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THE PROBLEMS OF THE NON-ARCHIMEDEAN ANALYSIS GENERATED BY QUANTUM PHYSICS

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Since 1987 large interest was shown in p-adic mathematical physics ,see [1]-[21] and references in these papers. Models of quantum mechanics ,field and superstring theory were investigated. Many mathematical problems were generated inside of these physical formalisms . This text is devoted to these problems.

1.Analytic Functions over Non-Archimedean Fields

Now we introduce spaces of analytic functions which will play a basic role in a theory of distributions. We study only the case of functions defined on balls with the centre at zero. It is much more simple to consider only the balls of radii $R \in \Gamma$, because only such balls are the "natural balls."

Denote by K a complete non-Archimedean field ,$CharK = 0$, $|\cdot|_K$ is the valuation on K ; $\Gamma = ||K|| = \{a = |x|_K : x \in K\}$. As usual $K^n = K \times \cdots \times K$. Denote by U_R a ball $U_R(0) = \{x : |x|_K \leq R\}$ in K^n. The function $f : U_R \to K$, $R \in \Gamma$, is said to be analytic if the series

$$f(x) = \sum_\alpha f_\alpha x^\alpha, \qquad f_\alpha \in K, \tag{1.1}$$

converges uniformly on U_R. Here as usual $\alpha = (\alpha_1, ..., \alpha_n), \alpha_i = 0, 1, 2, ...,$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. By virtue of the non-Archimedean Cauchy criterion of convergence (1.1), it is necessary and sufficient that

$$\lim_{|\alpha| \to \infty} |f_\alpha|_K R^{|\alpha|} = 0 \tag{1.2}$$

To prove that this condition is necessary we use $R \in \Gamma$. So there exists such $a_R \in K$ that $|a_R|_K = R$. The point $x_R = (a_R, ..., a_R) \in U_R$. And if (1.1) converges in this point, then (1.2) holds.

The topology in the space $\mathcal{A}(U_R) \equiv \mathcal{A}(U_R, K)$ ($\equiv \mathcal{A}_R$) of the functions analytic on the ball U_R is defined by the non-Archimedean norm $||f||_R = \max_\alpha |f_\alpha|_K R^\alpha$. This is a non-Archimedean Banach space.

The function $f : K^n \to K$ is *entire* if series (1.1) converges on the ball U_R for any $R \in \Gamma$. The topology in the space of entire functions $\mathcal{A}(K^n) \equiv \mathcal{A}(K^n, K)(\equiv \mathcal{A})$ is defined by a system on non-Archimedean norms $\{||\cdot||_R\}_{R \in R_+}$.

A sequence of entire functions $\{f_n\}$ converges in $\mathcal{A}$ if it converges uniformly on each ball U_R.

There is no problem to see that this topology could be defined with the aid of arbitrary sequence of norms $\{||\cdot||_{R_k}\}_{k=1}^\infty$, where $\lim_{k\to\infty} R_k = \infty$. It is a non-Archimedean Fréchet space. Such type of topology is known as the topology of projective limit of Banach spaces

$$\mathcal{A}(K^n) = \lim_{R \to \infty} \operatorname{proj} \mathcal{A}(U_R)$$

Remark 1.1. A sequence of functions $\{f_n\}$ converges in $\mathcal{A}$ to f iff $||f_n - f||_R \to 0$ for every norm $||\cdot||_R$.

The function f is said to be analytic at zero if there exists $R \in \Gamma$ such that $f \in \mathcal{A}(\mathcal{U}_R)$. The space of functions analytic at zero, $\mathcal{A}_0(K^n) \equiv \mathcal{A}_0(K^n, K)(\equiv \mathcal{A}_0)$ is provided with a topology of an inductive limit:

$$\mathcal{A}_0(K^n) = \lim_{R \to 0} \operatorname{ind} \mathcal{A}(\mathcal{U}_R)$$

2.Analytic Distributions, Gauss Distributions

We choose the spaces of analytic functions $\mathcal{A}(K^n)$ and $\mathcal{A}_0(K^n)$ as the spaces of test functions and the spaces of K-linear continuous functionals $\mathcal{A}'(K^n)$ and $\mathcal{A}'_0(K^n)$ as the spaces of distributions (generalized functions).

As usual, it is convenient to use the symbol of an integral to denote the effect produced by a distribution on the test function. The following symbols are convenient for us

$$J(f) = (J, f) = \int_{K^n} J(dx) f(x), \ J \in \mathcal{A}'_0, f \in \mathcal{A}_0,$$

$$\mu(f) = (f, \mu) = \int_{K^n} f(x) \mu(dx), \ \mu \in \mathcal{A}', f \in \mathcal{A}.$$

Thus,we write distributions of the class $\mathcal{A}'_0$ on the left side and distributions of the class $\mathcal{A}'$ on the right side to a test function.

We introduce an inner product on K^n, it is $(x, y) = \sum_{j=1}^n x_j y_j$.

Definition 2.1. *The Laplace transform (two-sided) of the distribution $g \in \mathcal{A}'_0$ is a function $L(g)(y) = (g, \exp\{(y, \cdot)\})$.*

Theorem 2.2. *The Laplace transformation $L : \mathcal{A}'_0 \to \mathcal{A}$ is an isomorphism.*

Thus we have a non-Archimedean Laplace calculus

$$\mathcal{A}'_0 \xrightarrow{L} \mathcal{A}, \qquad \mathcal{A}_0 \xrightarrow{L'} \mathcal{A}'. \tag{2.1}$$

The Laplace transformation has all standard properties of the usual Laplace transformation.

By the definition of a conjugate operator, we have *Parseval's equality*

$$\int_{K^n} L(g)(y) \mu(dy) = \int_{K^n} g(dx) L'(\mu)(x). \tag{2.2}$$

Definition 2.2. *The Gauss distribution on K^n (with the mean value $a \in K^n$ and the covariance matrix B) is $\gamma_{a,B} \in \mathcal{A}'$ with the Laplace transform $L'(\gamma_{a,B})(x) = \exp\{\frac{1}{2}(Bx, x) + (a, x)\}$.*

If we consider $\mathbf{R}$ instead of K, then we get ordinary Gaussian distribution for the matrix $B > 0$.

To denote the integral with respect to the Gaussian distribution, we use the symbol

$$\int_{K^n} \varphi(x) \exp\Big\{ -\frac{1}{2}(B^{-1}(x-a), (x-a))\Big\} dx.$$

The quadratic exponent multiplied to dx is only the symbol to denote the Gaussian distribution $\gamma_{a,B}$. But this symbol is sufficiently convenient in computations. Formally we can work with quadratic exponent as with usual density with respect to dx ,but we need to apply the

Laplace transform to justify such computations because at the moment the symbol dx is not defined. This symbolic expression for the Gaussian distribution $\gamma_{a,B}$ contains a convention on the unit normalization of the integral

$$\int_{K^n} \exp\Big\{-\frac{1}{2}(B^{-1}(x-a),(x-a))\Big\}\,\mathrm{d}x.$$

This normalization differs from the standard real normalization constant $\sqrt{2\pi\det B}$. But we cannot investigate the problem of the normalization on the basis of our definition of the Gaussian integral as the quadratic density is only convenient symbol in our computations. The separate question is a definition of a non-Archimedean analogue of π. There were attempts to define a non-Archimedean π, but these definitions were not connected in any way to our Gaussian integral.

Example 2.1 (*the canonical Gaussian distribution* $\nu(\mathrm{d}x) = \mathrm{e}^{-x^2}\mathrm{d}x$),

$$M_n = \int_K x^n \mathrm{e}^{-|x|^2}\,\mathrm{d}x = \frac{\partial^n}{\partial y^n}\mathrm{e}^{y^2/4}\Big|_{y=0}.$$

Consequently, $M_{2k+1} = 0$, $M_{2k} = (2k-1)!!/2^k$. Using proposition 2.1, we get a formula for the calculation of the Gaussian integral of any entire function $f(x) = \sum_{n=0}^{\infty} f_n x^n$:

$$\int_K f(x)\mathrm{e}^{-|x|^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} f_{2n}(2n-1)!!/2^n.$$

According to the Gaussian distribution, many functions "exotic" from the point of view of the theory of real functions, are integrable. Let $K = \mathbf{Q}_p$. Then the function $f(x) = \sum_{n=0}^{\infty} n!x^n$ is integrable and

$$\int_{\mathbf{Q}_p} f(x)\,\mathrm{d}x = \sum_{n=0}^{\infty}[(2n)!](2n-1)!!/2^n.$$

Let us introduce *Hermitian polynomials* over the field K : $H_n(x) = (-1)^n \mathrm{e}^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\mathrm{e}^{-x^2}$ and, as usual, $\int_K H_n(x)H_m(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = 0$, $n \neq m$, calculate $\int_K H_n^2(x)\mathrm{e}^{-x^2}\,\mathrm{d}x$. To do this, we take into account that the product of the test function $H_n \in \mathcal{A}$ by the distribution $\nu \in \mathcal{A}'$ is equal to the generalized derivative of ν : $(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\nu)$,

$$\int_K H_n^2(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = (-1)^n\int_K H_n(x)\frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{e}^{-x^2}\mathrm{d}x) =$$

$$= \int_K \frac{\mathrm{d}^n}{\mathrm{d}x^n}(H_n(x))\mathrm{e}^{-x^2}\,\mathrm{d}x = 2^n n!\int_K \mathrm{e}^{-x^2}\,\mathrm{d}x = 2^n n!.$$

Problems on the Gaussian distribution

Let us restrict to the case $K = Q_p$, one dimensional case and a=0 . Hence B is the number in Q_p

1. The Gaussian distribution $\gamma_B = \gamma_{0,B}$ was defined as the functional on the space of the entire analytical functions . The problem is to extend this functional to larger functional

spaces . It was proved [13] that it would be impossible to extend this functional to the space of continuous functions on Z_p . But this proof does not work in the case $p = 2$. Is it possible to prove this fact for $p = 2$?

2.Schikhof's idea . Professor W.Schikhof propose to try to extend γ_B on the space of C^1 or C^∞ . At the moment ,there is no results in this direction.

3. The analytical function $f(x)$ is said to be γ_B-negligible if

$$\int f(x)\phi(x)\gamma_B(dx) = 0$$

for every analytical function $\phi(x)$. Recently , it was proved by M.Endo and author that if $p \neq 2$ and $B \in Z_p$, then $f = 0$. What is about $p = 2$ and another B?

The generalization of this problem is to consider an arbitrary distibution μ and to try to find the class of μ-negligible functions.

4. All Gaussian measures are absolutely continuous in ordinary case . What is about p-adic case ?

3.Non-Archimedean Hilbert Space

The quantization over non-Archimedean number fields must be based on the non-Archimedean analog of a Hilbert space. In the mathematical literature the concept of a non-Archimedean Hilbert space adequate for physical applications has not yet been worked out. The concept of orthogonality in non-Archimedean spaces is based not on an inner product but on a norm.

Recall that the system of vectors $\{e_j\}_{j\in J}$ in the non-Archimedean normalized space E is said to be orthogonal if

$$\left\| \sum_{j\in S} x_j e_j \right\| = \max_{j\in S} |x_j|_K ||e_j||$$

for every finite set $S \subset J$ and any $x_j \in K$. The orthogonal system $\{e_j\}_{j\in J}$ is called an orthogonal basis in E if $x = \sum x_j e_j$ for any vector $x \in E$. In this case the space E is said to be orthogonalizable. By virtue of *J.-P. Serre's orthogonalization theorem*, every discrete normalized non-Archimedean Banach space is orthogonalizable.

There is no canonical way of defining the inner product $(\cdot,\cdot)$ in the orthogonalizable space E. Supposing that the vectors $\{e_j\}$ must be orthogonal not only with respect to the norm but also with respect to the inner product, we have

$$(x, x) = \sum \lambda_j x_j^2,$$

where $\lambda_j = (e_j, e_j)$. This series converges if and only if $\lim_{j\to\infty} |x_j|_K \sqrt{|\lambda_j|_K} = 0$. But the Banach space E consists of those x for which $\lim_{j\to\infty} |x_j|_K ||e_j|| = 0$. If $||e_j||^2 \in \Gamma$, then we can take as λ_j any elements of the field K such that $|\lambda_j|_K = ||e_j||^2$. Now if $||e_j||^2 \notin \Gamma$, then it is impossible, in general, to find $\lambda_j \in K$. It is natural to include the numbers λ_j into the definition of a non-Archimedean Hilbert space. In what follows we shall consider only countable sets of indices.

For the sequence $\lambda = (\lambda_n) \in K^\infty$, $\lambda_n \neq 0$ we set

$$\mathcal{H}_\lambda = \{f = (f_n) : \text{ the series } \sum f_n^2 \lambda_n \text{ converges}\}.$$

By virtue of the non-Archimedean Cauchy criterion we have

$$\mathcal{H}_\lambda = \{f = (f_n) : \lim_{n\to\infty} |f_n|_K \sqrt{|\lambda_n|_K} = 0\}.$$

In the space $\mathcal{H}_\lambda$ we introduce a norm relative to which the base vectors $e_j = (e_j^i) = (\delta_j^i)$, are orthogonal, $||f||_\lambda = \max_n |f_n|_K \sqrt{|\lambda_n|_K}$. The space $\mathcal{H}_\lambda$ is a non-Archimedean Banach space. On the space $\mathcal{H}_\lambda$ we introduce an inner product $(\cdot,\cdot)$ consistent with the length $|f|_\lambda^2 = \sum f_n^2 \lambda_n$ setting $(f,g)_\lambda = \sum f_n g_n \lambda_n$. The inner product $(\cdot,\cdot) : \mathcal{H}_\lambda \times \mathcal{H}_\lambda \to K$ is continuous and we have the *Cauchy-Buniakovski inequality*:

$$|(f,g)_\lambda|_K \leq ||f||\,||g||. \tag{3.1}$$

Definition 3.1. *The triplet* $(\mathcal{H}_\lambda,\ (\cdot,\cdot)_\lambda,\ ||\cdot||_\lambda)$ *is called a coordinate Hilbert space* .

An inner product on the non-Archimedean linear space E is an arbitrary nondegenerated symmetric bilinear form $(\cdot,\cdot) : E \times E \to K$. It is evidently impossible to introduce an analog of the positive definiteness of a bilinear form. For instance, for the field of p-adic numbers any element $\gamma \in \mathbf{Q}_p$ can be represented as $\gamma = (x,x)_\lambda$, $x \in \mathcal{H}_\lambda$.

The triplets $(E_j,\ (\cdot,\cdot)_j,\ ||\cdot||_j)$, $j = 1, 2$, where E_j are non-Archimedean Banach spaces, $||\cdot||_j$ are norms and $(\cdot,\cdot)_j$ are inner products satisfying (3.1), are isomorphic if the spaces E_1 and E_2 are algebraically isomorphic and the algebraic isomorphism $I : E_1 \to E_2$ is isometric and unitary, i.e., $||Ix||_2 = ||x||_1$, $(Ix, Iy)_2 = (x,y)_1$.

Definition 3.2. *The triplet* $(E,\ (\cdot,\cdot),\ ||\cdot||)$ *is a non-Archimedean Hilbert space if it is isomorphic to the coordinate Hilbert space* $(\mathcal{H}_\lambda,\ (\cdot,\cdot)_\lambda,\ ||\cdot||_\lambda)$ *for a certain* λ .

The isomorphic relation divides the class of Hilbert spaces into equivalence classes. we shall define the equivalence class of Hilbert spaces by some coordinate representative $\mathcal{H}_\lambda$.

Example 3.1. Let $\lambda = (1)$ and $\mu = (2^n)$. The spaces $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$ belong to the same class of equivalence for the field $K = \mathbf{Q}_p$, $p \neq 2$, and to different classes for the field $K = \mathbf{Q}_2$.

Non-Archimedean Hilbert spaces over quadratic extensions $Z = K(\sqrt{\tau})$ can be introduced by analogy. For the sequence $\lambda = (\lambda_n) \in K^\infty$, $\lambda_n \neq 0$, we set

$$\mathcal{H}_\lambda = \{f = (f_n) \in Z^\infty : \text{the series} \sum |f_n|^2 \lambda_n \text{ converges in the field } K\} =$$

$$\{f = (f_n) : \lim_{n\to\infty} |f_n|_K \sqrt{|\lambda_n|_K} = 0\};$$

$$||f||_\lambda = \max_n |f_n|_K \sqrt{|\lambda_n|_K};$$

$$(f,g) = \sum f_n \bar{g}_n \lambda_n; \qquad |f|_\lambda^2 = (f,f)_\lambda = \sum |f_n|^2 \lambda_n \in K.$$

The triplet $(\mathcal{H}_\lambda,\ (\cdot,\cdot)_\lambda,\ ||\cdot||_\lambda)$ is a non-Archimedean complex coordinate Hilbert space. The non-Archimedean complex coordinate Hilbert space $(E,\ (\cdot,\cdot),\ ||\cdot||)$ is defined as an isomorphic image of a coordinate Hilbert space.

We shall also use a non-Archimedean Hilbert space over the field of complex p-adic numbers $\mathbf{C}_p$ in some physical models. There is a possibility to compute all square roots $\sqrt{\lambda_j}$ and this is why we need not to consider these coefficients at all. Thus, we need only to define the standard sequence space :

$$\mathcal{H}(\mathbf{C}_p) = \{f = (f_n), f_n \in \mathbf{C}_p : \text{the series} \sum f_n^2 \text{ converges}\}.$$

But there is a serious difference with the usual complex case , because there is not any involution on the field $\mathbf{C}_p$. This is why the only possibility to define an inner product on $\mathcal{H}(\mathbf{C}_p)$ is the following one $(f,g) = \sum f_n g_n$. It is the hard problem in our further physical considerations that this inner product is valued in $\mathbf{C}_p$ and not in Q_p.

Problems on the Hilbert space

1.What are restrictions on the weight sequences $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ for the unitary isomorphism of $\mathcal{H}_\lambda$ and $\mathcal{H}_\mu$.

2. Let us consider a linear operator $U : \mathcal{H}_\lambda \to \mathcal{H}_\mu$. It can be realized as an infinite matrix . What are the restrictions on the coefficients to be a unitary isometry ? It would be interesting to study not only the canonical basis but an arbitrary orthogonal basis.

3. Does there exist some kind of the coordinate representation of the norm on $\mathcal{H}_\lambda$ in an arbitrary orthgonal basis ?

4. Is it possible to define the Hilbert space topology with the aid of only the inner product ?

4.The Space $L_2(K^n, e^{-|x|^2}dx)$ of Functions Square Summable with Respect to the Canonical Gaussian Distribution

On the space $\mathcal{A}$ of entire functions (K^n, Z) we consider a canonical Gaussian distribution

$$\nu(dx) = e^{-|x|^2}dx, \qquad |x|^2 = \sum_{j=1}^{n} x_j^2.$$

Using the fact that the space $\mathcal{A}$ is an algebra, we introduce on $\mathcal{A}$ an inner product

$$(f, g) = \int_{K^n} f(x)\bar{g}(x)e^{-|x|^2}\, dx \tag{4.1}$$

and the square of the length of the function f

$$|f|^2 = \int_{K^n} |f(x)|^2 e^{-|x|^2}\, dx \in K. \tag{4.2}$$

The algebra $\mathcal{A}$ is topological and, consequently, the inner product $(\cdot,\cdot)$ is continuous on $\mathcal{A}\times\mathcal{A}$. We use the symbols $H_\alpha(x)$ to denote the Hermitian polynomials corresponding to the Gaussian distribution ν,

$$H_{(\alpha_1,\dots,\alpha_n)}(x_1,\dots,x_n) = H_{\alpha_1}(x_1)\dots H_{\alpha_n}(x_n).$$

Proposition 4.1. *The Hermitian polynomials $H_\alpha(x)$ are orthogonal with respect to the inner product $(\cdot,\cdot)$, and*

$$\int_{K^n} H_\alpha^2(x)e^{-|x|^2}\, dx = 2^{|\alpha|}\alpha!.$$

This proposition is a direct corollary of Example 1.1

Theorem 4.1 (the estimate of the growth of the Hermitian coefficients for entire functions). *Let $f \in \mathcal{A}$. Then, for the Hermitian coefficients $\tilde{f}_\alpha = (f, H_\alpha)/(H_\alpha, H_\alpha)$, the estimate*

$$|\tilde{f}_\alpha|_K \le \frac{\|f\|_R}{|\alpha!|_K (R|2|_K)^{|\alpha|}} \tag{4.3}$$

is valid for all $R \ge 1/\sqrt{|2|_K}$.

Theorem 4.2. *The Hermitian polynomials $H_\alpha(x)$ form a basis in the space $\mathcal{A}$ of entire functions.*

Proposition 4.2. *Let $f,g \in \mathcal{A}$. Then*

$$(f,g) = \sum_{\alpha} \tilde{f}_\alpha \bar{\tilde{g}}_\alpha 2^{|\alpha|}. \tag{4.5}$$

In order to prove (4.5.), it is sufficient to use Theorem 4.1 and Proposition 4.1.

In the space $\mathcal{A}$ we introduce a norm relative to which the Hermitian polynomials are orthogonal:

$$||f|| = \max_{\alpha} |f_\alpha|_K \sqrt{|\alpha!|_K |2|_K^{|\alpha|}}.$$

The completion of the space of entire functions with respect to this norm is called the space of functions square summable by the Gaussian distribution ν and is denoted by $L_2(K^n, \nu)$.

Proposition 4.3. $L_2(K^n,\nu) = \{f = \sum_\alpha \tilde{f}_\alpha H_\alpha$: *the series* $|f|^2 = \sum |\tilde{f}_\alpha|^2 \alpha! 2^{|\alpha|}$ *converges*$\} = \{f = \sum \tilde{f}_\alpha H_\alpha : \lim_{|\alpha|\to\infty} |\tilde{f}_\alpha|_K |\alpha!|_K |2|_K^{|\alpha|} = 0\}$

The inner product (4.5) is continuous on $L_2(K^n,\nu)$ and the Cauchy–Buniakovski inequality (3.1) holds true. The triplet $(L_2(K^n,\nu), (\cdot,\cdot), ||\cdot||)$ is a non-Archimedean complex Hilbert space of the class $\mathcal{H}_{(\alpha!2^{|\alpha|})}$.

Proposition 4.4. *The injections*

$$\mathcal{A}(U_{\rho_1}, Z) \subset L_2(K^n, \nu) \subset \mathcal{A}(U_{\rho_2}, Z)$$

are valid, where

$$\rho_1 > \sqrt{\frac{b}{|2|_K}}, \qquad \rho_2 = \frac{1}{\sqrt{b|2|_K}}.$$

Remark 4.1. *Thus, in a non-Archimedean case all entire functions are square summable with respect to canonical Gaussian distribution and all square summable functions are analytical.*

Problems on L_2-theory

1. What is about L_2-space for the Gaussian measure $\gamma_{a,B}$? In particular, $a = 0, B \in Q_p$?
2. What are the conditions to B_1 and B_2 for the corresponding L_2- spaces to isomorphic ?

5. The Space $L_2(K^n, \mathrm{d}x)$ of Functions Square Summable with Respect the Lebesgue Distribution

We introduce a functional space

$$U(K^n, Z) = \{f(x) = \varphi(x)e^{-|x|^2} : \varphi(x) \in \mathcal{A}(K^n, Z)\}.$$

The topology in the space U is induced from the space $\mathcal{A}$ of entire functions by the isomorphism $I : U \to \mathcal{A}$, $I(f)(x) = f(x)e^{|x|^2}$. The space U is a reflexive non-Archimedean Fréchet space. We choose this space as the space of test functions and U' as the space of generalized functions. The space of generalized functions U' is isomorphic to the space $\mathcal{A}'$, $I' : \mathcal{A}' \to U'$.

Definition 5.1. *The Lebesgue distribution on the non-Archimedean space K^n is a generalized function* $\mathrm{d}x = I'(\nu) \in U'(K^n, Z)$.

For the Lebesgue distribution $\mathrm{d}x$ to act on the test function $f \in U$, we use the integral notation

$$(\mathrm{d}x, f) = \int_{K^n} f(x)\,\mathrm{d}x.$$

Note the obvious properties of the integral with respect to the Lebesgue distribution $\mathrm{d}x$.

1. Linearity:

$$\int_{K^n} (\lambda f(x) + \mu g(x))\,\mathrm{d}x = \lambda \int_{K^n} f(x)\,\mathrm{d}x + \mu \int_{K^n} g(x)\,\mathrm{d}x.$$

2. The limiting process under the integral sign:

$$\lim_{m\to\infty} \int_{K^n} f_m(x)\,\mathrm{d}x = \int_{K^n} f(x)\,\mathrm{d}x$$

if $f_m \to f$.

3. Fubini's theorem:

$$\int_{K^n} f(x)\,\mathrm{d}x = \int_K (\ldots \int_K f(x_1,\ldots,x_n)\,\mathrm{d}x_1 \ldots)\,\mathrm{d}x_n.$$

In order to prove Fubini's theorem, it is sufficient to note that the Lebesgue distribution $\mathrm{d}x$ can be represented as the product of distributions, i.e., $\mathrm{d}x = \mathrm{d}x_1 \otimes \ldots \otimes \mathrm{d}x_n$.

We introduce a functional space

$$W(K^n, Z) = \{f(x) = \varphi(x)e^{-|x|^2/2} : \varphi \in \mathcal{A}(K^n, Z)\}.$$

The topology in the space W is induced from $\mathcal{A}$ by means of the isomorphism $J : W \to \mathcal{A}$, $J(f) = fe^{|x|^2/2}$. Let $f, g \in W$, and then $f, g \in U$.

4. The formula for integration by parts ($f, g \in W$):

$$\int_{K^n} \frac{\partial f}{\partial x_j}(x) g(x)\,\mathrm{d}x = -\int_{K^n} f(x)\frac{\partial g}{\partial x_j}(x)\,\mathrm{d}x.$$

We use ordinary formulas to introduce the inner product and the square of the length on the functional space W:

$$(f, g) = \int_{K^n} f(x)\bar{g}(x)\,\mathrm{d}x, \tag{5.1}$$

$$|f|^2 = \int_{K^n} |f(x)|^2 \mathrm{d}x. \tag{5.2}$$

The inner product $(\cdot,\cdot) : W \times W \to Z$ is continuous (inequality (3.1)) and the Hermite functions $\varphi_\alpha(x) = H_\alpha(x)e^{-|x|^2/2}$ are orthogonal to (5.1.). By virtue of Theorem 4.1, the Hermite functions form the Schauder basis in the space W, i.e., for any function $f \in W$ the series

$$f(x) = \sum \tilde{f}_\alpha \varphi_\alpha(x),$$

where

$$\tilde{f}_\alpha = \int_{K^n} f(x)\varphi_\alpha(x)\,\mathrm{d}x \Big/ \int_{K^n} \varphi_\alpha^2(x)\,\mathrm{d}x$$

are Hermitian coefficients, converges in the space W, whence follows a coordinate representation for (5.1) and (5.2)

$$(f, g) = \sum_\alpha \tilde{f}_\alpha \bar{\tilde{g}}_\alpha \alpha! 2^{|\alpha|}, \tag{5.3}$$

$$|f|^2 = \sum_\alpha |f_\alpha|^2 \alpha! 2^{|\alpha|}. \tag{5.4}$$

On the functional space W we introduce a norm relative to which the Hermite functions are orthogonal:

$$||f|| = \max_\alpha |f_\alpha|_K \sqrt{|\alpha!|_K |2|_K^{|\alpha|}}.$$

The completion of the space $W(K^n, Z)$ according to this norm is called the space of functions square summable by the Lebesgue distribution and is denoted by

$$L_2(K^n, \mathrm{d}x) =$$

$$\{f(x) = \sum_\alpha \tilde{f}_\alpha \varphi_\alpha(x) : \quad \text{the series} \quad \sum_\alpha |f_\alpha|^2 \alpha! 2^{|\alpha|} \quad \text{converges}\} =$$

$$= \{f(x) = \sum_\alpha \tilde{f}_\alpha \varphi_\alpha(x) : \quad \lim_{|\alpha| \to \infty} |\tilde{f}_\alpha|_K \sqrt{|\alpha!|_K |2|_K^{|\alpha|}} = 0\}.$$

we shall use notations (5.1) and (5.2) for the inner product (5.3) and the square of the length (5.4).

The isomorphism $J : W \to \mathcal{A}$ is continued to the isomorphism of the Hilbert space $J : L_2(K^n, \mathrm{d}x) \to L_2(K^n, \nu)$. In particular, the following nested Hilbert space appears:

$$W(K^n, Z) \subset L_2(K^n, \mathrm{d}x) \subset L_2'(K^n, \mathrm{d}x) \subset W'(K^n, Z).$$

Proposition 5.1. *The injection*

$$L_2(K^n, \mathrm{d}x) \subset \mathcal{A}(U_\delta, Z), \qquad \delta < \sqrt{|2|_K / b}.$$

is valid.

Remark 5.1. *Thus, in a non-Archimedean case, all functions square summable by the Lebesgue distribution* $\mathrm{d}x$ *are analytic.*

Problems on the Lebesgue distribution

1. The Lebesgue distribution is an analogue of the uniform distribution . But the Volkenborn distribution is also the analogue of the uniform distribution . What is the connection ?

2. We can also introduce Lebesgue distribution on the basis of an arbitrary Gaussian distribution , γ_B. What is the connection between Lebesque distributions ,corresponding to different Gaussian distributions.

In applications I am also interested very much in the following question : Is it possible to introduce on Q such metric or topology t that every sequence A_n converging with respect to the real metric on Q will converge with respect to t and every B_n converging with respect to the p-adic metric will converge with respect to t ? What the completion of Q with respect to t ?

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