}| r^h$. But for each fixed $i \in \mathbb{N}$, we know that $\lim_{h\to\infty} |\lambda_{h,j}| r^h = 0$, hence $\lim_{h\to\infty} (\max_{1\leq j\leq N} |\lambda_{h,j}| r^h) = 0$. This finishes showing (10). Therefore (9) is proven and so is (2).

Now, we can apply the limits inversion theorem and, then, by (8), we have

$$(11) \quad x^n = \sum_{h=0}^{\infty} (\sum_{j=0}^{\infty} \alpha_j^n \lambda_{h,j}) x^h,$$

whenever $x \in d(0,r)$. Actually this is true for all $r \in]0,1[$ and therefore (11) holds for all $x \in d(0,1^-)$. Hence we have $\sum_{j=0}^{\infty} \alpha_j^n \lambda_{h,j} = 0$ whenever $n \neq h$ and $\sum_{j=0}^{\infty} \alpha_j^n \lambda_{n,j} = 1$. So (3) is satisfied.

Thus we have proven that $PM = I_\infty$. Now we check that $MP = I_\infty$. For every $h \neq j$, we have $g_j(\alpha_h) = g(\alpha_h) = 0$, hence $\sum_{h=0}^{\infty} \alpha_h^n \lambda_{h,j} = 0$. Besides, it is seen that $g_j(\alpha_j) = 1$, hence $\sum_{n=0}^{\infty} \alpha_j^n \lambda_{n,j} = 1$. So we conclude that $MP = I_\infty$ and this finishing proving (4).

Now, we will check that $P(\mathbf{b}) \in \mathcal{E}_1$ for all $\mathbf{b} \in \mathcal{E}_0$. Let $\mathbf{b} := (b_n) \in \mathcal{E}_0$, let $\mathbf{a} := (a_n) = P(\mathbf{b})$ and let $f(t) = \sum_{n=0}^{\infty} a_n t^n$. For each $j \in \mathbb{N}$ we put $f_j(t) = \sum_{m=0}^{j} b_m g_m(t)$. Then f_j belongs to $A(d(0,1^-))$ for all $j \in \mathbb{N}$. Let $r \in]0,1[$. Like the family $|\lambda_{n,j}| r^n$, the family $|\lambda_{n,j} b_j| r^n$ tends to zero uniformly with respect to n when j tends to $+\infty$. That way, in $H(d(0,r))$ we have $\lim_{j\to\infty} \|f - f_j\|_{d(0,r)} = 0$ and therefore f belongs to $H(d(0,r))$. This is true for all $r \in]0,1[$ and therefore f belongs to $A(d(0,1^-))$. Hence $P(\mathbf{b}) \in \mathcal{E}_1$. This shows (5).

Let us show (6). Let $\mathbf{b} := (b_0, \ldots, b_n, \ldots)$ be a bounded sequence. Let $\mathbf{a} = P\mathbf{b}$, and let $\mathbf{a} = (a_0, \ldots, a_n, \ldots)$. We will show

$$(12) \quad \limsup_{n\to\infty} |a_n|^{1/n} \leq 1.$$

Without loss of generality, we may assume $|b_j| \leq 1$, whenever $j \in \mathbb{N}$. Then we have $|a_n| \leq \sup_{j\in\mathbb{N}} |\lambda_{n,j}| = s_n$, therefore $\limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} s_n{}^{1/n} \leq 1$. Now, by (12), it is seen that for all $j \in \mathbb{N}$, the series $\sum_{n=0}^{\infty} a_n \alpha_j^n$ is convergent and therefore we may consider

$Ma = M(Pb)$. By definition, for each $i \in \mathbb{N}$, we have $a_i = \sum_{j=0}^{\infty} \lambda_{i,j} b_j$. Let $Ma = (x_h)_{h\in\mathbb{N}}$.

For each $h \in \mathbb{N}$ we have $x_h = \sum_{m=0}^{\infty} \alpha_h^m a_m = \sum_{m=0}^{\infty} \alpha_h^m (\sum_{j=0}^{\infty} \lambda_{m,j} b_j)$. Let $r = |\alpha_h|$. As we saw, the family $|\lambda_{m,j} b_j| r^m$ tends to 0 when m tends to $+\infty$, uniformly with respect to j. Hence by the Limits Inversion Theorem, we have

$$\sum_{m=0}^{\infty} \alpha_h^m (\sum_{j=0}^{\infty} \lambda_{m,j} b_j) = \sum_{j=0}^{\infty} b_j (\sum_{m=0}^{\infty} \lambda_{m,j} \alpha_h^m).$$

Hence by (3), we see that $x_j = b_j$ and this finishes proving (6). Then by (6) ψ_P is clearly injetive.

Finally we will prove the last statement of the theorem. Let $\phi(x) = \sum_{n=0}^{\infty} \nu_n x^n$. The function ϕ belongs to $A(d(0,1^-))$ and is invertible in $A(d(0,1^-))$ thanks to the inequality $|\nu_0| > |\nu_n|$ whenever $n > 0$. Hence the function $G(x) = g(x)\phi(x)$ is easily seen to satisfy i), ii), iii), iv) like g. Then G factorizes in $A(d(0,1^-))$ and can be written as $\phi_j(x)(1 - x/\alpha_j)$ with $\phi_j(x) = \psi_j(x)\phi(x)$. Hence we put $G_j(x) = \dfrac{\phi_j(x)}{\phi_j(\alpha_j)} = \dfrac{g_j(x)\phi(x)}{\phi(\alpha_j)}$. Now it is clearly seen that the power series of G_j is $\sum_{n=0}^{\infty} \mu_{n,j} x^n$. By definition, the matrix Q satisfies the same properties as P. But when ϕ is not a constant function, for each fixed $j \in \mathbb{N}$, we do not have $\mu_{n,j} = \lambda_{n,j}$ for all $n \in \mathbb{N}$. Hence Q is different from P. As a consequence we see that ψ_M is not surjective, it would be an automorphism of $\mathcal{E}_0$ and therefore ψ_P would also be an automorphism of $\mathcal{E}_0$ and it would be unique. This ends the proof of Theorem 3.

4. PROVING THEOREMS 4 AND 5

Notation. For each integer $q \in \mathbb{N}^*$, we will denote by $\mathcal{G}(q)$ the group of the q-roots of 1.

Lemma 7 : *Let (a_n) be a sequence in $d(0,1^-)$ such that $\lim_{n\to\infty} |a_n| = 1$. For each $s \in \mathbb{N}$, there exists a prime integer $q > p$ and $\zeta \in \mathcal{G}(q)$ such that $|\zeta^h a_s - a_j| = \max(|a_s|, |a_j|)$ for every $j \in \mathbb{N}$, for every $h = 1, \ldots, q-1$.*

Proof. Let $r = |a_s|$. Since $\lim_{n\to\infty} |a_n| = 1$, the circle $C(0,r)$ contains finitely many terms of the sequence (a_n). Without loss of generality we may assume $|a_n| < r$ whenever $n < l$, $|a_n| > r$ whenever $n > t$ and $|a_n| = r$, whenever $n = l, \ldots, t$ (with obviously $l \le s \le t$). Whatever $q \in \mathbb{N}$, $\zeta \in \mathcal{G}(q)$ are, it is seen that we have $|\zeta^h a_s - a_j| = |a_s|$ for all $j < l$ and $|\zeta^h a_s - a_j| = |a_j|$ for all $j > t$. In the residue class field k of $\mathbf{K}$, for every $j = l, \ldots, t$. let γ_j be the class of a_j/a_s. There does exist a prime integer $q > p$ such that the polynomial $p(x) = x^q - 1$ admits none of the γ_j $(l \le j \le t)$ as a zero. Hence, for every q-root ν of 1 in k, we have $\nu^h \neq \gamma_j$ whenever $j = l, \ldots, t$, whenever $h = 1, \ldots, q-1$. Now let ζ be a q-th root of 1 in $\mathbf{K}$. Then by classical properties of the polynomials, we have $\left|\dfrac{\zeta^h - a_j}{a_s}\right| = 1$, hence $|\zeta^h a_s - a_j| = |a_s| = r$ whenever $h = 1, \ldots, q-1$, whenever $j = l, \ldots, t$. This completes the proof of Lemma 7.

Lemma 8 : *Let (a_n) be a regular sequence and let $\rho = \inf_{n \neq m} |a_n - a_m|$. There exists a sequence (b_n) in $d(0, 1^-)$ satisfying :*

(1) $\lim_{n\to\infty} |b_n| = 1$.

(2) $|b_n - b_m| \geq \rho$ *whenever* $n \neq m$.

(3) (a_n) *is a subsequence of* (b_n),

(4) *There exists a sequence (q_n) of prime integers different from p satisfying* $\lim_{n\to\infty} q_n = +\infty$, *such that for every $m \in \mathbb{N}$, $\zeta \in \mathcal{G}(q_n)$, ζb_n is another term of the sequence (b_n),*

(5) *There exists $f \in A(d(0, 1^-))$ admitting each b_n as a simple zero and having no other zero in $d(0, 1^-)$, satisfying*

$$\lim_{\substack{|x|\to 1^- \\ x \in \Omega(b_n)}} |f(x)| = +\infty.$$

Proof. First we will construct a sequence (b'_n) satisfying (1), (2), (3), (4). Let (q_j) be a strictly increasing sequence of prime integers strictly bigger than p and, for each $j \in \mathbb{N}$, let $s_j = \sum_{i=0}^{j} q_i$, let $\zeta_j \in \mathcal{G}(q_j)\backslash\{1\}$ and let $b'_{\zeta_j + h} = \zeta_j^h a_j$ $(0 \leq h \leq q_j - 1)$. We will show that a good choice of the sequence (q_j) enables us to obtain

$$(6) \quad |b'_n - b'_m| = \max(|b'_n|, |b'_m|)$$

for every couple (n, m) satisfying $n \neq m$ and $(n, m) \neq (s_i, s_j)$ whenever $(i, j) \in \mathbb{N} \times \mathbb{N}$. In other words $|b'_n - b'_m| = \max(|b'_n|, |b'_m|)$ must be true all time except when $n = m$ and when (b'_n, b'_m) is equal to some couple (a_{s_i}, a_{s_j}). For each $t \in \mathbb{N}$, let $F_t = \{s_0, s_1, \ldots, s_t\}$ and let E_t be $\{0, 1, \ldots, s_t - 1\}\backslash F_t$. Assume that $q_0, q_1, \ldots, q_{t-1}$ have been chosen to satisfy the following properties (α_t) and (β_t)

(α_t) $\quad \left|b'_n - a_{s_j}\right| = \max(|b'_n|, \left|a_{s_j}\right|)$ for all $j \in \mathbb{N}$, for all $n \in E_t$.

(β_t) $\quad |b'_n - b'_m| = \max(|b'_n|, |b'_m|)$ for all $(n, m) \in E_t \times E_t$ such that $n \neq m$. We will choose q_t such that both (α_{t+1}), (β_{t+1}) are satisfied. Indeed, by Lemma 7 we can take a prime integer u such that, given $\zeta_t \in \mathcal{G}(u)$, we have $\left|\zeta_t^h a_t - a_j\right| = \max(|a_t|, |a_j|)$ for all $j \in \mathbb{N}$, for all $h = 1, \ldots, u-1$, $\left|\zeta_t^h a_t - b'_n\right| = \max(|a_t|, |b'_n|)$ for all $n < s_t$, for all $h = 1, \ldots, u-1$. Thus we can take $q_t = u$ and we see that both (α_{t+1}), (β_{t+1}) are satisfied. Hence we can construct the sequence (q_t) by induction and, therefore, the sequence (b'_n) satisfying (6) is now constructed. Then it is easily checked that the sequence (b'_n) so obtained satisfies (1), (2), (3), (4).

Now let $\{r_0, \ldots, r_n, \ldots\} = \{|a_j| : j \in \mathbb{N}\}$ and let $D = \Omega(b_n)$. The infinite product $g(x) = \prod_{j=0}^{\infty}(1-(x/a_j)^{q_j})$ converges in $A(d(0,1^-))$ and has no zero in $d(0,r) \cap D$ because, by construction of the sequence (b'_n), each zero of g is one of the points b'_m for some $m \in \mathbb{N}$. Hence it is seen that we have $|g(x)| \geq 1$ for every $x \in d(0,1^-) \setminus (\bigcup_{n=0}^{\infty} C(0,r_n))$. For each $n \in \mathbb{N}$, let $\Sigma_n = D \cap C(0,r_n)$, let $\tau_n = \inf_{x \in \Sigma_n} |g(x)|$, let $\sigma_n \in (r_n, r_{n+1}) \cap |\mathbf{K}|$, let $c_n \in C(0,\sigma_n)$, and let $u_n > \min(p,n)$ be a prime integer such that $\tau_n (\frac{r_{n+1}}{\sigma_n})^{u_n} > n+1$. Since $\lim_{n\to\infty} u_n = +\infty$, it is seen that the infinite product $h(x) = \prod_{n=0}^{\infty}(1-(x/c_n)^{u_n})$ converges in $A(d(0,1^-))$. Let $D' = \Omega((c_n),\rho)$ and let $D'' = D' \cap D$. Let $h(x) = \sum_{n=0}^{\infty} \lambda_n x^n$ and, for each $r \in (0,1)$, let $M(r) = \sup_{n\in\mathbb{N}} |\lambda_n| r^n$. Each pole of h is simple and is of the form ζc_n with $\zeta \in \mathcal{G}(u_n)$. Hence it is seen that h satisfies $|h(x)| \geq M|x|/\rho$ for all $x \in D'$. Hence if $x \in D'' \setminus (\bigcup_{n=0}^{\infty} \Sigma_n)$, then we have
$|g(x)h(x)| = M(r_n)\tau_n \geq (\frac{r_n}{r_{n-1}})^{u_{n-1}}\tau_n > n$ and finally we have

$$(7) \qquad \lim_{\substack{|x|\to 1 \\ x\in D''}} |g(x)h(x)| = +\infty.$$

Now let (b''_n) be the sequence of the zeros of g. Clearly (b''_n) satisfies (1) and (4) and also satisfies $|b''_n - b'_m| = \max(|b''_n|, |b'_m|)$ whenever $n, m \in \mathbb{N}$ and $|b''_n - b''_m| = \max(|b''_n|, |b''_m|)$ whenever $n \neq m$. Now we put $b_{2n} = b'_n$ and $b_{2n+1} = b''_n$. The sequence (b_n) clearly satisfies (1), (2), (3), (4) and also satisfies (5) because the zeros of h are the b''_n while those of g are the b'_n. Thus the zeros of f are just the b_n, and then, by (7), we have $\lim_{\substack{|x|\to 1 \\ x\in\Omega(b_n)}} |f(x)| = +\infty$.
This ends the proof of Lemma 8.

Proof of Theorem 4. Without loss of generality we may obviously assume $|\alpha_n| \leq |\alpha_{n+1}|$ whenever $n \in \mathbb{N}$. Let $\rho = |\alpha_0|$. Hence by hypothesis each disk $d(\alpha_q, \rho^-)$ contains no point α_n for each $n \neq q$. Let $D = \Omega((\alpha_n), \rho^-)$.

For each $n \in \mathbb{N}$, let T_n be the hole $d(\alpha_n, \rho^-)$ of D. Since $|\alpha_n| = 0$, it is shortly checked that the sequence $(T_n, 1)$ is a T-sequence of D ([8]). Then, since $\mathbf{K}$ is spherically complete, by [4], Theorem 4, there exists $g \in A(d(0,1^-))$ admitting each α_n as a simple zero and having no zero else in $d(0,1^-)$. Therefore, as $\prod_{n=0}^{\infty} |\alpha_n| = 0$, is is seen that g

satisfies $\lim_{\substack{|x|\to 1^- \\ x\in D}} |g(x)| = +\infty$. Now we can apply Theorem 3, which shows that the matrix $M = \mathcal{M}(a_n)$ admits inverses P. Then the sequence (a_n) satisfies $\sum_{n=0}^{\infty} a_n \alpha_j^n = b_j$ for every $j \in \mathbb{N}$ and this clearly ends the proof of Theorem 4.

Proof of Theorem 5. By Lemma 8, there exists a regular sequence (γ_n) of $d(0,1^-)$ such that (α_n) is a subsequence of (γ_n) together with an analytic function $g \in A(d(0,1^-))$ admitting each γ_m as a simple zero and having no other zero in $d(0,1^-)$, satisfying $\lim_{\substack{|x|\to 1^- \\ x\in\Omega(\gamma_n)}} |g(x)| = +\infty$ with $\rho = \inf_{n\neq m} |\gamma_n - \gamma_m|$. Then, by Theorem 3, the matrix $M = \mathcal{M}(\gamma_n)$ admits line-vanishing inverses M' satisfying $M(M'(\mathbf{b})) = \mathbf{b}$ for all bounded sequence $\mathbf{b} = (b_n)$. Let $\mathbf{a} := (a_n) = M'(\mathbf{b})$. Thus we have $M(\mathbf{a}) = \mathbf{b}$ and therefore $\sum_{n=0}^{\infty} a_n \gamma_j^n = b_j$ whenever $j \in \mathbb{N}$. This ends the proof of Theorem 5.

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