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EXOTIC SOLUTIONS OF THE CONFORMAL SCALAR CURVATURE EQUATION IN $\mathbb{R}^n$

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ABSTRACT. – We construct global exotic solutions of the conformal scalar curvature equation $\Delta u + [n(n-2)/4]Ku^{(n+2)/(n-2)} = 0$ in $\mathbb{R}^n$, with $K(x)$ approaching 1 near infinity in order as close to the critical exponent as possible. © 2001 Éditions scientifiques et médicales Elsevier SAS

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RÉSUMÉ. – Nous construisons des solutions globales exotiques de l'équation courbure scalaire conforme $\Delta u + [n(n-2)/4]Ku^{(n+2)/(n-2)} = 0$ dans $\mathbb{R}^n$, avec $K(x) \to 1$ quand $|x| \to \infty$. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

We consider a special class of positive solutions of the conformal scalar curvature equation

$$\Delta u + \frac{n(n-2)}{4} K u^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}^n. \tag{1.1}$$

Here Δ is the standard Laplacian on $\mathbb{R}^n$ equipped with Euclidean metric g_o, K a smooth function on $\mathbb{R}^n$, and $n \geqslant 3$ an integer. The solutions we construct breach a rather natural lower bound and have peculiar asymptotic property.

Eq. (1.1) is studied extensively by many authors in connection with the prescribed scalar curvature problem on a Riemannian manifold in general and on $\mathbb{R}^n$ and S^n in particular (on S^2, the Nirenberg problem; cf. [1,3–5,9,12,14,15,17,20,21,23,24,26] and the references within). As in the case of the Yamabe problem, recent studies indicate that the case when K is strictly positive affords many interesting and subtle developments.

Assume that K is bounded between two positive constants in $\mathbb{R}^n$. An important feature of Eq. (1.1) is the asymptotic behavior of $u(x)$ for large $|x|$ (cf. [2,5–8,10,12,16,18,19,

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22]). It is simpler to classify with the help of the Kelvin transformation:

$$y = \frac{x}{|x|^2} \quad \text{and} \quad w(y) := |y|^{2-n} u(y/|y|^2) \quad \text{for } x, y \in \mathbb{R}^n \setminus \{0\}. \tag{1.2}$$

From (1.2), w satisfies the equation

$$\Delta w(y) + \frac{n(n-2)}{4} \bar{K}(y) w^{\frac{n+2}{n-2}}(y) = 0 \quad \text{for } y \in \mathbb{R}^n \setminus \{0\}, \tag{1.3}$$

where $\bar{K}(y) := K(y/|y|^2)$ for $y \neq 0$ (see, for instance, [18]). w (and u) is said to have fast decay if w has a removable singularity at the origin. Otherwise, it is called a singular solution. In order to have reasonable control on the geometric and analytic behavior of singular solutions, it is crucial to obtain the upper bound or *slow decay*

$$w(y) \leqslant C_1 |y|^{-(n-2)/2} \quad \text{as } y \to 0, \quad \text{i.e.,} \quad u(x) \leqslant C_1 |x|^{-(n-2)/2} \quad \text{for } |x| \gg 1, \tag{1.4}$$

where C_1 is a positive constant. The question on slow decay is discussed in depth in [2, 5–8,16,18,19,22] (cf. also [27]; note that our definition of slow decay is slightly different from the one in [5] and [8]). Guided by the case when K is equal to a positive constant outside a compact subset of $\mathbb{R}^n$ (see [2,16]), it is natural to ask whether a singular positive solution u with slow decay also satisfies the lower bound

$$w(y) \geqslant C_2 |y|^{-(n-2)/2} \quad \text{as } y \to 0, \quad \text{i.e.,} \quad u(x) \geqslant C_2 |x|^{-(n-2)/2} \quad \text{for } |x| \gg 1, \tag{1.5}$$

where C_2 is a positive constant. If the lower bound holds, then the conformal metric $u^{4/(n-2)} g_o$ on $\mathbb{R}^n$ is complete and has bounded (sectional) curvature [8]. The radial Pohozaev number is an essential invariant in the study of equation (1.1) and is given by

$$P(u) := \lim_{R \to \infty} \int\limits_{B_o(R)} \left[x \cdot \nabla K(x)\right] u^{2n/(n-2)}(x)\, \mathrm{d}x, \tag{1.6}$$

provided the limit exists. Here $B_o(R)$ is the open ball with center at the origin and radius equal to $R > 0$. The following result is shown by Chen and Lin in [6] and [8], mindful of the slightly different notations we use.

THEOREM 1.7 (Chen-Lin). – *Let u be a positive smooth solution of Eq.* (1.1). *Assume that* $\lim_{|x| \to \infty} K(x)$ *exists and is positive, and there exist positive constants $l \geqslant (n-2)/2$ and C such that*

$$C^{-1} |x|^{-(l+1)} \leqslant |\nabla K(x)| \leqslant C |x|^{-(l+1)} \quad \textit{for all } |x| \gg 1.$$

Then u has slow decay and $P(u)$ exists and is non-positive. u has fast decay if and only if $P(u) = 0$ (*the Kazdan–Warner condition*). *Furthermore, if u is a singular solution, then we also have the lower bound $u(x) \geqslant C_2 |x|^{-(n-2)/2}$ for all $|x| \gg 1$ and for some positive constant C_2.*

More generally, under the condition that $\lim_{|x|\to\infty} K(x)$ exists and is positive, and $|\nabla K|$ is bounded in $\mathbb{R}^n$, for a positive smooth solution u of Eq. (1.1) with slow decay, we show in [10] (cf. also [5,8]) that $P(u) \leqslant 0$ if $P(u)$ exists. Moreover, $P(u) = 0$ if and only if

$$\liminf_{|x|\to\infty} |x|^{(n-2)/2} u(x) = 0. \tag{1.8}$$

In the latter case, the assumption on K is not strong enough to allow us to deduce that u has fast decay.

DEFINITION 1.9. – *We call a singular positive solution u of* Eq. (1.1) *with slow decay an* exotic solution *if* (1.8) *holds for u. That is, we* cannot *find a positive constant C_2 such that $u(x) \geqslant C_2 |x|^{-(n-2)/2}$ for all $|x| \gg 1$.*

Then it is necessary that $P(u) = 0$ if $P(u)$ exists. Exotic solutions are rather peculiar because from $P(u) = 0$ one would expect u to have fast decay. Instead, they decay slowly and the conformal metric $u^{4/(n-2)} g_o$ remains to be complete, but the (sectional) curvature is unbounded [8]. Theorem 1.7 leads to the observation that there are no exotic solutions if $|\nabla K|$ decays to zero near infinity fast enough.

(Local) Exotic solutions are first found by Chen and Lin in [8]. By a scaling and the Kelvin transform, we may consider the equation

$$\Delta u + \bar{K} u^{\frac{n+2}{n-2}} = 0 \quad \text{in } B_o(1) \setminus \{0\}. \tag{1.10}$$

Assume that $\bar{K}$ is radial and non-increasing in $(0, 1]$, and is given by

$$\bar{K}(r) = 1 - Ar^l + R(r) \tag{1.11}$$

for $r > 0$ close to zero. Here $A > 0$ and $0 < l < (n-2)/2$ are constants, and $R(r) = \mathrm{o}(r^l)$ and $R'(r) = \mathrm{o}(r^{l-1})$ for $r > 0$ close to zero. Given a positive number α, let $u(r, \alpha)$ be the unique solution of the initial value problem

$$\begin{cases} u''(r) + \frac{n-1}{r} u'(r) + \bar{K}(r) u^{\frac{n+2}{n-2}}(r) = 0, \\ u(0) = \alpha \quad \text{and} \quad u'(0) = 0. \end{cases}$$

Chen and Lin [8] show elegantly that there exists a sequence $\alpha_i \to \infty$ such that $u(r, \alpha_i)$ converges to an (local) exotic C^2-solution of Eq. (1.10) in $B_o(1) \setminus \{0\}$. Subsequently, Lin [22] obtains characterizations of exotic solutions in terms of the asymptotic expansion of $\bar{K}$ near the origin.

The exponent $(n-2)/2$ is found to be critical. For $l \geqslant (n-2)/2$, Theorem 1.7 shows that there are no exotic solutions of Eq. (1.1). In this paper we construct global exotic solutions of Eq. (1.1) in $\mathbb{R}^n$. As described above, in [8], an abstract existence argument is used to show the existence of (local) exotic solutions. Our construction is explicit by gluing the Delaunay–Fowler-type solutions. Given any positive number δ, we show that there is an exotic solution of Eq. (1.1) with $|K - 1| \leqslant \delta^2$ in $\mathbb{R}^n$. Moreover, with regard to the critical exponent $(n-2)/2$, we show that, given any positive function $\varphi(r)$ defined for $r \gg 1$ such that

$$r^{(n-2)/2}\varphi(r) \text{ is non-decreasing for } r \gg 1 \text{ and } \lim_{r\to\infty} r^{(n-2)/2}\varphi(r) = \infty, \tag{1.12}$$

(for example, $\varphi(r) = r^{-(n-2)/2}\ln(\ln r)$ for $r \gg 1$), we construct an exotic solution of Eq. (1.1) with

$$|K(x) - 1| \leqslant C_3\varphi(|x|) \quad \text{for all } |x| \gg 1, \tag{1.13}$$

where C_3 is a positive constant. The analytic property of exotic solutions resides in a neighborhood of infinity, or, by the Kelvin transformation, on a neighborhood of the origin. Our emphasis on the whole $\mathbb{R}^n$ reflects the geometric viewpoint of conformal deformations of Euclidean space $(\mathbb{R}^n, g_o)$. We follow the convention of using c, C, C', $C_1, \ldots$ to denote positive constants, whose actual values may differ from section to section.

2. Delaunay–Fowler-type solutions

Introduce polar coordinates (r, θ) in $\mathbb{R}^n$, where $r = |x|$ and $\theta = x/|x|$ for $x \in \mathbb{R}^n \setminus \{0\}$. Let $t = \ln r$ for $r > 0$ and

$$v(t, \theta) = r^{(n-2)/2}u(r, \theta) \quad \text{for } r > 0 \text{ and } \theta \in S^{n-1}. \tag{2.1}$$

By the above transformation, Eq. (1.1) can be re-written as

$$\frac{\partial^2 v}{\partial t^2} + \Delta_\theta v - \frac{(n-2)^2}{4}v + \frac{n(n-2)}{4}\tilde{K}v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R} \times S^{n-1}. \tag{2.2}$$

Here Δ_θ is the Laplacian on the standard unit sphere in $\mathbb{R}^n$ and $\tilde{K}(t, \theta) := K(x)$, where $|x| = \mathrm{e}^t$ and $x/|x| = \theta$. For the case $\tilde{K} \equiv 1$ in $\mathbb{R} \times S^{n-1}$, consider radial solutions v of (2.2) and the ODE

$$v'' - \frac{(n-2)^2}{4}v + \frac{n(n-2)}{4}v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}. \tag{2.3}$$

In connection with the study of surfaces of revolution of constant curvature by Delaunay [11] and a class of semilinear differential equations by Fowler [13], positive smooth solutions of Eq. (2.3) are known as Delaunay–Fowler-type solutions. We refer to [16,24, 25] for basic properties of the solutions. Eq. (2.3) is autonomous and the Hamiltonian energy

$$H(v, v') = (v')^2 - \frac{(n-2)^2}{4}\left[v^2 - v^{2n/(n-2)}\right] \tag{2.4}$$

is constant along solutions of (2.3). For a positive smooth solution v of (2.3), H is a non-positive constant in the interval $[-[(n-2)/n]^{n/2}(n-2)/2, 0]$ (see [16]). By shifting the parameter, we may normalize the solution so that

$$v(0) = \max_{t\in\mathbb{R}} v(t). \tag{2.5}$$

Let v_o be a positive solution of Eq. (2.3) with $H = 0$. Under the normalization, we have

$$v_o(t) = (\cosh t)^{(2-n)/2} \quad \text{for } t \in \mathbb{R}. \tag{2.6}$$

We note that, by the transformation in (2.1), v_o corresponds to

$$u_o(x) = \left(\frac{2}{1+|x|^2}\right)^{(n-2)/2} \quad \text{for } x \in \mathbb{R}^n, \tag{2.7}$$

which is a solution of Eq. (1.1) when $K \equiv 1$ in $\mathbb{R}^n$. In particular, u_o is smooth near 0, which corresponds to $s \to -\infty$ for v_o. The other extreme is when $H = -[(n-2)/n]^{n/2}(n-2)/2$, and the corresponding solution v is a constant function given by $v(t) = [(n-2)/n]^{(n-2)/4}$ for $t \in \mathbb{R}$.

For $H \in (-[(n-2)/n]^{n/2}(n-2)/2, 0)$, the solution can be indexed by the parameter $\varepsilon = \min_{t\in\mathbb{R}} v(t)$, which is called the *neck-size* of the solution, or the Fowler parameter. We have $\varepsilon \in (0, [(n-2)/n]^{(n-2)/4})$ and

$$H = H(\varepsilon) = \frac{(n-2)^2}{4}\left[\varepsilon^{2n/(n-2)} - \varepsilon^2\right]. \tag{2.8}$$

Denote the normalized positive solution by v_ε, where $0 < \varepsilon < [(n-2)/n]^{(n-2)/4}$. It is known that v_ε is periodic with period T_ε. Moreover, we always have [16]

$$\varepsilon \leqslant v_\varepsilon(t) \leqslant v_\varepsilon(0) < 1 \quad \text{for } t \in \mathbb{R}. \tag{2.9}$$

The following result is essentially proved in [24] (cf. also [16]).

LEMMA 2.10. *T_ε, the period of v_ε, is monotone in ε for $\varepsilon \in (0, [(n-2)/n]^{(n-2)/4})$. We have $T_\varepsilon \to 2\pi/\sqrt{n-2}$ as $\varepsilon \to [(n-2)/n]^{(n-2)/4}$ and $T_\varepsilon \to \infty$ as $\varepsilon \to 0^+$. Furthermore, there exists a positive constant C, independent on ε, such that*

$$-\frac{4}{n-2}\ln(C\varepsilon) \leqslant T_\varepsilon \leqslant -\frac{4}{n-2}\ln\left(C^{-1}\varepsilon\right) \quad \textit{as } \varepsilon \to 0^+. \tag{2.11}$$

It is also known that v_ε converges uniformly in compact subsets of $\mathbb{R}$ to the constant solution as $\varepsilon \to [(n-2)/n]^{(n-2)/4}$, and to $v_o(t) = (\cosh t)^{(2-n)/2}$ as $\varepsilon \to 0^+$ [16]. For applications in Section 3 , we study the order of the latter convergence in more detail. As H is constant along solutions, we have

$$H(v_\varepsilon, v'_\varepsilon) = -\frac{(n-2)^2}{4}\left(\varepsilon^2 - \varepsilon^{2n/(n-2)}\right) = -\frac{(n-2)^2}{4}\left[v_\varepsilon^2(0) - v_\varepsilon^{2n/(n-2)}(0)\right]$$

for $\varepsilon \in (0, [(n-2)/n]^{(n-2)/4})$. Thus we obtain

$$v_\varepsilon^2(0)\left[1 - v_\varepsilon^{4/(n-2)}(0)\right] = \varepsilon^2\left(1 - \varepsilon^{4/(n-2)}\right) = -\frac{4H}{(n-2)^2}. \tag{2.12}$$

As $v_\varepsilon(0) > \varepsilon$ when $\varepsilon \to 0^+$, it follows from (2.12) that $v_\varepsilon(0) \to 1$ and $\varepsilon \to 0^+$. Furthermore,

$$1 - v_\varepsilon^{4/(n-2)}(0) = \mathrm{O}\left(\varepsilon^2\right).$$

We have

$$v_\varepsilon(0) = \left[1 + \mathrm{O}\left(\varepsilon^2\right)\right]^{(n-2)/4} = 1 + \mathrm{O}\left(\varepsilon^2\right) \quad \text{as } \varepsilon \to 0^+. \tag{2.13}$$

Hence there exists a positive constant C_n which depends on n only, such that

$$|v_\varepsilon(0) - 1| \leqslant C_n \varepsilon^2 \quad \text{for } \varepsilon > 0 \text{ small.} \tag{2.14}$$

We use the following well-known inequalities a number of times; they can be derived by simple integration methods. For positive constants c and $\alpha \geqslant 1$, we have

$$\left|x^\alpha - y^\alpha\right| \leqslant C|x - y| \quad \text{for } 0 \leqslant x, y \leqslant c, \tag{2.15}$$

where $C = C(\alpha, c)$ is a positive constant; moreover, for $\beta > 0$,

$$(1 + z)^\beta = 1 + \mathrm{O}(|z|) \quad \text{as } z \to 0. \tag{2.16}$$

With v_o given by (2.6), it follows from (2.9) and (2.15) that

$$\left|v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)\right| \leqslant c_n |v_\varepsilon(t) - v_o(t)|, \tag{2.17}$$

where c_n is a positive constant depending on n only. Using Eq. (2.3) we have

$$\begin{aligned} \left|v_\varepsilon''(t) - v_o''(t)\right| &\leqslant \frac{(n-2)^2}{4} |v_\varepsilon(t) - v_o(t)| + \frac{n(n-2)}{4} \left|v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)\right| \\ &\leqslant \left[\frac{(n-2)^2}{4} + \frac{n(n-2)}{4} c_n\right] \left|v_\varepsilon(t) - v_o(t)\right| \\ &= \bar{C}_n |v_\varepsilon(t) - v_o(t)|, \end{aligned} \tag{2.18}$$

where $\bar{C}_n$ is the positive constant defined in the formula. We claim that

$$\left|v_\varepsilon''(t) - v_o''(t)\right| \leqslant 2C_n \bar{C}_n \varepsilon^2 \quad \text{for } t \in [0, \rho], \tag{2.19}$$

where $\rho := 1/(2C_n \bar{C}_n)$. Here C_n and C_n' are the positive constants in (2.14) and (2.18), respectively. Without loss of generality, we may assume that $\rho < C_n$. By (2.14) and (2.18), the bound holds on a neighborhood of 0. Suppose that it holds on $[0, \sigma]$ for some positive number σ less than ρ. As $v_\varepsilon'(0) = v_o'(0) = 0$, we have

$$\left|v_\varepsilon'(t) - v_o'(t)\right| \leqslant 2C_n \bar{C}_n \varepsilon^2 \sigma \leqslant \varepsilon^2 \quad \text{for } t \in [0, \sigma].$$

Hence

$$\left|v_\varepsilon(t) - v_o(t)\right| \leqslant (C_n + \sigma)\varepsilon^2 < 2C_n \varepsilon^2 \quad \text{for } t \in [0, \sigma]. \tag{2.20}$$

By (2.18) we have

$$\left|v_\varepsilon''(\sigma) - v_o''(\sigma)\right| < 2C_n \bar{C}_n \varepsilon^2.$$

Using an connectedness argument, we obtain (2.19) as claimed. A similar bound holds in $[-\rho, 0]$. Upon integration we obtain the following lemma.

LEMMA 2.21. – *Let v_ε and v_o be the solutions of Eq.* (2.3) *discussed above. There exists positive constants ρ and C_o which depend on n but not on* (*small enough positive*) *ε, such that*

$$\left|v_\varepsilon(t) - v_o(t)\right| \leqslant C_o\varepsilon^2, \quad \left|v'_\varepsilon(t) - v'_o(t)\right| \leqslant C_o\varepsilon^2 \quad \text{and} \quad v_\varepsilon(t) \geqslant 1/2 \tag{2.22}$$

for $t \in [-\rho, \rho]$ and $\varepsilon > 0$ close to 0.

3. Gluing solutions

We follow the notations used in Section 2 and consider (2.1) and Eq. (2.2). Slow decay for a positive smooth solution u of equation (1.1) corresponds to $v(s,\theta) \leqslant C$ for $s \gg 1$, $\theta \in S^{n-1}$ and a positive constant C. Moreover, u is an (global) exotic solution if and only if there exists a sequence $\{(s_i, \theta_i)\} \subset \mathbb{R} \times S^{n-1}$ such that $\lim_{i\to\infty} s_i = \infty$ and $\lim_{i\to\infty} v(s_i, \theta_i) = 0$, and, when the variable t is changed into r via $t = \ln r$, u is smooth across the origin. Let ϕ_1 be a smooth function on $\mathbb{R}$ such that $0 \leqslant \phi \leqslant 1$ in $\mathbb{R}$ and

$$\phi_1(t) = \begin{cases} 1 & \text{for } t \leqslant -\rho, \\ 0 & \text{for } t \geqslant \rho. \end{cases}$$

We also require that

$$|\phi'_1(t)| \leqslant 2/\rho \quad \text{and} \quad |\phi''_1(t)| \leqslant 2/\rho^2 \quad \text{for } t \in (-\rho, \rho). \tag{3.1}$$

Let $\phi_2 = 1 - \phi_1$ in $\mathbb{R}$. Define

$$v = \phi_1 v_o + \phi_2 v_\varepsilon \quad \text{in } \mathbb{R}, \tag{3.2}$$

where $\varepsilon > 0$ is close to zero. It follows that

$$\begin{aligned} -v''(t) + \frac{(n-2)^2}{4} v(t) = {} & \frac{n(n-2)}{4}\left[\phi_1 v_o^{\frac{n+2}{n-2}}(t) + \phi_2 v_\varepsilon^{\frac{n+2}{n-2}}(t)\right] + \phi'_1(t)\left[v'_\varepsilon(t) - v'_o(t)\right] \\ & + \phi''_1(t)\left[v_\varepsilon(t) - v_o(t)\right] \end{aligned} \tag{3.3}$$

for $t \in \mathbb{R}$. We also have

$$\begin{aligned} & \phi_1(t) v_o^{\frac{n+2}{n-2}}(t) + \phi_2(t) v_\varepsilon^{\frac{n+2}{n-2}}(t) \\ & \quad = \phi_1(t) v_o^{\frac{n+2}{n-2}}(t) + \phi_2(t) v_o^{\frac{n+2}{n-2}}(t) + \phi_2(t)\left[v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)\right] \\ & \quad = \left[\phi_1(t) v_o(t) + \phi_2(t) v_o(t)\right]^{\frac{n+2}{n-2}} + \phi_2(t)\left[v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)\right] \\ & \quad = \left\{v(t) + \phi_2(t)\left[v_o(t) - v_\varepsilon(t)\right]\right\}^{\frac{n+2}{n-2}} + \phi_2(t)\left[v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t)\right] \end{aligned}$$

for $t \in [-\rho, \rho]$. We obtain

$$\left|\left[-v''(t) + \frac{(n-2)^2}{4} v(t)\right]\left[\frac{n(n-2)}{4} v^{\frac{n+2}{n-2}}(t)\right]^{-1} - 1\right|$$

$$\leqslant \left| \left\{ 1 + \frac{\phi_2(t)}{v(t)} \left[v_o(t) - v_\varepsilon(t) \right] \right\}^{\frac{n+2}{n-2}} - 1 \right| + \frac{4}{n(n-2)} v^{-\frac{n+2}{n-2}}(t) \Big\{ \phi_2(t) \left| v_\varepsilon^{\frac{n+2}{n-2}}(t) - v_o^{\frac{n+2}{n-2}}(t) \right| + |\phi_1'(t)| \left| v_\varepsilon'(t) - v_o'(t) \right| + |\phi_1''(t)| \left[v_\varepsilon(t) - v_o(t) \right] \Big\} \tag{3.4}$$

for $t \in [-\rho, \rho]$. It follows from Lemma 2.21, (2.16), (2.17), (3.1) and (3.4) that v satisfies the equation

$$v'' - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} K v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R}, \tag{3.5}$$

where K is a smooth function on $\mathbb{R}$ such that

$$|K(t) - 1| = \left| \left[-v''(t) + \frac{(n-2)^2}{4} v(t) \right] \left[\frac{n(n-2)}{4} v^{\frac{n+2}{n-2}}(t) \right]^{-1} - 1 \right| \leqslant C_1 \varepsilon^2 \tag{3.6}$$

for $t \in [-\rho, \rho]$, and $K \equiv 1$ in $\mathbb{R} \setminus [-\rho, \rho]$. Here C_1 is a positive constant that depends on n only, so far as $\varepsilon > 0$ is close to zero.

Let $\{\varepsilon_i\}$ be a decreasing sequence of small positive numbers such that $\lim_{i\to\infty} \varepsilon_i = 0$. Denote the period of v_{ε_i} by T_{ε_i} for $i = 1, 2, \ldots$ With ε_1 small enough, we may assume that $T_{\varepsilon_1} \gg \rho$. We construct a positive smooth function by first gluing v_o and v_{ε_1} on $[-\rho, \rho]$ as described above and call the resulting positive smooth function v_1. Note that $v_1 = v_{\varepsilon_1}$ in $\mathbb{R}^+ \setminus (0, \rho)$. As $v_{\varepsilon_1}(t + T_{\varepsilon_1}) = v_{\varepsilon_1}(t)$ for $t \in \mathbb{R}$ and v_{ε_1} and v_{ε_2} are close to v_o near $[-\rho, \rho]$, we let

$$\tilde{v}_{\varepsilon_2}(t) = v_{\varepsilon_2}(t - T_{\varepsilon_1}) \quad \text{for } t \in \mathbb{R},$$

and glue $\tilde{v}_{\varepsilon_2}$ and v_1 (that is, v_{ε_1}) on $[T_{\varepsilon_1} - \rho, T_{\varepsilon_1} + \rho]$ in a process similar to the one described above. Call the resulting function v_2. We continue to glue the solutions on the intervals

$$[T_{\varepsilon_1} + T_{\varepsilon_2} - \rho, T_{\varepsilon_1} + T_{\varepsilon_2} + \rho], \ldots, \left[\sum_{k=1}^{i} T_{\varepsilon_k} - \rho, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho \right], \ldots$$

by $v_{\varepsilon_3}, \ldots, v_{\varepsilon_{i+1}}, \ldots,$ respectively, after shifting appropriately. In particular, in the $(i+1)$th step, let

$$\tilde{v}_{\varepsilon_i}(t) = v_{\varepsilon_i}\left(t - \sum_{k=1}^{i-1} T_{\varepsilon_k} \right) \quad \text{and} \quad \tilde{v}_{\varepsilon_{i+1}}(t) = v_{\varepsilon_{i+1}}\left(t - \sum_{k=1}^{i} T_{\varepsilon_k} \right) \quad \text{for } t \in \mathbb{R},$$

and glue $\tilde{v}_{\varepsilon_{i+1}}$ with $\tilde{v}_{\varepsilon_i}$ on the interval $[\sum_{k=1}^{i} T_{\varepsilon_k} - \rho, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho]$. Finally we obtain a positive smooth function v on $\mathbb{R}$ which satisfies the equation

$$v'' - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} K v^{\frac{n+2}{n-2}} = 0 \quad \text{in } \mathbb{R} \tag{3.7}$$

for some smooth function K such that

$$|K(t)-1| \leqslant C_2\varepsilon_1^2 \quad \text{for } t \in \mathbb{R}, \tag{3.8}$$

where C_2 is a positive constant depending on n only. We may choose $\varepsilon_1 > 0$ as small as we like. We also have

$$v\left(\sum_{k=1}^{i} T_{\varepsilon_k} - T_{\varepsilon_i}/2\right) = v_i(T_{\varepsilon_i}/2) = \varepsilon_i \to 0 \quad \text{and}$$
$$v\left(\sum_{k=1}^{i} T_{\varepsilon_k}\right) \to 1^- \quad \text{as } i \to \infty. \tag{3.9}$$

As $v(t) = v_o(t)$ for $t \leqslant -\rho$, by (2.6) and (2.7), the corresponding solution u related to v by (2.1) is smooth across the origin. Thus v corresponds to an exotic solution u of Eq. (1.1) through (2.1).

Given a positive function $\varphi(r)$ defined for $r \gg 1$ which satisfies (1.12), let $\psi(t) = \varphi(\mathrm{e}^t)$. It follows that ψ is defined for $t \gg 1$ and

$$\mathrm{e}^{(n-2)t/2}\psi(t) \tag{3.10}$$

is non-decreasing for $t \gg 1$ and unbounded from above. Let

$$\varpi(t) = \ln\left[\mathrm{e}^{(n-2)t/2}\psi(t)\right] \quad \text{for } t \gg 1. \tag{3.11}$$

We have $\lim_{t\to\infty} \varpi(t) = \infty$. Choose a decreasing sequence of numbers $\{\varepsilon_i\}$ such that ε_1 is small enough and the corresponding periods T_{ε_i} of v_{ε_i} satisfy the relation

$$\varpi(T_{\varepsilon_i}) \geqslant \frac{n-2}{2}\sum_{k=1}^{i-1} T_{\varepsilon_k} \quad \text{for } i = 2, 3, \ldots. \tag{3.12}$$

By gluing the solutions v_o, v_{ε_i}, $i = 1, 2, \ldots,$ as described above, we obtain a positive smooth function v which satisfies Eq. (3.7) for a smooth function K. Suppose that

$$t \notin [-\rho, \rho] \cup [T_{\varepsilon_1} - \rho, T_{\varepsilon_1} + \rho] \cup \cdots \cup \left[\sum_{k=1}^{i} T_{\varepsilon_k} - \rho, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho\right] \cup \cdots,$$

then $K(t) = 1$. Suppose that

$$t \in \left[\sum_{k=1}^{i} T_{\varepsilon_k} - \rho, \sum_{k=1}^{i} T_{\varepsilon_k} + \rho\right] \quad \text{for some } i \in \mathbb{N}.$$

According to the construction above and Lemma 2.10, we have

$$|K(t)-1| \leqslant C_3{\varepsilon_i}^2 \leqslant C_4 \exp\left(-\frac{n-2}{2}T_{\varepsilon_i}\right)$$

$$= C_4 \exp\left(-\frac{n-2}{2} T_{\varepsilon_i} - \varpi(t) + \varpi(t)\right)$$

$$\leqslant C_3 \exp\left(-\frac{n-2}{2} \sum_{k=1}^{i} T_{\varepsilon_k}\right)\left[e^{(n-2)t/2}\psi(t)\right] \leqslant C_4 \exp\left(\frac{n-2}{2}\rho\right)\psi(t),$$

where C_3 and C_4 are positive constants that depend on n only. Hence we obtain $|K(t) - 1| \leqslant C_5\psi(t)$ for $t \gg 1$ and for a positive constant C_5. The corresponding solution u is an exotic solution of Eq. (1.1) which satisfies (1.13). We note that $K(t)$ in this case is not monotonic for large t.

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