ANNALES MATHÉMATIQUES BLAISE PASCAL

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Annales mathématiques Blaise Pascal, tome 2, nº 1 (1995), p. 225-235
<http://www.numdam.org/item?id=AMBP_1995__2_1_225_0>

Ann. Math. Blaise Pascal, Vol. 2, N° 1, 1995, pp.225-235

RESTRICTED RANGE SIMULTANEOUS APPROXIMATION AND INTERPOLATION WITH PRESERVATION OF THE NORM

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Abstract. Let $(F, |\cdot|)$ be a complete non-archimedean non-trivially valued division ring, with valuation ring V. Let X be a compact 0-dimensional Hausdorff space, and let $D(X)$ be the ring of all continuous functions f from X into V equipped with the supremum norm. Let $A \subset D(X)$. Assume that for every ordered pair (s, t) of distinct elements of X, there is some multiplier of A, say φ, such that $\varphi(s) = 1$ and $\varphi(s) = 0$. Assume that A contains the constants. We show that A is uniformly dense in $D(X)$, and when A is an interpolating family then simultaneous approximation and interpolation, with preservation of the norm, by elements of A is always possible. We apply this to the case of von Neumann subsets and to the case of restricted range polynomial algebras.

1991 Mathematics subject classification: 46S10.

1. Introduction

Throughout this paper X is a compact Hausdorff space which is 0-dimensional *i.e.*, for any point x and any open set A containing x, there exists a closed and open set N with $x \in N \subset A$, and $(F, |\cdot|)$ is a complete, non-Archimedean non-trivially valued division ring. We denote by V the valuation ring of F, *i.e.*, $V = \{t \in F; |t| \leq 1\}$, and by $D(X)$ the set of all continuous functions from the space X into V, equipped with the topology of uniform convergence on X, determined by the metric d defined by

$$d(f, g) = \|f - g\| = \sup\{|f(x) - g(x)|; x \in X\}$$

for every pair, f and g, of elements of $D(X)$.
Our aim is to use the idea of T.J. Ransford (see [7]), to prove results in $D(X)$ that are analogous to those in $C(X; [0, 1])$ and $C(X; F)$, which were proved in [5] and [6], respectively. To avoid trivialities we assume that X has at least two points.

Definition 1 A non-empty subset $A \subset D(X)$ is said to be a **von Neumann subset** if $\varphi\psi + (1 - \varphi)\eta$ belongs to A, whenever φ, ψ and η belong to A .

Clearly, if $A \subset D(X)$ is a von Neumann subset containing the constant functions 0 and 1, then the following properties are true:

(i) if $\varphi \in A$, then $1-\varphi$ belongs to A;

(ii) if φ and ψ belong to A, then $\varphi\psi$ belongs to A.

When $A \subset D(X)$ has properties (i) and (ii), we say that A has **property** V. This definition is motivated by the similar one introduced by R. I. Jewett, who in [1] proved the variation of the Weierstrass - Stone Theorem stated by von Neumann in [8].

Definition 2 Let $A \subset D(X)$ be a non-empty subset. We say that $\varphi \in D(X)$ is a **multiplier** of A if $\varphi f + (1-\varphi)g$ belongs to A .

Clearly, if M is the set of all multipliers of A, then M satisfies property (i) above. The identity

$$\varphi\psi f + (1-\varphi\psi)g = \varphi[\psi f + (1-\psi)g] + (1-\varphi)g$$

shows that M satisfies condition (ii) as well. Hence M has property V.

Definition 3 A subset $A \subset D(X)$ is said to be **strongly separating** over X, if given any ordered pair $(x, y) \in X \times X$, with $x \neq y$, there exists a function $\varphi \in A$ such that $\varphi(x) = 1$ and $\varphi(y) = 0$.

Lemma 1 *Let $M \subset D(X)$ be a subset which has property V and is strongly separating over X. Let N be a clopen proper subset of X. For each $\delta > 0$, there is $\varphi \in M$ such that*

$$|\varphi(t) - 1| < \delta, \text{ for all } t \in N, \tag{1}$$

$$|\varphi(t)| < \delta, \text{ for all } t \notin N. \tag{2}$$

Proof. This result is essentially Lemma 1 of Prolla [6]. For the sake of completeness we include here its proof. Fix $y \in X$, $y \notin N$. Since M is strongly separating, for each $t \in N$, there is $\varphi_t \in M$ such that $\varphi_t(y) = 1$, $\varphi_t(t) = 0$. By continuity there is a neighborhood $V(t)$ of t such that $|\varphi_t(s)| < \delta$ for all $s \in V(t)$. By compactness of N there are $t_1, \dots, t_n \in N$ such that $N \subset V(t_1) \cup \dots \cup V(t_n)$. Consider the function $\psi_y = 1 - \varphi_{t_1}\varphi_{t_2} \cdot \dots \cdot \varphi_{t_n}$. Clearly $\psi_y \in M$ and $\psi_y(y) = 0$, while $|\psi_y(t) - 1| < \delta$ for all $t \in N$. Indeed, if $t \in N$, then $t \in V(t_i)$ for some index $i \in \{1, 2, \dots, n\}$. Hence

$$|\psi_y(t) - 1| = |\varphi_{t_i}(t)| \cdot \Pi_{j \neq i} |\varphi_{t_j}(t)| < \delta.$$

By continuity, there is a neighborhood $W(y)$ of y such that $|\psi_y(s)| < \delta$ for all $s \in W(y)$. By compactness of $K = X \setminus N$, there are $y_1, \dots y_m \in K$ such that $K \subset W(y_1) \cup \dots \cup W(y_m)$. Let $\varphi = \psi_{y_1} \cdot \psi_{y_2} \cdot \dots \cdot \psi_{y_m}$. Clearly $\varphi \in M$. We claim that for each $1 \leq k \leq m$ we have

$$|1 - \psi_{y_1}(t) \cdot \dots \cdot \psi_{y_k}(t)| < \delta, \text{for all } t \in N. \tag{3}$$

We prove (3) by induction. For $k = 1$, (3) is clear, since $|\psi_y(t) - 1| < \delta$ for all $t \in N$ and all $y \in K$. Assume (3) has been proved some k. To simplify notation we write $\psi_i = \psi_{y_i}$ for all $1 \leq i \leq m$. Then, for each $t \in N$

$$|1 - \psi_1(t) \cdot ... \cdot \psi_{k+1}(t)| =$$

$$|1 - \psi_{k+1}(t) + \psi_{k+1}(t) - \psi_1(t) \cdot ... \cdot \psi_k(t) \cdot \psi_{k+1}(t)|$$

$$\leq max \ \{|1 - \psi_{k+1}(t)|, |\psi_{k+1}(t)| \cdot |1 - \psi_1(t) \cdot ... \cdot \psi_k(t)|\} < \delta$$

because $|1 - \psi_{k+1}(t)| < \delta, |\psi_{k+1}(t)| \leq 1$, and $|1 - \psi_1(t) \cdot ... \cdot \psi_k(t)| < \delta$ by the induction hypothesis. Hence (3) is valid for $k + 1$.

Clearly (1) follows from (3) by taking $k = m$. It remains to prove (2). Now if $t \in K$ then $t \in W(y_i)$ for some $1 \leq i \leq m$. Hence $|\psi_i(t)| < \delta$, while $|\psi_i(t)| \leq 1$ for all $j \neq i$. Therefore $|\varphi(t)| < \delta$ and (2) is proved.

□

Remark. If $A \subset D(X)$ is a non-empty subset and $f \in D(X)$, the distance of f from A, denoted by dist(f, A), is defined as

$$\text{dist } (f, A) = \inf \{\|f - g\|; g \in A\}$$

Clearly, f belongs to the uniform closure of A in $D(X)$ if, and only if, dist$(f; A) = 0$.

If $S \subset X$ is a non-empty closed subset of X, we denote by $f_S \in D(S)$. Similarly, $A_S = \{\varphi_S; \varphi \in A\}$, for each $A \subset D(X)$. When S is a singleton set, say $S = \{x\}$, we identify f_S with its value $f(x)$, and A_S with $\{\varphi(x); \varphi \in A\} = A(x)$.

Lemma 2 *Let $A \subset D(X)$ be a non-empty subset. For each $f \in D(X)$, there exists a minimal closed and non-empty subset $S \subset X$ such that*

$$\text{dist}(f_S; A_S) = \text{dist}(f; A)$$

Proof. Since, for each $x \in X$, we have

$$\text{dist } (f(x); A(x)) \leq \text{dist } (f; A),$$

we see that when dist $(f; A) = 0$, any singleton set $S = \{x\}$ satisfies

$$\text{dist } (f_s; A_s) = \text{dist}(f; A)$$

Assume now dist $(f; A) > 0$. Let us put $d =$ dist $(f; A)$. Define

$$\mathcal{F}(X) = \{T \subset X; \ T \text{ is closed and non-empty}\}$$

and

$$\mathcal{F} = \{T \in \mathcal{F}(X);\ \text{dist}\ (f_T; A_T) = d\}.$$

Clearly $\mathcal{F} \neq \emptyset$. because $X \in \mathcal{F}$. Let us order $\mathcal{F}$ by set inclusion. Let $\mathcal{C}$ be a totally ordered non-empty subset of $\mathcal{F}$.

Let $S = \cap\{T: T \in \mathcal{C}\}$. Clearly, S is closed. If J is a finite subset of C, there is some $T_0 \in J$ such that $T_0 \subset T$ for all $T \in J$. Hence

$$T_0 = \cap\{T; T \in J\}.$$

Now $T_0 \neq \emptyset$ and by compactness $S \neq \emptyset$. Hence $S \in \mathcal{F}(X)$. We claim that $S \in \mathcal{F}$. Clearly, dist $(f_S; J_S) \leq d$. Suppose that dist $(f_S; A_S) < d$ and choose a real number r such that dist $(f_S; A_S) < r < d$. By definition of dist $(f_S; A_S)$ there exists $g \in A$ such that $|f(x) - g(x)| < r$ for all $x \in S$. Let

$$U = \{t \in X; |f(t) - g(t)| < r\}.$$

Then U is open and contains S. By compactness, there is finite subset $J \subset \mathcal{C}$ such that $\cap\{T: T \in J\} \subset U$. Let $T_0 \in J$ be such that $T_0 \subset T$ for all $T \in J$. Then $\cap\{T: T \in J\} = T_0$ and so $T_0 \subset U$. Hence $|f(t) - g(t)| < r$ for all $t \in T_0$, and so dist $(f_{T_0}); A_{T_0} \leq r < d$, which contradicts the fact that $T_0 \in \mathcal{F}$. This contradiction establishes our claim that dist $(f_S; A_S) = d$. Therefore S is a lower bound for $\mathcal{C}$ in $\mathcal{F}$. By Zorn's Lemma there exists a minimal element in $\mathcal{F}$, and this element satisfies all our requirements.

□

2. The Main Results

Theorem 1 *Let $A \subset D(X)$ be a non-empty subset, whose set of multipliers is strongly separating over X. For each $f \in D(X)$, there is some $x \in X$ such that*

$$(*) \qquad \text{dist}\ (f(x); A(x)) = \text{dist}\ (f; A)$$

Proof. By Lemma 2 above, there is a minimal closed and non-empty subset $S \subset X$ such that

$$\text{dist}\ (f_S; A_S) = \text{dist}\ (f; A)$$

We claim that $S = \{x\}$, for some $x \in X$. Since for any $x \in X$, dist $(f(x); A(x)) \leq$ dist $(f; A)$ we see that when dist $(f; A) = 0$, then $(*)$ is true for all $x \in X$. Hence we may assume $d =$ dist $(f; A)$ is strictly positive.

Assume that S contains at least two distinct points, say y and z. Let N be a clopen subset of X such that $y \in N$, while $z \notin N$. Define

$$Y = S \cap N,$$

$$Z = S \cap K,$$

where $K = X \setminus N$. Notice that both Y and Z are closed. $Y \cap Z = \emptyset$, and $Y \cup Z = S$. Since $y \in Y$ and $z \in Z$, both Y and Z are non-empty. Furthermore, $z \notin Y$ and $y \notin Z$. Hence both Y and Z are proper subsets of S. By the minimality of S we have

$$d_Y := \text{dist } (f_Y: A_Y) < d;$$

$$d_Z := \text{dist } (f_Z: A_Z) < d.$$

Choose a real number r such that

$$\max\{d_Y, d_Z\} < r < d.$$

Since $d_Y < r$, there is some $g \in A$ such that $|f(t) - g(t)| < r$, for all $t \in Y$. Similarly, since $d_Z < r$, there is some $h \in A$ such that $|f(t) - h(t)| < r$, for all $t \in Z$. Choose $0 < \delta < r$. By Lemma 1, there is a multiplier of A, say φ, such that

(1) $|1 - \varphi(t)| < \delta$, for all $t \in N$,

(2) $|\varphi(t)| < \delta$, for all $t \notin N$.

The function $k = \varphi g + (1 - \varphi)h$ belongs to A. We claim that $|f(t) - k(t)| < r$ for all $t \in S$. Let $t \in S$. There are two cases to consider, namely $t \in Y$ and $t \in Z$.

Case I. $t \in Y$

Let us write $g = \varphi g + (1 - \varphi)g$. Then

$$|k(t) - g(t)| = |1 - \varphi(t)| \cdot |h(t) - g(t)| \leq |1 - \varphi(t)| < \delta$$

because $Y \subset N$ implies, by (1), that $|1 - \varphi(t)| < \delta$, and $|h(t) - g(t)| \leq \max\{|h(t)|, |g(t)|\} \leq 1$. Hence

$$|f(t) - k(t)| = |f(t) - g(t) + g(t) - k(t)| \leq \max\ \{|f(t) - g(t)|, |g(t) - k(t)|\} < r$$

Case II. $t \in Z$

Let us write $h = \varphi h + (1 - \varphi)h$. Then

$$|k(t) - h(t)| = |\varphi(t)| \cdot |g(t) - h(t)| \leq |\varphi(t)| < \delta$$

because $Z \subset K = X \setminus N$ implies that $t \notin N$ and by (2), $|\varphi(t)| < \delta$.
Hence

$$|f(t) - k(t)| = |f(t) - h(t) + h(t) - k(t)| \leq \max\ \{|f(t) - h(t)|, |h(t) - k(t)|\} < r.$$

Therefore $|f(t) - k(t)| < r$, for all $t \in S$ and dist $(f_S, A_S) \leq r < d$, a contradiction.

□

Remark. If $A \subset D(X)$ is as in Theorem 1 and $A(x) \supset \{0, 1\}$, for every $x \in X$, then it follows that the closure of A contains the characteristic function of each clopen subset of X. Indeed, let $S \subset X$ be a clopen subset of X and let f be its characteristic function. Let $x \in X$ be given by Theorem 1. Now $f(x)$ is either 0 or 1 and therefore $A(x)$ contains $f(x)$ and so dist $(f, A) = 0$.

Corollary 1 *Let $A \subset D(X)$ be a von Neumann subset which is strongly separating over X. For each $f \in D(X)$, there is some $x \in X$ such that*

(*) $$\text{dist }(f(x); A(x)) = \text{dist }(f, A)$$

Proof. Let M be the set of all multipliers of A. Since A is a von Neumann subset, we see that $A \subset M$. Hence M is strongly separating too, and the result follows from Theorem 1.

□

Theorem 2 *Let $A \subset D(X)$ be a non-empty subset, whose set of multipliers is strongly separating over X. Let $f \in D(X)$ and $\varepsilon > 0$ be given. The following are equivalent:*

(1) there is some $g \in A$ such that $\|f - g\| < \varepsilon$.

(2) for each $t \in X$, there is some $g_t \in A$ such that $|f(t) - g_t(t)| < \varepsilon$.

Proof. Clearly (1) $\Rightarrow$ (2). Conversely, assume that (2) holds. Let $x \in X$ be given by Theorem 1, i.e.,

(*) $$\text{dist }(f; A) = \text{dist }(f(x); A(x)).$$

By (2) applied to $t = x$, there is some $g_x \in A$ such that $|f(x) - g_x(x)| < \varepsilon$. Hence dist $(f(x); A(x)) < \varepsilon$. By (*) above, dist $(f; A) < \varepsilon$, and therefore some $g \in A$ such that $\|f - g\| < \varepsilon$ can be found. Hence (1) is valid.

□

Corollary 2 *Let $A \subset D(X)$ be a von Neumann subset which is strongly separating over X. Let $f \in D(X)$ and $\varepsilon > 0$ be given. The following are equivalent:*

(1) there is some $g \in A$ such that $\|f - g\| < \varepsilon$,

(2) for each $t \in X$, there is $g_t \in A$ such that $|f(t) - g_t| < \varepsilon$

Proof. Corollary 2 follows from Corollary 1 in the same way that Theorem 2 follows from Theorem 1. Or else, note that $A \subset M$ if M denotes the set of all multipliers of A and then apply Theorem 2 to A, since M is strongly separating over X because it contains A.

□

Theorem 3 *Let $A \subset D(X)$ be a non-empty subset such that the set M of its multipliers is strongly separating, and for each $\lambda \in V$ and each $x \in X$, there is $\varphi \in A$ such that $\varphi(x) = \lambda$. Then A is uniformly dense in $D(X)$.*

Proof. Let $f \in D(X)$. By Theorem 1, there is some $x \in X$ such that

$$\text{dist } (f; A) = \text{dist } (f(x); A(x)).$$

Now, by hypothesis, $A(x) = V$. Hence $f(x) \in A(x)$ and so dist $(f(x); A(x)) = 0$ Hence dist $(f; A) = 0$ for all $f \in D(X)$, and A is uniformly dense in $D(X)$.

□

Remark. If $A \subset D(X)$ is as in Theorem 1 and contains all the constant functions with values in V, then Theorem 3 applies trivially and A is uniformly dense in $D(X)$.

Corollary 3 *Let $A \subset D(X)$ be a von Neumann subset which is strongly separating over X, and for each $\lambda \in V$ and $x \in X$ there is $\varphi \in A$ such that $\varphi(x) = \lambda$. Then A is uniformly dense in $D(X)$.*

Corollary 4 *Let W be a subring of $D(X)$ which is strongly separating over X and $W(x) = V$, for each $x \in V$. Then W is uniformly dense in $D(X)$.*

Proof. Clearly, every subring of $D(X)$ is a von Neumann subset.

□

Remark. The valuation ring V is a topological ring with unit, and has a fundamental system of neighborhoods of 0 which are ideals in V. Hence Theorem 32 of Kaplansky [2] applies, giving an alternate proof for Corollary 4.

3. Examples

Let us give some examples of von Neumann subsets of $D(X)$ which are strongly separating over X. Let us first remark that a separating subring of $D(X)$ is not necessarily strongly separating over X. The set $W = \{f \in D(X); |f(x)| < 1, \text{for all } x \in X\}$ is an example of a separating subring of $D(X)$ infact, it is a closed two-sided ideal of $D(X)$, which is not strongly separating. Indeed no function in W can take the value 1 at any point in X. Further examples can be found. Indeed, for a fixed point $a \in X$ let us define $W_a = \{f \in D(X); f(a) = 0\}$. Clearly, W_a is a subring of $D(X)$. Now W_a is separating over X. Indeed, let $x \neq y$ be given in X. If $x = a$ or $y = a$, the function $\varphi \in D(X)$ which is zero at a and one at the other point is such that $\varphi(x) \neq \varphi(y)$ and $\varphi \in W_a$. In case $x \neq a$ and $y \neq a$,let $\varphi \in D(X)$ be such that $\varphi(a) = 0$ and $\varphi(y) = 1$, and let $\psi \in D(X)$ be such that $\psi(x) = 0$ and $\psi(y) = 1$.

Then $\eta = \varphi\psi \in W_a$ and $\eta(x) = 0$ while $\eta(y) = 1$. On the other hand, W_a is not strongly separating over X. For every ordered pair (a, x), with $a \neq x$, there is no function $\varphi \in W_a$ such that $\varphi(a) = 1$ and $\varphi(x) = 0$. Indeed, $\varphi \in W_a$ implies $\varphi(a) = 0$, and so W_a is not strongly separating over X.

Example 1 The collection A of the characteristic functions of all the clopen subsets of X is a von Neumann subset of $D(X)$, containing 0 and 1, and moreover, since X is a 0-dimensional compact Hausdorff space, A is strongly separating over X.

Example 2 Let $X = V = \{t \in Q_p; |t|_p \leq 1\}$, where $(Q_p, |\cdot|_p)$ is the p-adic field. Then the unitary subalgebra W of all polynomials $q : Q_p \to Q_p$ is separating over X. By Proposition 1, Prolla [6], $A = \{q \in W; q(X) \subset V\}$ is strongly separating over X. Clearly, A is a von Neumann subset containing the constants in $D(X)$.

Example 3 Let $n \geq 1$ be an integer and let $V = \{t \in F; |t| \leq 1\}$ and assume that V is compact. Then the unitary subalgebra W of all polynomials $q : F^n \to F$ in n-variables is separating over $X = V^n$, because W contains all the n projections. By Proposition 1, Prolla [6], $A = \{q \in W; q(V^n) \subset V\}$ is a strongly separating von Neumann subset of $D(V^n)$, containing all constant functions with values in V.

Example 4 Let $\{S_i\}_{i\in I}$ be a finite partition of X into clopen subsets, *i.e.*, the set I of indices is finite, each S_i is a clopen set, $S_i \cap S_j = \emptyset$ for all $i \neq j$ and $X = \cup_{i\in I} S_i$. For each $i \in I$, let φ_i be the characteristic function of S_i and let $\lambda_i \in V$. Consider the function $\varphi \in D(X)$ defined by

$$\varphi(x) = \sum_{i\in I} \lambda_i \varphi_i(x)$$

for all $x \in X$. Let $A \subset D(X)$ be the collection of all functions φ defined as above. Then A satisfies all the hypothesis of Theorem 3 and therefore is uniformly dense in $D(X)$.

Definition 4 A non-empty subset $A \subset D(X)$ is said to be a **restricted range polynomial** algebra if for every choice $\varphi_1, ..., \varphi_n \in A$ and $q : F^n \to F$ a polynomial in n-variables such that $|q(\varphi_1(x), \varphi_2(x), ..., \varphi_n(x))| \leq 1$ for all $x \in X$, the mapping $x \to q(\varphi_1(x), ..., \varphi_n(x))$ belongs to A.

Notice that the polynomials $(u_1, u_2) \to u_1 + u_2$, $(u_1, u_2) \to u_1 u_2$ and $(u_1, u_2) \to u_1 - u_2$ are such that $V \times V$ is mapped into V, and therefore any restricted range polynomial algebra is a subring of $D(X)$, and a fortiori a von Neumann subset. Notice that any restricted range polynomial algebra contains all the constant functions with values in V.

Proposition 1 *Let $A \subset D(X)$ be a restricted range polynomial algebra which is separating over X. Then A is strongly separating over X.*

Proof. Let (s, t) be an ordered pair of distinct elements of X. By hypothesis, there exists $\varphi \in A$ such that $\varphi(s) \neq \varphi(t)$.

Let $q : F \to F$ be the linear function

$$u \to (\varphi(t) - \varphi(s))^{-1}(u - \varphi(s))$$

Then $q(\varphi(s)) = 0$ and $q((\varphi(t)) = 1$. Since q is continuous, $q(\varphi(X))$ is a compact subset of F. By Kaplansky's Lemma (see Kaplansky [3] or Lemma 1.23, Prolla [4]) there is a polynomial $p : F \to F$ such $p(1) = 1$ and $p(0) = 0$ and $|p(t)| \leq 1$ for all $t \in q(\varphi(X))$. Let $r = p \circ q$ then $r : F \to F$ is a polynomial such that $r(\varphi(X)) \subset V$. Hence $r \circ \varphi = \psi$ belongs to A. Now $\psi(s) = p(q(\varphi(s))) = p(0) = 0$ and $\varphi(t) = p(q(\varphi(t))) = p(1) = 1$. Hence A is strongly separating.

□

Corollary 5 *Let $A \subset D(X)$ be a restricted range polynomial algebra which is separating over X. Then A is uniformly dense in $D(X)$.*

Proof. By Proposition 1, A is strongly separating. On the other hand A contains all the constant functions with values in V. Hence $A(x) = V$, for every $x \in X$. Since A is a von Neumann set, the result follows from Corollary 3.Or else, notice that A is a subring and then apply Corollary 4.

□

4. Simultaneous Aproximation and Interpolation

Definition 5 A non-empty subset $A \subset D(X)$ is called an **interpolating family** for $D(X)$ if, for every $f \in D(X)$ and every finite subset $S \subset X$, there exists $g \in A$ such that $g(x) = f(x)$ for all $x \in S$.

Theorem 4 *Let $W \subset D(X)$ be is an interpolating family for $D(X)$, whose set of multipliers is strongly separating over X. Then, for every $f \in D(X)$ every $\varepsilon > 0$ and every finite set $S \subset X$, there exists $g \in A$ such that $\|f - g\| < \varepsilon$, $\|g\| = \|f\|$ and $g(t) = f(t)$ for all $t \in S$.*

Proof. Let $A = \{g \in W; g(t) = f(t)$ for all $t \in S\}$. Since W is an interpolating family for $D(X)$, the set A is non-empty. It is easy to see that every multiplier of W is also a multiplier of A. Hence the set of multipliers of A is strongly separating over X. Consider the point $x \in X$ given by Theorem 1, applied to A and f, *i.e.*,

$$(*) \qquad \text{dist } (f; A) = \text{dist } (f(x); A(x))$$

Consider the finite set $S \cup \{x\}$. Since W is an interpolating family for $D(X)$, there is some $g_x \in W$ such that $g_x(t) = f(t)$ for all $t \in S \cup \{x\}$. In particular, $g_x(t) = f(t)$ for all $t \in S$ and therefore $g_x \in A$. On the other hand $g_x(x) = f(x)$ implies that $f\{x\} \in A(x)$. By (*), dist $(f; A) = 0$. Choose $0 < \delta$ such that $\delta < \varepsilon$ and $\delta < \|f\|$.

There is some $g \in A$ such that $\|f - g\| < \delta$. From the definition of A, it follows that $g \in W$ and $g(t) = f(t)$ for all $t \in S$. Moreover, $\|f - g\| < \varepsilon$ and $\|g\| = \|g - f + f\| = \|f\|$, because $\|g - f\| < \delta < \|f\|$.

□

Corollary 6 *Let $W \subset D(X)$ be an interpolating family for $D(X)$ which is a von Neumann subset and which is strongly separating over X. Then, for every $f \in D(X)$, every $\varepsilon > 0$ and every finite set $S \subset X$, there exists $g \in W$ such that $\|f - g\| < \varepsilon$, $\|g\| = \|f\|$, and $g(t) = f(t)$ for all $t \in S$.*

Proof. The set W is contained in the set M of its multipliers and Corollary 6 follows from Theorem 4.

□

Remark. If $W \subset D(X)$ is an interpolating family for $D(X)$ which is strongly separating over X and which is a subring of $D(X)$, then Corollary 6 applies to it.

Corollary 7 *Let $W \subset D(X)$ be an interpolating family for $D(X)$ which is a restricted range polynomial algebra and which is separating over X. Then, for every $f \in D(X)$, every $\varepsilon > 0$, and every finite set $S \subset X$, there exists $g \in W$ such that $\|f - g\| < \varepsilon$, $\|g\| = \|f\|$ and $g(t) = f(t)$ for all $t \in S$.*

Proof. We know that every restricted range polynomial algebra is a von Neumann subset. By Proposition 1, W is strongly separating. The result now follows from the previous Corollary.

□

Acknowledgements

This paper was written while the first author was visiting the University of Santiago of Chile, July 1993, and the second author was partially supported by grant FONDECYT 91-0471, and DICYT - USACH.

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