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EXISTENCE RESULTS FOR SEMILINEAR ELLIPTIC EQUATIONS WITH SMALL MEASURE DATA

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ABSTRACT. – We give a smallness condition on $|m|$, and $\|f\|_q$ for the existence of a solution for the model problem: $-\Delta_p u = f(x)|u|^\gamma + m\mu$ with $u = 0$ on $\partial\Omega$, where Ω is a bounded open set of $\mathbb{R}^N$, $f(x) \in L^q(\Omega)$, $q \geqslant 1$, $m \in \mathbb{R}$ and μ is a Radon measure with bounded variation on Ω such that $|\mu|(\Omega) = 1$. © 2002 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous donnons une condition suffisante sur $|m|$, et $\|f\|_q$ pour l'existence de solution au problème modèle : $-\Delta_p u = f(x)|u|^\gamma + m\mu$ avec $u = 0$ sur $\partial\Omega$, où Ω est un ouvert borné de $\mathbb{R}^N$, $f(x) \in L^q(\Omega)$, $q \geqslant 1$, $m \in \mathbb{R}$ et μ est une mesure de Radon à variation bornée sur Ω telle que $|\mu(\Omega)| = 1$. © 2002 Éditions scientifiques et médicales Elsevier SAS

1. Introduction and main results

The main goal of this paper is to prove, if the data are small enough, the existence of a solution for the model problem

$$\begin{cases} -\Delta_p u = f(x)|u|^\gamma + m\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $N \geqslant 1$, Ω is a bounded open subset of $\mathbb{R}^N$, $-\Delta_p$ is the so called p-Laplace operator, $f(x) \in L^q(\Omega)$, $q \geqslant 1$, $\mu \in M_B(\Omega)$ (that is to say μ is a Radon measure with bounded variation in Ω) such that $|\mu|(\Omega) = 1$ and $m \in \mathbb{R}$.

In fact we study the more general problem

$$\begin{cases} -\mathrm{div}(a(x, Du)) = h(x, u) + m\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $u \mapsto -\mathrm{div}(a(x, Du))$ is a monotone operator defined on $W_0^{1,p}(\Omega)$ with values in $W^{-1,p'}(\Omega)$, $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$. We suppose more precisely that,

$$a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \quad \text{is a Caratheodory function,} \tag{1.3}$$

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that is to say $a(.,\xi)$ is measurable on Ω for every ξ in $\mathbb{R}^N$, and $a(x,.)$ is continuous on $\mathbb{R}^N$ for almost every x in Ω, that,

$$a(x,\xi)\xi \geqslant \alpha|\xi|^p, \tag{1.4}$$

for almost every x in Ω and for every ξ in $\mathbb{R}^N$, where $\alpha > 0$ is a constant, that,

$$|a(x,\xi)| \leqslant d\big(b(x)+|\xi|\big)^{p-1}, \tag{1.5}$$

for almost every x in Ω and every ξ in $\mathbb{R}^N$, where $d > 0$ is a constant and b is a nonnegative function in $L^p(\Omega)$, and that,

$$\big(a(x,\xi)-a(x,\xi')\big)(\xi-\xi') > 0, \tag{1.6}$$

for almost x in Ω, and for every ξ, ξ' in $\mathbb{R}^N$, $\xi \neq \xi'$. We also assume that,

$$h:\Omega\times\mathbb{R}\to\mathbb{R} \quad \text{is a Caratheodory function,} \tag{1.7}$$

that is to say $h(.,t)$ is measurable on Ω for every t in $\mathbb{R}$, and $h(x,.)$ is continuous on $\mathbb{R}$ for almost every x in Ω, and that,

$$\begin{cases} |h(x,t)| \leqslant f(x)|t|^\gamma, \\ \text{for some } 1 \leqslant \gamma < +\infty \text{ and some } f \in L^q(\Omega), \\ \text{where } 1 \leqslant q \leqslant +\infty, \end{cases} \tag{1.8}$$

for almost every x in Ω for every t in $\mathbb{R}$.
Observe that there is no sign assumption on $h(x,t)$, only the growth on t is considered.

We now recall some well known results about measures.
For every measure $\mu \in M_B(\Omega)$ there exists a unique pair of measures (μ_0, μ_s) such that $\mu = \mu_0 + \mu_s$ (see [5] and [10]) with μ_0 in $M_0(\Omega)$ (that is to say the set of all measures in $M_B(\Omega)$ which are absolutely continuous with respect to the p-capacity) and μ_s in $M_S(\Omega)$ (that is to say the set of all measures in $M_B(\Omega)$ which are singular with the p-capacity). In other words, μ_s is concentrated on a subset E of Ω with zero p-capacity, and μ_0 does not charge the set of zero p-capacity. Moreover it is equivalent for a measure to be in $M_0(\Omega)$ and to belong to $L^1(\Omega) + W^{-1,p'}(\Omega)$, that is to say every μ_0 can be written as $\mu_0 = f - \mathrm{div} g$ with $f \in L^1(\Omega)$ and $g \in (L^{p'}(\Omega))^N$. In short, every $\mu \in M_B(\Omega)$ can be decomposed as follows,

$$\mu = f - \mathrm{div} g + \mu_s^+ - \mu_s^-$$

where $f \in L^1(\Omega)$, $g \in (L^{p'}(\Omega))^N$, μ_s^+, μ_s^- (the positive part and negative part of μ_s) are two nonnegative measures in $M_s(\Omega)$ which are concentrated on two disjoint subsets E^+ and E^- of zero p-capacity. Recall also (see [3,7,8]) that if u is a measurable function defined on Ω, which is finite almost everywhere, and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$ (where $T_k(u)$ is the truncate at level k), then there exists a measurable function $v:\Omega \to \mathbb{R}^N$ such that $DT_k(u) = v\chi_{\{|u|\leqslant k\}}$ almost everywhere in Ω, for every $k > 0$, which is unique up to almost everywhere equivalence. We define the gradient Du of u as this function v.

Let us recall the definition of a renormalized solution (see [7,8]).

DEFINITION 1.1. – *We suppose* (1.3)–(1.6), $p > 1$, $\mu \in M_B(\Omega)$. *We say that u is a renormalized solution of*

$$\begin{cases} -\mathrm{div}(a(x, Du)) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

if,

- *the function u is measurable and finite everywhere and $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$,*
- *the gradient Du in the previous sense satisfies,*

$$|Du|^{p-1} \in L^q(\Omega), \quad \forall q,\ 1 \leqslant q < \frac{N}{N-1},$$

- *if w belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and if there exists $k > 0$ and $w^{+\infty}, w^{-\infty} \in W^{1,r}(\Omega) \cap L^\infty(\Omega)$ with $r > N$ such that,*

$$w = w^{+\infty} \quad \text{a.e. on the set } \{u > k\},$$

$$w = w^{-\infty} \quad \text{a.e. on the set } \{u < -k\},$$

then,

$$\int_\Omega a(x, Du) Dw \,\mathrm{d}x = \int_\Omega w \,\mathrm{d}\mu_0 + \int_\Omega w^{+\infty} \,\mathrm{d}\mu_s^+ - \int_\Omega w^{-\infty} \,\mathrm{d}\mu_s^-. \tag{1.10}$$

In [8] the authors give equivalent definitions of renormalized solutions. When $\mu \in M_0(\Omega)$, this definition is equivalent to the definition of an entropy solution (see [3] and [5]).

Let us observe that when $p > N$, the renormalized solution is just a usual weak solution and belongs to some $C^{0,\alpha}(\Omega)$; therefore the notion of renormalized solution is not really needed. This is also the case for example in the linear case where $a(x, \xi) = A(x)\xi$ when the matrix A has smooth coefficients. However, when the coefficients are not smooth, a new notion is necessary even in the linear case in order to obtain both existence and uniqueness results (see [16]). Observe in particular that the test function w which is used in (1.10) actually depends on the solution u itself, and that in some sense $u = +\infty$ on the set where μ_s^+ is concentrated, while $u = -\infty$ on the set where μ_s^- is concentrated since the action of μ_s on the set where $|u| \leqslant k$ does not appear in (1.10). For more comments on the notion of renormalized solutions, see [8]. These equations have been widely studied. Especially in [1,2,11], the authors give a sufficient and necessary condition for the existence of a solution of equations closed to (1.2) in the case $p = 2$, but their method doesn't extend to $p \neq 2$. See also [15] for the case of an

eigenvalue problem. Let us also quote [4] in which the authors give counter examples to the existence for the equation of the type (1.2). Quasilinear equations have been studied with more regular data in [9,12,14] for instance. In these papers existence results are obtained assuming that the data are small enough relatively to a convenient norm.
The main result of this paper is the following,

THEOREM 1.1. – *Assume* (1.3)–(1.8), *let* $m \in \mathbb{R}$ *and* $\mu \in M_B(\Omega)$, *such that* $|\mu|(\Omega) = 1$, $1 \leqslant \gamma < +\infty$, $1 \leqslant q \leqslant +\infty$ *with* $q \neq 1$ *if* $N = p$ *and* $\gamma q' < \frac{(p-1)N}{N-p}$ *if* $N > p$. *Then there exists a renormalized solution of* (1.2)

(1) *if* $1 \leqslant \gamma < p-1$ (*thus* $p > 2$)

$$\text{with no additionnal condition on} \|f\|_q, m;$$

(1) *if* $\gamma \geqslant p-1$ *then the condition is*

$$\|f\|_q |m|^{\frac{\gamma-p+1}{p-1}} \leqslant \frac{C}{|\Omega|^{\frac{1}{q'}+\frac{\gamma}{p-1}(-1+\frac{p}{N})}} \tag{1.11}$$

for some constant $C = C(N, p, \gamma)$.

Remarks. –

- First observe that when $p < N$, there exists some q with $1 \leqslant q \leqslant +\infty$ and some $\gamma \geqslant 1$ such that $\gamma q' < \frac{(p-1)N}{N-p}$ if and only if $p > \frac{2N}{N+1}$.
  This is a restriction on the values of γ and q, which is natural. Indeed, in order to define a renormalized solution of (1.2), we need $h(x, u)$ to belong to $L^1(\Omega)$. But even if $h(x, u) \equiv 0$, the renormalized solution u of (1.2) belongs to $L^r(\Omega)$ for any r, $1 \leqslant r < \frac{(p-1)N}{N-p}$ and is not in general in $L^{\frac{(p-1)N}{N-p}}(\Omega)$. Consequently if $\gamma q' \geqslant \frac{(p-1)N}{N-p}$ we shall not have $h(x, u) \in L^1(\Omega)$.
- If $\gamma = p-1$ condition (1.11) reads

  $$\|f\|_q \leqslant C|\Omega|^{\frac{1}{q}-\frac{p}{N}}$$

  with no condition on m. Actually, if u solves

  $$-\Delta_p u = f(x)|u|^{p-1} + m\mu,$$

  then for any $c > 0$, $v = cu$ solves

  $$-\Delta_p v = f(x)|v|^{p-1} + c^{p-1} m\mu.$$

  That is to say, if there is a solution for m and μ given, then there is a solution for every $|m|$.
- If $\mu \geqslant 0$ and $h \geqslant 0$, then a solution of (1.2) is nonnegative. Indeed, we can use $w = -T_k(u^-)$ as test function in the equation satisfied by u and then (observe that $\mu_s^- = 0$ and $w^{+\infty} = 0$)

  $$-\int_\Omega a(x, Du) DT_k(u^-)\,\mathrm{d}x = \int_\Omega h(x, u)\big(-T_k(u^-)\big)\,\mathrm{d}x + \int_\Omega -T_k(u^-)\,\mathrm{d}\mu_0 \leqslant 0,$$

from (1.4), we deduce that,

$$\alpha \| DT_k(u^-)\|_p \leqslant 0$$

for any $k > 0$, and then $u^- = 0$. It means that Theorem 1.1 gives conditions for the existence of a positive renormalized solution of

$$\begin{cases} -\Delta_p u = h(x,u) + \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2. Estimates and preliminary lemmas

Recall the following estimates,

LEMMA 2.1. – *We suppose* (1.3)–(1.6), $\mu \in M_B(\Omega)$, *such that* $|\mu|(\Omega) = 1$, $m \in \mathbb{R}$ *and* $p > 1$. *Let* u *be a renormalized solution of*

$$\begin{cases} -\operatorname{div}(a(x, Du)) = m\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

then the following estimate holds

$$\|u\|_r \leqslant C|\Omega|^{\frac{1}{r}+\frac{1}{p-1}(-1+\frac{p}{N})}|m|^{\frac{1}{p-1}}, \tag{2.1}$$

for some positive constant $C = C(N, p, r)$ *and for any* $r \in [1, +\infty]$ *if* $p > N$, $r \in [1, +\infty)$ *if* $p = N$, *and* $r \in [1, \frac{N(p-1)}{N-p})$ *if* $p < N$.

This estimate is proven in [13] for instance, where explicit value for C is explicitely given in a more general context. It can also be proven by symmetrization techniques (see [17]). We have to specify that in [13], the right-hand side is in $L^1(\Omega)$, but the proof extends to $\mu \in M_B(\Omega)$ without difficulty.

COROLLARY 2.1. – *Assume* (1.3)–(1.8), $1 \leqslant \gamma < +\infty$, $1 \leqslant q \leqslant +\infty$. *If* $v \in L^{\gamma q'}(\Omega)$, $m \in \mathbb{R}$ *and* $\mu \in M_B(\Omega)$ *such that* $|\mu|(\Omega) = 1$, *if* $q \neq 1$ *when* $N = p$ *and if* $\gamma q' < \frac{(p-1)N}{N-p}$ (*thus* $p > \frac{2N}{N+1}$) *when* $N > p$, *and if* u *is a renormalized solution of*

$$\begin{cases} -\operatorname{div}(a(x, Du)) = h(x, v) + m\mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

then,

$$\|u\|_{\gamma q'} \leqslant A + B\|v\|_{\gamma q'}^{\frac{\gamma}{p-1}}$$

where

$$A = C|\Omega|^{\frac{1}{\gamma q'}+\frac{1}{p-1}(-1+\frac{p}{N})}|m|^{\frac{1}{p-1}}, \qquad B = C|\Omega|^{\frac{1}{\gamma q'}+\frac{1}{p-1}(-1+\frac{p}{N})}\|f\|_q^{\frac{1}{p-1}},$$

for some positive constant $C = C(N, p, \gamma)$.

Proof. – We have

$$\big(|h(x,v)+m\mu|(\Omega)\big)^{\frac{1}{p-1}} \leqslant \big(\|h(x,v)\|_1+|m|\big)^{\frac{1}{p-1}},$$

then from (1.8), and Hölder inequality,

$$\big(|h(x,v)+m\mu|(\Omega)\big)^{\frac{1}{p-1}} \leqslant \big(\|v\|_{\gamma q'}^{\gamma}\|f\|_q+|m|\big)^{\frac{1}{p-1}}$$

and then,

- if $\frac{1}{p-1}<1$,

$$\big(|h(x,v)+m\mu|(\Omega)\big)^{\frac{1}{p-1}} \leqslant \|f(x)\|_q^{\frac{1}{p-1}}\|v\|_{\gamma q'}^{\frac{\gamma}{p-1}}+|m|^{\frac{1}{p-1}},$$

- if $\frac{1}{p-1}\geqslant 1$,

$$\big(|h(x,v)+m\mu|(\Omega)\big)^{\frac{1}{p-1}} \leqslant 2^{\frac{2-p}{p-1}}\|f(x)\|_q^{\frac{1}{p-1}}\|v\|_{\gamma q'}^{\frac{\gamma}{p-1}}+2^{\frac{2-p}{p-1}}|m|^{\frac{1}{p-1}}$$

and we get the corollary from (2.1) with $r=\gamma q'$.

We now study the function, $\varphi:\mathbb{R}^+\to\mathbb{R}$ defined by,

$$\varphi(X)=A+BX^{\frac{\gamma}{p-1}}-X,$$

where $A,\,B\geqslant 0$.

- If $\gamma>p-1$, then, $\varphi(0)=A\geqslant 0$ and $\lim_{X\to+\infty}\varphi(X)=+\infty$, moreover, by calculation of the derivative, we get that φ has a minimum at the point,

$$X_0=\left(\frac{p-1}{B\gamma}\right)^{\frac{p-1}{\gamma-p+1}}$$

with

$$\varphi(X_0)=A+\frac{1}{\gamma^{\frac{\gamma}{\gamma-p+1}}}\frac{(p-1)^{\frac{p-1}{\gamma-p+1}}}{B^{\frac{p-1}{\gamma-p+1}}}(p-1-\gamma),$$

then φ has at least one root if and only if $\varphi(X_0)\leqslant 0$ that is to say if,

$$AB^{\frac{p-1}{\gamma-p+1}}\leqslant\frac{1}{\gamma^{\frac{\gamma}{\gamma-p+1}}}(p-1)^{\frac{p-1}{\gamma-p+1}}(\gamma+1-p), \tag{2.3}$$

and φ has two roots if,

$$AB^{\frac{p-1}{\gamma-p+1}}<\frac{1}{\gamma^{\frac{\gamma}{\gamma-p+1}}}(p-1)^{\frac{p-1}{\gamma-p+1}}(\gamma+1-p).$$

- If $\gamma = p - 1$, then,

$$\varphi(X) = (B-1)X + A,$$

then φ has a root if

$$B < 1, \quad \forall A \geqslant 0. \tag{2.4}$$

- If $\gamma < p - 1$, then,

$$\varphi(X) = A + BX^{\frac{\gamma}{p-1}} - X$$

and then,

$$\lim_{X \to +\infty} \varphi(X) = -\infty \quad \text{and} \quad \varphi(0) \geqslant 0,$$

then φ has a root for any $A, B \geqslant 0$.

We henceforth denote (when it exists),

$$Y\text{: the smallest root of } \varphi. \tag{2.5}$$

3. Proof of Theorem 1.1

First observe that,

- if $\gamma > p - 1$, condition (2.3) is equivalent to

$$|\Omega|^{\frac{1}{q'} + \frac{\gamma}{p-1}(-1+\frac{p}{N})} |m|^{\frac{\gamma-p+1}{p-1}} \|f(x)\|_q \leqslant C$$

for some constant $C = C(N, p, \gamma)$.

- if $\gamma = p - 1$, condition (2.4) is equivalent to

$$|\Omega|^{-\frac{1}{q} + \frac{p}{N}} \|f(x)\|_q \leqslant C$$

for some constant $C = C(N, p)$, and we recognize the condition which appear in the second case of Theorem 1.1.

We set

$$h_n(s) = T_n\big(h(s)\big),$$

where T_n is the truncate at level n.

LEMMA 3.1. – *We suppose* (1.3)–(1.8), *let* $\mu \in M_B(\Omega) \cap W^{-1,p'}(\Omega)$, *such that* $|\mu|(\Omega) = 1$ *and* $m \in \mathbb{R}$, *we suppose that* Y *defined by* (2.5) *exists, that is to say if the previous conditions are fulfilled. Then, for any* $\mu_n \in W^{-1,p'}(\Omega) \cap M_B(\Omega)$ *such that,* $|\mu_n|(\Omega) \leqslant m$ *there exists a solution* $u \in W_0^{1,p}(\Omega)$ *of the equation*:

$$\begin{cases} \displaystyle\int_\Omega a(x, Du)Dw \,\mathrm{d}x = \int_\Omega h_n(x, u)w \,\mathrm{d}x + \langle \mu_n, w\rangle \\ \forall w \in W_0^{1,p}(\Omega), \end{cases} \tag{3.1}$$

such that,

$$\|u\|_{\gamma q'} \leqslant Y,$$

where γ, q' *satisfy the same conditions as in Corollary* 2.1.

Proof. – We shall use Schauder Fixed Point Theorem.

Let $v \in W_0^{1,p}(\Omega)$ then $h_n(x,v) + \mu_n \in W^{-1,p'}(\Omega)$ and there exists a unique $u \in W_0^{1,p}(\Omega)$, such that,

$$\begin{cases} \int_\Omega a(x,Du)Dw \,\mathrm{d}x = \int_\Omega h_n(x,v)w \,\mathrm{d}x + \langle \mu_n, w\rangle \\ \forall w \in W_0^{1,p}(\Omega). \end{cases} \tag{3.2}$$

Moreover since $|h_n(v)| \leqslant n$, using u as test function we easily get

$$\|Du\|_p \leqslant C_n, \tag{3.3}$$

where C_n is a constant which depends on n but not on v.
Let $v \in W_0^{1,p}(\Omega)$, we henceforth set $A_n(v) = u$ the solution of (3.2).
Let $E = \{v \in W_0^{1,p}(\Omega) \cap L^{\gamma q'}(\Omega), \|Dv\|_p \leqslant C_n, \|v\|_{\gamma q'} \leqslant Y\}$, then,

- E is a closed convex subset of $W_0^{1,p}(\Omega)$.
- Observe that from definition of Y, if $v \in E$ then

$$\|u\|_\gamma \leqslant A + B\|v\|_\gamma^{\frac{\gamma}{p-1}} \leqslant A + BY^{\frac{\gamma}{p-1}} = Y.$$

  Moreover we have already seen that

$$\|Du\|_p \leqslant C_n$$

  then,

$$A_n : E \to E.$$

- Suppose that (v_ε) is a sequence in E such that $v_\varepsilon \to v$ in $W_0^{1,p}(\Omega)$ strong and let $u_\varepsilon = A(v_\varepsilon)$. Since (v_ε) is bounded in $W_0^{1,p}(\Omega)$ there exists a subsequence still denoted (u_ε) such that,

$$u_\varepsilon \to u \ L^p(\Omega) \text{ strong, a.e. in } \Omega \text{ and } W_0^{1,p}(\Omega) \text{ weak.}$$

  Using $(u_\varepsilon - u)$ as test function in (3.2) we get,

$$\int_\Omega a(x,Du_\varepsilon)D(u_\varepsilon - u)\,\mathrm{d}x = \int_\Omega h_n(v_\varepsilon)(u_\varepsilon - u)\,\mathrm{d}x + \langle \mu_n, u_\varepsilon - u\rangle.$$

  We can easily see that the right-hand side tends to zero as ε tends to zero, then, since,

$$\int_\Omega \big(a(x,Du_\varepsilon)-a(x,Du)\big)D(u_\varepsilon-u)\,\mathrm{d}x$$
$$=\int_\Omega a(x,Du_\varepsilon)D(u_\varepsilon-u)\,\mathrm{d}x-\int_\Omega a(x,Du)D(u_\varepsilon-u)\,\mathrm{d}x$$

we have,

$$\lim_{\varepsilon\to 0}\int_\Omega \big(a(x,Du_\varepsilon)-a(x,Du)\big)D(u_\varepsilon-u)\,\mathrm{d}x=0$$

from a lemma of [6] it implies that,

$$\lim_{\varepsilon\to 0}\|D(u_\varepsilon-u)\|_p=0.$$

This implies that we can pass to the limit in the equation satisfied by u_ε, and we get $u=A(v)$. Consequently the whole sequence (u_ε) converges to u and finally it proves that A is continuous.

- With same arguments we can prove that $A(E)$ is precompact. Indeed if (u_ε) is a bounded sequence in $A(E)$ then $u_\varepsilon=A(v_\varepsilon)$ with (v_ε) or a subsequence is such that,

$$v_\varepsilon\to v \text{ a.e.}\quad \text{in } \Omega \text{ and } L^p(\Omega) \text{ strong}$$

and we deduce like previously that,

$$u_\varepsilon\to u\quad \text{in } W_0^{1,p}(\Omega) \text{ strong.}$$

End of the proof of Theorem 1.1.
Let $\mu\in M_B(\Omega)$ such that $|\mu|(\Omega)=1$ and $m\in\mathbb{R}$, then $m\mu$ can be decomposed as,

$$m\mu=f-\operatorname{div}g+\lambda^+-\lambda^-.$$

Let (μ_n) a sequence of measures in $M_B(\Omega)$ such that,

$$\mu_n=f_n-\operatorname{div}g+\lambda_n^\oplus-\lambda_n^\ominus$$

with,

$$f_n\in L^{p'}(\Omega) \text{ and } (f_n) \text{ converges to } f \text{ weakly in } L^1(\Omega), \tag{3.4}$$

$$\lambda_n^\oplus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \text{ that converges to } \mu_s^+ \text{ in the narrow topology of measures,} \tag{3.5}$$

$$\lambda_n^\ominus \text{ is a sequence of nonnegative functions in } L^{p'}(\Omega) \text{ that converges to } \mu_s^- \text{ in the narrow topology of measures,} \tag{3.6}$$

$$|\mu_n|(\Omega) \leqslant m, \tag{3.7}$$

then there exists a solution u_n of the corresponding Eq. (3.1) which satisfies

$$\|u_n\|_{\gamma q'} \leqslant Y.$$

Observe that in (3.1) the right-hand side is bounded in $M_B(\Omega)$, then it is proven in [8] that we can extract a subsequence which converges in measure and a.e. in Ω to a measurable function u which is finite almost everywhere. Moreover since the right-hand side in (3.1) is bounded in $M_B(\Omega)$, from Lemma 2.1 we have, if $q \neq 1$, with a small δ

$$\|u_n\|_{\gamma q'+\delta} \leqslant C,$$

where C is a constant which does not depend on n. We deduce that $(u_n^{\gamma q'})$ converges to $(u^{\gamma q'})$ in $L^1(\Omega)$ strong (see [3]). Moreover, we have,

$$\big\| f(x)|u_n|^\gamma - f(x)|u|^\gamma \big\|_{L^1(\Omega)} \leqslant \|f\|_q \left(\int_\Omega \big(|u_n|^\gamma - |u|^\gamma\big)^{q'} \right)^{1/q'} \tag{3.8}$$

but, $(|u_n|^\gamma - |u|^\gamma)^{q'}$ tends to 0 a.e. in Ω and,

$$\big(|u_n|^\gamma - |u|^\gamma\big)^{q'} \leqslant 2^{q'-1}|u_n|^{\gamma q'} + 2^{q'-1}|u|^{\gamma q'}.$$

The right-hand side converges in $L^1(\Omega)$ strong. Then by Vitali Lemma and (3.8), we deduce that,

$$f(x)|u_n|^\gamma \text{ tends to } f(x)|u|^\gamma \text{ in } L^1(\Omega) \text{ strong.} \tag{3.9}$$

We assert again that $h_n(x,u_n)$ converges a.e. in Ω to $h(x,u)$ and by (1.8) and (3.9), we deduce that $h_n(x,u_n)$ converges to $h(x,u)$ in $L^1(\Omega)$ strong. The same conclusion holds when $q = 1$. So $f_n + h_n(x,u_n)$ converges in $L^1(\Omega)$ weak, and with the additionnal assumptions (3.5), (3.6) on $\lambda_n^\ominus$ and $\lambda_n^\oplus$ we can apply Theorem 3.2 of [8] and conclude that u is a renormalized solution of (3.1).

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