

NONLINEAR MONOTONE SEMIGROUPS AND VISCOSITY SOLUTIONS

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ABSTRACT. – In a celebrated paper motivated by applications to image analysis, L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel showed that any monotone semigroup defined on the space of bounded uniformly continuous functions, which satisfies suitable regularity and locality assumptions is in fact a semigroup associated to a fully nonlinear, possibly degenerate, second-order parabolic partial differential equation. In this paper, we extend this result by weakening the assumptions required on the semigroup to obtain such a result and also by treating the case where the semigroup is defined on a general space of continuous functions like, for example, a space of continuous functions with a prescribed growth at infinity. These extensions rely on a completely different proof using in a more central way the monotonicity of the semigroup and viscosity solutions methods. Then we study the consequences on the partial differential equation of various additional assumptions on the semigroup. Finally we briefly present the adaptation of our proof to the case of two-parameters families. © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Dans un célèbre article motivé par les applications au traitement d’image, L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel ont montré qu’un semi-groupe monotone défini sur l’espace des fonctions bornées uniformément continues satisfaisant des hypothèses de régularité et de localité est en fait un semi-groupe associé à une équation aux dérivées partielles parabolique non linéaire éventuellement dégénérée. Dans le présent article, nous étendons ce résultat en affaiblissant légèrement les hypothèses nécessaires et en traitant le cas de semi-groupes définis sur des espaces généraux de fonctions continues. Ces extensions résultent d’une preuve totalement différente utilisant de manière plus centrale la monotonie du semi-groupe et des méthodes de solutions de viscosité. Nous étudions ensuite les conséquences d’hypothèses supplémentaires sur le semi-groupe. Finalement, nous présentons brièvement l’adaptation au cas d’une famille d’opérateurs à deux paramètres. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

In this article, we are interested in nonlinear semigroups $(T_t)_{t \geq 0}$ defined on some subspace $X \subset \mathcal{C}(\mathbb{R}^N)$ and satisfying the following monotonicity assumption: for any $f, g \in X$

$$f \leq g \Rightarrow T_t[f] \leq T_t[g] \quad \text{for any } t \geq 0,$$

where \leq denotes the partial ordering on $\mathcal{C}(\mathbb{R}^N)$ defined by

$$f \leq g \Leftrightarrow f(x) \leq g(x) \quad \text{for all } x \in \mathbb{R}^N.$$

In [1], L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel show that if $X = BUC(\mathbb{R}^N)$, the space of bounded uniformly continuous functions, and if $(T_t)_{t \geq 0}$ satisfies in addition suitable regularity and locality assumptions, such a semigroup is associated to a second-order parabolic partial differential equation. More precisely, they prove that, for any $u_0 \in BUC(\mathbb{R}^N)$, the continuous function $u(t, x) := T_t[u_0](x)$ is a viscosity solution of an initial value problem of the form

$$\begin{cases} \frac{\partial u}{\partial t} + F(x, u, Du, D^2u) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where Du and D^2u denote respectively the gradient and the Hessian matrix of u , and F is a continuous function on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$, $\mathcal{S}(N)$ being the space of the $N \times N$ symmetric matrices. We recall that Eq. (1) is said to be (degenerate) parabolic if the function F satisfies the so-called “ellipticity” condition: for all $x \in \mathbb{R}^N, r \in \mathbb{R}, p \in \mathbb{R}^N$ and $M, N \in \mathcal{S}(N)$,

$$M \leq N \Rightarrow F(x, r, p, M) \geq F(x, r, p, N). \tag{2}$$

Their proof is essentially done in three steps. The main step is the first one where, using only the regularity assumption on the semigroup together with a contraction property, they show the existence of a nonlinear infinitesimal generator

$$A[f] = \lim_{t \rightarrow 0^+} \frac{T_t[f] - f}{t}$$

which is well-defined if f is smooth. Then, using the monotonicity and the locality assumption, they prove the existence of a continuous function F such that, for any smooth function f ,

$$A[f](x) = -F(x, f(x), Df(x), D^2f(x)) \quad \text{in } \mathbb{R}^N.$$

Finally, using again the monotonicity assumption, a classical argument in viscosity solutions theory yields that u is a viscosity solution of (1).

The first contribution of this article is to provide a completely different and, to our opinion, far simpler proof of this result, using in a more fundamental way the

monotonicity of the semigroup. From the technical point of view, this will mean also that we are also going to use in a more central way viscosity solutions methods. This new proof allows us to weaken slightly the assumptions made in [1] on $(T_t)_{t \geq 0}$, to remove the use of a Banach space structure on X and thus to extend the result to a large class of spaces X .

Before going further in this introduction, we recall that viscosity solutions are weak solutions for second-order degenerate elliptic partial differential equations and we refer the reader to M.G. Crandall, I. Ishii and P.-L. Lions [6], W. Fleming and H.M. Soner [7] for a complete presentation of this theory and to M. Bardi and I. Capuzzo-Dolcetta [2], G. Barles [3] for an introduction to this theory in the case of first-order equations.

To be more specific on our results, we prove in Section 3, under localized versions of the assumptions used in [1], that there exists a continuous function F such that, for all $x \in \mathbb{R}^N$,

$$\lim_{h \rightarrow 0^+} \frac{T_t[f](x) - f(x)}{h} = -F(x, f(x), Df(x), D^2f(x)) \quad (3)$$

for any smooth function f of X and simultaneously that the function $u(t, x) := T_t[u_0](x)$ is a viscosity solution of (1) for every $u_0 \in X$.

Obviously, we need some assumptions on X but it is worth noticing that they are very weak and that they hold for a large class of spaces $X \subset \mathcal{C}(\mathbb{R}^N)$, like $\mathcal{C}(\mathbb{R}^N)$ itself, $BUC(\mathbb{R}^N)$, the space of bounded continuous functions or of continuous functions with growth conditions at infinity, the space of uniformly continuous functions ... etc.

In Section 4, we study the consequences of various additional assumptions on $(T_t)_{t \geq 0}$ and describe in particular the consequence of a finite speed of propagation property for $T_t[f]$, i.e.

$$\begin{aligned} f(x) \leq g(x) \text{ for } x \in B(x_0, r) &\Rightarrow \exists L_{x_0, f, g} > 0 \\ \text{such that } T_t[f](x_0) \leq T_t[g](x_0) &\text{ for } t \leq L_{x_0, f, g} r^s. \end{aligned}$$

The fact that a semigroup satisfying such kind of property is a semigroup associated to an Hamilton–Jacobi equation was first proved by P.-L. Lions in [12]. Nevertheless, we give a simplified proof using the Theorem of Section 3 which works in a slightly more general context.

Then we address the following natural question: do the assumptions used on $(T_t)_{t \geq 0}$ to connect it to the initial value problem (1) ensure that u is the unique solution of this problem? In [1], the answer was yes; indeed, because of the applications to image analysis, it was natural to assume that the semigroup commutes with translations and additions of constants (which yields an x and u -independent F). In this context, a comparison result holds for the viscosity solutions of (1) in $BUC(\mathbb{R}^N)$ (see for example M.G. Crandall, I. Ishii, P.-L. Lions [6] or Y. Giga, S. Goto, I. Ishii and M.-H. Sato [8]).

Here, on the contrary, the answer is no in general, even for a linear semigroup if the assumption of commutation with translations is removed. In Section 5, we build an example of a semigroup defined on $BUC(\mathbb{R}^N)$ which satisfy the assumptions of Section 2 and which is associated to a transport equation of the form

$$\frac{\partial u}{\partial t} + b(x) \cdot Du = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N.$$

But, uniqueness fails for the associated initial value problem.

This example is taken from M.G. Crandall and P.-L. Lions [5] and based on “pathological” situations for flows in \mathbb{R}^N studied by A. Beck in [4]. We give nevertheless all the details of the construction for the convenience of the reader and in order to ensure that assumptions of Section 2 hold even in the case $X = BUC(\mathbb{R}^N)$.

Finally, in Section 6, we briefly present, as in [1], some necessary adjustments in order to extend the results of Section 3 to the case of two-parameters monotone families of operators $(T_{t,s})_{t,s>0}$. We refer the reader to the book of W. Fleming and H.M. Soner [7] for developments about this kind of families in the context of viscosity solutions and optimal control and we just recall here that they are related to time-dependent equations

$$\frac{\partial u}{\partial t} + F(t, x, u, Du, D^2u) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N,$$

and that the semigroup property is replaced by

$$T_{t,r} = T_{t,s} \circ T_{s,r} \quad \text{for all } t \geq s \geq r \geq 0 \text{ and } T_{t,t} = Id_X.$$

2. Notations and assumptions on the semigroup

We first describe the functional spaces we use throughout this work and the related notations.

In the sequel, X will denote a subspace of $\mathcal{C}(\mathbb{R}^N)$ satisfying the following conditions.

- (H1) X contains $\mathcal{D}(\mathbb{R}^N)$, the space of C^∞ -functions with compact support in \mathbb{R}^N .
- (H2) For every $f \in X$ and $y \in \mathbb{R}^N$, the function $x \mapsto f(x + y)$ belongs to X .
- (H3) For every $u \in X$, there exists $g \in C_X^\infty(\mathbb{R}^N)$ such that $u \leq g$, where $C_X^\infty(\mathbb{R}^N) = C^\infty(\mathbb{R}^N) \cap X$.

It is worth noticing that most of the classical subspaces of $\mathcal{C}(\mathbb{R}^N)$ used in non-linear analysis satisfy this three assumptions: $\mathcal{C}(\mathbb{R}^N)$, $BUC(\mathbb{R}^N)$, $W^{1,\infty}(\mathbb{R}^N)$, bounded continuous functions, uniformly continuous functions, continuous functions with growth conditions at infinity... etc.

We give now the assumptions we use throughout this work on the family of mappings $(T_t)_{t \geq 0}$ defined from X into X and make some comments about it. They are all more or less slightly weak versions of these used in [1] thus we use the same terminology.

[Causality]. –

$$T_{t+s} = T_t \circ T_s \quad \text{for all } t, s \geq 0 \quad \text{and} \quad T_0 = Id_X.$$

[Monotonicity]. – For all $f, g \in X$

$$f \leq g \Rightarrow T_t[f] \leq T_t[g] \quad \text{for all } t \geq 0.$$

[Continuity]. – For every $u_0 \in X$, the function $(t, x) \mapsto T_t[u_0](x)$ is continuous and for all $b > a \geq 0$ there exists $f_{a,b,u_0} \in C_X^\infty(\mathbb{R}^N)$ such that

$$|T_t[u_0]| \leq f_{a,b,u_0} \quad \text{for all } t \in [a, b].$$

This last assumption may appear as being unusual and even restrictive. But, if X is for example defined by growth conditions at infinity, it may be equipped with a norm. The assumption can be seen in this case as a relaxed version of the classical continuity of $t \mapsto T_t[f]$ in the norm sense.

[Locality]. – For every $f, g \in C_X^\infty(\mathbb{R}^N)$ and for any fixed x in \mathbb{R}^N , if $f \equiv g$ on some $B(x, r)$ then

$$T_t[f](x) - T_t[g](x) = o(t) \quad \text{as } t \rightarrow 0^+.$$

In order to state the next assumption, we introduce some particular subsets of $\mathcal{D}(\mathbb{R}^N)$ already used in [1]. If $d = (d_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers, we set

$$Q_d = \{f \in \mathcal{D}(\mathbb{R}^N), \|D^\alpha f\| \leq d_n \text{ for } \alpha \in \mathbb{N}^p \text{ with } |\alpha| \leq n\}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_p$ if $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$.

The assumption (H1) together with the vector spaces structure of X allows us to formulate the following assumption.

[Regularity]. – For any sequence of positive numbers d , for any compact subset $K \subset \mathbb{R}^N$ and for every $f \in C_X^\infty(\mathbb{R}^N)$, there exists a positive function $m_{K,f,d}(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ with $m_{K,f,d}(0^+) = 0$ such that

$$|T_t[f + \lambda g](x) - T_t[f](x) - \lambda g(x)| \leq m_{K,f,d}(\lambda)t$$

for any $(x, g) \in K \times Q_d$ and any $\lambda, t \geq 0$.

Compared to [1], and if we restrict ourselves to the *BUC*-framework, it is the assumption which is the most relaxed. More precisely, we use only g with compact support and do not specify the dependence in f for $m_{K,f,d}$. Moreover, $m_{K,f,d}$ can be an arbitrary modulus and not only a linear one.

If $x \in \mathbb{R}^N$, we denote by τ_x the translation operator on $\mathcal{C}(\mathbb{R}^N)$ defined by

$$\tau_x \cdot f(y) = f(x + y)$$

for $f \in \mathcal{C}(\mathbb{R}^N)$. Since $\tau_x \cdot X = X$ from (H2), we also denote by τ_x the restriction of τ_x to X .

[Translation]. – For any compact subset $K \subset \mathbb{R}^N$ and every $f \in \mathcal{D}(\mathbb{R}^N)$, there exists a function $n_{K,f}(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $n_{K,f}(0^+) = 0$ such that

$$|\tau_x \cdot T_t[f](y) - T_t(\tau_x \cdot f)(y)| \leq n_{K,f}(|x|)t$$

for any $y \in K, t \geq 0$.

We recall that in [1], the semigroup $(T_t)_t$ was supposed to be invariant by translation i.e. $\tau_x \cdot T_t[f] = T_t(\tau_x \cdot f)$ for any $f \in X$ and $x \in \mathbb{R}^N$ and to take in account semigroups which do not satisfy such invariance property is also a main contribution of this article.

3. Generation of the parabolic P.D.E

The main result is the:

THEOREM 3.1. – *Let X be a subspace of $C(\mathbb{R}^N)$ for which (H1), (H2) and (H3) hold. Let $(T_t)_{t \geq 0}$ a family of mappings from X into X satisfying [Causality], [Monotonicity], [Continuity], [Locality], [Regularity], and [Translation]. Then there exists a continuous function F defined on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ such that (3) holds for any $f \in C_X^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Moreover, F satisfies the ellipticity condition (2) and the function $u(t, x) := T_t[u_0](x)$ is a continuous viscosity solution of (1) for every initial data $u_0 \in X$.*

Proof. – For any $f \in C_X^\infty(\mathbb{R}^N)$ and $t > 0$, we set

$$\delta_t[f] = \frac{T_t[f] - f}{t}.$$

In [1], in order to prove the existence of the infinitesimal generator, the main step was to show that, for t small enough, $(\delta_t[f])_{t \geq 0}$ was a Cauchy sequence in $BUC(\mathbb{R}^N)$. A rather difficult and technical task where [Regularity] together with the contraction property of $(T_t)_{t \geq 0}$ in $BUC(\mathbb{R}^N)$ were playing the main roles.

Here the key idea is to avoid this step by introducing and studying the mappings \bar{A} and \underline{A} defined on $C_X^\infty(\mathbb{R}^N)$ by setting

$$\bar{A}[f](x) = \limsup_{t \rightarrow 0^+} \delta_t[f](x)$$

and

$$\underline{A}[f](x) = \liminf_{t \rightarrow 0^+} \delta_t[f](x).$$

The functions \bar{A} and \underline{A} can be seen respectively as the “upper infinitesimal generator” and the “lower infinitesimal generator” for the semigroup. The following lemma shows that they are well-defined.

LEMMA 3.1 (Boundedness of the upper and lower infinitesimal generator). – *For any $x \in \mathbb{R}^N$ and $f \in C_X^\infty(\mathbb{R}^N)$, $\bar{A}[f](x)$ and $\underline{A}[f](x)$ are finite.*

The proof of Lemma 3.1 is based on the following technical result whose proof is postponed to Appendix A. We recall that $\mathcal{P}^{2,+}[u](t_0, x_0)$ and $\mathcal{P}^{2,-}[u](t_0, x_0)$ denote classically (see, for example, [6]) the second order parabolic semi-jets of the real-valued function u at (t_0, x_0) .

LEMMA 3.2. – *Let $u_0 \in X$ and $u(t, x) := T_t[u_0](x)$. If $(a, p, M) \in \mathcal{P}^{2,+}[u](t_0, x_0)$ (respectively $\mathcal{P}^{2,-}[u](t_0, x_0)$) then there exists a function $\tilde{\phi} : (t_0 - a, t_0 + a) \times \mathbb{R}^N \mapsto \mathbb{R}$ such that $\tilde{\phi}(t, \cdot) \in C_X^\infty(\mathbb{R}^N)$ for any $t \in (t_0 - a, t_0 + a)$ with*

$$\left(\tilde{\phi}(t_0, x_0), \frac{\partial \tilde{\phi}}{\partial t}(t_0, x_0), D\tilde{\phi}(t_0, x_0), D^2\tilde{\phi}(t_0, x_0) \right) = (u(t_0, x_0), a, p, M)$$

and

$$\overline{A}[\tilde{\phi}(t_0, \cdot)](x_0) \geq \underline{A}[\tilde{\phi}(t_0, \cdot)](x_0) \geq \frac{\partial \tilde{\phi}}{\partial t}(t_0, x_0)$$

(respectively $\frac{\partial \tilde{\phi}}{\partial t}(t_0, x_0) \leq \underline{A}[\tilde{\phi}(t_0, \cdot)](x_0) \leq \overline{A}[\tilde{\phi}(t_0, \cdot)](x_0)$).

It is worth noticing that, a priori, the previous lemma does not say that the lim sup or lim inf are finite.

Proof of Lemma 3.1. – 1) The continuity of $u(t, x) = T_t[0](x)$ implies that there exists $(t_1, x_1) \in (0, +\infty) \times \mathbb{R}^N$ (respectively $(t_2, x_2) \in (0, +\infty) \times \mathbb{R}^N$) such that $\mathcal{P}^{2,+}[u](t_1, x_1) \neq \emptyset$ (respectively $\mathcal{P}^{2,-}[u](t_2, x_2) \neq \emptyset$). Using Lemma 3.2, we deduce the existence of a function $\phi_1 \in C_X^\infty(\mathbb{R}^N)$ (respectively $\phi_2 \in C_X^\infty(\mathbb{R}^N)$) such that

$$\underline{A}[\phi_1(t_1, \cdot)](x_1) \geq \frac{\partial \phi_1}{\partial t}(t_1, x_1) \tag{4}$$

and

$$\overline{A}[\phi_2(t_2, \cdot)](x_2) \leq \frac{\partial \phi_2}{\partial t}(t_2, x_2). \tag{5}$$

2) Using a standard truncation argument together with [Locality], we get the same inequalities for every $\Phi_1, \Phi_2 \in \mathcal{D}(\mathbb{R}^N)$ such that $\Phi_1 \equiv \phi_1(t_1, \cdot)$ (respectively $\Phi_2 \equiv \phi_2(t_2, \cdot)$) on some neighbourhood of x_1 (respectively x_2).

3) Using [Regularity] with $f = 0, g = \Phi_i$ and $\lambda = 1$, we get, for $i = 1, 2$

$$|\delta_t[0](x_i) - \delta_t[\Phi_i](x_i)| \leq m_{x_i, \Phi_i}(1) \tag{6}$$

we deduce from (6), together with (4) and (5) that

$$\underline{A}[0](x_1) \geq \frac{\partial \phi_1}{\partial t}(t_1, x_1) - m_{x_1, \phi_1}(1) = C_1 \tag{7}$$

and

$$\overline{A}[0](x_2) \leq \frac{\partial \phi_2}{\partial t}(t_2, x_2) + m_{x_2, \phi_2}(1) = C_2. \tag{8}$$

4) Writing [Translation] for $f \equiv 0$ gives for any $t > 0$ and $y \in \mathbb{R}^N$

$$|\delta_t[0](x_i + y) - \delta_t[0](x_i)| \leq n(x_i, y) \quad \text{for } i = 1, 2$$

and together with inequalities (7) and (8) this implies that, for any $x \in \mathbb{R}^N, \overline{A}[0](x)$ and $\underline{A}[0](x)$ are finite.

5) We consider now $g \in \mathcal{D}(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. Using [Regularity] as in step 3) and the previous result, we get a bound for $\overline{A}[g](x)$ and $\underline{A}[g](x)$. We conclude that the same property holds for $g \in C_X^\infty(\mathbb{R}^N)$ using again [Locality] together with a standard truncation arguments. \square

Now we turn to further properties of \overline{A} and \underline{A} .

LEMMA 3.3 (Structure of the upper and lower infinitesimal generator). – *There exist two functions $\underline{F}, \overline{F} \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N))$ satisfying (2) such that, for every $f \in C_X^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$,*

$$\overline{A}[f](x) = -\overline{F}(x, f(x), Df(x), D^2f(x))$$

and

$$\underline{A}[f](x) = -\underline{F}(x, f(x), Df(x), D^2f(x)).$$

We postpone the proof of this lemma to the end of the present section and observe that in view of them, the notion of viscosity solution is well-defined for the parabolic problems associated to \overline{F} and \underline{F} . The following lemma is then nothing that a direct consequence of Lemma 3.2 together with the definition of continuous viscosity solutions.

LEMMA 3.4 (Semigroup, infinitesimal generators and parabolic equations). – *Let $u_0 \in X$. Then the function $u(t, x) := T_t[u_0](x)$ is a continuous viscosity solution of*

$$\frac{\partial u}{\partial t} + \underline{F}(x, u, Du, D^2u) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N$$

and

$$\frac{\partial u}{\partial t} + \overline{F}(x, u, Du, D^2u) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^N.$$

We have now, in order to complete the proof of Theorem 3.1, to show that $\overline{F} = \underline{F}$.

LEMMA 3.5 (Existence of the infinitesimal generator). – *For any $(x, r, p, M) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$, we have*

$$\underline{F}(x, r, p, M) = \overline{F}(x, r, p, M).$$

In particular, if we set $F := \overline{F} = \underline{F}$ then, for any $f \in C_X^\infty(\mathbb{R}^N)$, we have

$$\lim_{t \rightarrow 0^+} \frac{T_t[f](x) - f(x)}{t} = -F(x, f(x), Df(x), D^2f(x)).$$

The proof of Theorem 3.1 is indeed complete since it shows that there exists a function $F := \underline{F} = \overline{F}$ defined on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ such that (3) holds for any $f \in C_X^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. The function F is continuous and elliptic since, by Lemma 3.3, \underline{F} and \overline{F} are continuous and elliptic and from Lemma 3.4, the function $u(t, x) := T_t[u_0](x)$ is a continuous viscosity solution of (1) for every initial data $u_0 \in X$.

It remains to prove Lemmas 3.5 and 3.3.

Proof of Lemma 3.5. – To any $(x, r, p, M) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$, we associate the function $f_{x,r,p,M}$ defined for $y \in \mathbb{R}^N$ by

$$f_{x,r,p,M}(y) = \left(r + \langle p, y - x \rangle + \frac{1}{2} \langle M \cdot (y - x), (y - x) \rangle \right) v(y - x), \quad (9)$$

where ν is a fixed function in $\mathcal{D}(\mathbb{R}^N)$ with a compact support in $B(0, 1)$ such that $0 \leq \nu \leq 1$ and $\nu \equiv 1$ on $B(0, 1/2)$.

In fact, because of Lemma 3.3, we have to prove that

$$\limsup_{h \rightarrow 0^+} \frac{T_h[f](x) - f(x)}{h} \leq -\underline{F}(x, f(x), p, M),$$

where $f = f_{x,r,p,M}$. We set $u(t, y) := T_t[f][y]$ and proceed in several steps.

1) We introduce the family of functions $(W_{\eta,\delta})_{\eta,\delta>0}$ defined in $(0, \infty) \times \mathbb{R}^N$ by

$$W_{\eta,\delta}(t, y) = f(y) + \eta|y - x|^2 + t[\underline{F}(x, f(x), p, M) + \delta]$$

and claim that for every $\delta > 0$, there exists $\eta, T, r > 0$ such that

$$u(t, y) \leq W_{\eta,\delta}(t, y) \quad \text{on } [0, T] \times \overline{B}(x, r) = \overline{\Omega}_{T,r}. \tag{10}$$

To prove our claim, we set $\phi_{\eta,\delta} = u - W_{\eta,\delta}$ and

$$M_{\eta,\delta,T,r} = \sup_{\overline{\Omega}_{T,r}} \phi_{\eta,\delta}.$$

We have to show that $M_{\eta,\delta,T,r} \leq 0$ for a suitable choice of parameters.

2) To do so, we first remark that, since u and \underline{F} are continuous, for every $\delta > 0$, there exists $\eta, T, r > 0$ such that

$$\frac{\partial W_{\eta,\delta}}{\partial t} > -\underline{F}(y, u(t, y), D W_{\eta,\delta}, D^2 W_{\eta,\delta}) \quad \text{on } \overline{\Omega}_{T,r}. \tag{11}$$

3) It is clear that (10) holds on $\{0\} \times B(x_0, r)$ for every $\eta, r > 0$ with a strict inequality for $x \neq x_0$ and thus on $\{0\} \times \partial B(x_0, r)$. Using the continuity of u and the previous observation, we conclude that for every η, r there exists $T(\eta, r) > 0$ such that (10) holds on the lateral boundary $[0, T(\eta, r)] \times \partial B(x_0, r)$.

4) We choose the parameter η, T, r according to point 2) and 3) above and we assume by contradiction that $M_{\eta,\delta,T,r} > 0$. Then necessarily, this maximum is achieved at an interior point $(\bar{t}, \bar{x}) \in \Omega_{T,r}$ or for $\bar{t} = T$. Since u is a viscosity solution of the initial value problem (1) with $F = \underline{F}$ and since $W_{\eta,\delta} \in C^2([0, +\infty) \times \mathbb{R}^N)$, it follows, by definition, that

$$\frac{\partial W_{\eta,\delta}}{\partial t}(\bar{t}, \bar{y}) \leq -\underline{F}(\bar{y}, u(\bar{t}, \bar{y}), D_{\eta,\delta}(\bar{t}, \bar{y}), D^2 W_{\eta,\delta}(\bar{t}, \bar{y}))$$

even if $\bar{t} = T$ (see [3]). But, this inequality contradicts the property (11) above and therefore (10) holds.

5) We conclude by writing (10) at $y = x$. For t sufficiently small

$$\frac{T_t[f](x) - f(x)}{t} \leq [-\underline{F}(x, f(x), Df(x), D^2 f(x)) + \delta].$$

Taking the lim sup for $t \rightarrow 0^+$ and then letting $\delta \rightarrow 0^+$, we complete the proof of Lemma 3.5 and also the proof of Theorem 3.1. \square

We conclude this section by giving the *proof of Lemma 3.3*. We provide the proof for \overline{A} , the proof for \underline{A} being essentially the same with straightforward adaptations. We follow the idea of [1].

1) To obtain that $\overline{A}[f](x) = \overline{F}(x, f(x), Df(x), D^2f(x))$, we take $x \in \mathbb{R}^N$ and consider two functions $f, g \in C_X^\infty(\mathbb{R}^N)$ with $D^\alpha f(x) = D^\alpha g(x)$ for $|\alpha| \leq 2$. We introduce a function f_ε by setting

$$f_\varepsilon(y) = f(y) + \varepsilon|y - x|^2 \cdot v(y - x)$$

recalling that v is the smooth truncation defined in in the proof of Lemma 3.5.

By Taylor’s formula, f_ε is greater than g on $B(x, r_\varepsilon)$ for a suitable r_ε . But in order to use [Monotonicity], we need an inequality in the whole space. To this end, we use the function $v_\varepsilon(y) = v((y - x)/r_\varepsilon)$ and define

$$\overline{f}_\varepsilon = f_\varepsilon v_\varepsilon, \quad \overline{g}_\varepsilon = g v_\varepsilon.$$

We clearly have

$$\overline{f}_\varepsilon \geq \overline{g}_\varepsilon$$

and $\overline{f}_\varepsilon, \overline{g}_\varepsilon \in C_X^\infty(\mathbb{R}^N)$ by (H1).

2) Applying [Monotonicity] to the previous inequality we get

$$T_t(\overline{f}_\varepsilon) \geq T_t(\overline{g}_\varepsilon).$$

Since $\overline{f}_\varepsilon(x) = f_\varepsilon(x) = f(x) = g(x)$, we have

$$T_t(\overline{f}_\varepsilon)(x) - \overline{f}_\varepsilon(x) \geq T_t(\overline{g}_\varepsilon)(x) - g(x).$$

Then, since $\overline{f}_\varepsilon \equiv f_\varepsilon$ and $g \equiv \overline{g}_\varepsilon$ on $B(x, r_\varepsilon/2)$, we can use [Locality] to obtain

$$T_t(f_\varepsilon)(x) - f_\varepsilon(x) + o(\varepsilon, t) \geq T_t(g)(x) - g(x).$$

Using now [Regularity] with $f \in C_X^\infty(\mathbb{R}^N)$, $w = |\cdot - x|^2 v(\cdot - x) \in \mathcal{D}(\mathbb{R}^N)$ and $\lambda = \varepsilon$ we get

$$T_t(f)(x) - f(x) + o(\varepsilon, t) \geq T_t(g)(x) - g(x) - m_{x, f, w}(\varepsilon)t.$$

Dividing by t and taking the lim sup for $t \rightarrow 0^+$, and then letting $\varepsilon \rightarrow 0$ we obtain

$$\overline{A}[f](x) \geq \overline{A}[g](x).$$

Since the previous computations are symmetric in f and g , we get the equality.

3) Here and therein (x, r, p, M) will ever denote an arbitrary element of $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ and $f_{x,r,p,M} \in \mathcal{D}(\mathbb{R}^N)$ the function defined by (9). We define the function \overline{F} on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ by setting

$$-\overline{F}(x, r, p, M) = \overline{A}[f_{x,r,p,M}](x) \tag{12}$$

and thanks to the previous point we have for any $f \in C_X^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$

$$\overline{A}[f](x) = -\overline{F}(x, f(x), Df(x), D^2 f(x)).$$

The ellipticity of \overline{F} is then a direct consequence of [Monotonicity] together with the obvious following fact

$$N \geq M \Rightarrow f_{x,r,p,N} \geq f_{x,r,p,M}.$$

4) To complete the proof of the proposition, we need to see that the function \overline{F} is continuous.

Let $(x, r_0, p_0, M_0) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ and set $f_0 = f_{x_0,r_0,p_0,M_0}$. We have, using [Translation], that

$$\begin{aligned} |T_t[f_0](x_0) - T_t[\tau_{-x} \cdot f_0](x_0 + x)| &= |\tau_{-x} \cdot T_t[f_0](x_0 + x) - T_t[\tau_{-x} f_0](x_0 + x)| \\ &\leq n_{x_0, f_0}(|x|)t. \end{aligned}$$

and thus,

$$|\overline{F}(x_0, r_0, p_0, M_0) - \overline{F}(x_0 + x, r_0, p_0, M_0)| \leq n_{x_0, f_0}(|x|). \tag{13}$$

Moreover, for any $R > 0$ and any positive sequence d , we can rewrite [Regularity] under the form

$$|\overline{A}[f_0 + \varepsilon g](x_0 + x) - \overline{A}[f_0](x_0 + x)| \leq m_{x_0, R, f_0, d}(\varepsilon)$$

for any $(x, g) \in B(0, R) \times Q_d$. We apply the previous inequality with

$$g_\varepsilon = f_{x_0+x, r/\varepsilon, p/\varepsilon, M/\varepsilon}$$

noticing that the g_ε are in a same Q_d for $|r|, |p|, |M| \leq \varepsilon$ and get that there exists an $R > 0$ such that

$$|\overline{F}(x_0 + x, r_0, p_0, M_0) - \overline{F}(x_0 + x, r_0 + r, p_0 + p, M_0 + M)| \leq m_{x_0, R, f_0, d}(\varepsilon) \tag{14}$$

for any $|x|, |r|, |p|, |M| \leq \varepsilon$.

From (13) and (14), we complete the proof of the continuity. □

Remark 3.1. – We want to point out that our proof works if [Regularity] is stated only for functions f appearing in [Continuity] or which belong to $\mathcal{D}(\mathbb{R}^N)$ instead of $f \in C_X^\infty(\mathbb{R}^N)$. In particular, if X is a subspace of bounded functions as in [1], it suffices to write the assumption only for $f \in C_b^\infty(\mathbb{R}^N)$.

4. Consequences of additional properties for the semigroup

We start with elementary facts. The proofs of these facts are straightforward using Theorem 3.1, so we will omit them.

PROPOSITION 4.1. – *Let F be the function which appears in Theorem 3.1.*

- (1) If T_t is linear for any t , there exists continuous functions a, b, c from \mathbb{R}^N to $\mathcal{S}^+(N), \mathbb{R}^N$ and \mathbb{R} respectively, such that

$$F(x, r, p, M) = -Tr[a(x)M] + \langle b(x), p \rangle + c(x)r,$$

where $\mathcal{S}^+(N)$ denote the space of the positive symmetric matrices.

- (2) If $T_t[f + C] = T_t[f] + C$ for any $f \in X$ and $C \in \mathbb{R}$ then F is independent of r .
 (3) If $\tau_x[T_t(f)] = T_t[\tau_x(f)]$ for all $(t, x) \in [0, +\infty) \times \mathbb{R}^N$ and $f \in \mathcal{D}(\mathbb{R}^N)$ then F is independent of x .

It is well-known that, under suitable assumptions, first-order Hamilton–Jacobi equations satisfy properties of “finite speed of propagation” or “domain of dependence”. Such properties were first proved by M.G. Crandall and P.-L. Lions in [5] (see also I. Ishii [9] or O. Ley [10] for a different proof).

We address now the question of the converse property: is a semigroup satisfying some domain of dependence-type property associated to a first-order equation? The answer is yes and a first result in this direction was first proved by P.-L. Lions in [12] under slightly stronger but less numerous assumptions than here.

We introduce the following formulation of a domain of dependence property for $(T_t)_{t \geq 0}$

[Strong-Locality]. – For any $f, g \in X$ and any compact subset $K \subset \mathbb{R}^N$ there exists a positive constant $L_{K,f,g}$ and $0 < s < 2$ such that

$$f \leq g \text{ in } B(x_0, R) \Rightarrow T_t[f](x_0) \leq T_t[g](x_0) \text{ for } t \leq L_{K,f,g} R^s$$

for any $x_0 \in K$ and any $R > 0$.

THEOREM 4.1. – Let $(T_t)_{t \geq 0}$ satisfy the assumptions of Theorem 3.1 where [Locality] and [Monotonicity] are replaced by [Strong-Locality]. There exists a function $F \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N)$ such that, for any $u_0 \in X$, the function $(t, x) \mapsto T_t[u_0](x)$ is a viscosity solution of the Hamilton–Jacobi Equation associated to F , i.e.

$$\frac{\partial u}{\partial t} + F(x, u, Du) = 0 \text{ in } (0, +\infty) \times \mathbb{R}^N. \tag{15}$$

Proof of Theorem 4.1. – First it is clear that [Strong-Locality] implies [Monotonicity] and [Locality] and thus Theorem 3.1 holds. Therefore we have only to prove that

$$F(x, r, p, A) = F(x, r, p, B) \text{ for any } (x, r, p, A, B) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)^2.$$

To this end, we consider the functions $f_{x,r,p,A}$ and $f_{x,r,p,B}$ and $1/2 > \eta > 0$. We have

$$f_{x,r,p,A} \leq f_{x,r,p,B} + \frac{|B - A|}{2} \eta^2 \nu(\cdot - x) \text{ in } B(x, \eta)$$

for η sufficiently small, where ν is the smooth truncation function defined in the proof of Lemma 3.5.

Using [Strong-Locality] and [Regularity], this implies

$$T_t[f_{x,r,p,A}](x) \leq T_t[f_{x,r,p,B}](x) + \frac{|B - A|}{2} \eta^2 + m_{x,r,p,A,B}(\eta^2)t \quad \text{for } t \leq L\eta^s$$

with L independent of η . Subtracting r , we use this inequality for $t = L\eta^s$, divide by $L\eta^s$ and finally we let $\eta \rightarrow 0^+$. We obtain

$$-F(x, r, p, A) \leq -F(x, r, p, B)$$

and we conclude by the symmetry of A and B in the previous arguments. \square

Remark 4.1. – More regularity on F occurs with respect to p if we add more restrictions on the term LR^s in [Strong-Locality]. For example, if L is independent of (x, f, g) and $s = 1$, the nonlinearity F is Lipschitz continuous in p uniformly with respect to the other variables.

5. On the uniquenesses for the associated p.d.e

We will show in this section that uniqueness may fail in general for viscosity solutions of the partial differential equation associated to a semigroup which satisfies only assumptions of Theorem 3.1 or Theorem 4.1 even if $X = BUC(\mathbb{R}^N)$.

Our example is entirely taken from [5] where it was used by Crandall and Lions as a counter-example to uniqueness for viscosity solution of a transport equation when the natural assumptions are not satisfied. This is related to the non-uniqueness for flows of the associated dynamic studied by Beck in [4].

LEMMA 5.1. – *There exist two different continuously differentiable homeomorphism f, h on \mathbb{R} such that*

$$f'(f^{-1}(x)) = h'(h^{-1}(x)) \quad \text{for any } x \in \mathbb{R}. \tag{16}$$

In addition, one can impose that f, f^{-1}, h, h^{-1} are uniformly continuous on \mathbb{R} and f', f'', f''' bounded.

Proof. – We follow the ideas of [4] and just introduce slightly specifications in order to ensure that the assumptions of Section 2 hold.

1) We consider a Cantor set $K \subset [0, 1]$ with a strictly positive Lebesgue measure and a smooth function g such that $0 \leq g \leq 1$, $g(x) = 0$ if and only if $x \in K$, g', g'' bounded on \mathbb{R} and $g \geq \eta > 0$ on $(-\infty, -1] \cup [2, +\infty)$. Then we define a function f by setting

$$f(x) = \int_0^x g(\tau) d\tau.$$

Clearly, f is a Lipschitz continuous differentiable homeomorphism from \mathbb{R} to \mathbb{R} . Moreover the last requirement on g ensures that f^{-1} is Lipschitz continuous on

$(-\infty, f^{-1}(-1)] \cup [f^{-1}(2), +\infty)$ and thus is uniformly continuous on \mathbb{R} since it is continuous on \mathbb{R} .

2) We construct an other differentiable homeomorphism h by setting successively

$$\alpha(x) = x + \mu(K \cap [0, x]),$$

where μ denote the Lebesgue measure on \mathbb{R} and then

$$h = f \circ \alpha^{-1}.$$

The function α is strictly increasing continuous and thus is an homeomorphism from \mathbb{R} to \mathbb{R} . Moreover it is a Lipschitz continuous function on \mathbb{R} . Finally we have the following inequality

$$\alpha(y_2) - \alpha(y_1) \geq y_2 - y_1 \quad \text{for any } y_2 \geq y_1 \tag{17}$$

which ensures that α^{-1} is also Lipschitz continuous.

It follows finally that h and h^{-1} are also uniformly continuous.

3) We check the property (16).

(a) We first consider $x \in \mathbb{R}$ such that $f^{-1}(x) \notin K$. We want to check that h is differentiable at $h^{-1}(x)$. But, since $f^{-1}(x) \notin K$, there is a neighborhood of $f^{-1}(x)$ on which α is nothing but $Id_{\mathbb{R}} + c$ for some constant c , and therefore α is differentiable at $f^{-1}(x)$ with derivative 1.

We deduce from this fact that α^{-1} is differentiable at $\alpha(f^{-1}(x)) = h^{-1}(x)$ with derivative 1 and then that h is differentiable at $h^{-1}(x)$ with

$$h'(h^{-1}(x)) = 1 \times f'(\alpha^{-1}(h^{-1}(x))) = f'(f^{-1}(x)).$$

(b) Let $x \in \mathbb{R}$ such that $y = f^{-1}(x) \in K$. We have

$$\begin{aligned} h(h^{-1}(x)) - h(z) &= f[\alpha^{-1}(h^{-1}(x))] - f[\alpha^{-1}(z)] \\ &\leq \sup_{I_z} |f'| \cdot K_{\alpha^{-1}} \cdot |(h^{-1}(x)) - z|, \end{aligned} \tag{18}$$

where $K_{\alpha^{-1}}$ denote the Lipschitz constant for α^{-1} and $I_X = [\alpha^{-1}(h^{-1}(x)), \alpha^{-1}(z)]$. Since f is C^1 and $f'(\alpha^{-1}(h^{-1}(x))) = f'(f^{-1}(x)) = 0$ recalling $f^{-1}(x) \in K$ we obtain dividing by $(h^{-1}(x)) - X$ and letting $X \rightarrow h^{-1}(x)$ that h is differentiable at $h^{-1}(x)$ with

$$h'(h^{-1}(x)) = 0 = f'(f^{-1}(x)). \quad \square$$

Following [5] we now define two flows by setting $Y_f(t, x) = f(t + f^{-1}(x))$ and $Y_h(t, x) = h(t + h^{-1}(x))$. There flows are distinct and provide two semigroups on $BUC(\mathbb{R})$ by setting

$$T_t^f[u_0](x) = u_0(Y_f(t, x)) \quad (\text{respectively } T_t^h[u_0](x) = u_0(Y_h(t, x)))$$

for every $u_0 \in BUC(\mathbb{R})$.

PROPOSITION 5.1. – *The linear semigroup $(T_t^f)_{t \geq 0}$ satisfies all the assumptions Theorem 4.1 with $X = BUC(\mathbb{R})$. Moreover T_t^f and T_t^h give both viscosity solutions of*

$$\frac{\partial u}{\partial t} + b(x) \cdot Du(t, x) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}, \tag{19}$$

where $b(x) = f'(f^{-1}(x)) = h'(h^{-1}(x))$.

Sketch of proof of Proposition 5.1. – It is easy to check that $(T_t^f)_{t \geq 0}$ (respectively $(T_t^h)_{t \geq 0}$) is defined on $BUC(\mathbb{R})$ using and that f^{-1} (respectively h^{-1}) is uniformly continuous. Then, straightforward computations involving the linearity of the semigroups and the differentiability of f and h show that [Continuity] [Strong-Locality] and [Regularity] hold for $(T_t^f)_{t \geq 0}$ and $(T_t^h)_{t \geq 0}$.

Moreover $(T_t^f)_{t \geq 0}$ satisfies [Translation]. Indeed, for $u_0 \in \mathcal{D}(\mathbb{R})$, we write

$$\begin{aligned} & |\tau_y \cdot T_t^f[u_0](x) - T_t^f[\tau_y \cdot u_0](x)| \\ &= |u_0[f(-t + f^{-1}(x + y))] - u_0[f(-t + f^{-1}(x)) + y]| \\ &\leq \|Du_0\|_\infty \cdot |f(-t + f^{-1}(x + y)) - f(-t + f^{-1}(x)) - y|. \end{aligned}$$

But, using a Taylor’s formula, we have

$$\begin{aligned} & |f(-t + f^{-1}(x + y)) - f(-t + f^{-1}(x)) - y| \\ &\leq t|b(x + y) - b(x)| + t^2 \int_0^1 \frac{(1-h)^2}{2} |f''(f^{-1}(x + y) - ht) - f''(f^{-1}(x) - ht)| dh \end{aligned}$$

and we get that [Translation] holds for $(T_t^f)_{t \geq 0}$ using the boundedness of f''' together with the uniform continuity of $b = f' \circ f^{-1}$.

But it is worth mentioning that it is not clear that the semigroup $(T_t^h)_t$ satisfies the [Translation] property. Anyway, as in [5], $u_f(t, x) := T_t^f[u_0](x)$ and $u_h(t, x) := T_t^h[u_0](x)$ are both continuous viscosity solutions of (19), this fact being also a consequence of Theorem 3.1 only for u_f . \square

This example shows in particular that, in general, uniqueness may fail for viscosity solutions of the initial value problems deduced from semigroups satisfying only the assumptions of Section 2.

6. The case of two-parameters families

We give briefly the generalization of Theorem 3.1 for a two-parameters family $(T_{t,s})_{t \geq s \geq 0}$ from X into X . As in [1], we use adaptations of the assumptions used for the one-parameter case.

[Causality]. –

$$T_{t,r} = T_{t,s} \circ T_{s,r} \quad \text{for any } t \geq s \geq r \geq 0 \quad \text{and} \quad T_{t,t} = Id_X.$$

[Monotonicity]. – For any $f, g \in X, t \geq h \geq 0$,

$$f \geq g \Rightarrow T_{t,t-h}[f] \geq T_{t,t-h}[g].$$

[Continuity]. – For any $u_0 \in X, t \geq 0$, the function $u(h, x) := T_{t+h,t}[u_0](x)$ is continuous and for any $b \geq a \geq 0$, there exists $f_{u_0,a,b} \in C_X^\infty(\mathbb{R}^N)$ such that

$$|T_{t+h,t}[u_0]| \leq f_{t,u_0,a,b} \quad \text{for any } h \in [a, b].$$

[Locality]. – For any $f, g \in C_X^\infty(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$, if $f \equiv g$ in $B(x, R)$ for some $R > 0$ then

$$T_{t,t-h}[f](x) - T_{t,t-h}[g](x) = o(h) \quad \text{as } h \rightarrow 0^+.$$

[Regularity]. – For any sequence of positive numbers d , any compact subset $K \subset \mathbb{R}^N \times [0, +\infty)$ and every $f \in C_X^\infty(\mathbb{R}^N)$ there exists a positive function $m_{K,f,d}(\cdot)$ with $m_{K,d,f}(0^+) = 0$ such that

$$|T_{t,t-h}[f + \lambda g](x) - T_{t,t-h}[f](x) - \lambda g(x)| \leq m_{K,f,d}(\lambda)h$$

for every $((x, t), g) \in K \times Q_d, \lambda \geq 0, 0 \leq h \leq t$.

[Translation]. – For any compact subset $K \subset \mathbb{R}^N \times [0, +\infty)$ and any $f \in \mathcal{D}(\mathbb{R}^N)$ there exists a function $n_{K,f}(\cdot)$ with $n_{K,f}(0^+) = 0$ such that

$$|\tau_y \cdot T_{t,t-h}[f](x) - T_{t,t-h}(\tau_y \cdot f)(x)| \leq n_{K,f}(|y|)h$$

for every $((x, t), f) \in K \times Q_d, y \in \mathbb{R}^N, 0 \leq h \leq t$.

In addition, we need as in [1] the following property:

[Stability]. – For every sequence of positive numbers d and any compact subset $K \subset \mathbb{R}^N$, there exists a function $p_{K,d}(\cdot)$ with $p_{K,d}(0^+) = 0$ such that

$$|T_{t,t-h}[f](x) - T_{t',t'-h}[f](x)| \leq p_{K,d}(|t - t'|)h$$

for any $(x, f) \in K \times Q_d, t, t' \geq h \geq 0$.

We have then the:

THEOREM 6.1. – Let $(T_{t,s})_{t \geq s \geq 0}$ a family of mappings defined from X into X where $X \subset C(\mathbb{R}^N)$, satisfying [Causality], [Monotonicity], [Continuity], [Locality], [Regularity], [Translation] and [Stability] and X satisfies (H1)–(H3). There exists a continuous function F on $[0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S}(N)$ such that for any $f \in C_X^\infty(\mathbb{R}^N), x \in \mathbb{R}^N$

$$\lim_{h \rightarrow 0^+} \frac{T_{t,t-h}[f](x) - f(x)}{h} = A_t[f](x) = -F(t, x, f(x), Df(x), D^2 f(x)).$$

Moreover, the function F satisfies the ellipticity condition (2) and for any $u_0 \in X$, the function $u(t, x) := T_{t,0}[u_0](x)$ is a continuous viscosity solution of the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} + F(t, x, u, Du, D^2u) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{20}$$

Proof. – First, we observe that for every fixed $t \geq 0$ the mappings $(T_{t,t-h})_{t \geq h \geq 0}$ satisfy the same properties as $(T_h)_{h \geq 0}$ except [Causality]. Thus the proof is nothing else than a straightforward adaptation of those we have given in the one parameter case. Therefore we do not mention every details and just give the essential adjustments.

Again, we define two time-dependent mappings by

$$\limsup_{h \rightarrow 0^+} \frac{T_{t,t-h}[f](x) - f(x)}{h} = \limsup_{h \rightarrow 0^+} \delta_{t,t-h}[f](x) = \overline{A}_t[f](x)$$

and

$$\liminf_{h \rightarrow 0^+} \frac{T_{t,t-h}[f](x) - f(x)}{h} = \liminf_{h \rightarrow 0^+} \delta_{t,t-h}[f](x) = \underline{A}_t[f](x).$$

1) For Lemma 3.2, we have first to obtain the equivalent of (24). Using [Continuity], we construct a function $\tilde{\phi}$ as in Lemma 3.2 and write

$$T_{t_0-h,0}[u_0] \leq \tilde{\phi}(t_0 - h, \cdot) \quad \text{for } h < t_0.$$

Applying T_{t_0,t_0-h} , [Monotonicity] and [Causality], we get

$$T_{t_0,0}[u_0] - u(t_0, x_0) \leq T_{t_0,t_0-h}[\tilde{\phi}(t_0 - h, \cdot)] - \tilde{\phi}(t_0, x_0).$$

By the same computation, using [Regularity], we obtain

$$\begin{aligned} 0 &\leq T_{t_0,t_0-h}[\tilde{\phi}(t_0, \cdot)](x_0) - \tilde{\phi}(t_0, x_0) - h \left[\frac{\partial \tilde{\phi}}{\partial t}(t_0, x_0) + \varepsilon(h, x_0) \right] \\ &\quad + m \left(t_0, x_0, \tilde{\phi}(t_0, \cdot), \frac{\partial \tilde{\phi}}{\partial t}(t_0, \cdot) + \varepsilon(h, \cdot), h \right) h. \end{aligned} \tag{21}$$

This inequality provides the equivalent one of Lemma 3.2.

2) The proof of Lemma 3.1 is the same and in proof of Lemma 3.3, we just have to use [Stability] in order to obtain the time-continuity for \underline{F} and \overline{F} deduced from \underline{A} and \overline{A} .

3) For Lemma 3.4 we observe in addition that the same arguments show that if t_0 is fixed and $u_0 \in X$, $(h, x) \mapsto T_{t_0+h,t_0}[u_0](x)$ is a continuous viscosity solution of

$$\frac{\partial u}{\partial h} + \overline{F}(t_0 + h, x, u(h, x), Du(h, x), D^2u(h, x)) = 0 \tag{22}$$

and also of the same equation with \overline{F} replaced by \underline{F} .

4) In the proof of the Lemma 3.5, we fix $(t_0, x_0, r, p, M) \in [0, +\infty) \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}(N)$ and use the function

$$(h, x) \mapsto T_{t_0+h, t_0}[f](x),$$

where $f = f_{x_0, r, p, M}$ which gives a viscosity solution of (22). Using this fact, we obtain that for $h > 0$ small enough,

$$T_{t_0+h, t_0}[f](x_0) \leq f(x_0) + h[-\underline{E}(t_0, x_0, r, p, M) + \delta].$$

We then use [Stability] with $t = t_0$ and $t' = t_0 - h$ to get

$$T_{t_0, t_0-h}[f](x_0) - f(x_0) - h.p_{x_0, f}(h) \leq h[-\underline{E}(t_0, x_0, r, p, M) + \delta]$$

and we obtain the result dividing by h , taking the lim sup and letting δ go to 0. \square

Appendix A

We give here the proof of Lemma 3.2 we have postponed in Section 3.

Proof of Lemma 3.2. – We make the proof for $(a, p, M) \in \mathcal{P}^{2,+}[u](t_0, x_0)$, the one for $(a, p, M) \in \mathcal{P}^{2,-}[u](t_0, x_0)$ being the same with straightforward adaptations.

1) We first construct a suitable function. Since $(a, p, M) \in \mathcal{P}^{2,+}[u](t_0, x_0)$, by classical results, there exists $\phi \in C^\infty((0, +\infty) \times \mathbb{R}^N)$ such that $u - \phi$ has a local maximum point at (t_0, x_0) with $\phi(t_0, x_0) = u(t_0, x_0)$, $D\phi(t_0, x_0) = p$, $D^2\phi(t_0, x_0) = M$ and $\frac{\partial \phi}{\partial t}(t_0, x_0) = a$ (see for example [3]). There exist $\alpha, r > 0$ such that

$$u \leq \phi \quad \text{on } [t_0 - \alpha, t_0 + \alpha] \times \overline{B(x_0, r)}.$$

Recalling that $u(t, x) = T_t[u_0](x)$, [Continuity] provides a function $f_{u_0, t_0, \alpha}$ such that

$$u(t, x) \leq f_{u_0, t_0, \alpha}(x) \quad \text{for } t \in [t_0 - \alpha, t_0 + \alpha], x \in \mathbb{R}^N.$$

Taking v as in proof of Lemma 3.5, we construct $v_r(x) := v((x - x_0)/r)$, and get

$$u \leq \phi v_r + (1 - v_r) f_{u_0, t_0, \alpha} = \tilde{\phi} \quad \text{on } [t_0 - \alpha, t_0 + \alpha] \times \mathbb{R}^N. \tag{23}$$

Since X is assumed to be a vector space containing $\mathcal{D}(\mathbb{R}^N)$, $\tilde{\phi}(t, \cdot) \in C_X^\infty(\mathbb{R}^N)$ for $t \in [t_0 - \alpha, t_0 + \alpha]$. Moreover, $\tilde{\phi} \equiv \phi$ on $[t_0 - \alpha, t_0 + \alpha] \times B(x_0, r/2)$. Thus

$$\left(\tilde{\phi}(t_0, x_0), \frac{\partial \tilde{\phi}}{\partial t}(t_0, x_0), D\tilde{\phi}(t_0, x_0), D^2\tilde{\phi}(t_0, x_0) \right) = (u(t_0, x_0), a, p, M).$$

2) We rewrite (23), for h sufficiently small, under the form

$$T_{t_0-h}[u_0] \leq \tilde{\phi}(t_0 - h, \cdot).$$

Then using [Monotonicity] and subtracting $u(t_0, x_0) = \tilde{\phi}(t_0, x_0)$ we obtain, for $h > 0$ sufficiently small

$$T_h [T_{t_0-h}(u_0)] - u(t_0, x_0) \leq T_h [\tilde{\phi}(t_0 - h, \cdot)] - \tilde{\phi}(t_0, x_0).$$

Using now [Causality] and applying the inequality at x_0 , this yields

$$0 \leq T_h [\tilde{\phi}(t_0 - h, \cdot)](x_0) - \tilde{\phi}(t_0, x_0).$$

Then by Taylor's formula for $\tilde{\phi}(t_0 - \cdot, x)$

$$0 \leq T_h \left[\tilde{\phi}(t_0, \cdot) - h \frac{\partial \tilde{\phi}}{\partial t}(t_0, \cdot) - h\varepsilon(h, \cdot) \right](x_0) - \tilde{\phi}(t_0, x_0).$$

We notice that, by construction of $\tilde{\phi}$, $w(h, \cdot) = \frac{\partial \tilde{\phi}}{\partial t}(t_0, \cdot) + \varepsilon(h, \cdot) \in \mathcal{D}(\mathbb{R}^N)$ and apply [Regularity] in the last inequality. We get

$$0 \leq T_h [\tilde{\phi}(t_0, \cdot)](x_0) - \tilde{\phi}(t_0, x_0) - h \left[\frac{\partial \tilde{\phi}}{\partial t}(t_0, x_0) + \varepsilon(h, x_0) \right] + m_{x_0, \tilde{\phi}(t_0, \cdot), w(h, \cdot)}(h)h. \quad (24)$$

Remarking that the functions $(w(h, \cdot))_{t_0 \geq h \geq 0}$ are in a same Q_d independent of h for some sequence d , we deduce from (24) that

$$\limsup_{h \rightarrow 0^+} \delta_h [\tilde{\phi}(t_0, \cdot)](x_0) \geq \liminf_{h \rightarrow 0^+} \delta_h [\tilde{\phi}(t_0, \cdot)](x_0) \geq \frac{\partial \tilde{\phi}}{\partial t}(t_0, x_0).$$

And the proof is complete. \square

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