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SEMINORMED SPACES

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Abstract. Several functional characterizations of the equivalence of non-archimedean seminorms on groups are given in this paper. This study allows us to define a "non-archimedean seminorm" on the group of continuous homomorphims between non-archimedean seminormed groups, extending the usual and well-known norm defined on the space of continuous linear maps between non-archimedean normed vector spaces.

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0. INTRODUCTION.

In 1936 G.BIRKHOFF [1] and J.KAKUTANI [3] proved that a topological group $(G, \cdot, \tau)$ has a countable basis of neighbourhoods at the identity 1_G if and only if τ is defined by a left (right)-invariant semimetric on G, or equivalentely, by a seminorm on G (a map $\| \cdot \| : G \longrightarrow [0,+\infty)$ is called a seminorm if :
i) $\|1_G\| = 0$,
ii) $\|x^{-1}\| = \|x\| \; \forall x \in G$,
iii) $\|x \cdot y\| \leq \|x\| + \|y\| \; \forall x, y \in G$).

Later, A.MARKOV [5] applied the result of G.BIRKHOFF and J.KAKUTANI to conclude that the topology of a topological group is defined by a family of seminorms.

The non-archimedean counterpart of the Birkhoff-Kakutani result was given in [6] by G.RANGAN, who proved that a topological group $(G, \cdot, \tau)$ has a countable basis of neighbourhoods at the identity of G consisting of subgroups of G if and only if τ is defined by a non-archimedean left(right)-invariant semimetric d on G (i.e., satisfying the strong triangle inequality $d(x,y) \leq \max\{d(x,z), d(z,y)\} \; \forall x, y, z \in G$), or equivalentely, τ is defined by a non-archimedean seminorm (i.e., satisfying the strong triangle inequality $\|x \cdot y\| \leq \max\{\|x\|, \|y\|\} \; \forall x, y \in G$).

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Similarly to A.MARKOV in [5], we can also apply the result of G.RANGAN to derive that the topology of a non-archimedean topological group is defined by a family of non-archimedean seminorms (G is called non-archimedean if its topology has a basis B of neighbourhoods at the identity such that $V \cdot V \subseteq V$ for all $V \in B$).

The aim of this paper is to give some functional characterizations of the topological equivalence of non-archimedean seminorms on groups (Section 2). As an application, we define a "non-archimedean seminorm" on the group of continuous homomorphisms, extending the well-known definition of norm on the space of continuous linear maps between normed vector spaces (Section 3). Also, several examples of non-archimedean seminormed groups are given in Sections 1 and 4 (this last one devoted to describe the non-archimedean seminorms on the additive group of integers numbers).

1. PRELIMINARIES AND EXAMPLES.

Throughout this paper G,G' will be multiplicative groups and $1_G, 1_{G'}$ will be its corresponding identity elements.

1.1 Definition. (See e.g. [2] and [7]). A *non-archimedean seminorm* on G is a map $\| \ \| : G \longrightarrow [0, +\infty)$ with the properties

i) $\|1_G\| = 0$.

ii) $\|x^{-1}\| = \|x\|$ for all $x \in G$.

iii) $\|x \cdot y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in G$.

If in addition, $\|x\| = 0 \iff x = 1_G$, then we say that $\| \ \|$ is a *non-archimedean norm* on G.

The pair $(G, \| \ \|)$ is called a *non-archimedean (semi)normed group.*

Similarly to [8], 1.1.5, we have that if $x, y \in G$ and $\|x\| \neq \|y\|$, then $\|x \cdot y\| = \max\{\|x\|, \|y\|\}$.

A non-archimedean seminorm $\| \ \|$ on G is called *trivial* if $\| \ \| \equiv 0$ or $inf\{\|x\| : x \in G, \|x\| > 0\} > 0$.

For each $r > 0$, U_r (resp. B_r) will denote the open ball $\{x \in G : \|x\| < r\}$ (resp. the closed ball $\{x \in G : \|x\| \leq r\}$). And we will call $B_0 := \{x \in G : \|x\| = 0\}$.

A non-archimedean seminorm induces two non-archimedean semimetrics d_1, d_2 on G by

$$d_1(x,y) = \|x^{-1}y\| \quad \text{(left invariant)},$$
$$d_2(x,y) = \|xy^{-1}\| \quad \text{(right invariant)}, \quad (x, y \in G),$$

and thereby two zerodimensional topologies on G, which will be denoted by τ_1 and τ_2 respectively (observe that the map $x \to x^{-1}$ is an isometry from (G, d_1) onto (G, d_2)).

Relative to these topologies, the map $G \times G \longrightarrow G : (x,y) \to xy$ is continuous at $(1_G, 1_G)$ and the map $G \longrightarrow G : x \to x^{-1}$ is continuous at 1_G. Moreover $B := \{V_n : n \in N^*\}$ (where $V_n = U_{1/n}$ or $V_n = B_{1/n}$, $N^* = N - \{0\}$) is a countable basis of neighbourhood of 1_G for both topologies and satisfies the following properties :

1) For all $n \in N^*$ V_n is a subgroup.

2) For all $n, m \in N^*$ there exists $p \in N^*$ such that $V_p \subseteq V_n \cap V_m$.

Conservely, if $B := \{V_n : n \in N^*\}$ is a countable collection of subsets of a group G verifying the above properties 1) and 2), we can derive, like in the proof of Theorem 1 of [6], the existence of a non-archimedean seminorm $\| \ \|$ on G such that B is a neighbourhood basis at 1_G for the topologies τ_1 and τ_2 associated with $\| \ \|$.

One can easily see that $\tau_1 = \tau_2$ if and only if (G, τ_j) (j = 1 or 2) is a topological group, i.e., multiplication and inversion are continuous maps. This, for example, happens when G is an abelian group. But in general we can have $\tau_1 \neq \tau_2$ and also $\tau_1 = \tau_2$ while $d_1 \neq d_2$, as it is showed in the following examples.

1.2 Examples.

1.- Let $G := \{f : [0,2] \longrightarrow [0,2] :$ f is a biyection $\}$ endowed with the composition operation $\circ$, and let $\| \ \| : G \longrightarrow [0, +\infty)$ given by

$$\|f\| := \begin{cases} sup\{x \in [0,2] : f(x) \neq x\} & \text{if } f \neq I_d \\ 0 & \text{otherwise} \end{cases}$$

(where I_d denotes the identity map from [0,2] onto [0,2]). Then, $(G, \| \ \|)$ is a non-archimedean normed group for which $\tau_1 \neq \tau_2$.

Indeed, one can easily see that $\| \ \|$ is a non-archimedean norm on G. To prove that $\tau_1 \neq \tau_2$, let $f, g_n \in G$ $(n \in N^*)$ given by

$$f(x) := 2 - x,$$

$$g_n(x) := \begin{cases} 2 - x & \text{if } x < 2 - \frac{1}{n} \\ 2 - (2 - \frac{1}{n}) & \text{otherwise} \end{cases}$$

(where $x \in [0, 2]$).

Then, $\|g_n \circ f\| = 1/n \to 0$ $(n \to +\infty)$, whereas $\|f \circ g_n\| = 2$ for all $n \in N^*$. Hence, $\tau_1 \neq \tau_2$.

2.- Let $(K, | \ |)$ be a non-archimedean valued field of characteristic different from 2, and let

$$G := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in K \ and \ ad - bc = 1 \right\}$$

endowed with the usual matrix multiplication. Let

$$H_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : \max\{|a-1|, |b|, |c|, |d-1|\} < 1 \right\}.$$

Then, $\| \ \| : G \longrightarrow [0, +\infty)$, given by

$$\|A\| := \begin{cases} 1 & \text{if } A \notin H_1 \\ \max\{|a-1|, |b|, |c|, |d-1|\} & \text{otherwise} \end{cases}$$

$$(where \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G)$$

defines a non-archimedean norm on G for which $\tau_1 = \tau_2$ and $d_1 \neq d_2$.

Indeed, it is straightforward to check that $\| \ \|$ is a non-archimedean norm on G.

To see that $\tau_1 = \tau_2$, observe that if $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $r := \max\{|a-1|, |b|, |c|, |d-1|\}$, then for each $n \in N^*$ one verifies that

$$A^{-1} \cdot U_{1/m} \cdot A \subseteq U_{1/n} \quad \forall m \geq \max\{n, nr, nr^2\}.$$

Now, let (a_n) be a sequence in K with $\lim a_n = 0$. For each $n \in N$, choose $x_n, y_n \in G$ by

$$x_n = \begin{pmatrix} a_n & a_n \\ 0 & a_n^{-1} \end{pmatrix}, \quad y_n = \begin{pmatrix} a_n^{-1} & a_n \\ 0 & a_n \end{pmatrix}.$$

Then, $\lim x_n y_n = \lim \begin{pmatrix} 1 & 2a_n^2 \\ 0 & 1 \end{pmatrix} = 1_G$, whereas $y_n x_n = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ does not converge to 1_G. This proves that $d_1 \neq d_2$.

Before going into the subject of this paper we present a few more examples.

Clearly, every non-archimedean seminormed vector space is an abelian additive non-archimedean seminormed group.

Now, let $(G, \| \ \|)$ be a non-archimedean (semi)normed group and let X be any set. Then, as a natural extension of 3.A and 3.B in [8] we obtain that

$$\{f : X \longrightarrow G : \text{f is bounded map }\}$$

and

$$\{f : X \longrightarrow G : \text{for every } \epsilon > 0, \text{ there exist only finitely many elements x of X for which } \|f(x)\| \geq \epsilon\}$$

are non-archimedean (semi)normed groups, when we consider on them the supremun (semi)norm.

In a similar way, extending the examples considered in 3.D - 3.G of [8] we can construct more examples of non-archimedean seminormed groups consisting of continuous functions, when we provide X with a topology.

If $\{(G_i, \| \ \|_i) : i \in I\}$ is a family of non-archimedean (semi)normed groups, and ΠG_i is the corresponding cartesian product, then

$$\{a := (a_i) \in \Pi G_i : \{\|a_i\| : i \in I\} \text{ is bounded }\}$$

is a non-archimedean (semi)normed group, endowed with the supremum (semi)norm.

If $(G, \| \ \|)$ is a non-archimedean (semi)normed group and H is a subgroup of G, then H is again in a natural way a non-archimedean (semi)normed group. If, in addition, H is a normal subgroup of G, then

$$\| \ \|_H : G/H \longrightarrow [0, +\infty) \text{ given by } \|x \cdot H\|_H := inf\{\|xz\| : z \in H\} \ (x \in G),$$

is a non-archimedean seminorm on G/H, and such that $\| \ \|_H$ is a norm iff H is a $\tau_1 - closed$ (or equivalentely, $\tau_2 - closed$) subgroup of G.

2. FUNCTIONAL CHARACTERIZATION OF EQUIVALENT SEMINORMS

Throughout this section $\| \ \|$ (resp. $\| \ \|'$) will be a non-archimedean seminorm on G and τ_j (resp. τ_j') (j=1,2) will be the associated topologies. Also, for every $r > 0$, U_r and B_r (resp. U_r' and B_r') will denote the corresponding open and closed balls on $\| \ \|$ (resp. $\| \ \|'$).

2.1 Definition. We say that $\| \ \|$ is *weaker* than $\| \ \|'$ if $\tau_1 \leq \tau_1'$ (or equivalentely $\tau_2 \leq \tau_2'$). If $\tau_1 = \tau_1'$ (or equivalentely $\tau_2 = \tau_2'$) we say that $\| \ \|$ and $\| \ \|'$ are *equivalent* seminorms.

As it is well known, if G is a vector space over a non-archimedean valued field $(K, |.|)$ and $\| \ \|, \| \ \|'$ are non-archimedean seminorms on G satisfying $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in K, x \in G$ (and analogously for $\| \ \|'$), then

(*) $\| \ \|$ is weaker than $\| \ \|'$ iff there exists $M \in (0, +\infty)$ such that $\|x\| \leq M \cdot \|x\|'$ for all $x \in G$.

In the case of non-archimedean seminorms on arbitrary groups, property (*) is not true in general (see 2.5.3). In this case, property (*) has to be reformulated by using some classes of real functions instead of real numbers. This is the objective of this section. The following result will be crucial to our purpose.

2.2 Lemma : *Suppose that $\| \ \|$ is weaker than $\| \ \|'$.*

a) Let $A := \{s \geq 0 : \exists r > 0$ such that $B_s' \subseteq B_r\}$, and let J be the convex hull of A. Then, the map $f : J \longrightarrow [0, +\infty)$ given by

$$f(s) := \begin{cases} inf\{r > 0 : B_s' \subseteq B_r\} & if\, s > 0 \\ 0 & if\, s = 0 \end{cases}$$

satisfies the following properties :

1) f(0) = 0 and f is continuous at s=0.

2) f is an increasing map.

3) $\|x\| \leq f(\|x\|')$ for all $x \in G$ with $\|x\|' \in J$.

4) f is the smallest map among all maps from J into $[0, +\infty)$ that verify 1),2) and 3).

b) Let I be the convex hull of $\{\|x\| : x \in G\}$. Then, the map $g : I \longrightarrow [o, +\infty)$, given by

$$g(r) := \begin{cases} \max\{s > 0 : U_s' \subseteq U_r\} & if\, r > 0 \\ 0 & if\, r = 0 \end{cases}$$

satisfies the following properties :

1) $g(r) = 0 \iff r = 0$.

2) g is an increasing map.

3) $g(\|x\|) \leq \|x\|'$ for all $x \in G$.

4) g is the biggest map among all maps from I *into* $[0,+\infty)$ *that verify 1),2) and 3).*

Proof : We only prove a). The proof of b) is similar.

Firstly observe that since $\| \ \|$ is weaker than $\| \ \|'$, A is non-empty set and so f is well defined.

Obviously the map f satisfies 2) anf f(0) = 0.

Now, suppose that there exists a sequence (s_n) in J with lim $s_n = 0$ and inf $f(s_n) > \epsilon$ for some $\epsilon > 0$. Then, for all n = 1,2,... we obtain that B'_{s_n} is not included in B_ϵ, which contradicts the fact that $\| \ \|$ is weaker than $\| \ \|'$. This proves that f is continuous at s = 0.

To prove 3), let $x \in G$ with $\|x\|' \in J - \{0\}$ (if $\|x\|' = 0$ then $\|x\| = 0$ and the result is trivial). If $t \in \{r > 0 : B'_{\|x\|'} \subseteq B_r\}$, then $\|x\| \leq t$, and so $\|x\| \leq f(\|x\|')$.

Finally, assume that $h : J \longrightarrow [0,+\infty)$ satisfies 1), 2) and 3), and there exists $s \in J$ for which $h(s) < f(s)$. Then, there is $x \in G$ with $\|x\|' \leq s$ and $\|x\| > h(s)$, a contradiction (observe that if $\|x\|' \leq s$, then $\|x\| \leq h(\|x\|') \leq h(s)$).

If $K \in [0,+\infty]$, by I_K we indiscriminately denote the real interval [0,K) or [0,K] (being in the first case $K > 0$ and in the second case $K < +\infty$). If K = 0 we agree that $I_K = \{0\}$. With this notation and applying 2.2 we can now formulate the main result of this section.

2.3 Theorem : *The following are equivalent :*

i) $\| \ \|$ *is weaker than* $\| \ \|'$.

ii) There exists a $H \in (0,+\infty]$ *and an increasing map* $f : I_H \longrightarrow [0,+\infty)$ *vanishing at s=0 and continuous at s=0, such that* $\|x\| \leq f(\|x\|')$ *for all* $x \in G$ *with* $\|x\|' \in I_H$.

iii) There exists a $M \in [0,+\infty]$ *and an increasing map* $g : I_M \longrightarrow [0,+\infty)$ *vanishing only at r=0, such that* $g(\|x\|) \leq \|x\|'$ *for each* $x \in G$.

Proof :

$i) \Rightarrow ii)$ Let A be as 2.2.a). Then, take H := sup A and f the map defined in 2.2.b) with $J = I_H$.

$i) \Rightarrow iii)$ Take $M := sup\{\|x\| : x \in G\}$ and g the map defined in 2.2.b) with $I = I_M$.

$ii) \Rightarrow i)$ Given $r > 0$, there exists $s \in I_H - \{0\}$ such that $f(s) < r$. Thus, for $x \in G$ with $\|x\|' < s$, we have $\|x\| \leq f(\|x\|') \leq f(s) < r$, and we conclude that $U'_s \subseteq U_r$.

$iii) \Rightarrow i)$ We can assume $M > 0$ (if M=0, then $\| \ \| \equiv 0$ and i) follows trivially). Let $r \in I_M - \{0\}$ be given and take $s = g(r) > 0$. If $x \in U'_s$ then $g(\|x\|) \leq \|x\|' < s = g(r)$ and so $\|x\| < r$, which implies that $U'_s \subseteq U_r$.

2.4 Corollary : *The following are equivalent :*

i) $\| \ \|$ *and* $\| \ \|'$ *are equivalent seminorms.*

ii) There exist $H, H' \in (0,+\infty]$ *and incresing maps* $f : I_H \longrightarrow [0,+\infty)$, $f' : I_{H'} \longrightarrow [0,+\infty)$ *vanishing and continuous at 0, such that* $\|x\| \leq f(\|x\|')$ *for all* $x \in G$ *with* $\|x\|' \in I_H$ *and* $\|x\|' \leq f'(\|x\|)$ *for all* $x \in G$ *with* $\|x\| \in I_{H'}$.

iii) There exist $M, M' \in [0,+\infty]$ *and increasing maps* $g : I_M \longrightarrow [0,+\infty), g' : I_{M'} \longrightarrow [0,+\infty]$ *vanishing only at 0, such that* $g(\|x\|) \leq \|x\|'$ *and* $g'(\|x\|') \leq \|x\|$ *for each* $x \in G$.

2.5 Remarks :

1.- Let $f : I_H \longrightarrow [0, +\infty)$ be the map of 2.2.a) with H = sup $A > 0$. Then,

i) $f(s) = min\{r \geq 0 : B'_s \subseteq B_r\}$ $(s \in I_H)$.

ii) For all $K > 0$ with $I_K \subseteq I_H$, the restriction of f to $I_K, f|I_K$, is the smallest map among all maps $h : I_K \longrightarrow [0, +\infty)$ verifying 1), 2) and $\|x\| \leq h(\|x\|')$ for all $x \in G$ with $\|x\|' \in I_K$.

2.- In 2.3.ii) we can not assure in general the existence of a map f satisfying $\|x\| \leq f(\|x\|')$ for all $x \in G$ (compare with 2.3.iii)).

Indeed, if G is a group endowed with a non-bounded non-archimedean seminorm $\| \ \|$, then for every $\epsilon > 0$, the formula

$$\|x\|_\epsilon = min\{\|x\|, \epsilon\} \quad (x \in G)$$

defines a bounded non-archimedean seminorm on G, which is equivalent to $\| \ \|$.

3.- The group G considered in the above Remark shows also that two equivalent seminorms on a group do not have, in general, the same bounded sets (compare with the well-known situation in the case of normed vector spaces).

We finish this section showing some relations between the maps f and g defined in 2.2. To do that we need to introduce some auxiliary functions.

Let $\alpha \in [0, +\infty]$ and let $h : I_\alpha \longrightarrow [0, +\infty)$ be an increasing map vanishing at 0. Let $\beta := sup\{h(x) : x \in I_\alpha\}$ and I_β the convex hull of $h(I_\alpha)$. Clearly, we have the following three possibilities :

a) $I_\alpha = [0, \alpha)$, $I_\beta = [0, \beta)$.
b) $I_\alpha = [0, \alpha)$, $I_\beta = [0, \beta]$.
c) $I_\alpha = [0, \alpha]$, $I_\beta = [0, \beta]$.

Then we define the following maps :

i) $h_i : I_\beta \longrightarrow I_\alpha$ given by

$$h_i(y) := inf\{x \in I_\alpha : h(x) \geq y\} \quad (y \in I_\beta).$$

ii) $h_s : I_\beta \longrightarrow I_\alpha$ given by

$$h_s(y) := sup\{x \in I_\alpha : h(x) \leq y\} \quad (y \in I_\beta) \quad \text{in cases a) and c)},$$

$$h_s(y) := \begin{cases} sup\{x \in I_\alpha : h(x) \leq y\} & \text{if } y < \beta \\ inf\{x \in I_\alpha : h(x) = \beta\} & \text{if } y = \beta \end{cases} \quad \text{in case b)}.$$

It is straightforward to verify that the maps h_i and h_s, previously defined satisfy the following properties.

2.6 Lemma :

i) h_i *and* h_s *are increasing maps.*

ii) h_i *is left continuous and* h_s *is right continuous in* I_β.

iii) $h_i(h(x)) \leq x \quad \forall x \in I_\alpha$ *and* $h_s(h(x)) \geq x \quad \forall x \in h^{-1}([0, \beta))$.

iv) $h_i(y) \leq h_s(y) \quad \forall y \in I_\beta$.

v) $y < y' \Longrightarrow h_s(y) \leq h_i(y') \quad (y, y' \in I_\beta)$.

2.7 Proposition : *Suppose $\| \;\|$ is weaker than $\| \;\|'$. Let $H, M, f : I_H \longrightarrow [0,+\infty)$ and $g : I_M \longrightarrow [0,+\infty)$ be as in 2.2 and 2.3. Let $M' := sup\{f(s) : s \in I_H\}$ and $H' := sup\{g(r) : r \in I_M\}$. Then,*

i) $M' \leq M$ and $f_i(r) \leq g(r) \leq f_s(r) \quad \forall r < M'$

(If $M' = 0$, then $f_i \equiv f_s \equiv 0$ on $I_{M'} = \{0\}$, and $g \equiv 0$ on $I_M = \{0\}$).

ii) $H' \leq H$ and $g_i(t) \leq f(t) \leq g_s(t) \quad \forall t < H'$

(If $H' = 0$, then $g_i \equiv g_s \equiv 0$ on $I_{H'} = \{0\}$, and $f \equiv 0$ on $[0,+\infty)$).

Proof : We only prove i). The proof of ii) is similar.

Observe that, for each $s \in I_H - \{0\}$

$$f(s) = inf\{r > 0 : B'_s \subseteq U_r\}, \tag{1}$$

and for each $r \in I_M - \{0\}$

$$g(r) = sup\{s > 0 : B'_s \subseteq U_r\}. \tag{2}$$

If $M' > M$, there exist $t > M$ and $s \in I_H - \{0\}$ such that $f(s) > t$. By (1), B'_s is not included in $U_t = G$, a contradiction. Hence, $M' \leq M$.

Choose $r < M'$ and assume $f_i(r) > g(r)$. Then, there exists a $t > g(r)$ (and hence B'_t is not included in U_r, by (2)) such that $t < f_i(r)$. It follows from 2.6.i),iii), that $f(t) < r$, and so $B'_t \subseteq U_r$ (see (1)), a contradiction. Hence, $f_i(r) \leq g(r)$. Analogously, one can see that $g(r) \leq f_s(r)$.

CONTINUOUS HOMOMORPHISMS BETWEEN SEMINORMED GROUPS

We shall apply the results proved in section 2 to define a "seminorm" on the group of continuous homomorphisms (see 3.5), extending the usual norm defined on the space of continuous linear maps between non-archimedean normed vector spaces. To do that we need some preliminary machinery.

For every increasing map $h : I_\alpha \longrightarrow [0,+\infty)$ ($\alpha \in (0,+\infty]$) vanishing at 0, we will consider the corresponding upper and lower right Dini derivatives of h at 0 (see e.g. p.101 of [4]), given by

$$D^+h(0) = inf\{sup\{\frac{h(x)}{x} : 0 < x < s\} : s \in I_\alpha - \{0\}\} \in [0,+\infty]$$

$$D_+h(0) = sup\{inf\{\frac{h(x)}{x} : 0 < x < r\} : r \in I_\alpha - \{0\}\} \in [0,+\infty]$$

If $\alpha = 0$, we define $D^+h(0) = 0$ and $D_+h(0) = +\infty$.

One can easily check that given $h, h' : I_\alpha \longrightarrow [0,+\infty)$ then

- $h \leq h' \implies D^+h(0) \leq D^+h'(0)$ and $D_+h(0) \leq D_+h'(0)$. (3)
- $D^+(\max\{h,h'\})(0) = \max\{D^+h(0), D^+h'(0)\}$. (4)
  $D_+(min\{h,h'\})(0) = min\{D_+h(0), D_+h'(0)\}$.

3.1 Proposition : *Suppose that $\| \;\|$ and $\| \;\|'$ are two non-archimedean seminorms on G such that $\| \;\|$ is weaker than $\| \;\|'$. Let H, M, H', M', f, g be as in 2.7. Then,* *i)* $D^+g_i(0) = D^+f(0) = D^+g_s(0)$.

ii) $D_+f_i(0) = D_+g(0) = D_+f_s(0)$.

iii) $D_+g(0) = \frac{1}{D^+f(0)}$ *(with the criterion* $\frac{1}{+\infty} = 0$ *and* $\frac{1}{0} = +\infty$*).*

Proof : i) By 2.7.ii), $D^+g_i(0) \le D^+f(0) \le D^+g_s(0)$.

Assume $D^+g_i(0) < D^+g_s(0)$ and choose $u > 0$ such that $D^+g_i(0) < u < D^+g_s(0)$. Since $D^+g_i(0) < u$, there exists $s' \in I_{H'} - \{0\}$ such that

$$g_i(s) < u \cdot s \quad \forall s \in I_{H'}, \quad s < s'. \tag{5}$$

On the other hand, since $u < D^+g_s(0)$, then $u < sup\{\frac{g_s(t)}{t} : 0 < t < s'\}$ and therefore there exists $t < s'$ such that

$$u \cdot t < g_s(t). \tag{6}$$

Now, take $s \in I_{H'}$, with $t < s < s'$, such that $u \cdot t < u \cdot s < g_s(t)$. Then $g_i(s) < u \cdot s < g_s(t)$, which is a contradiction (see 2.6.v)).

The proof of ii) is similar.

iii) Assume that $D_+g(0) \neq 0$ and $D^+f(0) \neq 0$, and there is $u > 0$ such that $D_+g(0) > u > \frac{1}{D^+f(0)}$. By ii), $D_+f_s(0) > u$, and so there exists $t' \in I_{M'} - \{0\}$ such that

$$f_s(t) > ut \quad \forall t \in I_{M'} - \{0\}, t < t'. \tag{7}$$

Also, since $D^+f(0) > \frac{1}{u}$, $sup\{\frac{f(t)}{t} : 0 < t < s\} > \frac{1}{u}$ for all $s \in I_H - \{0\}$, and therefore

$$f(t) > \frac{t}{u} \quad \forall t \in I_{H'} - \{0\}. \tag{8}$$

Now, take $t \in I_{M'} - \{0\}$ such that $\max\{t, \frac{t}{u}\} < t'$. Then, by (7), $f_s(\frac{t}{u}) > t$, and applying 2.6.i),iii) we obtain that $f(t) \le \frac{t}{u}$, which contradicts (8). Hence, $D_+g(0) \le \frac{1}{D^+f(0)}$.

Analogously, we can prove that $D_+g(0) \ge \frac{1}{D^+f(0)}$.

3.2 Definition. Let $(G, \| \ \|)$ and $(G', \| \ \|')$ be a non-archimedean seminormed groups and let τ_j (resp. τ'_j) (j=1,2) be the corresponding topologies associated with $\| \ \|$ (resp. $\| \ \|'$). We say that a homomorphism $\phi : G \longrightarrow G'$ from G into G' is *continuous* if $\phi : (G, \tau_1) \longrightarrow (G', \tau'_1)$ is continuous (or equivalentely, $\phi : (G, \tau_2) \longrightarrow (G', \tau'_2)$ is continuous).

Observe that, if $\phi : G \longrightarrow G'$ is a homomorphism, then the formula

$$\|x\|_\phi := \|\phi(x)\|' \quad (x \in G)$$

defines a non-archimedean seminorm on G. Also, ϕ is continuous iff $\| \ \|_\phi$ is weaker than $\| \ \|$. Taking account this fact in conjuction with 2.2 and 2.3, we obtain the following characterization of the continuous homomorphisms between seminormed groups.

3.3 Theorem : *Let* $(G, \| \ \|)$ *and* $(G', \| \ \|')$ *be non-archimedean seminormed groups and let* $\phi : G \longrightarrow G'$ *be a homomorphism. Then, the following are equivalent :*

i) ϕ *is continuous.*

ii) There exist $H_\phi \in (0, +\infty]$ *and* $f_\phi : I_{H_\phi} \longrightarrow [0, +\infty)$ *such that*

1) $f_\phi(0) = 0$ *and* f_ϕ *is continuous at s=0.*

2) f_ϕ *is an increasing map.*

3) $\|\phi(x)\|' \leq f_\phi(\|x\|)$ *for all* $x \in G$ *with* $\|x\| \in I_{H_\phi}$.

Moreover, for all $K > 0$ *with* $I_K \subseteq I_{H_\phi}$, *the restriction* $f|I_K$ *is the smallest map among all maps* $h : I_K \longrightarrow [0, +\infty)$ *verifying 1), 2) and* $\|\phi(x)\|' \leq h(\|x\|)$ *for all* $x \in G$ *with* $\|x\| \in I_K$, *(see 2.5.1).*

iii) There exist $M_\phi \in [0, +\infty]$ *and* $g_\phi : I_{M_\phi} \longrightarrow [0, +\infty)$ *such that*

1) $g_\phi(r) = 0 \iff r = 0$.

2) g_ϕ *is an increasing map.*

3) $g_\phi(\|\phi(x)\|') \leq \|x\|$ *for all* $x \in G$.

Moreover, g_ϕ *is the biggest map among all the maps from* I_{M_ϕ} *into* $[0, +\infty)$ *that verify 1), 2) and 3).*

3.4 Theorem : *Let* $(G, \| \|)$ *and* $(G', \| \|')$ *be non-archimedean seminormed groups and assume thar G' is abelian. Then,*

$$\mathrm{C}(G, G') := \{\phi : G \longrightarrow G' : \phi \text{ is a continuous homomorphism } \}$$

is an abelian group with respect to the product operation $((\phi \cdot \psi)(x) := \phi(x) \cdot \psi(x),\ x \in G)$.

Also, the map $\| \| : \mathrm{C}(G, G') \longrightarrow [0, +\infty]$ *given by*

$$\|\phi\| := D^+ f_\phi(0) \quad (= \tfrac{1}{D_+ g_\phi(0)}, \text{ by 3.1.iii}))$$

(where $\phi \in \mathrm{C}(G, G')$, f_ϕ, g_ϕ *as in 3.3) satisfies the following properties :*

i) $\|1\| = 0$ *(where* $1 : G \longrightarrow G'$ *is the identity element of* $\mathrm{C}(G, G')$*).*

ii) $\|\phi^{-1}\| = \|\phi\|$ *for all* $\phi \in \mathrm{C}(G, G')$.

iii) $\|\phi \cdot \psi\| \leq \max\{\|\phi\|, \|\psi\|\}$ *for all* $\phi, \psi \in \mathrm{C}(G, G')$.

Proof : It is easy to see that $(\mathrm{C}(G, G'), \cdot)$ is an abelian group.

i) Obviously $f_1 \equiv 0$ on $I_{H_1} = [0, +\infty)$ and so $\|1\| = D^+ f_1(0) = 0$.

ii) It follows from the fact that

$$\|\phi(x)\|' = \|\phi(x)^{-1}\|' = \|\phi^{-1}(x)\|' \quad \text{for all } x \in G.$$

iii) Let $\phi, \psi \in \mathrm{C}(G, G')$ and let $H := min\{H_\phi, H_\psi\} \leq H_{\phi\cdot\psi}$ (where H_ϕ, H_ψ and $H_{\phi\cdot\psi}$ are as in 3.3.ii)). Then, for all $x \in G$ with $\|x\| \in I_H$ one verifies $\|(\phi \cdot \psi)(x)\|' = \|\phi(x) \cdot \psi(x)\|' \leq \max\{\|\phi(x)\|', \|\psi(x)\|'\} \leq \max\{f_\phi(\|x\|), f_\psi(\|x\|)\} = (\max\{f_\phi, f_\psi\})(\|x\|)$.

By 3.3.ii), $f_{\phi\cdot\psi}(s) \leq \max\{f_\phi(s), f_\psi(s)\}$ for all $s \in I_H$. Applying properties (3) and (4), we deduce

$$\|\phi \cdot \psi\| = D^+ f_{\phi\cdot\psi}(0) \leq D^+(\max\{f_\phi, f_\psi\})(0) = \max\{D^+ f_\phi(0), D^+ f_\psi(0)\} = \max\{\|\phi\|, \|\psi\|\}.$$

3.5 Remarks :

1.- We can have $\|\phi\| = 0$ and $\phi \neq 1$, even when $(G, \| \|)$ and $(G', \| \|')$ are normed groups.

Example : Let p be a prime number and take $G := G' := Q_p$, $\| \| := | |_p$, $\|.\|' := | |_p^2$, and $\phi : (G, \| \|) \longrightarrow (G', \| \|')$ given by $\phi(x) = x$ (where $| |_p$ denotes the p-adic valuation on Q_p, see [8]).

Since $\| \|$ and $\| \|'$ have the same bounded sets, $H_\phi = +\infty$. Also, by 3.3.ii), $f_\phi : [0, +\infty) \longrightarrow [0, +\infty)$ verifies that $f_\phi(s) \leq s^2$ for all $s \in [0, +\infty)$. Applying (3) we obtain that $D^+ f_\phi(0) \leq D^+(s^2)(0) = 0$. Hence $\|\phi\| = D^+ f_\phi(0) = 0$.

2.- We can have $\|\phi\| = +\infty$.

Using the above example with now $\| \| := | |_p$ and $\| \|' := | |_p^{\frac{1}{2}}$, it is not dificult to see that $D^+ f_\phi(0) = D^+(s^{\frac{1}{2}})(0) = +\infty$.

3.- In order to obtain a non-archimedean seminormed group we can consider the following possibilities :

i) Define $\mathrm{CB}(G, G') := \{\phi \in \mathrm{C}(G, G') : \|\phi\| < +\infty\}$. Then, $\mathrm{CB}(G, G')$ is a subgroup of $\mathrm{C}(G, G')$, and $(\mathrm{CB}(G, G'), \| \|)$ is a non-archimedean seminormed group.

ii) Define $\| \|_1 : \mathrm{C}(G, G') \longrightarrow [0, +\infty)$ given by $\|\phi\|_1 := min\{\|\phi\|, 1\}$. Then $(\mathrm{C}(G, G'), \| \|_1)$ is a non-archimedean seminormed group (see 2.5.2).

4.- Let $(E, \| \|)$ and $(F, \| \|')$ be non-archimedean normed spaces over a non-archimedean valued field. As it is well known, the space $\mathrm{L}(E, F)$ of all continuous linear maps from E into F is again a non-archimedean normed space with the norm

$$|||\phi||| = sup\{\frac{\|\phi(x)\|'}{\|x\|} : x \in E, x \neq 0\} \quad (\phi \in \mathrm{L}(E, F))$$

(see [8], p.59).

On the other hand, thinking of E and F as additive normed groups, for every $\phi \in \mathrm{L}(E, F)$ we can define $\|\phi\|$ according with 3.4. But, happily we have that

$$\|\phi\| = |||\phi||| \quad \text{for all } \phi \in \mathrm{L}(E, F).$$

Indeed, observe that since $\|\phi(x)\|' \leq |||\phi||| \cdot \|x\|$ for all $x \in E, H_\phi = +\infty$ (see 2.3). Also, by 3.3.ii), $f_\phi : [0, +\infty) \longrightarrow [0, +\infty)$ verifies that $f_\phi(s) \leq |||\phi||| s$ for all $s \in [0, +\infty)$ and therefore $D^+ f_\phi(0) \leq |||\phi|||$.

Suppose that $D^+ f_\phi(0) < |||\phi|||$ and take $c \in R$ such that $D^+ f_\phi(0) < c < |||\phi|||$. Then there exists $\delta > 0$ such that $f_\phi(s) \leq c \cdot s$ for all $s \in (0, \delta)$ and therefore $\|\phi(x)\|' \leq c \cdot \|x\|$ for each $x \in E$ with $\|x\| < \delta$.

However, as $|||\phi||| = min\{d > 0 : \|\phi(x)\|' \leq d \cdot \|x\| \ \forall x \in E\}$, there exists $x \in E$ such that $\|\phi(x)\|' > c \cdot \|x\|$. Taking $\lambda \in K$, with $0 < |\lambda| < 1$, and $z := \lambda^n \cdot x \in E$ such that $\|z\| < \delta$. Then, $\|\phi(z)\|' > c \cdot \|z\|$, and this is a contradiction.

4. NON-ARCHIMEDEAN SEMINORMS ON

Let Z be the additive group of integers numbers. In this section we are going to describe the non-archimedean seminorms on Z. As an application, we obtain a new formula for $\|\phi\|$ (see Theorem 3.4) when ϕ is a continuous homomorphism from Z into a non-archimedean seminormed and abelian group G'.

First, we fix some notation. By $|\ |$ we denote the usual absolute value on Z. Also, if $n, m \in Z$, by $n|m$ we mean that m is a multiple of n; otherwise we write $n \nmid m$.

4.1 Proposition : *Let*

$\mathtt{N} := \{\|\ \| : Z \longrightarrow [0,+\infty) : \|\ \|$ *is a non-archimedean seminorm on* $Z\}$ *and* $\mathtt{S} = \mathtt{S}' \cup \mathtt{S}''$, *where*

$\mathtt{S}' := \{((n_1,a_1),(n_2,a_2),...) : n_m \in N^*, n_1 = 1, n_m|n_{m+1}, n_m \neq n_{m+1}\ \forall m \in N^*$ *and* (a_m) *is a strictly decreasing sequence in* $[0,+\infty)\}$

and

$\mathtt{S}'' := \{((n_1,a_1),(n_2,a_2),...) : n_m \in N^*, a_m \in [0,+\infty)\ \forall m \in N^*, n_1 = 1$ *and there exists* $m_0 \in N^*$ *such that* $n_{m-1}|n_m, n_{m-1} \neq n_m, a_{m-1} > a_m$ *if* $m \leq m_0$, *and* $n_m = n_{m_0}, a_m = a_{m_0}$ *if* $m > m_0\}$

(with the criterion $n_0 = a_0 = +\infty, +\infty|1$ *and* $+\infty > a_1$*).*

Then, the map $\gamma : \mathtt{N} \longrightarrow \mathtt{S}$ *given by*

$$\gamma(\|\ \|) := ((n_1,a_1),(n_2,a_2),...) \quad \textit{with}$$
$$n_1 := 1$$
$$n_{m+1} := min\{n \in N^* : \|n\| < \|n_m\|\} \quad \textit{if this set is not empty,}$$
$$\textit{and } n_{m+1} := n_m \quad \textit{otherwise}$$
$$a_m := \|n_m\| \quad (m \in N^*) \tag{9}$$

is a bijection.

Proof :

1) First we prove that γ is well-defined. For each $m \in N^*$, let n_m, a_m defined as in (9). Obviously $n_m \leq n_{m+1}$ and $a_m \geq a_{m+1}$ for all m.

Without loss of generality we can assume that $\{n \in N^* : \|n\| < \|n_m\|\}$ is a non-empty set for all m. We clearly have in this case that $n_m \neq n_{m+1}$ for all m and (a_m) is a strictly decreasing sequence in $(0,+\infty)$. Also, if there exists $m > 1$ for which $n_m \nmid n_{m+1}$, then we can choose $k \in N^*, k < n_m$ and $h \in N$ such that $n_{m+1} = hn_m + k$. Hence

$$\|hn_m\| \leq \|n_m\| < \|n_{m-1}\| \leq \|k\|$$

and so (see the Preliminaries),

$$\|n_{m+1}\| = \max\{\|hn_m\|, \|k\|\} = \|k\|,$$

a contradiction.

2) Next, we prove that for every $x \in N^*$ there exists an unique $s \in N^*$ and $p \in N^*$ such that $x = pn_s$ and $\|x\| = a_s$. To see that, take $s := \max\{h \in N^* : n_h|x$ and $n_{h-1} \neq n_h\}$ and choose $p \in N^*$ such that $x = pn_s$. Then, obviously $\|x\| \leq \|n_s\| = a_s$. On the other hand, if $n_{s+1} \nmid x$, we can prove, like in the proof of the last part of 1) that

$$\|x\| \geq \|n_s\| = a_s \tag{10}$$

and if $n_{s+1}|x$, then $n_s = n_{s+t}$ (and hence, $a_s = a_{s+t}$) for all $t \in N$, wich implies that (10) again holds. Thus, $\|x\| = a_s$.

3) Now, the fact that γ is injective follows directly from 2).

4) Finally, we prove that γ is surjective. Let $((n_1,a_1),(n_2,a_2),...) \in \mathtt{S}$ and let $\|\ \| : Z \longrightarrow [0,+\infty)$ given by

$\|0\| := 0$

$\|x\| := a_s$, if $x \in Z^*$, where $|x| = pn_s$ ($p \in N^*, s = \max\{h \in N^* : n_h \| |x|, n_{h-1} \neq n_h\}$, see the proof of 2)).

Clearly $\| \ \|$ satisfies properties i) and ii) of 1.1. Also, given $x, y \in Z^*$ with $|x| = pn_s$ and $|y| = qn_t$ with $s \leq t$, we have that $n_s | n_t$ and so $n_s \| x + y|$, which implies that $\|x + y\| \leq a_s = \max\{\|x\|, \|y\|\}$.

4.2 Remarks :

1.- Let $\| \ \| \in \mathtt{N}$ and $((n_1, a_1), (n_2, a_2), ...) = \gamma(\| \ \|)$. Then $\{\|x\| : x \in Z\} = \{a_1, a_2, ...\} \cup \{0\}$. Moreover, if $r > 0$, we have

i) $r \geq a_1 \implies B_r = Z = B_{a_1}$.

ii) $a_{m-1} > r \geq a_m$ for some $m \geq 2 \implies B_r = \{n_m y : y \in Z\} = B_{a_m}$.

iii) $r < a_m$ for all $m \in N^* \implies B_r = \{0\}$.

2.- The restriction of γ to $\mathtt{N}_1 := \{\| \ \| : Z \longrightarrow [0, +\infty) : \| \ \|$ is a non-trivial non-archimedean norm on $Z\}$ provides a bijection from $\mathtt{N}_1$ onto $\mathtt{S}_1 := \{((n_1, a_1), (n_2, a_2), ...) \in \mathtt{S}' : \lim a_m = 0\}$.

3.- Analogously, the restriction of γ to $\mathtt{N}_2 := \{\| \ \| : Z \longrightarrow [0, +\infty) : \| \ \|$ is a trivial non-archimedean seminorm on $Z\}$ provides a bijection from $\mathtt{N}_2$ onto $\mathtt{S}_2 := \{((n_1, a_1), (n_2, a_2), ...) \in \mathtt{S}' : \lim a_m > 0\} \cup \mathtt{S}''$.

Taking into account 4.1 and the results proved in the previous sections, we can now give the following formula for $\|\phi\|$ (see Theorem 3.4) when $\phi \in \mathtt{C}(Z, G')$.

4.3 Corollary : *Let $(Z, \| \ \|)$ be the additive group of integers numbers endowed with a non-archimedean seminorm $\| \ \|$ and let $(G', \| \ \|')$ be an abelian non-archimedean seminormed group. If $\phi \in \mathtt{C}(Z, G')$, then*

$$\|\phi\| := \begin{cases} \limsup_m \frac{f_\phi(a_m)}{a_m} & \text{if } \| \ \| \text{ is a non-trivial norm.} \\ 0 & \text{otherwise} \end{cases}$$

(where f_ϕ is as in 3.4, and $((n_1, a_1), (n_2, a_2), ...)$ is the sequence associated with $\| \ \|$ according to 4.1).

Proof : Firstly, recall that if $\phi \in \mathtt{C}(Z, G')$, then the formula

$$\|x\|_\phi = \|\phi(x)\|' \quad (x \in Z)$$

defines a non-archimedean seminorm on Z, such that ϕ is continuous if and only if $\| \ \|_\phi$ is weaker than $\| \ \|$. Also, observe that since $\| \ \|_\phi$ is bounded ($\|x\|_\phi \leq \|1\|_\phi \ \forall x \in G$) we obtain that $I_{H_\phi} = [0, +\infty)$.

Then, according with 2.2, we have that $f_\phi : [0, +\infty) \longrightarrow [0, +\infty)$ is given by

$$f_\phi(s) = inf\{r > 0 : B_s \subseteq B_r^\phi\} = inf\{r > 0 : \{x \in Z : \|x\| \leq s\} \subseteq \subseteq \{x \in Z : \|x\|_\phi \leq r\}\}. \tag{11}$$

Now, assume that $\| \ \|$ is a non-trivial norm on Z and call $b := \limsup_m \frac{f_\phi(a_m)}{a_m}$. Clearly $b \leq D^+ f_\phi(0) = \|\phi\|$ (see (3) and 4.2.2).

Suppose there is a $t \in R$ with $b < t < D^+ f_\phi(0)$. Then, $t < sup\{\frac{f_\phi(s)}{s} : 0 < s < r\}$ for all $r \in [0, +\infty)$, and therefore we can choose $m_0 \in N^*$ and a strictly decreasing sequence (s_n) in $(0, +\infty)$ with $\lim s_n = 0$ verifying

$$\frac{f_\phi(a_m)}{a_m} < t \quad \forall m \geq m_0 \qquad (12)$$
$$t < \frac{f_\phi(s_n)}{s_n} \quad \forall n \in N^*.$$

Take $m > m_0$ and k such that $a_m \leq s_k < a_{m-1}$ (and hence $B_{s_k} = B_{a_m}$ by 4.2.1.ii)). Then,

$$t < \frac{f_\phi(s_k)}{s_k} \leq \frac{f_\phi(s_k)}{a_m} = \frac{f_\phi(a_m)}{a_m} < t,$$

a contradiction.

Finally, assume that $\| \ \|$ is not a non-trivial norm on Z. We have the following possiblities :

a) (a_m) is a strictly decreasing sequence in $(0, +\infty)$ with $\lim a_m > 0$.

b) There exists $m_0 \in N^*$ such that $a_{m-1} > a_m$ if $m \leq m_0$ and $a_m = a_{m_0} > 0$ if $m > m_0$.

c) There exists $m_0 \in N^*$ such that $a_{m-1} > a_m$ if $m \leq m_0$ and $a_m = a_{m_0} = 0$ if $m > m_0$.

In case a), b) we can choose $\alpha > 0$ with $\alpha < a_m$ for all $m \in N^*$. Then, for each $s \in (0, \alpha), B_s = \{0\}$ (4.2.1.iii)) and so $f_\phi(s) = 0$ (see (11)). Hence, $\|\phi\| = D^+ f_\phi(0) = 0$.

In case c), we can suppose that $m_0 > 1$ (if $a_1 = 0$, then $\| \ \| \equiv \| \ \|_\phi \equiv 0$, which gives that $f_\phi \equiv 0$). Take $s \in (0, +\infty)$ with $a_{m_0} = 0 < s < a_{m_0 - 1}$. We shall prove that $f_\phi(s) = 0$ (and so $\|\phi\| = D^+ f_\phi(0) = 0$). Suppose $f_\phi(s) > 0$. Then, there exists $\beta > 0$ with $0 < \beta < f_\phi(s)$. It follows from 4.2.1.ii) and (11) that $B_s = B_0 = \{x \in Z : \|x\| = 0\}$ is not included in B^ϕ_β. On the other hand, since $\| \ \|_\phi$ is weaker than $\| \ \|$, there exists $t > 0$ such that $B_0 \subseteq B_t \subseteq B^\phi_\beta$, a contradiction.

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